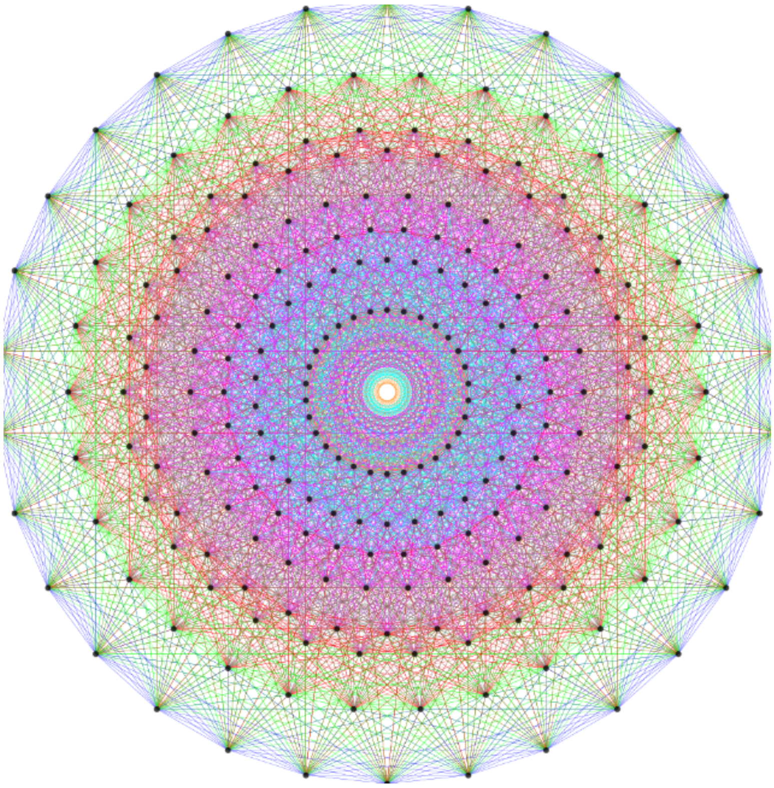

Notes: Sketchy live-tex'd notes, any errors are my own!

Automorphic Forms Beyond GL2

Arizona Winter School 2022 Lectures

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Remark 1.0.1: A fairy tale: see Langlands Elephant. The point of this talk is to see the entire elephant!

Remark 1.0.2: In the analogy between number fields and 3-manifolds, automorphic forms are on the number field side – what is the manifold analog? Some history:

- Mazur 63/64, in conversations with Artin: $\operatorname{Spec} \mathbb{Z}/p\mathbb{Z} \to \operatorname{Spec} \mathbb{Z}$ is like a knot in a (simply connected) 3-manifold.
- Weil 49: Weil conjectures, there should be an "algebraic" cohomology theory $H^*(-)$ for $X_{/k}$ for $k = \bar{k}$ where $H^*(X(\mathbb{C}))$ recovers singular cohomology.
  - Artin/Grothendieck: for finite coefficients, realized this using **étale cohomology** $H^*_{\text{ét}}(X)$.
- Tate 62, Poitou 61: looked at Galois cohomology, showed $H^*_{\text{ét}}(\operatorname{Spec} \mathbb{Z})$ (or other number rings) has a "Poincare duality" $H^i \rightleftharpoons H^{3-i}$, making it look like a 3-manifold. Recovers results in class field theory.

Example 1.0.3*(Varieties with automorphisms):* Let $X = V(x^3 + y^3 + z^3) \subseteq \mathbb{P}^2_{/\mathbb{C}}$; there are automorphisms

- $x \mapsto y, x \mapsto z$, etc
- $x \mapsto \bar{x}, y \mapsto \bar{y}$, etc
- $x, y, z \mapsto \sigma(x), \sigma(y), \sigma(z)$ for $\sigma \in S_3$

These should all act on $H^*(X)$ for any such H^*, but conjugation is quite discontinuous.

Example 1.0.4*(Manifolds vs schemes):* Let $X \in \mathsf{Mfd}$, how does one compute $H^*_{\text{sing}}(X; \mathbb{Z})$? Reduce to a linear-algebraic problem by triangulating and forming a chain complex of simplices. However, for $X \coloneqq \operatorname{Spec} R, R \coloneqq \mathbb{Z}[\frac{1}{2}]$,

$$H^1_{\text{ét}}(X; C_2) = R^\times / R^{\times}{}_{\square} = \{\pm 1, \pm 2\} \cong {C_2}^{\times^2}$$
$$H^2_{\text{ét}}(X; C_2) = \mathsf{QuatAlg}_{/R} = \left\{ \operatorname{Mat}_{2\times 2}(R), R[i,j,k] / \left\langle i^2 = j^2 = k^2 = -1 \right\rangle \right\}.$$

so H^2 classifies quaternion algebras over R. So computing this is very different to the case of manifolds! Also note that these are not dual on-the-nose, since H^1, H^2 have different orders.

Remark 1.0.5: Comparing duality for number rings vs 3-manifolds. For us, a number ring will be

- $\mathcal{O}_{K,S}$ for K a number field, where S is a finite number of primes we can invert (the **S-integers of K**), or
- functions on a smooth curve over $\mathbb{F}_q$.

For simplicity, we'll take $R = \mathbb{Z}\left[\frac{1}{p}\right], X = \operatorname{Spec} R$. For $M \in \mathsf{AbGrp}$ a p-torsion group (where we need the order to be invertible in R), we'll consider

$$H^i(X; M) := H^i_{\text{ét}}(\operatorname{Spec} \mathbb{Z}\left[\tfrac{1}{p}\right]; M).$$

There is a LES involving $M^\vee := \underset{\mathsf{Grp}}{\operatorname{Hom}}(M, S^1) = \underset{\mathsf{Grp}}{\operatorname{Hom}}(M, \mu_{p^\infty})$ where μ_{p^∞} are the p-power roots of 1:

$$\cdots \to H^{i-1}(\cdots) \to H^{3-i}(\mathbb{Z}\left[\tfrac{1}{p}\right]; M^\vee)^\vee \longrightarrow H^i(\mathbb{Z}\left[\tfrac{1}{p}\right]; M) \longrightarrow H^i(\mathbb{Q}_p; M) \to H^{i+1}(\cdots) \to \cdots$$

Link to Diagram

On the other hand, let $X \in \mathsf{Mfd}^3$ be a manifold with boundary, then there is a LES

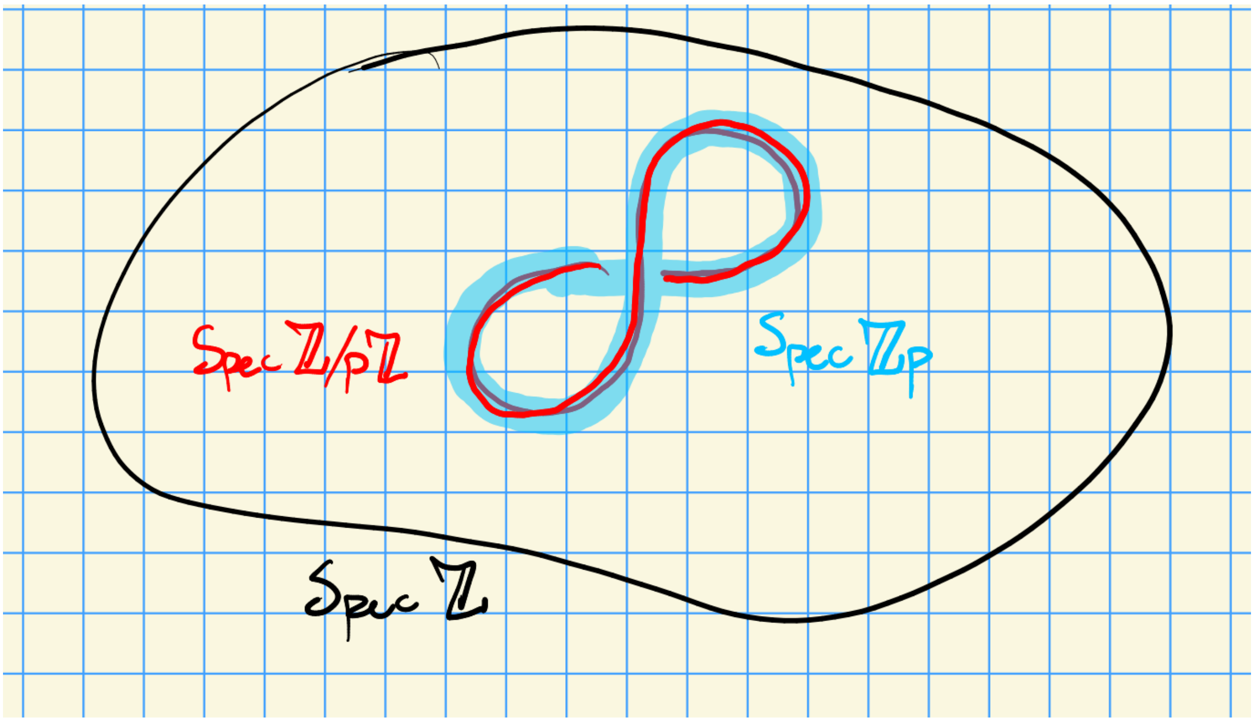
$$\cdots \to H^{i-1}(\cdots) \to H^i(X, \partial X; M) \cong H^{3-i}(X; M^\vee)^\vee \longrightarrow H^i(X; M) \longrightarrow H^i(\partial X; M) \to H^{i+1}(\cdots) \to \cdots$$

Link to Diagram

So $X \approx \mathbb{Z}\left[\frac{1}{p}\right]$ is a (nonorientable!) 3-manifold in this analogy, with $\partial X \approx \mathbb{Q}_p$, which is now like a 2-manifold. The nonorientable assumption here is related to the need to twist the Galois actions on the scheme side. How to realize this: delete a tubular neighborhood of the knot, then

- $\partial X = \partial \nu(\operatorname{Spec} \mathbb{Z}/p\mathbb{Z}) \approx \mathbb{Q}_p$.
- $\operatorname{Spec} \mathbb{Z}_p \approx \nu$,

- $\partial \operatorname{Spec} \mathbb{Z}_p \approx \partial \operatorname{Spec} \mathbb{Z}\left[\frac{1}{p}\right] \sim \operatorname{Spec} \mathbb{Q}_p$.



Remark 1.0.6:

- The 3-dimensional objects: $\operatorname{Spec} R$ for $R = \mathbb{Z}, \mathbb{Z}\left[\frac{1}{p}\right], \mathbb{Z}[\sqrt{2}]$, or $\mathbb{F}_p(t)$, $\mathbb{Z}_p$, or projective smooth curves over $\mathbb{F}_p$.
- The 2-dimensional objects: $\mathbb{Q}_p, \mathbb{F}_p((t))$, or projective smooth curves over $\overline{\mathbb{F}}_p$.

Remark 1.0.7: Relating to automorphic forms: for $G = \mathrm{SL}_2$,

- For $\mathbb{Z}$, we study the vector space $\mathcal{A}_{\mathbb{Z}} = \left\{ \text{functions on } G(\mathbb{Z}) \backslash G(\mathbb{R}) \right\}$.
- For $\mathbb{Z}\left[\frac{1}{p}\right]$, we instead look at $\mathcal{A}_{\mathbb{Z}\left[\frac{1}{p}\right]} = \left\{ \text{functions on } G(\mathbb{Z}\left[\frac{1}{p}\right]) \backslash G(\mathbb{R}) \times G(\mathbb{Q}_p) \right\}$.

We would like some association that works similarly:

$$\begin{aligned} \mathsf{Mfd}^3 &\to \mathsf{Vect}_{/k} \\ M &\mapsto \mathcal{A}_M. \end{aligned}$$

Example 1.0.8 ***(?):*** A non-example is $M \mapsto H^*_{\text{sing}}(M; \mathbb{C})$, which behaves nothing like $\mathbb{Z} \mapsto \mathcal{A}_{\mathbb{Z}}$. E.g. it has the wrong type of functoriality: the map $\mathbb{Z} \to \mathbb{Z}[\sqrt{2}]$ is like a branched double cover, but for manifolds there are maps both ways and here it is difficult to find wrong-way maps. Moreover the corresponding spaces of automorphic forms $\mathcal{A}_{\mathbb{Z}}, \mathcal{A}_{\mathbb{Z}\left[\frac{1}{p}\right]}$ differ by far more than just a part coming from $\mathbb{Q}_p$.

Note also that $H^*(M \cup N) \cong H^*(M) \oplus H^*(N)$, but for automorphic forms we have $\mathcal{A}_{\mathbb{Z}\oplus\mathbb{Z}} \cong \mathcal{A}_{\mathbb{Z}} \otimes \mathcal{A}_{\mathbb{Z}}$, where the source is like a degenerate quadratic extension.

Remark 1.0.9: A heavily studied piece of the analogy:

$$\text{Automorphic forms} \rightleftharpoons \mathsf{TQFT},$$

coming from the Kapustin-Witten 2006, where a (4-dimensional) TQFT is a essentially a monoidal functor:

$$\mathsf{TQFT}^3 = [(\mathsf{Bord}^3, \coprod) \to (\mathsf{Vect}_{/k}, \otimes_k)].$$

.

See Atiyah TQFT, section 2.

2 | Closing: Akshay Venkatesh

Remark 2.0.1: Automorphic forms using TFQTs as a metaphor. We'll consider $\mathsf{TQFT}_4 = [(\mathsf{Bord}^3, \coprod), (\mathsf{Vect}_{/\mathbb{C}}, \otimes_{\mathbb{C}})]$ where Bord^3 is the category whose objects are 3-manifolds and morphisms $M \to N$ are 4-manifolds W with $\partial W = M \coprod N$. These are meant to extract invariants of 4-manifolds that are amenable to cut-and-paste arguments. The correspondences:

- Manifolds $M \rightsquigarrow$ vector spaces A_M,
- Bordisms $(M \to N) \rightsquigarrow$ linear maps $A_M \to A_N$,
- A 4-manifold Z with boundary $M \rightsquigarrow$ a vector $v_Z \in A_M$
- A 4-manifold Z with empty boundary $\rightsquigarrow \lambda_Z \in \mathbb{C}$
- A decomposition X without boundary into $M \coprod_Z N \rightsquigarrow$ vector $\ell \in A_M, r \in {A_M}^\vee$, where $\langle \ell,\ r \rangle = \lambda_Z$.

Example 2.0.2 ***(A*** TQFT_2 ***due to Dijkgraaf-Witten):*** Fix $G \in \mathsf{FinGrp}$, and a correspondence

- $S^1 \rightsquigarrow \mathbb{C}[G]^{\text{conj}}$, conjugacy-invariant functions in the group algebra,
- Pants $\rightsquigarrow$ multiplication
- Σ_g a genus g surface $\rightsquigarrow \dfrac{1}{k!}N$ where N is the number of ways to write $e \in G$ as a product of g commutators.

Remark 2.0.3: An informal definition of **extended TQFTs**, in particular TQFT_4:

- 4-manifolds $\rightsquigarrow \mathbb{C}$
- 3-manifolds $\rightsquigarrow \mathsf{Vect}_{/\mathbb{C}}$
- 2-manifolds $\rightsquigarrow$ categories enriched over $\mathsf{Vect}_{/\mathbb{C}}$

This should yield

- 3-manifolds without boundary $\rightsquigarrow A_S$ objects in C
- $X = M\coprod_Z N \rightsquigarrow \mathrm{Hom}(A_M, A_N)$

Remark 2.0.4: Last time we said $\operatorname{Spec}\mathbb{Z}\left[\frac{1}{p}\right]$ is like a 3-manifold with boundary $\mathbb{Q}_p$, which is like a 2-manifold. A philosophy is to put all places on the same footing – note that we haven't included the places at p and ∞ here, so really we should have $\partial\operatorname{Spec}\mathbb{Z}\left[\frac{1}{p}\right] = \operatorname{Spec}\mathbb{R}, \operatorname{Spec}\mathbb{Q}_p$, and $\partial\operatorname{Spec}\mathbb{Z} = \operatorname{Spec}\mathbb{R}$. So our new picture should be:

(todo finish)

2.1 Automorphic Forms as TQFT_4

Remark 2.1.1: What should the automorphic form correspondence be in this analogy? Our 3-dimensional objects:

- $\mathbb{Z} \rightsquigarrow \mathcal{A}_\mathbb{Z}$ functions on $G_\mathbb{Z}\diagdown G_\mathbb{R}$,
- $\mathbb{Z}\left[\frac{1}{p}\right] \rightsquigarrow \mathcal{A}_{\mathbb{Z}\left[\frac{1}{p}\right]}$ functions on $?\diagdown {G_\mathbb{R}}^{\times 2}$,
- $X_{/\mathbb{F}_p}$ a smooth projective curve $\rightsquigarrow$ functions on $\mathsf{Bun}_G(X)$,
- $\mathbb{Z}_p \rightsquigarrow$ something more difficult.

The 2-dimensional objects:

- $\mathbb{Q}_p \rightsquigarrow \mathsf{Rep}G(\mathbb{Q}_p)$,
- $\mathbb{R} \rightsquigarrow \mathsf{Rep}G(\mathbb{R})$,
- $X_{/\bar{\mathbb{F}}_p}$ a smooth projective curve $\rightsquigarrow \mathsf{Sh}(\mathsf{Bun}_G(X))$.

Really for a TQFT, we should assign something to objects in the category of its boundary – here e.g. the vector space $\mathcal{A}_\mathbb{Z}$ is an object in $\mathsf{Rep}G(\mathbb{Q}_p)$. Idea: functions on Bun_G are hard to deal with, e.g. Hecke operators turn into infinite sums. Make things robust to passing to algebraic closures by passing from functions to sheaves!

Remark 2.1.2: Last time: thinking of $\operatorname{Spec}\mathbb{Z}$ as a 3-manifold,

$$\operatorname{Spec}\mathbb{Z} = \operatorname{Spec}\mathbb{Z}_p \coprod_{\operatorname{Spec}\mathbb{Q}_p} \operatorname{Spec}\mathbb{Z}\left[\tfrac{1}{p}\right] \approx M^3\coprod_{Z^2} N^3.$$

We want the following:

$$\operatorname*{Hom}_{G(\mathbb{Q}_p)}(\mathcal{A}_{\mathbb{Z}_p}, \mathcal{A}_{\mathbb{Z}\left[\frac{1}{p}\right]}) =_? \mathcal{A}_\mathbb{Z}.$$

We regard $\mathcal{A}_\mathbb{Z}$ as elements of $\mathcal{A}_{\mathbb{Z}\left[\frac{1}{p}\right]}$ which are also unramified at p, and by Frobenius reciprocity this should yield

$$\mathcal{A}_{\mathbb{Z}} = \operatorname*{Hom}_{G(\mathbb{Q}_p)} \left(\left\{ \text{Functions on } G(\mathbb{Q}_p) \diagup G(\mathbb{Z}_p), \mathcal{A}_{\mathbb{Z}\left[\frac{1}{p}\right]} \right\} \right),$$

which should encode a Hecke algebra at p.

Question 2.1.3
What is the Langlands correspondence in this language? What should an "arithmetic TQFT" be?

Remark 2.1.4: Let O be a category of arithmetic ring and A_O be a category or vector space, and call an associated $O \to A_O$ an *arithmetic field theory.* Let $X_{/\mathbb{F}_p}$ be a smooth projective curve and $G = \mathrm{GL}_n$. The Langlands correspondence here (due to Drinfeld and Lafforgue) yields

$$\left\{ \begin{matrix} \text{Cuspidal functions on } n\text{-dimensional} \\ \text{vector bundles on } X \end{matrix} \right\} \rightleftharpoons \left\{ \begin{matrix} \text{Functions on } n\text{-dimensional} \\ \text{irreducible Galois reps} \end{matrix} \right\}$$
$$T_x \curvearrowright x \rightleftharpoons \mathrm{Frob}_x \curvearrowright ?$$

We regard the LHS A as "automorphic forms".

This suggests the following viewpoint on the Langlands correspondence: we are only seeing one level, and there is a *second* arithmetic field theory $B^{(G^\vee)}$ built out of Galois representations of the Langlands dual $G^\vee$, so $\mathbb{Z}$ yields a vector space and $\mathbb{Q}_p$ yields a category, and an equivalence of arithmetic field theories $A^{(G)} \rightleftharpoons B^{(G^\vee)}$. Often B is a category of coherent sheaves. This should package local, global, and geometric Langlands into a single theory!

Remark 2.1.5: The abstract correspondence between automorphic forms and Galois reps isn't so useful; the real utility comes from matching structures and numerical invariants on both sides, e.g. Fourier coefficients, Rankin-Selberg or doubling integrals, the Θ correspondence, etc which all match with something on the Galois side (usually an L-function). This yields a panoply of matching invariants! E.g. for $E/\mathbb{Q}$ an elliptic curve,

$$L(\operatorname{Sym}^2 E, 1) = \prod_p \frac{p^2}{(1 - 1/p)\sharp E(\mathbb{F}_{p^2})} \in \pi \cdot \mathrm{Area}(E_{\mathbb{C}})\mathbb{Q},$$

where the area is of the fundamental parallelogram of E, which is hard to prove without automorphic forms. How can we interpret this in terms of TQFTs?

Remark 2.1.6: Consider numerical invariants of automorphic forms and Galois reps landing in $\mathbb{C}$. Let $\mathcal{O}$ be a 3-dimensional ring of integers over X.

The numerical invariants of Galois reps should be elements of $B_{\mathcal{O}}^{(G^\vee)}$, and numerical invariants of automorphic forms should come from $A_{\mathcal{O}}^{(G)}$ where given P, one takes $\varphi \mapsto \langle P,\ \varphi \rangle$. To find matching invariants, we want to match elements in $A_{\mathcal{O}}$ to elements in $B_{\mathcal{O}}$. More ambitiously, we can ask for matching *boundary conditions* in $A^{(G)}$ and $B^{(G^\vee)}$.

Definition 2.1.7 (Boundary conditions, informal definition)
A **boundary condition** in TQFT_4 is a coherent assignment:

- 3-manifolds $M \rightsquigarrow v \in A_M$ a distinguished vector
- 2-manifolds $S \rightsquigarrow X_S$ a distinguished object in A_S

See Kasputin's 2010 ICM address for a nice overview.

Remark 2.1.8: Joint work with David Ben-Zvi, Sakellaridis, an informal summary:

- A variety G-variety gives a boundary condition for both $A^{(G)}$ and $B^{(G^\vee)}$,
- For suitable choice of Y, this recovers familiar invariants of automorphic forms mentioned above,
- On the Galois side this recovers L-functions,
- There is a proposed specific class of dual pairs $(G, Y) \rightleftharpoons (G^\vee, Y^\vee)$ which give matching/dual invariants. E.g. each periodic integral should have a dual.

Question 2.1.9
For the next generation of number theorists: why are there such similarities between TQFTs and automorphic forms? This is something deep that we barely understand at all.

Remark 2.1.10: Extending to 1-dimensional objects: these should be 2-categories which are categorical reps of a loop group.

3 | Ellen Eischen, Automorphic Forms on Unitary Groups, Talk 1

Remark 3.0.1: Overall plan:

- Introduce automorphic forms on unitary groups,
- Techniques to study *algebraic* aspects of L-functions,
- Using unitary groups as a convenient setting – a large enough class of groups to be interesting, but confined enough to be tractable.

Today:

- Motivations from modular forms,
- Fundamental definitions.

If the previous talk was a "fairy tale", this will flesh out the "based on a true story" part!

3.1 Motivation from modular forms

Example 3.1.1 *(?):* Consider $\zeta(2k)$ for $k \in \mathbb{Z}_{\geq 0}$; known to Euler as

$$\zeta(2k) = (-1)^k \pi^{2k} \frac{2^{2k-1}}{(2k-1)!} \left(-\frac{B_{2k}}{2k} \right) = (-1)^k \pi^{2k} \frac{2^{2k-1}}{(2k-1)!} \zeta(1-2k),$$

where B_{2k} is the $2k$th **Bernoulli number**, whose exponential generating function is

$$\frac{ze^z}{e^z - 1} = \sum_{k \geq 0} B_k \frac{z^k}{k!} \in \mathbb{Q}[[z]].$$

Proving rationality of $\zeta(2k)$ (up to powers π^n) involves the normalized Eisenstein series

$$G_{2k}(q) = \zeta(1-2k) + 2 \sum_{k \geq 1} \sigma_{2k-1}(n) q^n, \quad q := e^{2\pi i z}, \quad \sigma_k(n) := \sum_{d \mid n} d^k,$$

and one can use similar techniques to prove rationality for

- Dedekind zeta functions for $K \in \mathsf{Field}_{/\mathbb{Q}}$ totally real:

  $$\zeta_K(s) = \sum_{\mathfrak{a} \trianglelefteq \mathcal{O}_K} \frac{1}{N(\mathfrak{a})^s}, \qquad N(I) := \mathrm{Nm}_{K/\mathbb{Q}}(I),$$

  where rationality was proved by realizing it as the constant term of a Fourier expansion of an Eisenstein series and studying spaces of modular forms.

- L-functions $L(\chi, s)$ for χ a **Hecke character** of a totally real field, and their p-adic analogs.

All of these correspond to **Artin L-functions** $L(s, \rho)$ for ρ a Galois representation. Dimensions $n = 1$ and (partially) $n = 2$ are handled class field theory. Can we generally show special values are algebraic? And if so, what do these values mean?

Remark 3.1.2: More generally, one can ask about algebraicity or rationality of special values of L-functions attached to modular forms. Our first tool for constructing such things will be **Rankin-Selberg convolution**. Why care about special values: Kummer used congruences for ζ, checking if $p \mid \mathrm{cl}(K)$ is equivalent to checking if p divides numerators of Bernoulli numbers, which can be used to prove special cases of Fermat. Picked up later for Iwasawa theory, controls behavior of towers of towers of cyclotomic extensions in $\mathsf{G_K}$-Mod.

Remark 3.1.3: Conjectures

- Meanings of L-function values, e.g. Deligne's conjecture that they come from **motives**.
- Langlands: connections between Galois reps ρ and automorphic reps.

Warning 3.1.4
It might seem like GL_n for $n \geq 3$ is the next step, but this turns out to be too general! Even SL_n in these ranges is difficult. Instead we'll move to **unitary groups**, where we'll have Shimura varieties to work with.

3.2 Unitary groups

Remark 3.2.1: Fix $K \in \mathsf{CMField}$, so $K/K^+/\mathbb{Q}$ with $K^+/\mathbb{Q}$ totally real and K/K^+ quadratic imaginary, and $V \in \mathsf{Vect}_{/K}$ with a nondegenerate Hermitian pairing $\langle -,\ - \rangle$, which can be extended linearly to $V_R := V \otimes_{K^+} R$ for any $R \in \mathsf{Alg}_{/K^+}$.

Definition 3.2.2 (General Unitary Groups)
The **general unitary group** is the algebraic group $G := \mathrm{GU}(V, \langle -,\ - \rangle)$ which is defined for each $R \in \mathsf{Alg}_{/K^+}$ as

$$R \mapsto \left\{ g \in \mathrm{GL}_{K_R}(V_R) \;\middle|\; \langle gv,\ gw \rangle = \nu \langle v,\ w \rangle \text{ for some } \nu \in R \right\}.$$

The **unitary group** is the subgroup for which $\nu = 1$ is enforced.

Remark 3.2.3: If $R = \mathbb{R}$, choose an ordered basis for B to define the **signature**

$$\langle v,\ w \rangle = vA^t(w), \qquad A = \begin{bmatrix} \mathbb{1}_a & 0 \\ 0 & -\mathbb{1}_b \end{bmatrix}, \qquad \mathrm{sig}(A) := (a, b).$$

For the remainder of today, assume $K^+ = \mathbb{Q}$.

3.3 Automorphic forms on unitary groups, connections to modular forms

Remark 3.3.1: On the modular form side:

1. $f : \mathfrak{h} \to \mathbb{C}, f(z) = (cz + d)^{-k} f(\gamma z)$, holomorphic at cusps, etc
2.
$$\varphi_f : \mathrm{SL}_2(\mathbb{R}) \to \mathbb{C}, \mathrm{SL}_2(\mathbb{R}) \curvearrowright \mathfrak{h}.$$

   transitively fixing i,

$$\varphi_f(g) = j(g, i)^{-k} f(gi),$$

and

$$\varphi_f : \Gamma \backslash G(\mathbb{R}) \to \mathbb{C}$$

$$\varphi_f(g(\mathrm{rot}(\theta)) = e^{ki\theta}\varphi_f(g) \qquad \mathrm{rot}(\theta) := \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix},$$

extend to

$$\varphi : \Gamma Z(G) \backslash G(\mathbb{R}) \to \mathbb{C}$$

for $G = \mathrm{GL}_2, \mathrm{SL}_2, \mathrm{GL}_2^+$, etc

3. Adelic interpretation: $\mathrm{GL}_2(\mathbb{A}) = \mathrm{GL}_2(\mathbb{Q})\,\mathrm{GL}_2^+(\mathbb{R})\tilde{K}$ where $\tilde{K} := \prod_p K_p$is a compact open subgroups of $\mathrm{GL}_2(\mathbb{Q}_p)$ with determinant ${\mathbb{Z}_p}^\times$ and equal to $\mathrm{GL}_2(\mathbb{Q}_p)$ for all but finitely many places.

- Can recover $\Gamma := \mathrm{GL}_2(\mathbb{Q}) \cap (\mathrm{GL}_2^+(\mathbb{R}) \times \mathcal{K})$
- Match up

$$\Gamma \backslash \mathrm{GL}_2(\mathbb{R}) \rightleftharpoons \mathrm{GL}_2(\mathbb{Q}) \backslash \mathrm{GL}_2(\mathbb{A}) / \mathcal{K}.$$

- Get functions

$$\varphi_f : \mathrm{GL}_2(\mathbb{Q}) \backslash \mathrm{GL}_2(\mathbb{A}) \to \mathbb{C}.$$

On the automorphic side:

1. Replace $\mathfrak{h}$ with $G/\mathcal{K}_\infty = \mathrm{U}_{n,m}(\mathbb{R})/{\mathrm{U}_n}^{\times 2}$, a quotient by a compact.
2. Writing $G := \mathrm{GU}(n, m)$ for a form of signature (n, m), replace with $\Gamma Z(G) \backslash G$ where $G \supseteq \mathcal{K}_\infty := U(n)^{\times 2}$, and analogously $\Gamma Z(G) \backslash G(\mathbb{R}) \to \mathbb{C}$.
3. For $G(\mathbb{A}_f) = \coprod_i G(\mathbb{R})^{-1} ? \mathcal{K}$.

Remark 3.3.2: An **automorphic form** on $\mathrm{U}_{n,n}$ is a holomorphic function $f \in \mathfrak{h}_n \to V$ where $\rho \curvearrowright V$ is a representation of $\mathrm{GL}_n(\mathbb{C})^{\times 2}$ where

$$f(z) = \rho(cz + d, {}^t(\bar{c})z + \bar{d})^{-1} f(\gamma z) \qquad \forall \gamma \in \Gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{U}_{n,n}(\mathcal{O}_K)$$

where

$$\gamma z = (az + b)(cz + d)^{-1}, \qquad \mathfrak{h}_n := \left\{ z \in \mathrm{Mat}_n(\mathbb{C}) \;\middle|\; i({}^t(\bar{z}) - z) > 0 \right\}.$$

Remark 3.3.3: One thing we haven't mentioned yet: modular forms as sections of line bundles over modular curves, so moduli of elliptic curves with level structure, and the generalized setup will be vector bundles over (unitary) Shimura varieties.

4 | Ellen Eischen, Talk 2

Remark 4.0.1: Today: more on automorphic forms, and approaches to studying $L(s, \pi)$ for π a cuspidal representation of a unitary group. Note that we've been taking the adelic approach, see Wee Tek's talk for how to define L-functions in this setting.

Remark 4.0.2: Several perspectives on automorphic forms on unitary groups:

- Functions on generalizations of $\mathfrak{h}$,
- Functions on $G(\mathbb{R})$ and $G(\mathbb{A})$ for $G = \mathrm{U}_{a,b}$ a unitary group,

Today: sections of a vector bundle over certain moduli spaces.

See Shimura's first paper on Rankin-Selberg convolutions! See also two papers by Siegel's student that define the generalizations $\mathfrak{h}_n$.

4.1 Modular Forms

Definition 4.1.1 (Modular form, geometric definition)
Let $\mathcal{M}$ be a modular curve (parameterizing curves with some level structure) and let $\xi \to \mathcal{M}$ be the universal elliptic curve. Write $\Omega^1_{\xi/\mathcal{M}}$ for the relative differentials, and define

$$\omega := \pi_* \Omega^1_{\xi/\mathcal{M}}.$$

A **modular form** is section of a tensor power of ω, i.e. an element of $H^0(\mathcal{M}; \omega^{\otimes^k})$.

Remark 4.1.2: We can regard a modular form as a rule $(E, \omega) \mapsto F(E, \omega) \in \mathbb{C}$ that transforms like

$$F(E, \lambda\omega) = \lambda^{-k} F(E, \omega) \qquad \forall \lambda \in \mathbb{C}^\times.$$

Equivalently, a rule $\tilde{F}$ that maps E to some $\omega \in \Omega_{E/\mathbb{C}}$, e.g.

$$\tilde{F}(E) = F(E, \omega)\omega^{\otimes^k}.$$

Connecting this with last time:

$$\begin{aligned} (E, \omega) &\rightleftharpoons \Lambda_{(E,\omega)} \rightleftharpoons \mathbb{Z} + \tau\mathbb{Z} \\ F(E, \omega) &\longrightarrow \cdots \longrightarrow f_F(\tau), \end{aligned}$$

i.e. such a rule can be regarded as a function on lattices.

Remark 4.1.3: Similarly, automorphic forms arise as global sections of a vector bundle over a unitary Shimura variety $\mathcal{M}$ parameterizing abelian varieties with

- A polarization,
- An endomorphism, and
- A level structure.

One can similarly identify

$$\mathcal{M}(\mathbb{C}) \cong G(\mathbb{Q})\backslash G(\mathbb{A}) / \mathcal{K} \cdot \mathcal{K}_\infty,$$

which will be a finite disjoint unions of copies of symmetric spaces (e.g. $\mathfrak{h}_n$) for $\mathrm{U}_{a,b}$.

Remark 4.1.4: Write $\underline{A}$ for an abelian variety with some extra structure, one can then also view an automorphic form as a function

$$F(\underline{A}, \ell) = \rho(tg)^{-1} F(\underline{A}, \ell) \qquad \forall G \in \mathrm{GL}_a \times \mathrm{GL}_b,$$

where $\ell = (\ell_+, \ell_-)$ is an ordered basis for $\Omega_{A/\mathbb{C}}$ that decomposes according to the signature.

Let $\mathcal{A} \xrightarrow{\pi} \mathcal{M}$ be the universal family and define the sheaf $\omega := \pi_* \Omega_{\mathcal{A}/\mathcal{M}}$; one can then build a sheaf of automorphic forms ω^p in much the same way. This reformulates the notion of an automorphic form in terms of lattices and functions on symmetric spaces like $\mathfrak{h}_n$.

4.2 ?

Remark 4.2.1: Goal for today: introduce an approach to studying certain L-functions using the *doubling method.* Note that the example of looking at the constant term of an Eisenstein series from yesterday turns out to be deceptively simple, hence a different approach today.

Example 4.2.2 ***(Motivating example):*** Let - $f(q) = \sum_{k \geq 1} a_k q^k$ be a weight k cusp form and - $g(q) = \sum_{k \geq 0} b_k q^k$ be a weight ℓ modular form,

where $a_k, b_k \in \overline{\mathbb{Q}}$. The **Rankin-Selberg product** is

$$D(s, f, g) = \sum_{n \geq 1} a_n b_n n^{-s}.$$

Shimura proved that

$$\frac{D(m, f, g)}{\langle f,\ f \rangle_{\mathrm{Pet}}} \in \pi^k \overline{\mathbb{Q}} \qquad \text{for } \ell < k, \ \frac{k + \ell - 2}{2} < m < k.$$

This prove relies on the realization

$$D(k-1-r, f, g) = c\pi^k \left\langle \tilde{f},\ g\delta_\lambda^{(r)} E \right\rangle_{\text{Pet}},$$

where

- E is an Eisenstein series of weight $\lambda = k - \ell - 2r$
- δ is a Maass-Shimura differential operator that raises weights by $2r$,
- $\tilde{f}$ is f with conjugate Fourier coefficients, so $\tilde{f}(q) := \sum_{k\geq 0} \overline{a_k} q^k$
- $\langle f,\ g\rangle_{\text{Pet}}$ is the Petersson pairing: for F a fundamental domain

$$\langle f,\ g\rangle_{\text{Pet}} := \int_F \bar{f}(z) g(z)\, y^{l-2}\, dx\, dy.$$

Note that the weight-raising operator doesn't preserve holomorphicity, which can be bad for algebraicity results, but it turns out that the result is "almost holomorphic".

4.3 Proving Algebraicity: A Recipe

Remark 4.3.1: A general strategy:

- Find a Petersson-style pairing of automorphic forms, e.g. integrating against an Eisenstein series, which looks like an L-function:
  - Factors into an Euler product
  - Has a functional equation
  - Can be meromorphically continued to $\mathbb{C}$
- Prove appropriate rationality results for E, e.g. the higher order Fourier coefficients are rational, or E transforms nicely under a differential operator. Fourier coefficients are almost always important in this step.
- Express a familiar automorphic L-function in terms of this pairing.

Note that each step is highly nontrivial, and in some contexts, some steps haven't even been completed yet. The third step often involves working one place at a time. Even having all three may not be enough, sometimes the results one gets aren't amenable to algebraic/geometric study and are instead only good for analytic purposes.

4.4 Doubling

Remark 4.4.1: Setup:

- $K \in \mathsf{Field}_{/\mathbb{Q}}$ a quadratic extension
- $V \in \mathsf{Vect}_{/K}^{\dim=n}$ with a nondegenerate Hermitian pairing $\langle -,\ - \rangle_V$
- $G = \mathrm{U}(V, \langle -,\ - \rangle_V)$
- $W = V^{\oplus^2}$ with the induced Hermitian pairing

$$\langle (u,v),\ (u',v') \rangle_W := \langle u,\ u' \rangle_V - \langle v,\ v' \rangle_V.$$

- A *doubled* group $H := \mathrm{U}(W, \langle -,\ - \rangle_W)$
- An embedding

$$\mathrm{U}(V, \langle -,\ - \rangle) \times \mathrm{U}(V, -1 \cdot \langle -,\ - \rangle) \hookrightarrow H,$$

which in terms of signatures is $\mathrm{sig}(a,b) \times \mathrm{sig}(b,a) \to \mathrm{sig}(a+b, a+b)$

Remark 4.4.2: Next time: we'll introduce the doubling integral after pairing with an Eisenstein series.

5 Ellen Eischen, Talk 3

Remark 5.0.1: Some references on doubling:

- PIatestski-Shapiro, Rallis, L-functions for classical groups
- Harris, Shimura varieties for unitary groups and the doubling method
- Garrot, Pullbacks of Eisenstein series

5.1 Eisenstein Series

Remark 5.1.1: We'll continue with the previous setup. Let $P \leq H$ be parabolic preserving $V^\Delta := \left\{ (v,v) \;\middle|\; v \in V \right\}$. We get a decomposition $W = V^\Delta \oplus V_\Delta$ where $V_\Delta = \left\{ (v,-v) \;\middle|\; v \in V \right\}$. Then

$$P = \left\{ \begin{bmatrix} A & * \\ 0 & {}^t(\overline{A}^{-1}) \end{bmatrix} \;\middle|\; A \in \mathrm{GL}_n \right\}.$$

Given a Hecke character

$$\chi : K^\times \backslash \mathbb{A}_K^\times \to \mathbb{C},$$

view this as a character of P via

$$P \to \mathrm{GL}_n \xrightarrow{\det} \mathbb{A}_K^\times \to \mathbb{C}$$

$$\begin{bmatrix} A & * \\ 0 & {}^t(\overline{A}^{-1}) \end{bmatrix} \mapsto A \mapsto \det(A) \xrightarrow{\chi} \chi(\det(A)).$$

Definition 5.1.2 (Siegel Eisenstein series)
For $s \in \mathbb{C}$, let

$$f_{S,\chi} = \operatorname{Ind}_{P(\mathbb{A})}^{H(\mathbb{A})}(\chi \cdot |-|^{-s}) = \left\{ f : H(\mathbb{A}) \to \mathbb{C} \;\middle|\; f(ph) = \chi(p)|p|^{-s} f(h) \right\}.$$

Define the **Siegel Eisenstein series** for $g \in H$ by

$$E_{F_{s,\chi}}(g) = \sum_{\gamma \in P(\mathbb{Q}) \diagdown H(\mathbb{Q})} f_{s,\chi}(\gamma g).$$

5.2 Doubling Integral

Remark 5.2.1: Setup:

- π a cuspidal automorphic representation of G
- $\tilde{\pi}$ the contragradient (dual) representation of π
- $\varphi \in \pi$
- $\tilde{\varphi} \in \tilde{\pi}$

Define a zeta integral

$$Z(\varphi, \tilde{\varphi}, f_{s,\chi}) := \int_{(G \times G)(\mathbb{Q}) \diagdown (G \times G)(\mathbb{A})} E_{f_{s,\chi}}(g_1, g_2) \varphi(g_1) \tilde{\varphi}(g_2) \chi^{-1}(\det g_2)\, dg_1\, dg_2,$$

which is against an appropriately normalized Haar measure.

Remark 5.2.2: Most analytic properties of E will carry over to Z, e.g. the functional equation and meromorphic continuation – this is a common theme! In the case of $G = \mathrm{GU}_1$ or GU_n a definite unitary group, one can express Z as a *finite* sum.

Theorem 5.2.3 *(?)*.

$$Z(\varphi, \tilde{\varphi}, f_{s,\chi}) = \int_{G(\mathbb{A})} f_{s,\chi}((g,1)) \langle \pi(g)\varphi,\ \tilde{\varphi} \rangle\, dg, \qquad \langle \varphi,\ \tilde{\varphi} \rangle = \int_{G(\mathbb{Q}) \diagdown G(\mathbb{A})} \varphi(g) \tilde{\varphi}(g)\, dg,$$

where this pairing is G-invariant which is unique up to a constant multiple.

Corollary 5.2.4 *(?)*.
If there is a restricted tensor product representation

$$\pi = \bigotimes_v^{\text{res}} \pi_v, \qquad \tilde{\pi} = \bigotimes_v^{\text{res}} \tilde{\pi}_v,$$

where $\operatorname{Ind}(\chi|-|^v) = \bigotimes O_v$ with

- $\varphi = \bigotimes_v^{\text{res}} \varphi_v,$
- $\tilde{\varphi} = \bigotimes_v^{\text{res}} \tilde{\varphi}_v,$
- $f_{s,\chi} = \bigotimes_v^{\text{res}} f_v,$

then there is an Euler product decomposition

$$Z(\varphi, \tilde{\varphi}, f_{s,\chi}) = \prod_v Z_v(\varphi_v, \tilde{\varphi}_v, f_v)$$

where

$$Z_v(\varphi_v, \tilde{\varphi}_v, f_v) := \int_{G(\mathbb{Q}_v)} f_v((g,1)) \langle \pi_v(g)\varphi_v,\ \tilde{\varphi}_v \rangle \, dg.$$

Proof (of corollary).
By the uniqueness of the invariant pairing, there must exist a decomposition

$$\langle \varphi,\ \tilde{\varphi} \rangle = \prod_v \langle \varphi_v,\ \tilde{\varphi}_v \rangle.$$

■

Remark 5.2.5: Shimura computed coefficients in many cases.

Remark 5.2.6: Outline proof of the theorem: we'll analyze the orbits of $G \times G \curvearrowright X := P\backslash H$.
Setup:

- Fix $X := P\backslash H$ and write $\gamma \in X$, identifying it with its coset $H\gamma$.
- For each $\gamma \in X$, write $[G \times G]^\gamma := \mathrm{Stab}_{G\times G}(\gamma)$.
- Write $[\gamma]$ for the orbit of $P\gamma$ under the $G \times G$ action.

Remark 5.2.7: Idea: write E as a sum and rearrange, then reduce to a known computation. Reexpress it as

$$\begin{aligned} E_{f,\chi}(h) &= \sum_{[\gamma] \in P(\mathbb{Q})\backslash H(\mathbb{Q}) / (G\times G)(\mathbb{Q})} \left(\sum_{(G\times G)(\mathbb{Q})\backslash (G\times G)(\mathbb{Q})} f_{s,\chi}(\gamma h) \right) \\ &= \sum_{[\gamma] \in P(\mathbb{Q})\backslash H(\mathbb{Q}) / (G\times G)(\mathbb{Q})} \left(\sum_{(G\times G)(\mathbb{Q})\backslash (G\times G)(\mathbb{Q})} \int_{(G\times G)(\mathbb{Q})\backslash [G\times G](\mathbb{A})} f_{s,\chi}((g,h))\varphi(g \right. \\ &:= \sum_{[\gamma] \in P(\mathbb{Q})\backslash H(\mathbb{Q}) / (G\times G)(\mathbb{Q})} I(\gamma), \end{aligned}$$

where

$$I(\gamma) := \int_{[G\times G]^\gamma(\mathbb{Q})\diagdown [G\times G](\mathbb{A})} f_{s,\chi}(\gamma(g,h))\varphi(g)\tilde{\varphi}(g)\chi^{-1}\det(g)\,dg\,dh.$$

We'll have to analysis the $\gamma = 1$ and $\gamma \neq 1$ cases separately, they're quite different:

- For $\gamma = 1$, $I(\gamma)$ will be the RHS in the theorem statement.
- For $\gamma \neq 1$, $I(\gamma) = 0$.

Rewrite the stabilizer in a more convenient way:

$$\begin{aligned}[G\times G]^\gamma &= \left\{(g,h)\in G\times G \;\middle|\; P\gamma(g,h) = P\gamma\right\}\\ &= \left\{(g,h)\in G\times G \;\middle|\; \gamma(g,h)\gamma^{-1}\in P\right\},\end{aligned}$$

so

$$[G\times G]^1 = P\cap (G\times G) = \left\{(g,g) \;\middle|\; g\in G\right\} := G^\Delta.$$

Thus

$$\begin{aligned}f_{s,\chi}(1\cdot(g,h)) &= f_{s,\chi}(g,h)\\ &= f_{s,\chi}((h,h)\cdot(h^{-1}g,1))\\ &= \chi(\det h) f_{s,\chi}(h^{-1}g,1)\end{aligned}$$

and

$$I(1) = \int_{G^\Delta(\mathbb{Q})\diagdown (G\times G)(\mathbb{A})} f_{s,\chi}(h^{-1}g,1)\varphi(g)\tilde{\varphi}(g)\,dg\,dh.$$

We have an identification

$$\begin{aligned}G\times G &\rightleftharpoons G^\Delta\times(G\times 1)\rightleftharpoons G\times G\\ (g,h) &\rightleftharpoons (h,h)\cdot(h^{-1}g,1)\rightleftharpoons (h,h^{-1}g) := (g,g_1),\end{aligned}$$

which we can use to write

$$\begin{aligned}I(1) &= \int_{G(\mathbb{A})}\int_{G(\mathbb{Q})\diagdown G(\mathbb{A})} f_{s,\chi}(g_1,1)\pi(g_1)\pi(h)\tilde{\varphi}(h)\,dh\,dg_1\\ &= \int_{G(\mathbb{A})} f_{s,\chi}(g_1,1)\langle\pi(g_1)\pi,\ \tilde{\varphi}\rangle\,dg_1.\end{aligned}$$

All other orbits $[\gamma]\neq[1]$ decompose to products including terms of the form

$$\int_{N_i(\mathbb{Q})\diagdown N_i(\mathbb{A})} \varphi_i(n\cdot g)\,dn$$

for $i = 1,2$, $\varphi_1 = \varphi, \varphi_2 = \tilde{\varphi}$ and N_i unipotent radicals of a parabolic subgroup of G which is nontrivial for at least on term – however, these are cuspidal, so such integrals vanish (essentially by definition), making the entire thing vanish.

Remark 5.2.8: This falls into step 1 of the overall strategy – we found a pairing. So the next question is step 2: can we choose $f_{s,\chi}, \varphi, \tilde{\varphi}$ so that we nice multiples of Langlands L-functions $L(s,\pi,\chi)$? This will rely on reducing to computations to local integrals that were computed by Godement and Jacquet for GL_n.

Next: pulling back automorphic forms to smaller groups, what does this look like for $n=1$?

6 Ellen Eischen, Talk 4: Revisiting the Doubling Method for $n=1$

6.1 Reducing to Finite Sums

Remark 6.1.1: Goal: see what happens if we do the doubling method in the following setup. Let

- $n=1$
- $K \in \mathsf{Field}_{/\mathbb{Q}}$ be an imaginary quadratic field
- $V \in \mathsf{Vect}_{/K}^{\dim=1}$
- $W = V^{\oplus^2}$
- $G \coloneqq \mathrm{U}(V, \langle -,\ - \rangle_V) \cong \mathrm{U}_1 = \left\{ g \in \mathrm{GL}_2 \;\middle|\; g\bar{g} = 1 \right\}$. Note that $G \subseteq \mathrm{GU}(V, \langle -,\ - \rangle_V) \cong \mathrm{GU}_1 \cong \mathrm{GL}_1$
- $H \coloneqq \mathrm{U}(W, \langle -,\ - \rangle_V) \cong \mathrm{U}_{1,1}$. Note that $H \subseteq \mathrm{GU}(W, \langle -,\ - \rangle_V) \cong \mathrm{GU}_{1,1}$

Remark 6.1.2: Spoiler: we'll get an expression for the L-function $L(s,\chi)$ for a Hecke character $\chi : K^\times \backslash \mathbb{A}_K^\times \to CC^\times$ as a finite sum of values $E_\chi(A)\chi(A)$ for some elliptic curves A with CM by $\mathcal{O}_K$, and we'll obtain an algebraicity result.

Remark 6.1.3: Note that

$$\mathrm{GU}_{1,1} \cong \mathrm{GL}_2 \times \mathrm{Res}_{K/\mathbb{Q}} \mathbb{G}_m / \mathbb{G}_m,$$

and the associated symmetric space consists of copies of the upper half plane $\mathfrak{h}_1$. The associated modular form is a modular form, possibly with mild additional conditions on each component.

Remark 6.1.4: Reminder of the doubling method: we had an integral

$$Z(s,\chi,\varphi,\tilde{\varphi}) = \int_{G^{\times^2}(\mathbb{Q}) \backslash \mathbb{G}^{\times^2}(\mathbb{A})} E_{f_S,\chi}(g,h)\varphi(g)\tilde{\varphi}(h)\chi^{-1}(\det h)\, dg\, dh.$$

Some properties:

- Z is an automorphic form on $\mathrm{GU}_1 = \mathrm{GL}_1$, and thus a Hecke character.
- If one chooses $\varphi = \chi^{-1}$ so $\varphi^{-1} = \chi$ (plus some compatibility conditions), Z collapses to a finite sum:

$$Z(s, \chi, \varphi\tilde{\varphi}) = \sum_{G^{\times^2}(\mathbb{Q})\backslash G^{\times^2}(\mathbb{A})/\mathcal{K}} E_{S,\chi}(g,h)\chi^{-1}(g).$$

Remark 6.1.5: There is a diagram:

$$\begin{array}{ccc} G(\mathrm{U}_1{}^{\times^2}) & \hookrightarrow & \mathrm{GU}_{1,1} \\ & & \uparrow \\ \mathrm{U}_1{}^{\times^2} & \hookrightarrow & \mathrm{U}_{1,1} \end{array}$$

Link to Diagram

Moreover there is an embedding:

$$\begin{array}{ccc} \mathrm{GU}(V) \times \mathrm{GU}(V) & & \mathrm{GU}(W) \\ & \nwarrow \qquad \nearrow & \\ & \mathsf{G}(\mathrm{U}(V) \times \mathrm{U}(-V)) = \left\{ (g,h) \in \mathrm{GU}^{\times^2} \;\middle|\; \nu(g) = \nu(h) \right\} & \end{array}$$

Link to Diagram

These induce embeddings of corresponding Shimura varieties:

- $\mathcal{M}_{G(\mathrm{U}_1{}^{\times^2})} \to \mathcal{M}_{\mathrm{GU}_1{}^{\times^2}}$ which classifies produces $A_1 \times A_2$ 1-dimensional AVs with PEL structures, so elliptic curves with CM by $\mathcal{O}_K$,
- $\mathcal{M}_{G(\mathrm{U}_1{}^{\times^2})} \to \mathcal{M}_{\mathrm{GU}_{1,1}}$ which classifies *certain* 2-dimensional AVs with PEL structures.

Remark 6.1.6: Recall that the adelic points of our quotients are $\mathbb{C}$-points of unitary Shimura varieties, and $\mathcal{M}_{\mathrm{GU}_{1,1}}(\mathbb{C}) = \coprod \Gamma_K \backslash \mathfrak{h}_1$ where we mod out by some level. Any $z \in \mathfrak{h} = \mathfrak{h}_1$ corresponds to some $\mathbb{C}^{\times^2} / \left\langle (z\bar{a} + \bar{b}, za + b) \right\rangle$ where a, b are in some $\mathcal{O}_K$ lattice. Note the similarity to $\mathbb{C}/\langle \mathbb{Z} + \tau\mathbb{Z} \rangle$ for elliptic curves.

Remark 6.1.7: An upshot is that there are three special things in this case:

- Integral is a finite sum,
- There are only characters,
- We're evaluating at special points!

Remark 6.1.8: Let $Z(s, \chi) \coloneqq Z(s, \chi, \varphi, \tilde{\varphi})$. We can choose $f_{S,\chi}$ such that

$$Z(s, \chi) = cL(s, \chi)$$

for c a scalar, i.e. they differ by a multiple. This expresses $L(s, \chi)$ as a finite sum of values of $E(s, \chi) \cdot \chi(-)$ for E an automorphic form on $\mathrm{U}_{1,1}$, so a special kind of modular form. There is a variant of *Damerell's formula*, which expresses $L(s, \chi)$ as such a finite sum where E is an Eisenstein series in a space of Hilbert modular forms.

6.2 Rationality Properties for Eisenstein Series

Remark 6.2.1: We can obtain an Eisenstein series on $\mathfrak{h} = \mathfrak{h}_1$ of the form

$$\sum_{(c,d) \in \Lambda} \frac{\chi(d)}{(cz + d)^k (cz + d)^s}$$

where Λ is an appropriate $\mathcal{O}_K$ lattice, and for certain characters will converge for $\Re(s) + k > 2 = 2n$. This will have rational Fourier coefficients, and is holomorphic for $s = 0$. As in the case of modular forms, there is a q-expansion (or more generally in other signatures, a Fourier-Jacobi expansion) principle:

Slogan 6.2.2
Automorphic forms on $\mathrm{U}_{n,n}$ are determined by their q-expansions.

In particular, if the coefficients of the q-expansion are contained in R, then f is in fact defined over R. Kai-Wen Lan proved a more general version of this principle for $\mathrm{U}_{a,b}$ with any signature, and showed that algebraic q-expansions and analytic (i.e. Fourier) expansions agree.

So things look good for $s = 0$!

Question 6.2.3
What about $s \neq 0$, i.e. when the Eisenstein series is not holomorphic?

Answer 6.2.4
We use Mass-Shimura differential operators $\delta_K^{(r)}$ to relate E at $s \neq 0$ to E at $s = 0$, where here δ

raises weights by $2r$. For F a modular form defined over $\overline{\mathbb{Q}}$, Shimura proved the following:

$$\frac{(\delta_k^{(r)} F)(A)}{\Omega^{k+2r}} \in \overline{\mathbb{Q}}$$

for each CM point A. These operators have incarnations in $\mathrm{U}_{n,m}$ and there are analogous algebraicity results. In fact,

$$E(z, -r, \chi) = c(-4\pi y)^r \delta_k^{(r)} E(z, 0, \chi).$$

where c is a nice rational factor. Combining these results yields

$$\frac{L(r, \chi)}{\Omega^{k+2r}} \in \mathbb{Q}.$$

Remark 6.2.5: A word about this operator:

$$\delta_k f = \frac{1}{2\pi i}\left(\frac{k}{2iy} + \frac{\partial}{\partial z}\right) f = \frac{1}{2\pi i} y^{-k} \frac{\partial}{\partial z}(y^k f), \qquad \delta_k^{(r)} = \delta_k \circ \delta_k \circ \cdots \circ \delta_k.$$

Katz's idea: reexpress this operator geometrically over a moduli space of elliptic curves, or more generally AVs, in terms of the **Gauss-Manin connection** and the **Kodaira morphism**, and a splitting

$$H^1_{\mathrm{dR}} = \omega \oplus H^{0,1}$$

which preserves algebraicity at CM points.

7 Wee Teck Gan: Automorphic forms and the theta correspondence (Talk 1)

Remark 7.0.1: Goal: reformulating the Ramanujan-Petersson conjecture in terms of representation theory.

7.1 The Ramanujan-Petersson conjecture

Remark 7.1.1: Let $f : \mathfrak{h} \to \mathbb{C}$ be a holomorphic cusp form of weight k and level 1. Suppose f is an eigenvector for the Hecke operator T_p, then f has a Fourier expansion

$$f(z) = \sum_{k \geq 1} a_k(f) q^n, \qquad q := e^{2\pi i z},$$

which can be normalized so that $a_1(f) = 1$. The remaining coefficients are then the Hecke eigenvalues, so

$$T_p f = a_p(f) f.$$

Conjecture 7.1.2 ***(Ramanujan-Petersson).***

$$|a_p(f)| \leq 2p^{\frac{k-1}{2}}.$$

This was proved by Deligne as a consequence of the Weil conjectures. There is an analog for **Maass forms**, which involves the **hyperbolic Laplacian**, which similarly bounds Fourier coefficients.

Remark 7.1.3: Error terms come from the cusp forms here. There is a bridge that takes holomorphic modular forms and Maass forms to the world of automorphic forms.

Remark 7.1.4: Setup: let $k \in \mathsf{Field}_{/\mathbb{Q}}, v \in \mathrm{Places}(K)$ so that $k_v \in \mathsf{LocField}$. Define the adeles as $\mathbb{A} := \prod^{\mathrm{res}} k_v$ which admits a diagonal embedding $k \hookrightarrow \mathbb{A}$ with $k \backslash \mathbb{A}$ compact. Let $G \in \mathsf{AlgGrp}_{/k}$ be reductive, e.g. $\mathrm{SL}_n, \mathrm{U}_n$, we then similarly have

$$G(k) \hookrightarrow G(\mathbb{A}) = \prod_v^{\mathrm{res}} G(k_v)$$

with $\{k_v\}$ an open compact subgroup. Write $[G] = G(k) \backslash G(\mathbb{A})$, and note the there is a right action $[G] \curvearrowleft G(\mathbb{A})$.

7.2 Automorphic Reps

Definition 7.2.1 (Automorphic forms on reductive groups)
An **automorphic form** on G is a function $f : [G] \to \mathbb{C}$ satisfying

- Regularity conditions: e.g. at worst polynomial growth $f \sim z^k$, smoothness, and derivatives $f^{(n)} \sim z^k$ for the same exponent.
- Finiteness conditions: K-finiteness for $K = \prod_v k_v$, or more generally $Z(\mathfrak{g})$-finiteness.

Write $\mathcal{A}(G)$ for the vector space of automorphic forms on G. Note that this carries at left $G(\mathbb{A})$ action:

$$(g_0.f)(g) = f(gg_0).$$

Remark 7.2.2: The finiteness condition will guarantee that f will come from the kernel of a differential operator, e.g. the CR equations for holomorphy. Requiring K-finiteness only gives an action on finite adeles.

Definition 7.2.3 (Automorphic representation)
An **automorphic representation** is an irreducible representation of $\mathcal{A}(A)$.

7.3 Cusp Forms

Definition 7.3.1 (Cusp forms)
A form $f \in \mathcal{A}(G)$ is **cuspidal** iff for all parabolic subgroups P with $P = MN$, the constant term of f along N is zero, where the constant term is defined as

$$f_N(g) = \int_{[N]} = f(ng)\, dn.$$

This yields a subspace of cusp forms $\mathcal{A}_{\text{cusp}}(G) \leq \mathcal{A}(G)$ which is stable under the $G(\mathbb{A})$ action.

Remark 7.3.2: One can take a character $\psi : [N] \to \mathbb{C}^\times$, then there is a (N, ψ)-Fourier coefficient of f:

$$f_{N,\psi}(g) = \int_{[N]} \overline{\psi(n)} \cdot f(ng)\, dn.$$

Remark 7.3.3: Uniform moderate growth and being cuspidal imply that $f \in \mathcal{A}_{\text{cusp}}(G)$ rapidly decays at ∞, i.e. faster than $1/p$ for any polynomial, so that $f \in L^2$:

$$\int_{[G]} |f|^2 < \infty.$$

So define the Hilbert space of square-integrable automorphic forms

$$\mathcal{A}_2(G) := \left\{ f \in \mathcal{A}(G) \,\middle|\, \|f\|_{L^2} < \infty \right\}.$$

There is a containment

$$\mathcal{A}_{\text{cusp}}(G) \subseteq \mathcal{A}_2(G) \subseteq \mathcal{A}(G),$$

where there are a decomposition into irreducible reps

- $\mathcal{A}_{\text{cusp}}(G) = \bigoplus_\pi m(\pi)\pi$ for some cuspidal multiplicities m,
- $\mathcal{A}_2(G) = \bigoplus_\pi m(\pi)\pi$ for some L^2 multiplicities m.

Question 7.3.4
A main question for automorphic representations: for which π is $m(\pi) > 0$? I.e. which representations occur as cuspidal or L^2 reps? Moreover, what do all of the irreducible reps of $G(\mathbb{A})$ look like?

Remark 7.3.5: Recall that since $G(\mathbb{A}) = \prod_v^{\text{res}} G(k_v)$, we expect a representation π of $G(\mathbb{A})$ to break up as $\pi = \bigotimes_v^{\text{res}} \pi_v$ with

- $\pi_v \in \mathsf{Irr}(G(k_v))$,
- $\pi_v^{k_v} \neq 0$ for almost all v, so k_v is **unramified** or **spherical**.

7.4 Unramified Reps

Remark 7.4.1: There is a containment $G_v \supseteq K_v$ where G_v is unramified, i.e. quasi-split (so has a Borel) and split by an unramified extension of k_v, and K_v is a **hyper-special** subgroup, which is a maximal compact. This yields $G_v \supseteq B_v \supseteq T_v N_v$, and there is a bijection

$$\begin{aligned} \mathsf{IrrRep}(G_v)(K_v\text{-unramified}) &\rightleftharpoons \{\text{Unramified characters of } T_v\}/W \\ I(\chi) = \operatorname{Ind}_{B_v}^{G_v} \chi &\hookleftarrow \chi, \end{aligned}$$

where we mod out by a Weyl group action W. Note that $I(\chi)$ is the unique unramified subquotient.

There is a further correspondence

$$\begin{aligned} \{\text{Unramified characters of } T_v\}/W & \\ \xrightleftharpoons[\text{Langlands}]{} & \{\text{Semisimple conjugacy classes in } G^\vee(\mathbb{C})\} \\ \chi \mapsto S_\chi, & \end{aligned}$$

so there is some semisimple conjugacy class associated to characters χ.

Remark 7.4.2: Thus for $\pi \in \mathcal{A}_{\text{cusp}}(G)$ with $\pi = \bigotimes_v^{\text{res}} \pi_v$, one gets a collection $\left\{ S_{\pi_v} \;\middle|\; v \notin S \right\} \subseteq G^\vee$. For $R : G^\vee \to \mathrm{GL}_N(\mathbb{C})$, we can form an L-function

$$L^S(s, \pi R) := \prod_{v \notin S} L(s, \pi_v, R), \qquad L(s, \pi_v, R) := \frac{1}{\det 1 - q_v^{-s} R(S_{\pi_v})}, \qquad q :=?.$$

These generalize Hecke L-functions and those attached to modular forms.

7.5 Tempered Reps

Remark 7.5.1: A character of the torus $\chi : T_v \to \mathbb{C}^\times$ yields π_χ a K_v-unramified irrep. Say π_χ is **tempered** iff χ is unitary, i.e. it factors as $\chi : T_v \to S^1$ so that $|\chi| = 1$. Tempered reps naturally occur as regular representations.

Remark 7.5.2: Note that tempered reps are *weakly* contained in $L^2(G_v)$, but not e.g. the trivial representation of SL_2 is not in $L_2(\mathbb{R})$, but $\mathrm{SL}_2(\mathbb{R})$ does not have finite volume. In general, the trivial representation is not tempered unless the group is compact.

Conjecture 7.5.3 ***(Ramanujan-Petersson, reformulated but false).***
Let $\pi = \bigotimes_v^{\mathrm{res}} \subseteq \mathcal{A}_{\mathrm{cusp}}(G)$ for G quasi-split (or split), then π_v is tempered for almost all v.

Remark 7.5.4: This conjecture is false! There is a counterexample for $G = \mathrm{SP}_4$, and a goal for this course is to construct a counterexample for $G = \mathrm{U}_3$.

Conjecture 7.5.5 ***(Ramanujan-Petersson, reformulated and fixed).***
If $\pi \subseteq \mathcal{A}_{\mathrm{cusp}}(G)$ and π is **globally generic** (a certain big enough Fourier coefficient), then π_v is tempered for almost all v.

7.6 Unitary groups

Definition 7.6.1 (Unitary Groups)
Let

- E/F be a quadratic extension,
- $\mathsf{Gal}(E/F) = \langle c \rangle$ is cyclic,
- $V \in \mathsf{Vect}_{/E}$,
- $\langle -,\ - \rangle : V^{\times^2} \to E$ which is ε-Hermitian for $\varepsilon = \pm 1$, i.e.

$$\langle av,\ bw \rangle = a \langle v,\ w \rangle b^c, \qquad \langle v,\ w \rangle = \varepsilon \langle w,\ v \rangle.$$

- $$\delta \in E_0^\times \coloneqq \left\{ x \in E^\times \;\middle|\; \mathrm{Tr}(x) = 0 \right\}.$$
- $\delta \cdot \langle -,\ - \rangle$ is $(-\varepsilon)$-Hermitian.

Then define the **unitary group** as

$$\mathrm{U}(V) \coloneqq \mathrm{Aut}(V, \langle -,\ - \rangle).$$

Remark 7.6.2: There are some invariants:

- $n = \dim V$,
- $$\mathrm{disc}(V) = (-1)^{\binom{n}{2}} \det(V) \in F^\times / \mathrm{Nm}(E^\times),$$

  where the quotient by the image of the norm map is needed to make it well-defined.

Henceforth we'll take V to be Hermitian and W to be skew-Hermitian.

8 | Wee Teck Gan (Talk 2)

Remark 8.0.1: Correction from last time:

$$\langle v_2,\ v_1\rangle = \varepsilon\langle v_1,\ v_2\rangle^c.$$

Notation from last time:

- V Hermitian, W skew-Hermitian
- An invariant $\operatorname{disc}(V) := (-1)^m \det V \in F^\times / \operatorname{Nm}(E^\times)$ where $m := \binom{n}{2}$ and $n := \dim V$
- $\operatorname{disc}(W) = \operatorname{disc}(\delta^{-n} V)$ where $\delta \in E_0^\times$.

Fact 8.0.2
Over p-adic fields, $\operatorname{disc}(V)$ determines V. By composing with a quadratic character $w_{E/F}$, we obtain

$$\operatorname{disc} \circ w_{E/F} = (V \xrightarrow{\operatorname{disc}} F^\times / \operatorname{Nm}(E^\times)) \xrightarrow{w_{E/F}} \langle \pm 1\rangle\,,$$

so there are exactly two classes of Hermitian vector spaces of a given dimension, which we'll denote V^+, V^-.

Remark 8.0.3: Over a real field, this is not enough – one also needs the signature $\operatorname{sig}(V) = (p, q)$ where $p + q = n$, in which case

$$\operatorname{disc}(V_{p,q}) = (-1)^q (-1)^{\binom{p}{2}}.$$

For E/K an extension of number fields, there is a local-global principle:

$$\begin{aligned} \mathsf{HermVect}_{/K} &\hookrightarrow \prod_{v \in \operatorname{Places}(K)} \mathsf{HermVect}_{/K_v} \\ V &\mapsto \{V \otimes_K K_v\}_{v \in \operatorname{Places}(K)}\,. \end{aligned}$$

We'll call spaces in the image of this correspondence **coherent**.

Fact 8.0.4
V is coherent iff for almost every place v,

$$V \text{ is coherent} \iff \varepsilon(V_v) = 1 \text{a.e. and } \prod_v \varepsilon(V_v) = 1.$$

Example 8.0.5 ***(of classification):*** Let k be a p-adic field.

In rank 1:

- $E_0^\times / \operatorname{Nm}(E^\times) = \{\delta, \delta'\}$,
- $W_1^+ = \langle \delta \rangle$,
- $W_1^- = \langle \delta' \rangle$.

In rank 2:

- $\mathbb{H} = W_2^+ = Ee_1 + Ee_2$,
- $\langle e_i,\ e_i \rangle = 0$ and $\langle e_1,\ e_2 \rangle = 1$, which yields matrix $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$,
- W_2^- is described by a quaternionic division algebra.

In rank $2n$:

- $W_{2n}^+ = \mathbb{H}^{\oplus n}$
- $W_{2n}^- = W_2^- \oplus \mathbb{H}^{\oplus (n-1)}$

In rank $2n+1$:

- $W_{2n+1}^+ = \langle \delta \rangle \oplus \mathbb{H}^{\oplus n}$,
- $W_{2n+1}^- = \langle \delta' \rangle \oplus \mathbb{H}^{\oplus n}$.

8.1 Howe-PS: Counterexample to the Ramanujan-Petersson Conjecture

Remark 8.1.1: Let $\dim W = 3$, so $U(W) = \mathrm{U}_3$, then $\underset{E/K}{\operatorname{Res}}(W) \in \mathsf{Vect}_{/k}^{\dim = 6}$. The trace to K yields a symplectic form:

$$\omega(-,-) := \operatorname{Tr}_{E/k} \langle -,\ - \rangle_W.$$

There is an embedding $U(W) \hookrightarrow \mathsf{Sp}(\underset{E/k}{\operatorname{Res}}(W))$, so $\mathrm{U}_3 \hookrightarrow \mathsf{Sp}_6$ There is a simple something:

$$\Omega \subseteq \mathcal{A}_2(\mathsf{Sp}(-)),$$

which we'll call **theta functions**. Note that $ZU(W) = E^1 := \left\{ x \in E^\times \;\middle|\; \operatorname{Nm}(x) = 1 \right\}$. Consider $i^*\Omega \subseteq \mathcal{A}(U(W))$; there is a central character decomposition

$$i^*(\Omega) = \bigoplus_\chi \Omega_\chi,$$

where the sum is over automorphic characters of E^1.

Claim: Ω_χ is an irreducible cuspidal representation, with at most one exception χ, and this Ω_χ produced a counterexample for the RP conjecture.

⚠ Warning 8.1.2
A complication: the theta functions don't live on Sp_6, but rather on a double cover, and this leads to many technicalities.

Remark 8.1.3: Howe-PS produces a correspondence:

$$\left\{\text{Automorphic characters on } E^1 \cong \mathrm{U}_1\right\} \rightleftharpoons \{\text{Automorphic reps of } \mathrm{U}_3\}$$
$$\chi \mapsto \Omega_\chi.$$

Question 8.1.4
How can one produce an injective map

$$\mathsf{Irr}(G) \hookrightarrow \mathsf{Irr}(H)?$$

Answer 8.1.5
Recall that

$$\mathsf{Irr}(G \times H) = \left\{\pi \otimes \sigma \;\middle|\; \pi \in \mathsf{Irr}(G), \sigma \in \mathsf{Irr}(H)\right\}.$$

The idea to produce this map: find $(G \times H)$-reps Ω and produce a subset

$$\Sigma_\Omega = \left\{(\pi, \sigma) \;\middle|\; \underset{G\times H}{\mathrm{Hom}}(\Omega, \pi \otimes \sigma) \neq 1\right\} \subseteq \mathsf{Irr}(G) \times \mathsf{Irr}(H).$$

Question 8.1.6
Is the correspondence Σ_Ω a graph?

Remark 8.1.7: There is a decomposition

$$\begin{aligned}
\Omega_{G\times H} &= \bigoplus_\pi \bigoplus_\sigma m(\pi, \sigma)\pi \otimes \sigma \\
&= \bigoplus_\pi \left(\bigoplus_\sigma m(\pi, \sigma)\sigma\right) \otimes \pi \\
&\coloneqq \bigoplus_\pi \Theta(\pi) \otimes \pi.
\end{aligned}$$

Is $\Theta(\pi)$ an irreducible rep, or zero? If so, this produces a map

$$\Theta : \mathsf{Irr}(G) \to \mathsf{Irr}(H) \cup \{0\}.$$

Remark 8.1.8: Upshot: one needs $\dim \Omega$ to be small. Suppose $G \times H \to E$, take the smallest non-trivial representation Ω of E and pull it back to $G \times H$. If $G \times H \subseteq E$, this can be done by restriction.

Remark 8.1.9: The **theta correspondence** is an instance of all of these ideas.

8.2 The Theta Correspondence

Remark 8.2.1: Let

- $F \in \mathsf{Field}$ be a p-adic,
- E/F a quadratic extension,
- V Hermitian and W skew-Hermitian so that $V \otimes_E W$ is skew-Hermitian under the symplectic form induced by the trace,

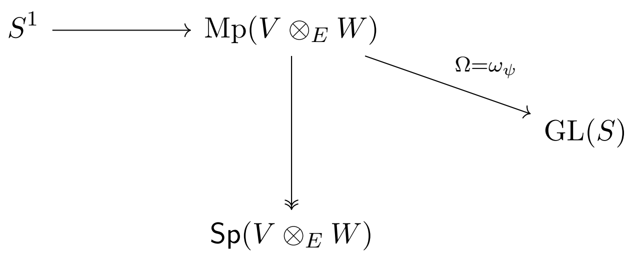
This yields a map of the form $G \times H \to E$:

$$\mathrm{U}(V) \times \mathrm{U}(W) \to \mathsf{Sp}(V \otimes_E W).$$

What is Ω? To get small enough weights, one needs to pass to the **metaplectic cover** Mp.

8.3 Metaplectic Groups and Weil Reps

Remark 8.3.1: For $\psi : F \to \mathbb{C}^\times$ a nontrivial character:



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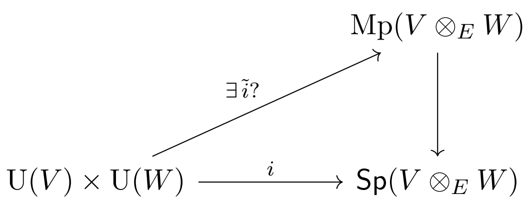
Here $\{\omega_\psi\}$ is the smallest infinite-dimensional representation of Mp and referred to as the **Weil representation**.

Remark 8.3.2: On where this comes from: QM. One looks at the Heisenberg group, uses the Stone-von-Neumann theorem, see 2.3 and 2.4 in the notes.

Remark 8.3.3: One needs a lift of the following form:



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By Xudle, $\tilde{\imath}$ exists and is determined by a pair of characters (χ_V, χ_W) of $E^\times$ such that

- $\chi_V|_{F^\times} = \omega_{E/F}^{\dim V}$
- $\chi_W|_{F^\times} = \omega_{E/F}^{\dim W}$

Such a χ_V gives $\tilde{\mathrm{U}}(W) \to \mathrm{Mp}$, and similarly for W.

Remark 8.3.4: Set

$$\Omega_{V,W,\chi_V,\chi_W,\psi} := \tilde{\imath}^*_{\chi_V,\chi_W}(\omega_\psi),$$

which has properties described in the lecture notes.

Definition 8.3.5 (The big theta lift as a multiplicity space)
For $\pi \in \mathsf{Irr}\mathrm{U}(V)$, define

$$\Theta(\pi) := \mathrm{coinv}_{\mathrm{U}(V)}(\Omega \otimes \pi^\vee).$$

Remark 8.3.6: Note that there is a $\mathrm{U}(W)$ action on both sides. Moreover,

$$\mathrm{Hom}(\mathrm{coinv}_G(\Omega \otimes \pi^\vee), \mathbb{C}) \cong \underset{G}{\mathrm{Hom}}(\Omega \otimes \pi^\vee, \mathbb{C}) \cong \underset{G}{\mathrm{Hom}}(\Omega, ?).$$

Theorem 8.3.7 ***(Howe-Kudla).***

- $\Theta(\pi)$ has finite length as a $\mathrm{U}(W)$ rep, and thus has finitely many irreducible quotients.
- For any pair (π, σ),

$$\dim \underset{\mathrm{U}(V)\times\mathrm{U}(W)}{\mathrm{Hom}}(\Omega, \pi \otimes \sigma) < \infty.$$

Definition 8.3.8 (Small theta lift)
Define $\theta(\pi)$ to be the maximal semisimple quotient of $\Theta(\pi)$. This is a finite length semisimple rep.

Theorem 8.3.9 ***(Howe Duality).***

- $\theta(\pi)$ is irreducible if $\Theta(\pi) \neq 0$.
- Uniqueness: $\theta(\pi) \cong \theta(\pi') \implies \pi \cong \pi'$. Thus $\theta : \mathsf{Irr}\mathrm{U}(V) \to \mathsf{Irr}\mathrm{U}(W) \setminus \{0\}$ is injective on $\operatorname{supp}\theta$, those reps which are not sent to zero.

Question 8.3.10
Is $\theta(\pi)$ zero or not?

9 | Wee Teck Gan (Talk 3)

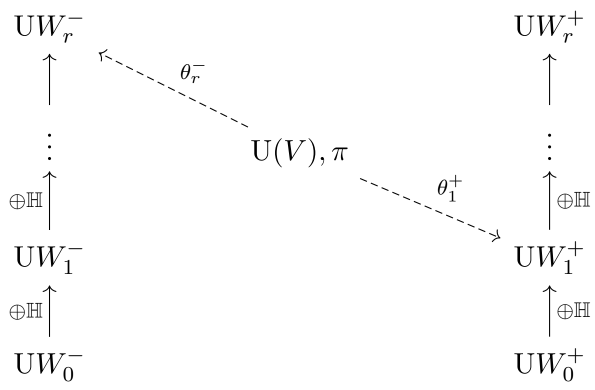
Remark 9.0.1: Last time: we describe the Howe-PS correspondence

$$\text{Automorphic characters of } \mathrm{U}_1 \rightleftharpoons \text{Automorphic reps of } \mathrm{U}_3$$
$$\chi \mapsto \Omega_\chi.$$

A correction: it's not true that Ω_χ is cuspidal except for at most one χ; instead if can be cuspidal for many χ. We defined $\Omega, \Theta(\pi)$ with a $\mathrm{U}(W)$ action, and Howe duality which took $\Theta(\pi) \neq 0$ to a unique irreducible quotient $\theta(\pi)$. Thus $\Theta : \mathsf{Irr}\mathrm{U}(V) \hookrightarrow \mathsf{Irr}\mathrm{U}(W) \coprod \{0\}$ is injective away from the zero locus.

Question 9.0.2
When is $\Theta(\pi) \neq 0$?

Remark 9.0.3: Let $\dim W$ be odd, and label $W_r^\varepsilon = 2r+1$. We know all skew-Hermitian spaces of a particular dimension, so we obtain towers:



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Note that $W_{r+1}^+ = W_r^+ \oplus \mathbb{H}$.

Question 9.0.4
Which $\theta_r^\varepsilon(\pi)$ are nonzero?

Theorem 9.0.5 *(?).*

- For $\pi \in \mathsf{Irr}\mathrm{U}(V)$ and a fixed $\varepsilon = 1$, there is a smallest $r_0^\varepsilon(\pi) \leq \dim V$ such that this is the first occurrence of π in the ε tower, i.e. $\theta^\varepsilon_{r_0^\varepsilon(\pi)}(\pi) \neq 0$.

- For all $r > r_0, \Theta^{\varepsilon}(\pi) \neq 0$,
- If π is a supercuspidal rep, then by Kudla, $\Theta_r^{\varepsilon}(\pi)$ is irreducible and is s.c. at the first occurrence but not after.

Remark 9.0.6:

- Nonvanishing is reduced to determining $r_0^+(\pi)$ and $r_0^-(\pi)$.
- If $r \geq \dim V$, so r is in the stable range and $\Theta_r^{\varepsilon}(\pi) \neq 0$.

Thus reduces checking infinitely many nonzero conditions to just computing the values of these two numbers. We can reduce this further to just checking *one* number by the following:

Theorem 9.0.7 ***(Conservation relation (B.Y. Sun, C.B. Zhu, Kudla-Rallis)).***

$$\dim W^+_{r_0^+(\pi)} + \dim W^-_{r_0^-(\pi)} = 2\dim V + 2.$$

Corollary 9.0.8 ***(Dichotomy).***
If $\dim W^+ + \dim W^- = 2\dim V$, then for any $\pi \in \mathsf{Irr}\,\mathrm{U}(V)$, exactly one of $\Theta_{W^+}(\pi)$ or $\Theta_{W^-}(\pi)$ is nonzero.

Example 9.0.9 ***(?):*** Take $\mathrm{U}_1 \times \mathrm{U}_1 = \mathrm{U}(V) \times \mathrm{U}(W_0)$ where $\mathrm{U}(V) = E^1$, and let $\chi \in \mathsf{Irr}\,E^1$. Then

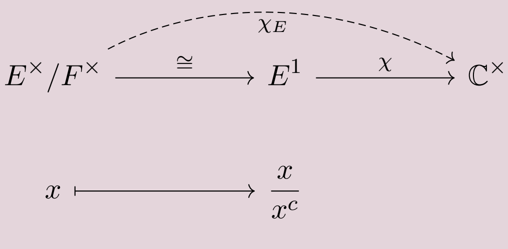
$$\dim W^+_{r^+(\chi)} + \dim W^-_{r^-(\chi)} = 4,$$

These two dimensions are numbers in $\{1, 3\}$, and exactly one of $\theta_0^{\pm}(\chi)$ is nonzero, and for $r > 0$ we have $\theta_r^{\varepsilon}(\chi) \neq 0$. Which $\theta_0^{\varepsilon}(\chi)$ are nonzero?

Theorem 9.0.10 ***(Moen, Rogawski?, Hams-Kudla-Sweat).***

$$\theta_{V,W_0,\psi}(\pi) \neq 0 \iff \varepsilon(v)\varepsilon(W_0) = \varepsilon_E\left(\frac{1}{2}, \chi_E\chi_W^{-1}, \psi(\mathrm{Trace}_{E/F}(\delta - 1))\right)$$

where ε_E is the local epsilon factor defined in Tate's thesis. Here χ_E is the composite character $\chi_E(x) = \chi\left(\frac{x}{x^?}\right)$ defined by



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The $\delta \in E_0^\times$ appears because a Hermitian space depends on a choice of a traceless element.

Example 9.0.11 *(?)*: Applying Howe-PS to $U_1 \times U_3$: let $V = \langle 1 \rangle = V_0^+$ and $\chi \in \mathsf{Irr} E^1 = \mathsf{Irr} U(V)$. Since $\dim W^\varepsilon = 3$, Ω^ε is semisimple and decomposes as

$$\Omega^\varepsilon = \bigoplus_{\chi \in \mathsf{Irr} E^1} \chi \otimes \Theta^\varepsilon(\chi).$$

- Since $\dim V = 1$, we're in the stable range and thus $\Theta^\varepsilon(\chi) \neq 0$ for all χ.
- $\Theta^\varepsilon(\chi)$ is irreducible by Howe duality and s.c.
- If $\varepsilon = \varepsilon_E \left(\frac{1}{2}, \cdots \right)$ as in the theorem, $\Theta^\varepsilon(\chi)$ is non-supercuspidal and $\Theta^{-\varepsilon}(\chi)$ is supercuspidal.

In fact, $\Theta^\varepsilon(\chi) \hookrightarrow \mathrm{Ind}_B^{U(W)} \left(\chi_v |-|^{-\frac{1}{2}} \otimes \chi \right)$ where $B = \mathrm{diag}(a, b, (a^c)^{-1}) + N^+$ (upper triangular) with $a \in E^\times$ and $b \in E^1$.

9.1 Global Setting

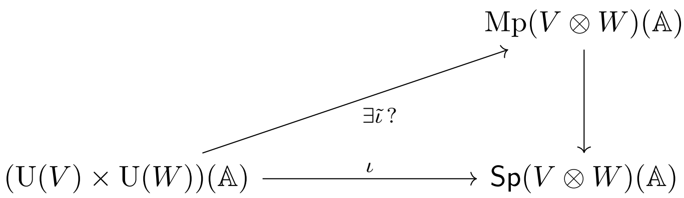
Remark 9.1.1: For $K \in \mathsf{Field}_{/\mathbb{Q}}$, writing $\theta = \prod_v \theta_v$, one might hope for a map $\mathsf{Irr} U(V)(\mathbb{A}) \to U(W)(\mathbb{A})$. Instead, we'll want a map

$$\theta : \{\text{Automorphic reps of } U(V)\} \to \{\text{Automorphic reps of } U(W)\},$$

i.e. a concrete way to transfer functions from a space X to a space Y. If $K \in C(X \times Y)$, we can define

$$\begin{aligned} T_K &: C(X) \to C(Y) \\ T_k(f)(y) &:= \int_X K(x, y) f(x)\, dx, \end{aligned}$$

so K acts like a matrix. In our case, we'll want a lift



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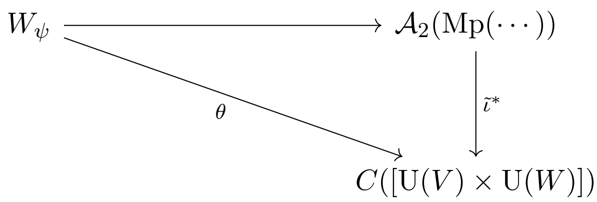
Here $\Omega = \tilde{\iota}^* W_\psi$. For $\pi \in \mathcal{A}_{\mathrm{cusp}}(U(V))$, we have a map



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This yields

$$\begin{aligned} w_\psi \otimes \pi &\to \mathcal{A}(\mathrm{U}(V)) \\ \varphi \otimes f &\mapsto \theta(\varphi, f), \qquad \theta(\varphi, f)(g) := \int_{[\mathrm{U}(V)]} \theta(\varphi)(g, h)\overline{f(h)}\, dh. \end{aligned}$$

So define the **global theta lift of** π as

$$\Theta(\pi) := \left\langle \theta(\varphi, f) \;\middle|\; \varphi \in w_\varphi, f \in \pi \right\rangle \subseteq \mathcal{A}(\mathrm{U}(W)).$$

Question 9.1.2

- Is $\Theta(\pi)$ nonzero?
- Does it land in $\mathcal{A}_2$ or $\mathcal{A}_{\text{cusp}}$?
- What is the relation with the local picture?

Proposition 9.1.3 ***(?).***
If $\Theta(\pi) \subset \mathcal{A}_2(\mathrm{U}(W))$ is a proper subset, then $\Theta(\pi)$ is either zero or isomorphic to $\bigotimes_v \theta(\pi_v)$.

Theorem 9.1.4 ***(?).***
Let $\pi \subseteq \mathcal{A}_{\text{cusp}}(\mathrm{U}(V))$,

1. There exists a smallest $r_0 = r_0^\varepsilon(\pi)$ such that $\Theta_{r_0}^\varepsilon(\pi) \neq 0$. In this case, $\Theta_{r_0}^\varepsilon(\pi) \subseteq \mathcal{A}_{\text{cusp}}(\mathrm{U}(W))$.
2. For all $r > r_0$,$\Theta_r^\varepsilon(\pi) \neq 0$ and is noncuspidal, i.e. not contained in $\mathcal{A}_{\text{cusp}}(\mathrm{U}(W))$.
3. For all $r \geq \dim V$ in the stable range, $0 \neq \Theta_r^\varepsilon(\pi) \subseteq \mathcal{A}_2(\mathrm{U}(W))$. Note that being nonzero follows from 1 and 2.

10 | Wee Teck Gan (Talk 4)

Remark 10.0.1: Take $V = \langle 1 \rangle$ and $W_r^\varepsilon = W_0^\varepsilon \oplus \mathbb{H}^r$ which has dimension $2r+1$, and let $\chi \in \mathcal{A}(\mathrm{U}(V))$. We know $\Theta_r^\varepsilon(\chi) \neq 0$ for all $r > 0$, which is the stable range. Note that $\Theta_r^\varepsilon(\chi) \subseteq \mathcal{A}_2(\mathrm{U}(W_r^\varepsilon))$, i.e. these are square-integrable. What happens when $r = 0$?

Theorem 10.0.2 ***(?).***
$\Theta_{W_0^\varepsilon}(\chi) \neq 0 \iff$ several conditions hold:

- For all v, $\Theta_{W_0}^{\varepsilon_v}(\chi_v) \neq 0$, so it is controlled by local conditions,
- $L(1/2, \chi_E \chi_W^{-1}) \neq 0$, a global condition.

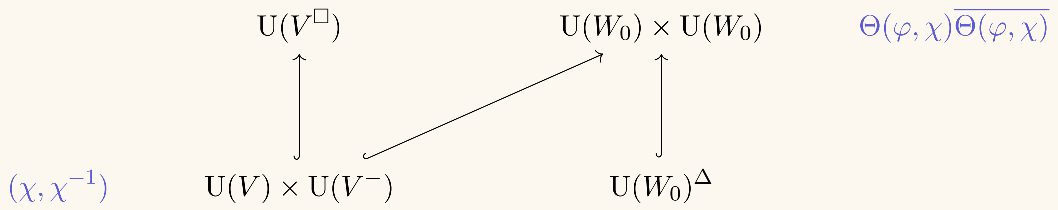
Remark 10.0.3: Note that

$$\varepsilon_v = \varepsilon(1/2, \chi_E \chi_{W,v}^{-1}, \varphi(\text{Trace?}))$$
$$1 = \prod_v \varepsilon_v = \varepsilon(1/2, \chi_E \chi_W^{-1}).$$

Proof (of theorem, sketch).
For $\psi \in W_\varphi$, we produce $\Theta(\varphi)$ and obtain an integral

$$\Theta(\varphi, \chi)(g) = \int_{[\mathrm{U}(V)]} \varphi(g, h)\chi(h)^{-1}\, dh \in \mathcal{A}(\mathrm{U}(W_0)).$$

Is this function nonzero for some φ? There isn't a good notion of Fourier expansion here, so one instead computes $\langle \Theta(\varphi, \chi),\ \Theta(\varphi, \chi)\rangle$. Write $V^\square = V \oplus -V$, where $-V$ is V with the form negated.



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One can them map $\mathrm{U}(V^\square) \to \mathrm{U}(W_0)^\square$; this diagram is called the **doubling see-saw**. Combining this with Siegel-Weil associates to the above inner product the doubling zeta integral $Z(0, \varphi, \chi)$. By Ellen's lectures, this reduces to computing the central value of an L-function, $cL(1/2, \chi_E \chi_W^{-1})$, up to a fudge factor c. The process is the **Rallis inner product formula**:

$$\langle \Theta(\varphi, \chi),\ \Theta(\varphi, \chi)\rangle \rightsquigarrow Z(0, \varphi, \chi) \rightsquigarrow L(1/2, \chi_E \chi_W^{-1}).$$

∎

10.1 Howe-PS

Remark 10.1.1: Setup:

- $V = \langle 1 \rangle$ a 1-dim space
- $W = W_0 \oplus \mathbb{H}$

- A nonzero irreducible theta lift $\Theta_W(1) \subseteq \mathcal{A}_2(\mathrm{U}(W))$

We know that the local components are contained in non-tempered principal series, i.e.

$$\Theta_W(1)_v \hookrightarrow \mathrm{Ind}_?^{\mathrm{U}(W_v)} \left|-\right|_v^+ \otimes 1_v?.$$

It only remains to check that happens when this is not cuspidal. If it is not, then $\Theta_{W_0}(1) \neq 0$, so pick 2 places v_1, v_2 of K and swap the signs on W_{0,v_i} to produce W_0', and run the above argument on $W' = W_0' \oplus \mathbb{H}$.

10.2 Arthur's Conjecture

Remark 10.2.1: Goal: classify constituents of $\mathcal{A}_2(G)$, i.e. describe this as a $G(\mathbb{A})$-module. We'll make a basic hypothesis (global Langlands for GL_n) that there exists a group L_F (thought of as $\mathsf{Gal}(\overline{F}/F)$) such that there is a bijection

$$\mathsf{IrrRep}^{\dim = n} L_F \rightleftharpoons \mathsf{Rep}_{\mathrm{cusp}} \mathrm{GL}_n,$$

where for all v there is a Weil-Deligne group $L_{F_v} \approx \mathsf{Gal}(\overline{F}_v/F_v)$ with a map $L_{F_v} \hookrightarrow L_F$.

Definition 10.2.2 (Near equivalence)
Two adelic representations $\pi = \bigotimes_v \pi_v$ and $\pi' = \bigotimes_v \pi_v'$ are **nearly equivalent** iff $\pi_v \cong \pi_v'$ for almost all places v.

Remark 10.2.3: We can decompose into near-equivalence classes $\mathcal{A}_w(G) = \bigoplus_\psi \mathcal{A}_\psi$, where $\psi : L_F \times \mathrm{SL}_2 \to {}^L G$ is a map to the Langlands L group ${}^L G = G^\vee \rtimes \mathsf{Gal}(\overline{F}/F)$, such that

- A tempered condition: the image is big, $\psi(L_F)$ is bounded, and
- Centralizers are small: $Z_{G^\vee}/Z_{G^\vee}^{\Gamma_F}$ is finite.

This has something to do with elliptic A-parameters.

Question 10.2.4
Given ψ, how can we describe $\mathcal{A}_\psi$?

Remark 10.2.5: From ψ we'll obtain

- a global component group/centralizer $S_\psi = Z_{G^\vee}/Z_{G^\vee}^{\Gamma_F}$,
- local factors $\psi_v : L_{F_v} \times \mathrm{SL}_2 \hookrightarrow L_f \times \mathrm{SL}_2 \xrightarrow{\psi} {}^L G$,
- Local component groups $\pi_0 \left(Z_{G^\vee}(\psi_v)/Z(G^\vee)^{\Gamma_F} \right)$ which are finite?

- $S_\psi \xrightarrow{\Delta} \prod_v S_{\psi_v} \coloneqq S_{\psi/\Delta}$ which is compact
- Quadratic characters $\varepsilon_\psi : S_\psi \to \langle \pm 1 \rangle$.

Remark 10.2.6: For all v, we should have a finite set of unitary reps of $G(F_v)$,

$$\prod_{\psi_v} = \left\{ \pi_{\eta_v} \;\middle|\; \eta_v \in \mathsf{Irr} S_{\psi_v} \right\},$$

i.e. for almost all v, π_{1_v} is irreducible unramified with Satake parameters

$$\psi_v \left(\mathrm{Frob}_v, \mathrm{diag}(q_v^{\frac{1}{2}}, q_v^{-\frac{1}{2}}) \right) \in {}^L G.$$

Observation 10.2.7
The key point: if $\psi(\mathrm{SL}_2) = 1$, then π_{1_v} is tempered. If not, ψ_{1_v} is non-tempered.

This explains how rigidity obstructs the Ramanujan-Petersson conjecture?

Remark 10.2.8: Set $\pi_\psi = \bigotimes_v \pi_{\psi_v}$ and let

$$\mathcal{A}_\psi = \bigoplus_{\eta \in \mathsf{Irr} S_{\psi,?}} m_\eta \pi_\eta, \qquad m_\eta = \dim \operatorname{Hom}_{S_\psi}(\varepsilon_\psi, \eta).$$

To define ε_ψ, define a map

$$(L_F \times \Omega_2) \times S_\psi \xrightarrow{\psi \times \mathrm{id}} {}^L G / Z(G^\vee)^{\Gamma_F} \curvearrowleft \mathrm{Ad} \mathfrak{g}^\vee$$

where $\mathfrak{g}^\vee = \mathsf{Lie}(G^\vee) = \bigoplus_{i \in I} \rho_i \otimes S_{r_i} \otimes \eta_i$ for some index set I, and S_r are r-dimensional irreps of SL_2. Set $T \subseteq I$ to be the indices such that r_i is even, η_i is orthogonal, and ρ_i is symplectic, and $\varepsilon(1/2, \varphi_i) = -1$. Then define

$$\begin{aligned} \varepsilon_\psi : S_\psi &\to \langle \pm 1 \rangle \\ s &\mapsto \prod_{i \in T} \eta_i(s). \end{aligned}$$

Example 10.2.9*(?)*: For $\psi(\mathrm{SL}_2) = \{1\}$, $\varepsilon_\psi = 1$ and $T = \emptyset$.

10.3 Specializing to U_n

Remark 10.3.1: Fix $G = \mathrm{U}_n$, let E/F be an extension, and

$$G^\vee = \mathrm{GL}_n(\mathbb{C}) \trianglelefteq {}^L G = \mathrm{GL}_n(\mathbb{C}) \rtimes \mathsf{Gal}(E/F).$$

An L-parameter is a map

$$\psi : L_F \times \mathrm{SL}_2 \to {}^L G$$

where the subset $L_E \times \mathrm{SL}_2$ maps to $G^\vee$. By an email comment of Benedict Gross, ψ is determined by this restriction. Not every such map extends, but conjugate self-dual reps of sign $(-1)^{n-1}$ will.

Example 10.3.2*(?)*: For U_3, $\psi|_{L_E} : L_E \times \mathrm{SL}_2 \to \mathrm{GL}_3(\mathbb{C})$ which decomposes as $\psi|_{L_E} = \mu \oplus \chi \otimes S_2$ such that χ are not characters of L_E, but rather automorphic characters of $E^\times \diagdown \mathbb{A}_E^\times$ with $\mu|_{\mathbb{A}_F^\times} = 1$ and $\chi|_{\mathbb{A}_F^\times} = \omega_{E/F}$. For $S_\psi = \mu_2 \xrightarrow{\Delta} \prod_v S_{\psi_v}$, we have $S_{\psi_v} = \mu_2$ if v is inert in E and 1 otherwise. Then $\varepsilon_\psi : \mu_2 \to \langle \pm 1 \rangle$ which is trivial when $\varepsilon(1/2, \chi\mu^{-1}) = 1$ and nontrivial if this is -1. So $\prod_{\psi_v} = \left\{ \pi_v^+, \pi_v^- \right\}$ if v is inert, and just $\left\{ \pi_v^+ \right\}$ otherwise, meaning

$$m(\pi^\varepsilon) = \begin{cases} 1 & \prod_v \varepsilon_v = \varepsilon(1/2) \\ 0 & \text{otherwise.} \end{cases}.$$

For almost every v, $\pi_{1_v} = \pi_v^+$. Something about $\mathrm{Ind}_{B_v}^{U_?} \chi|-|_v^{-\frac{1}{2}} \otimes \tilde{\mu}$. Something about Howe-PS.

11 Aaron Pollack: Modular forms on exceptional groups (Lecture 1)

Remark 11.0.1: Plans for lectures:

1. What is G_2, what are modular forms on it?
2. Fourier expansions of modular forms on G_2.
3. Examples and theorems about modular forms on G_2.
4. Beyond G_2, possibly $\mathbf{E}_8$.

Remark 11.0.2: First generalize modular forms to modular functions: let $f : \mathfrak{h} \to \mathbb{C}$ be a modular form of level Γ and weight $\ell > 0$. Define

$$\begin{aligned} \varphi_f &: \mathrm{SL}_2(\mathbb{R}) \to \mathbb{C} \\ \varphi_f(g) &:= j(g, z)^{-\ell} f(gz) \end{aligned}$$

$$j\left(g = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, z\right) := cz + d.$$

Some properties:

1. Growth: φ_f is of moderate growth.

2. Invariance: $\varphi_f(\gamma g) = \varphi_f(g)$ for all $\gamma \in \Gamma \leq \mathrm{SL}_2(\mathbb{Z})$.
3. Equivariance on a compact: $k_\theta := \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \in \mathrm{SO}_2(\mathbb{R})$ satisfies $\varphi_f(gk_\theta) = e^{-i\ell\theta}\varphi_f(g)$
4. Operator equation: $D_{\mathrm{CR}}\varphi_f \equiv 0$ where we decompose the complexified Lie algebra

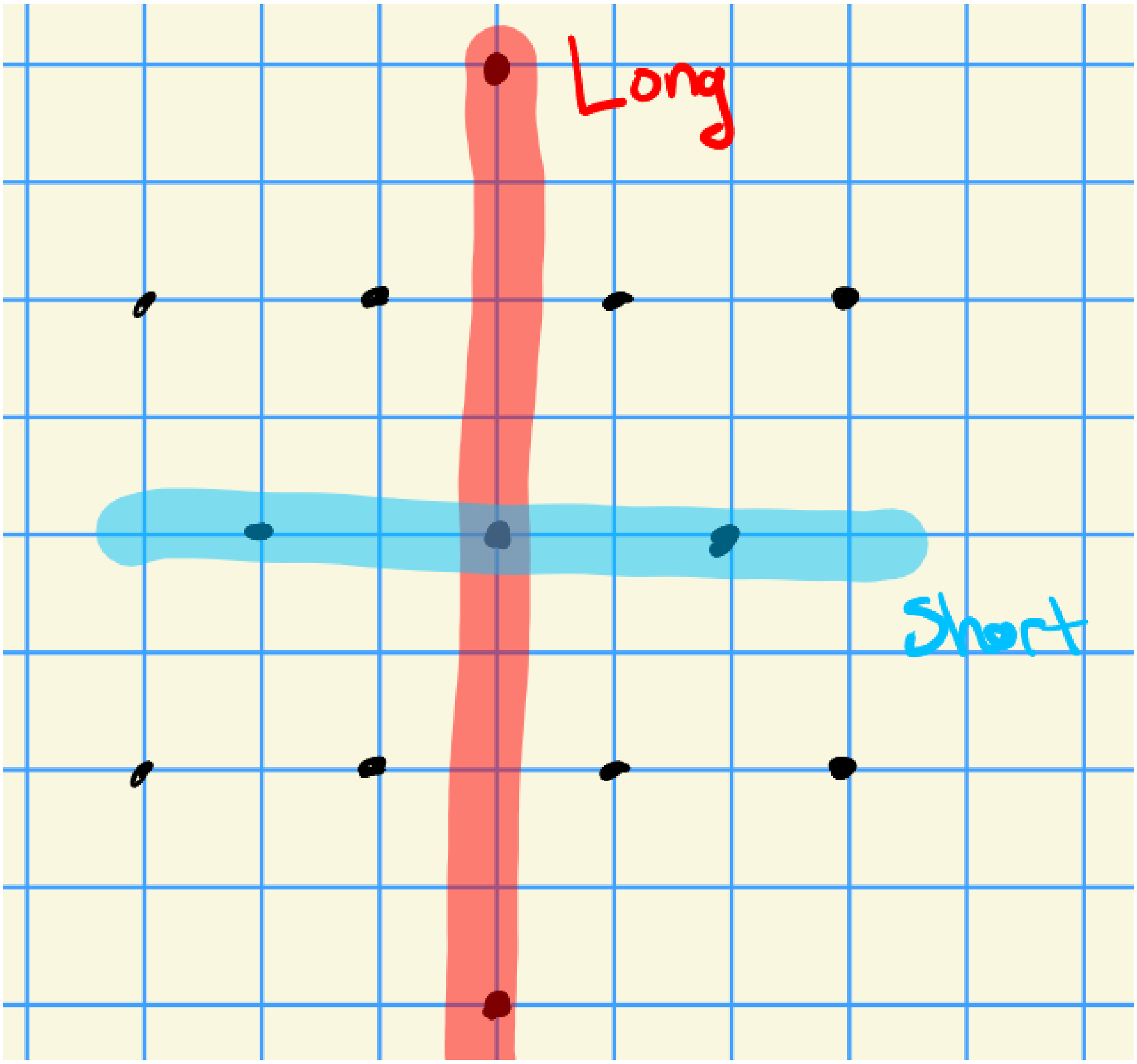
$$\mathfrak{sl}_3(\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C} \cong (k_0 \otimes \mathbb{C}) + (\varphi_0 \otimes \mathbb{C})$$

as antisymmetric and symmetric parts, then $p_0 \otimes \mathbb{C} = \mathbb{C}X_+ + \mathbb{C}X_-$ where $X_\pm = \begin{bmatrix} 1 & \pm i \\ \pm i & -1 \end{bmatrix}$, and $D_{\mathrm{CR}}\varphi_f := X_- f$.

Conversely, if $\varphi : \mathrm{SL}_2(\mathbb{R}) \to \mathbb{C}$ satisfies these properties, then $f(z) = j(g_z, w^\ell)\varphi(g_z)$ where $g_z \cdot c = z$ is well-defined, holomorphic, weight ℓ, level Γ modular forms.

11.1 Modular forms on G_2

Remark 11.1.1: Recall that G_2 is a simple noncompact Lie group of dimension 14, with maximal compact $K = (\mathrm{SU}_2 \times \mathrm{SU}_2)/\langle \pm I \rangle$. Write the first factor as SU_2^l for "long" and the second as SU_2^s for "short", then the root system looks like the following:



There is an action of K on $V_\ell := \mathrm{Sym}^\ell(\mathbb{C}^2) \otimes \mathbb{1}$, and the diagonal acts trivially.

Definition 11.1.2 (Modular forms on G_2)
Suppose $\Gamma \leq \mathsf{G}_2$ is a congruence subgroup, so $\Gamma = \mathsf{G}_2(\mathbb{Q}) \cap K_f$ where $K_f \subseteq \mathsf{G}_2(M_f)$, and let $\ell \in \mathbb{Z}_{>0}$,
A **modular form** of weight ℓ and level Γ is a map $\varphi : \mathsf{G}_2 \to V_\ell$ such that

1. Growth: φ has moderate growth.
2. Invariance: $\varphi(\gamma g) = \varphi(g)$ for all $\gamma \in \Gamma$.
3. Equivariance on a compact: $\varphi(gk) = k^{-1}\varphi(g)$ for all $k \in K$.
4. Operator equation: $D_\ell \varphi = 0$.

Equivalently, a map $\varphi : \mathsf{G}_2(\mathbb{Q}) \backslash \mathsf{G}_2(\mathbb{A})$ satisfying similar conditions.

Remark 11.1.3: The upshot: modular forms on G_2 have a classical Fourier expansion and Fourier coefficients, which appear very arithmetic.

11.2 What is G_2?

Remark 11.2.1: Todos:

- What is G_2?
- What is D_ℓ?
- What are some examples/theorems about modular forms on G_2?

Remark 11.2.2: We'll define a C_3-graded Lie algebra over $\mathbb{Q}$:

$$\mathfrak{g}_2 = \mathfrak{sl}_3[0] + V_g(\mathbb{Q})[1] + {V_3}^\vee(\mathbb{Q})[2],$$

where $\mathfrak{sl}_3$ are the traceless matrices as usual and V_3 is the 3-dimensional standard representation of $\mathfrak{sl}_3$. The grading will mean that $[x, y]$ will land in degree $|x| + |y|$. The bracket is defined as follows:

$$\begin{aligned} [\varphi, \varphi'] &\coloneqq \varphi\varphi' - \varphi'\varphi & \varphi, \varphi' &\in \mathfrak{sl}_3 \\ [\varphi, v] &\coloneqq \varphi(v), & v &\in V_3 \\ [\varphi, \delta] &\coloneqq \varphi(\delta), & \delta &\in {V_3}^\vee. \end{aligned}$$

Observation 11.2.3 (Constructing G_2)

$$\bigwedge^3 V_3 = \mathbb{1} \implies \bigwedge^2 V_3 = {V_3}^\vee \implies \bigwedge^2 (V^\vee) = V_3.$$

Fix a basis $V_3 = \langle v_1, v_2, v_3 \rangle$ and ${V_3}^\vee = \langle \delta_1, \delta_2, \delta_2 \rangle$ its dual basis, then

- $v_i \vee v_{i+1} = \delta_{i-1}$
- $\delta_i \vee \delta_{i+1} = \delta_{i-1}$

Moreover,

$$\begin{aligned} [v, v'] &= 2v \vee v' \in \bigwedge^2 V_3 \cong {V_3}^\vee \\ [\delta, \delta'] &= 2\delta \vee \delta' \in \bigwedge^2 {V_3}^\vee \cong V_3 \\ [\delta, v] &= 3v \otimes \delta - \delta(v)\mathbb{1} \in \mathfrak{sl}_3, \end{aligned}$$

noting that the last is traceless and $3v \otimes \delta \in V_3 \otimes {V_3}^\vee \cong \operatorname{End}(V_3)$. All other brackets are determined by antisymmetry and linearity

Proposition 11.2.4 ***(Construction of*** G_2***).***
The algebra $\mathfrak{g}_2$ as defined above is a simple Lie algebra, i.e. the Jacobi identity holds and there are no nontrivial ideals. Moreover

$$\mathrm{Aut}(\mathfrak{g}_2) = \left\{ g \in \mathrm{GL}(\mathfrak{g}_2) \;\middle|\; [gx, gy] = g[x, y] \,\forall x, y \in \mathfrak{g}_2 \right\}$$

and $\mathsf{G}_2 \cong \overset{0}{\mathrm{Aut}}(\mathfrak{g}_2)$ is the connected component.

Remark 11.2.5: Note: a similar procedure can be used to define all of the exceptional groups, see notes.

Remark 11.2.6: What is the root diagram for $\mathfrak{g}_2$? Let $\mathfrak{h} \le \mathfrak{sl}_3$ be the diagonal elements, i.e.

$$\mathfrak{h} = \left\{ \sum_{1 \le i \le 3} \alpha_i E_{ii} \;\middle|\; \sum \alpha_i = 0 \right\},$$

and let $r_1, r_2, r_3 : \mathfrak{h} \to \mathbb{Q}$ be such that

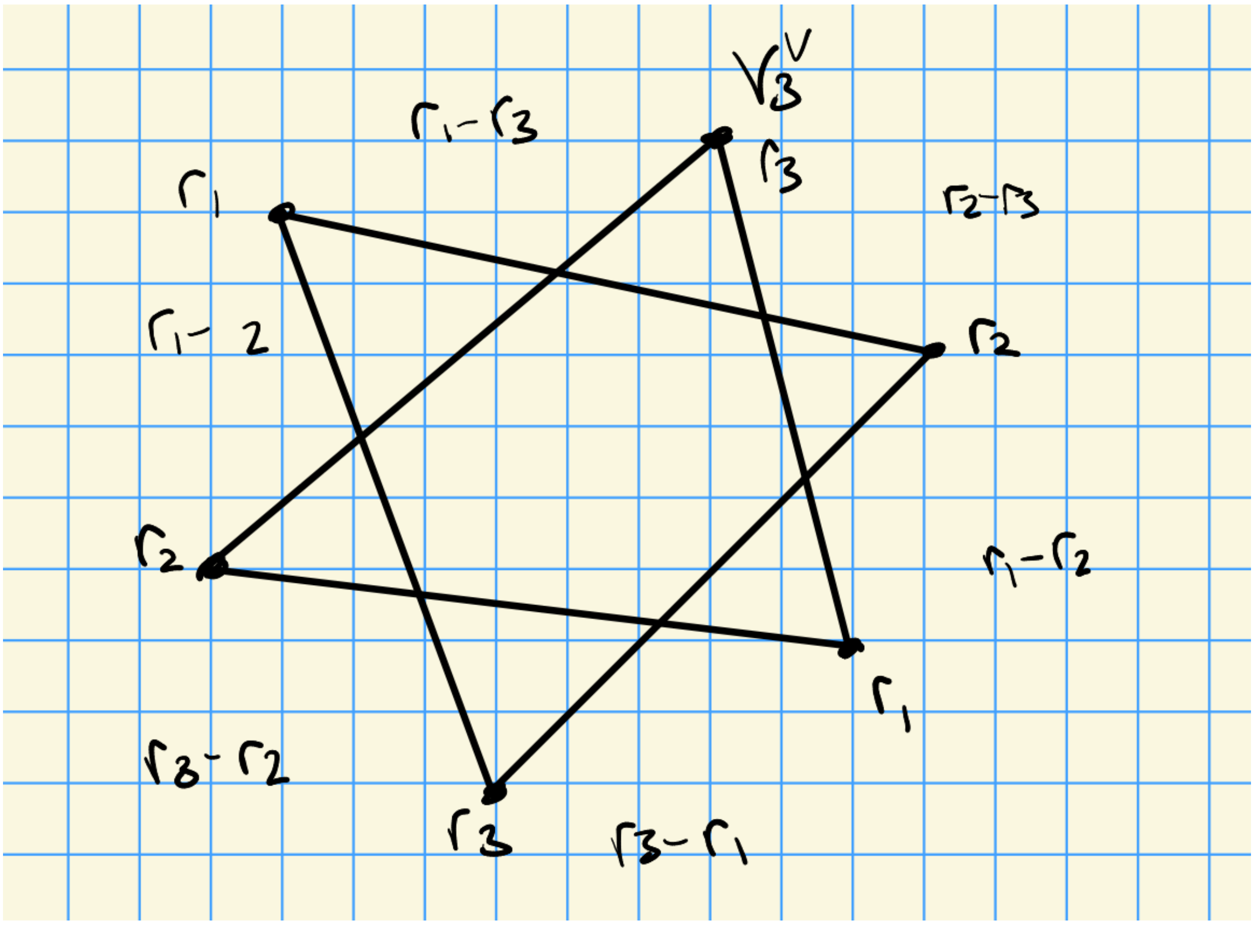
$$r_j \sum_{1 \le i \le 3} \alpha_i E_{ii} = \alpha_j,$$

i.e. projection onto the jth component. Note that $\sum r_i = 0$.

What are the weights of $\mathfrak{h}$ on $\mathfrak{g}_2$? Since $\mathfrak{g}_2 = \mathfrak{sl}_3 + V_3 + V_3^\vee$, the actions are:

- On V_3 it acts by r_1, r_2, r_3.
- On $V_3^\vee$ it acts by $-r_1, -r_2, -r_3$.
- On $\mathfrak{sl}_3$ it acts by $\left\{ r_i - r_j \;\middle|\; i \ne j \right\}$.

This yields a root diagram:



Remark 11.2.7: On the differential operator: take the Cartan involution

$$\Theta : \mathfrak{g}_2 \otimes \mathbb{R} \to \mathfrak{g}_2 \otimes \mathbb{R}.$$

Explicitly,

- On $\mathfrak{sl}_3$, this acts as $X \mapsto {}^{-t}X$
- On V_3, it's $V_3 \mapsto {V_3}^\vee$ by $v_j \mapsto \delta_j$.

Define

- $k_0 = (\mathfrak{g}_2 \otimes \mathbb{R})^{G=\mathrm{id}}$
- $p_0 = (\mathfrak{g}_2 \otimes \mathbb{R})^{G=-\mathrm{id}}$
- $K = \left\{ g \in \mathsf{G}_2 \;\middle|\; \mathrm{Ad}_g \circ \Theta = \Theta \circ \mathrm{Ad}_g \right\}$
- $k = k_0 \otimes \mathbb{C}$, something about $\mathfrak{sl}_3 + \mathfrak{sl}_2$
- $p = p_0 \otimes \mathbb{C}$, something about $V_2 \otimes \mathrm{Sym}^3(V_2)$.
- $D_\ell = \mathrm{pr}\tilde{D}_\ell$, which we'll define.

Suppose $\varphi : \mathsf{G}_2 \to V_\ell = \mathrm{Sym}^{2\ell}(\mathbb{C}^2) \otimes \mathbb{1}$ such that $\varphi(gk) = k^{-1}\varphi(g)$ for all $k \in K$. Let $\{X_\alpha\}$ be a basis of p, $\{X_\alpha{}^\vee\}$ basis of $p^\vee$, then

$$\tilde{D}_\ell \varphi = \sum_\alpha X_\alpha p \otimes X_\alpha{}^\vee \in V_\ell \otimes p^\vee.$$

where $X_\alpha \varphi$ is the derivative of the right regular action, i.e. if $X \in p_0$,

$$(X_p)(g) = \frac{\partial}{\partial t} \varphi(g \exp(tx))\Big|_{t=0}.$$

Then

$$\begin{aligned} V_\ell \otimes \varphi^\vee &= (S^{2\ell} \otimes \mathbb{1}) \times V_\ell \boxtimes \operatorname{Sym}^3(V_2) \\ &= (S^{2\ell+1} + S^{2\ell-1}) \boxtimes S^3(V_2) \\ &\xrightarrow{\text{pr}} S^{2\ell-1}(V_\ell) \boxtimes S^3(V_3). \end{aligned}$$

This relates

- $\mathsf{G}_2 \rightsquigarrow \mathrm{SL}_2$
- ?

12 | Aaron Pollack, Talk 2

Remark 12.0.1: Last time: modular forms on G_2. Note that G_2 over K does not have a G_2-invariant complex structure, while $\mathrm{SL}_2(\mathbb{R})/\mathrm{SO}_2 = \mathfrak{h}$ has an $\mathrm{SL}_2(\mathbb{R})$-invariant complex structure.

Remark 12.0.2: Today: let $f(z) = \sum_{k \geq 0} a_f(k) q^k$ of weight ℓ where $\varphi_f(g) = j(g,i)^{-\ell} f(gi)$ where $\varphi_f : \mathrm{SL}_2(\mathbb{R}) \to \mathbb{C}$. Define

$$\begin{aligned} W_n : \mathrm{SL}_2(\mathbb{R}) &\to \mathbb{C} \\ g &\mapsto j(g,i)^{-\ell} \exp(2\pi i n(gi)). \end{aligned}$$

Some properties:

- $W_n\left(\begin{bmatrix} 1 & * \\ 0 & 1 \end{bmatrix} g\right) = e^{2\pi i n x} W_n(g)$
- $W_n(gk_0) = e^{-i\ell\theta} W_n(g)$ where $k_\theta = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$,
- $X_- W_n = 0$
- $W_n \operatorname{diag}(y^{\frac{1}{2}}, y^{-\frac{1}{2}}) = y^{\frac{\ell}{2}} e^{2\pi i n y}$ is complete explicit
- $\varphi_f(g) = \sum a_f(k) W_k(g)$ is the Fourier expansion.

Remark 12.0.3: What will happen: we'll define $\varphi : \Gamma\diagdown \mathsf{G}_2 \to V_\ell$ where $V_\ell = \mathrm{Sym}^2(\mathbb{C}^2) \otimes \mathbb{1}$ which admits an action by $K = \mathrm{SU}_2 \times \mathrm{SU}_2/\pm I$. In this case, we'll essentially have $\varphi \approx \sum_{f\in ?} a_\varphi(f) W_f(g)$ where the $a_\pi(f) \in \mathbb{C}$ are Fourier coefficients and W_f satisfies similar properties.

Remark 12.0.4: Recall that $\mathfrak{g}_2 = \mathfrak{sl}_3 + V_3 + V_3{}^\vee$, spanned by $\{E_{ij}\}, \{v_1, v_2, v_3\}, \{\delta_1, \delta_2, \delta_3\}$ respectively.

Note that

- G_2 has 2 conjugacy classes of maximal parabolics,
- P will be the parabolic where $\mathsf{Lie}(P)$ yields the top 3 layers of the root diagram
- $P = MN$ where $M \cong \mathrm{GL}_2$ and $N \supseteq Z = [N, N]$ with N/Z abelian.

We want to define a Fourier expansion along the unipotent radical of P.

Remark 12.0.5: Some facts:

- $Z = \exp(\mathbb{R}E_{13})$
- $W = \mathbb{R}E_{12} + \mathbb{R}v_1 + \mathbb{R}\delta_3 + \mathbb{R}E_{23}$
- $N/Z = \exp(W)$
- $M \curvearrowright Z$ by the determinant
- $M \curvearrowright N/Z$ as $\mathrm{Sym}^3(V_3) \otimes \det(V_3)^{-1}$
- There is a symplectic form on W where $[w, w'] = \langle w,\ w'\rangle E_{13}$
- Explicitly, one can write $w = \sum aE_{12} + \frac{b}{3}v_1 + \frac{c}{3}\delta_3 + dE_{13}$ and w' similarly, then $\langle w,\ w'\rangle = ad' - \frac{bc'}{3} + \frac{cb'}{3} - da'$, and $\langle mw,\ mw'\rangle = \det(m)\langle w,\ w'\rangle$.

Remark 12.0.6: What are the characters of N? Suppose

- φ is an automorphic form on $\mathsf{G}_2(\mathbb{A})$
- $\psi : \mathbb{Q}\diagdown \mathbb{A} \to \mathbb{C}^\times$ is a fixed adelic character
- $w \in W(\mathbb{Q})$

Define

$$\varphi_w(g) := \int_{[N]} \psi^{-1}(\langle w,\ \bar{n}\rangle)\varphi(ng)\, dn,$$

where $\bar{n}$ is the image of n in N/Z which we identify with W via the exponential. Similarly define

$$\varphi_Z(g) = \int_{[Z]} \varphi(zg)\, dz, \qquad \varphi_N(g) = \int_{[N]} \varphi(zg)\, dz.$$

Then

$$\varphi_Z(g) = \varphi_N(g) + \sum_{w\in W(\mathbb{Q})} \varphi_w(g),$$

and we'll produce a refinement.

Proposition 12.0.7 *(?).*

$$\varphi_Z(g) \equiv 0 \implies \varphi(g) \equiv 0.$$

12.1 Generalized Whittaker Functions

Definition 12.1.1 (Generalized Whittaker functions)
Suppose $\varphi : \mathsf{G}_2(\mathbb{Q}) \diagdown \mathsf{G}_2(\mathbb{A}) \to V_\ell$ is a modular form of weight ℓ. These satisfy

- $\varphi_w(g)$ is of moderate growth.
- $\varphi_w(g) = \psi(\langle w,\ \bar{n}\rangle)\psi_W(g)$ (equivariant for the Heisenberg parabolic)
- $\varphi_w(gk) = k^{-1}\varphi_w(g)$ (equivariant for K)
- $D_\ell \varphi_w = 0$.

Call such functions satisfying these properties **general Whittaker functions of type** (w, ℓ).

Remark 12.1.2: We'll show that such functions are uniquely determined up to a scalar multiple, i.e. for some explicit W_w,

$$\varphi_w(g) = \lambda W_w(g).$$

From this, we'll obtain a Fourier expansion for φ a modular form of weight ℓ:

$$\varphi_Z(g) = \varphi_N(g) + \sum_{w \neq 0} a_\varphi(w) W_w(g).$$

Remark 12.1.3: Identify W as a space B of binary cubics under

$$W \to B$$
$$w := aE_{12} + \frac{b}{3}v_1 + \frac{c}{3}\delta_3 + dE_{23} \mapsto f_w := au^3 + bu^2v + cuv^2 + dv^3.$$

For $w \in W(\mathbb{R}) \setminus \{0\}$, for $m \in \mathrm{GL}_2(\mathbb{R})$ define

$$\beta_w(m) := \left\langle w,\ m \cdot (u{-}?v)^3 \right\rangle,$$

which will appear in Fourier expansions.

Proposition 12.1.4 *(?).*
TFAE:

- $\beta_w(m) \neq 0$ for all $m \in \mathrm{GL}_2(\mathbb{R})$,
- $f_w(z, 1) \neq 0$ for $z \in \mathfrak{h}$,

- f_w splits into linear factors over $\mathbb{R}$.

Definition 12.1.5 (PSD)
If w satisfies these properties, say w is **positive semidefinite** and write $w \geq 0$.

Example 12.1.6 ***(of PSD binary cubics):***

- $f_w(u, v) := au^3 \geq 0$.
- $f_w(u, v) := -u^3 + uv^2 = u(v-u)(v+u) \geq 0$.
- $f_w(u, v) := u^3 + v^2 \ngeq 0$.

Definition 12.1.7 (?)
For $m \in \mathrm{GL}_2(\mathbb{R}) = M(\mathbb{R})$ and $w \geq 0$ PSD,

$$W_w(m) = |\det w| \det(w)^\ell \sum_{-\ell \leq v \leq 0} \left(\frac{|\beta_w(m)|}{\beta_w(m)} \right)^v K_v(|\beta_w(m)|) \frac{x^{\ell+v} y^{\ell-v}}{(\ell+v)!(\ell-v)!}.$$

where x, y are a fixed basis of V_ℓ, and K_v is a classical K-Bessel function

$$K_v(y) := \frac{1}{?} \int_0^N e^{-\frac{y(t+t^{-1})}{2}} t^v \frac{dt}{t},$$

which diverges at $y = 0$.

Remark 12.1.8: These functions $W_w : M(\mathbb{R}) \to V_\ell$ extend uniquely to $\mathsf{G}_2 \to V_\ell$, viz

- $W_w(ng) = e^{2\pi i \langle w, \overline{n} \rangle} W_w(g)$ for all $n \in N(\mathbb{R})$
- $W_w(gk) = k^{-1} W_w(g)$ for all $k \in K$.

Theorem 12.1.9 ***(?).***
Suppose $w \neq 0$ and F is a generalized Whittaker function of type (w, ℓ). Then

- $w \ngeq 0 \implies F = 0$
- $w \geq 0 \implies F(g) = \lambda W_w(g)$ for some $\lambda \in \mathbb{C}$

Consequently, if φ is a modular form on G_2 of weight ℓ, there exist $a_\varphi(w) \in \mathbb{C}$ with

$$\varphi_Z(g) = \varphi_N(g) + \sum_{w \geq 0 \text{ integral}} a_\varphi(w) W_w(g).$$

Moreover, φ_N can be explicitly described in terms of modular forms of weight 3ℓ on GL_2.

Definition 12.1.10 (?)
The terms $a_\varphi(w)$ are by definition the Fourier coefficients of φ.

Remark 12.1.11: Gan-Gross-Savim used a multiplicity 1 result of Wallach to define the Fourier coefficients without using the explicit function $W_w(s)$.

13 Aaron Pollack, Talk 3: Examples of (and theorems about) modular forms on G_2

13.1 Degenerate Eisenstein Series

Remark 13.1.1: Recall $\mathsf{G}_2 \supseteq P$ a Heisenberg parabolic, with $P = MN$ where $M \cong \mathrm{GL}_2$. Write ν for the composition $P \to M \xrightarrow{\det} \mathrm{GL}_1$. Suppose $\ell > 0$ is even, and recall that $V_\ell = \mathrm{Sym}^2(\mathbb{C}^2) \otimes \curvearrowleft K \leq \mathsf{G}_2$ for K a maximal compact. Let

$$f_{\ell,\infty}(g;s) = \mathrm{Ind}_{P(\mathbb{R})}^{\mathsf{G}_2(\mathbb{R})} |\nu|^s \otimes V_\ell$$

be defined by

$$\begin{aligned} f_{\ell,\infty}(pg^j s) &= N(\mu)^s f_{e\ll,\infty}(g;s) \quad \forall p \in P(\mathbb{R}) \\ f_{\ell,\infty}(gk;s) &= k^{-1} f_{\ell,\infty}(g) \quad \forall k \in K. \end{aligned}$$

By the Iwasawa decomposition $C_3(\mathbb{R}) = P(\mathbb{R})K$, f is uniquely determined one we set

$$f_{\ell,\infty}(1) = x^\ell y^\ell \in V_\ell = \left\langle x^?, x^?, \cdots, y^{2\ell} \right\rangle.$$

Let $f_?$ be a flat section in $\mathrm{Ind}_{P(\mathbb{A}_f)}^{\mathsf{G}_2(\mathbb{A}_f)}(|\nu|^3)$, and let $f_g(g,s) = f_?(gf,s) f_{\ell,\infty}(g_?;s) \in \mathsf{G}_2(\mathbb{A})$. Define

$$E_\ell(g,f,s) = \sum_{\gamma \in P(\mathbb{Q}) \diagdown \mathsf{G}_2(\mathbb{Q})} f_\ell(\gamma g, s).$$

If $\Re(s) > 3$, set $E_\ell(g) := E_\ell(g,f,s = \ell+1)$.

Theorem 13.1.2 *(?)*.
If $\ell > 0$ is even and $\ell \geq 4$, then $E_\ell(g)$ is a quaternionic modular form on G_2 of weight ℓ.

Proof (?).
$f_{\ell,?}(g, s = \ell+1)$ is annihilated by D_ℓ, so $E_\ell(g)$ is as well by absolute convergence. ■

Remark 13.1.3: If π is a cuspidal automorphic representation of $\mathrm{GL}_2 = M$ associated to a holomorphic weight 3ℓ modular form which is cuspidal,

- $f_\pi \in \mathrm{Ind}_{P(\mathbb{A})}^{\mathsf{G}_2(\mathbb{A})}(\pi)$
- $E(g, f_\pi) = \sum_{\gamma \in P(\mathbb{Q}) \backslash \mathsf{G}_2(\mathbb{Q})} f_\pi(\gamma g)$
- If $\ell \geq 6$ this is a weight ℓ modular form on G_2.
- If $\ell = 4$, one can make sense of $E(g, f_\pi)$ using analytic continuation to produce a weight 4 modular form on G_2 associated to the Ramanujan Δ.

Fact 13.1.4
If φ is a level 1 quaternionic modular form on G_2,

- $a_\varphi(w) \neq 0 \implies f_w(u,v) = au^3 + \cdots + dv^3$ with $a, \cdots, d \in \mathbb{Z}$ (integrality)
- $a_p(w\gamma) = \det(\gamma)^\ell a_\varphi(w)$ for $r \in \mathrm{GL}_2(\mathbb{Z})$, so Fourier coefficients are constant on orbits of binary cubic forms.

Fact 13.1.5
There is a bijection

$$\{\text{Integral binary cubic forms}\} \,/\, \mathrm{GL}_2(\mathbb{Z}) \rightleftharpoons \{\text{Cubic rings}\}_{/\sim},$$

where cubic rings are free rank 3 $\mathbb{Z}$-algebras. Thus if $\ell > 0$ is even, φ is a level 1 weight ℓ modular form on G_2, and A is a cubic ring, there is a well-defined map $a_\varphi(A) = a_\varphi(w)$ if $A \rightleftharpoons f_w$.

Remark 13.1.6: If f_w is nondegenerate, the cubic ring $A(f_w)$ associated to f_w is totally real $\iff f$ is positive semidefinite.

Theorem 13.1.7*(?)*.
Suppose A is the maximal order in a totally real cubic etale $\mathbb{Q}$-algebra E. There exists a constant $c_\ell \in \mathbb{C}$, independent of A, such that

$$a_{E_\ell}(A) = c_\ell \zeta_E(1-\ell),$$

where ℓ is even. The LHS are Fourier coefficients of modular forms on G_2.

Remark 13.1.8: It is not known that c_ℓ is nonzero.

Question 13.1.9
An open question: $E(g, f_\pi)$ is Eisenstein, can anything be said about its Fourier coefficients.

13.2 Cusp Forms

Theorem 13.2.1 ***(?).***
Suppose $\ell \geq 16$ is even. There exist nonzero cusp forms on G_2 of weight ℓ, all of whose Fourier coefficients are algebraic integers.

Proof (of theorem).
Steps:

1. Start with a holomorphic Siegel modular form f on SP_4 of weight ℓ, so f has Fourier coefficients in $\overline{\mathbb{Z}}$
2. Take a G-lift of f to $\mathrm{SO}_{4,4}$ to obtain $G(f)$, and define
$$\Theta(f)(g) = \int_{[\mathrm{SP}_4]} \theta(g,h)\overline{f(h)}\, dh,$$
then Θ on $\mathrm{SO}_{4,4} \times \mathrm{SP}_4$ is a θ function.
3. There is a good theory of quaternionic modular forms on $\mathrm{SO}_{4,n}$, so choose $\theta(g,h)$ such that $\Theta(f)$ is a one of weight ℓ (and cuspidal).
4. Express the Fourier coefficients of $\Theta(f)$ in terms of classical Fourier coefficients of f, showing that the Fourier coefficients of $G(f)$ are in $\overline{\mathbb{Z}}$.
5. Use $\mathsf{G}_2 \overset{\iota}{\hookrightarrow} \mathrm{SO}_{4,4}$ and pullback to obtain $i^*(\Theta(f))$, which is still cuspidal and has Fourier coefficients that are sums of the original coefficients, so still in $\overline{\mathbb{Z}}$.

■

Theorem 13.2.2 ***(R. Dalal).***
There is an explicit dimension formula for the level 1 cuspidal quaternionic modular forms of weight ℓ. In particular, the smallest is a level 1 cusp form of weight 6.

Theorem 13.2.3 ***(Cicek-Dadivdoff,Dijok,Hammonds, P, Roy).***
Suppose φ is a level 1 cuspidal quaternionic modular form on G_2 associated to a cuspidal automorphic representation π on $\mathsf{G}_2(\mathbb{A})$. Suppose that the Fourier coefficient $a_\varphi(\mathbb{Z}^{\times^3}) \neq 0$, then

1. The complete standard L-function of π has a functional equation:
$$\Lambda(\pi, \mathrm{std}, s) = \Lambda(\pi, \mathrm{std}, 1-s).$$
2. There exists a Dirichlet series for this L-function expressing the Fourier coefficients in terms of an L-function:
$$\sum_{T \subseteq \mathbb{Z}^{\times^3}, n \geq 1} \frac{a_\varphi(\mathbb{Z}+nT)}{[\mathbb{Z}^{\times^3} : T]^{s-\ell+1}} n^{-s} = a_p(\mathbb{Z}^{\times^3}) \frac{L(\pi, \mathrm{std}, s - z\ell + 1)}{\zeta(s-2\ell+2)^2 \zeta(2s-4\ell+2)}.$$

Proof (?).
Carry out a refined analysis of a Rankin-Selberg integral (due to Gurevich-Segal). ■

13.3 A Theorem

Remark 13.3.1: There is a theory of *half*-integral weight modular forms on G_2. These have a good notion of Fourier coefficients taking values in $\mathbb{C}/\{\pm 1\}$.

Suppose $R \subseteq E$ is a cubic ring in a totally real cubic field. Let ∂_R be the different, and let Q_R be the square roots of ∂_R^{-1} in the narrow class group of E. Say (I, μ) is balanced

- I is a fractional ideal in R.
- $\mu \in E_{>0}^\times$ is totally positive
- $I\mu^2 \subseteq \partial_R^{-1}$
- $N(I)^2 N(\mu)\mathrm{disc}(R) = 1$.

Note that if R is the maximal order, (I, μ) is balanced iff $I^2\mu = \partial_R^{-1}$. Define an equivalence relation by

$$(I, \mu) \sim (I', \mu') \iff \exists \beta \in E^\times, I' = \beta I, \mu' = \beta^{-2}\mu$$

and set Q_R to be the balanced pairs mod equivalence.

Remark 13.3.2:
- Q_R can be empty
- For Q_R nonempty and R a maximal order in E,

$$|Q_R| = \sharp \mathrm{Cl} E^\times[2].$$

Theorem 13.3.3 ***(Leslie-P).***
There exists a weight $1/2$ modular form θ' on G_2 whose Fourier coefficients include the numbers $\pm|Q_R|$ for R *even monogenic*, i.e.

$$R = \mathbb{Z}[y]/\left\langle y^3 + cy^2 + by + a, a, b, c \in 2\mathbb{Z}\right\rangle.$$

Proof (?).
Define Θ on $\tilde{\mathrm{F}}_4$? Then let Θ' be a pullback along $\mathsf{G}_2 \to \mathrm{F}_3$. ■

14 | Aaron Pollack, Talk 4: Beyond G_2

Remark 14.0.1: Upshot for today: there exist groups $\mathsf{G}_3, \mathrm{F}_4, \mathbf{E}_{n,4}$ for $n = 6, 7, 8$ where G_2 is split and the $\mathbf{E}$ groups are rank 4 over $\mathbb{R}$. These admit modular forms with Fourier expansions and coefficients similar to the G_2 story. We'll define these exceptional groups today.

14.1 Exceptional Algebras

Remark 14.1.1: Let be a composition algebra over k, where $\operatorname{ch} k = 0$, with a multiplication $C^{\otimes_k^2} \to C$ which is not necessarily commutative or associative. There exists a norm map $n_C : C \to K$ given by a nondegenerate quadratic form with $n_C(xy) = n_C(x)n_C(y)$.

Example 14.1.2*(?)*:

- $C = k$ and

$$\begin{aligned} n_c : k &\to k \\ x &\mapsto x^2. \end{aligned}$$

- $C = E/k$ for E a quadratic etale extension, $n_C = \mathrm{Nm}_{E/k}$.
- $C = B/k$ for B a quaternion algebra with $n_C = n_{B,\mathrm{red}}$
- $C = \Theta$ an octonion algebra, with $\Theta = B \bigoplus B$. There is an involution $C \to C$ with
  - $x + x^* = \mathrm{Trace}_C(x)1 \in k1$,
  - $xx^* = n_C(x)1 \in k1$

Definition 14.1.3 (?)
Let $J_C = H_3(C)$ be Hermitian 3×3 matrices with coefficients in C, so

$$J_C = \left\{ \begin{bmatrix} c_1 & x_3 & x_2^* \\ x_3^* & c_2 & x_1 \\ x_2 & x_1^* & c_3 \end{bmatrix} \;\middle|\; c_i \in k, c_i \in C \right\}.$$

This has dimension $3 + 3C$ over k.

Example 14.1.4*(?)*: For $C = k, H_3(K)$ are symmetric 3×3 matrices and there is a determinant map

$$\begin{aligned} \det : J_C &\to k \\ X &\mapsto c_1c_2c_3 - \sum c_i n_C(x_i) + \mathrm{Trace}_C(x_1(x_2x_3)). \end{aligned}$$

If $C = k$ this is the usual determinant. Note that $M'_J = \left\{ g \in \mathrm{GL}(J_C) \,\middle|\, \det(gX) = \det(X) \forall X \in J_C \right\}$ has positive dimension, and is thus infinite, making it an interesting algebra.

Remark 14.1.5: Idea: there exists a group G_{J_C} such that

- $C = \mathbb{Q} \rightsquigarrow \mathrm{F}_4$,
- $C = K$ quadratic imaginary $\rightsquigarrow \mathbf{E}_{6,4}$,
- $C = B$ a quaternionic algebra $\rightsquigarrow \mathbf{E}_{7,4}$,
- $C = \Theta$ an octonionic algebra $\rightsquigarrow \mathbf{E}_{8,4}$.

All have a good notion of quaternionic modular forms and Fourier expansions/coefficients.

A degree map $J_C \xrightarrow{\deg} k$ commuting with $x \mapsto x^3$ and det recovers $G_{J_{C=k}} = \mathsf{G}_2$. Recall

$$\mathfrak{g}_2 = \mathfrak{sl}_{2,\ell}[0] + \mathfrak{sl}_{2,s}[0] + V_2 \otimes W[1]$$

which is C_2-graded; we'll mimic this to construct $\mathfrak{g}_{J_C}$.

14.2 Freudenthal Construction

Definition 14.2.1 (Quaternionic exceptional groups)
Let $J = J_C/\mathbb{Q}$ and $k = \mathbb{Q}$ and

$$W_J = \mathbb{Q} \bigoplus J \oplus J^\vee \oplus \mathbb{Q}.$$

There is a symplectic form

$$\langle [a, b, c, d],\ [a, b, c, d] \rangle = ad' - (b, c') - (c, b') - dc'.$$

There is a degree 4 polynomial map $q : W_J \to \mathbb{Q}$. Define

$$H_J^1 = \left\{ g \in \mathrm{GL}(W_J) \,\middle|\, \langle gw,\ gw' \rangle = \langle w,\ w' \rangle \forall w, w' \in W_J,\ q(gw) = q(w) \right\}.$$

This recovers:

C	H_J^1	$\mathfrak{g}_J$
$\mathbb{Q}$	C_3	F_4
K	A_5	$\mathbf{E}_6$
B	D_6	$\mathbf{E}_7$
Θ	E_7	$\mathbf{E}_8$

Define

$$\mathfrak{g}_J = \mathfrak{sl}_2[0] + \mathfrak{h}_J^0[0] + (V_2 \otimes W_J)[1],$$

where $\mathfrak{h}_J^0 = \mathsf{Lie}(H_J^1)$, and define

$$G_J := \overset{0}{\mathrm{Aut}}(\mathfrak{g}_J).$$

If $n_C : C \otimes \mathbb{R} \to \mathbb{R}$ is positive definite, we say G_J is a **quaternionic exceptional group**.

Fact 14.2.2
If $K_J \subseteq \mathsf{G}_J(\mathbb{R})$ is a maximal compact, then

$$K_J = \frac{\mathrm{SU}_2 \times L_J'}{\mu_2}$$

where L_J^1 is a compact form of H_J^1.

There is a Cartan involution $\theta : \mathfrak{g}_J \to \mathfrak{g}_J$, which over $\mathbb{C}$ yields

$$\mathfrak{g}_J^{\theta=\mathrm{id}} = k_0 \otimes \mathbb{C} \cong \mathfrak{sl}_2 + \mathfrak{h}_J^0$$
$$\mathfrak{g}_J^{\theta=-\mathrm{id}} = p_0 \otimes \mathbb{C} \cong V_2 + W_J.$$

Remark 14.2.3: There is an action $K_J \curvearrowright V_\ell = \mathrm{Sym}^{2\ell}(\mathbb{C}^2) \otimes \mathbb{1}$

Definition 14.2.4 (Modular forms)
A **modular form** on G_3 of weight ℓ is an automorphic form

$$\varphi : G_J(\mathbb{R}) \backslash G_J(\mathbb{A}) \to V_\ell$$

such that

1. $\varphi(gk) = k^{-1}\varphi(g)$ for all $k \in K_J$
2. $D_\ell \varphi \equiv 0$

Here D_ℓ is defined as in the G_2 case, replacing $\mathrm{Sym}^3(V_2) = W$ with W_J.

Remark 14.2.5: There is a Heisenberg parabolic $P = MN \leq G_J$ with $M = H_J$, and $N \supseteq Z$ a two-step filtration with Z 1-dimensional and $N/Z \cong W_J$ abelian.

Theorem 14.2.6 *(?).*
Modular forms on G_J of weight ℓ

1. Have Fourier coefficients and expansions along N/Z:

$$\varphi_Z(g) = \varphi_N(g) + \sum_{w \in W_J(\mathbb{Q}), w \geq 0} a_\varphi(w) W_w(g),$$

where $a_\varphi(w) \in \mathbb{C}$ are the Fourier coefficients of φ and W_w are completely explicit.

2. Under appropriate embeddings

$$\mathsf{G}_2 \hookrightarrow \mathrm{F}_4 \hookrightarrow \mathbf{E}_{6,4} \hookrightarrow E_{7,4} \hookrightarrow E_{8,4},$$

for a modular form φ of weight ℓ of one groups, the pullbacks $i^*\varphi$ to a smaller group are again modular forms of weight ℓ whose Fourier coefficients are sums of Fourier coefficients of φ.

Theorem 14.2.7 *(?).*

1. There exists a nonzero weight 4 modular form $\theta_{\min}$ on $\mathbf{E}_{8,4}$ with rational Fourier coefficients.
2. There exists a nonzero weight 8 modular form $\tilde{\theta}_{\min}$ on $\mathbf{E}_{8,4}$ with rational Fourier coefficients.

Proof (Sketch).

- Construct $\tilde{\theta}_{\min}$ using Eisenstein series
- Savim: most of the Fourier coefficients are zero, particularly the ones that are harder to compute explicitly
- By explicit computations, the remaining coefficients are rational.

■

Definition 14.2.8 (?)
Say a modular form φ on G_3 is **distinguished** iff

- There exists a $w_0 \in W_J(\mathbb{Q})$ such that $q(W_0) \neq 0$ and $a_\varphi(w_0) \neq 0$
- If $w \in W_J(\mathbb{Q}), a_\varphi(w) \neq 0$, then $q(w) \equiv q(w_0) \bmod (\mathbb{Q}^\times)^{\times 2}$.

Theorem 14.2.9 *(?).*
Suppose $K/\mathbb{Q}$ is quadratic imaginary, then there exists a distinguished modular form of weight 4 Θ_K on $G_{J_K} = \mathbf{E}_{6,4}$.

Proof (?).
Set $\Theta_K i^*(\Theta_{\min})$, pullback to $\mathbf{E}_{6,4}$. By arithmetic invariant theory, one shows it is distinguished.
■

15 Zhiwei Yun: Rigidity method for automorphic forms over function fields (Lecture 1)

Remark 15.0.1: Goal: construct automorphic data over function fields and working out the Langlands correspondence for such examples. Setup:

- $k = \mathbb{F}_q$,
- $X_{/k}$ a projective, smooth, geometrically connected algebraic curve.[1]
- $F \coloneqq k(X)$ its function field, noting that $\operatorname{trdeg} F = 1$.
- $|X| \coloneqq \mathrm{Places}(X)$, and $\mathcal{O}_x \twoheadrightarrow k_x$ with $F_x \subseteq \mathcal{O}_x$, $F_x = k_x((t_x))$ will be formal Laurent series and $\mathcal{O}_x \cong k[[t_x]]$.
- $G_{/k}$ a split semisimple group, e.g. $G = \mathrm{SL}_n, \mathrm{PGL}_n, \mathsf{Sp}_{2n}, \mathsf{G}_2, \mathbf{E}_8, \cdots$
- The adeles $\mathbb{A} = \prod_{x \in |X|}^{\text{res}} F_x$, where the F_x will be nonarchimedean fields,
- $G(\mathbb{A}) = \prod_{x \in |X|}^{\text{res}} G(F_x)$ where almost all components are in $G(\mathcal{O}_x)$.
- Level groups:
  - $K = \prod_{x \in |X|}^{\text{res}} K_x$ where $K_x \subseteq G(F_x)$ is a compact open and almost all $K_x = G(\mathcal{O}_x)$.
  - $K^\natural = \prod_{x \in |X|} G(\mathcal{O}_x)$
- Automorphic functions

$$\mathcal{A}_K = C^0\left(G(F) \backslash G(\mathbb{A}) / K \to \mathbb{C} \right).$$

Typically $\dim \mathcal{A}_K = \infty$, admits a left action by the Hecke algebra

$$\mathcal{H}_K \coloneqq C_c^0\left(K \backslash G(\mathbb{A}) / K \to \mathbb{C} \right)$$

[1] This is already interesting in the case of $X = \mathbb{P}^1$.

(the compactly supported functions) equipped with convolution with unit given by the characteristic function χ_K and is defined as

$$\begin{aligned}(f * g) : \mathbb{G}(\mathbb{A}) &\to \mathbb{C} \\ x &\mapsto \sum_{g \in G(\mathbb{A}) / K} f(xg)h(g^{-1}),\end{aligned}$$

where this sum is finite due to the compact support condition.

- $\mathcal{A}_{K,c}$ the compactly supported functions in $\mathcal{A}_K$.
- Cusp forms $\mathcal{A}_{K,\text{cusp}} = \left\{ f \in \mathcal{A}_{K,c} \;\middle|\; \dim_{\mathbb{C}} \mathcal{H}_K \cdot f < \infty \right\}$.
- Eigenforms f: f such that for almost all places x, f is an eigenvector for the action of

$$\mathcal{H}_{K_x} = C_c(K_x \backslash G(F_x) / K_x \to \mathbb{C}).$$

The goal of this theory is to study $\mathcal{A}_K$ as an object of $\mathcal{H}_{\mathsf{K}}$-Mod.

Definition 15.0.2 (Bun_G)
Define

$$\mathsf{Bun}_G = \{G\text{-bundles on } X\} \cong G(F) \backslash G(\mathbb{A}) / K^\natural, \qquad K^\natural \cong \prod_{x \in |X|} G(\mathcal{O}_x).$$

Example 15.0.3*(?)*: For $G = \mathrm{GL}_n$, passing from a vector bundle to its frame bundle yields a bijection

$$\begin{aligned}\{\mathrm{GL}_n\text{-bundles}\} &\rightleftharpoons \{\text{Vector bundes of rank } n\} \\ \mathrm{Isom}(\mathcal{O}^{\oplus^n}, \mathcal{E}) \curvearrowleft \mathrm{GL}_n &\mapsfrom \mathcal{E},\end{aligned}$$

where the Isom sheaf is regarded as principal G-bundles over X. This generalizes the frame bundle construction.

Observation 15.0.4
Due to Weil: enrich to sets with automorphisms, i.e. groupoids. Then there is an equivalence of groupoids

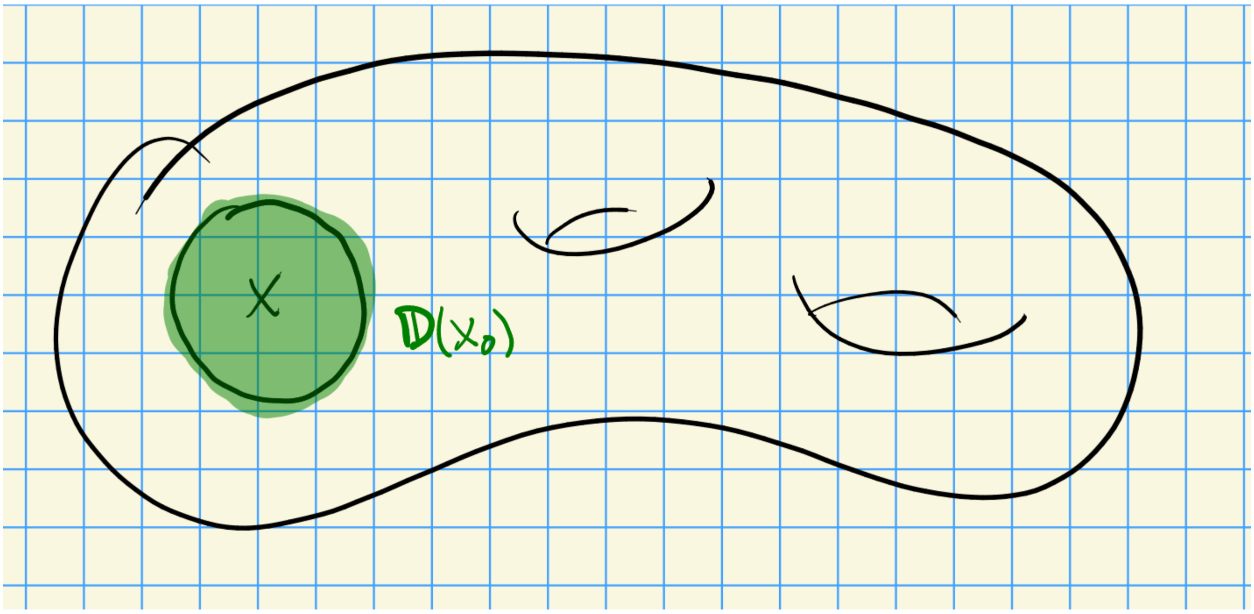
$$\mathrm{GL}_n(F) \backslash \mathrm{GL}_n(\mathbb{A}) / K^\natural \xrightarrow{\sim} \mathsf{VectBun}_n(X).$$

Remark 15.0.5: Let $(g_x) \in \mathrm{GL}_n(\mathbb{A})$ and assume $g_x = 1$ for all $x \neq x_0$. Assign a lattice in the local field

$$\Lambda_{x_0} \coloneqq g_{x_0} \mathcal{O}_{x_0}{}^{\oplus^n} \subseteq F_{x_0}{}^{\oplus^n},$$

which is an $\mathcal{O}_{x_0}$ submodule of rank n. Now construct a bundle by gluing with the trivial bundle on X away from x_0, so glue Λ_{x_0} with ${\mathcal{O}_{X\setminus x_0}}^{\oplus n}$ in the following way: let $j : X \setminus \{x_0\} \to X$ and form $j_* {\mathcal{O}_{X\setminus x_0}}^{\oplus n}$, which is no longer coherent and it quasicoherent, so looks like meromorphic functions but with no control on the poles. For $U \subseteq X$ an affine open, take the functions regular away from x_0 and constrain its behavior at x_0 and take the sheaf associated to the following:

$$U \mapsto \Gamma\left(U \setminus \{x_0\}\, ;\, {\mathcal{O}_X}^{\oplus n}\right) \cap \Lambda_{x_0} \subseteq {F_{x_0}}^{\oplus n}.$$



Example 15.0.6 *(?)***:** For t_{x_0} a uniformizer, set $g_{x_0} = \mathrm{diag}(t_{x_0}, 1, 1, \cdots, 1)$. The construction above yields the bundle $\mathcal{O}(-x_0) \oplus \mathcal{O}^{\oplus(n-1)}$.

Conversely, starting a vector bundle, you can get a double coset in $\mathrm{GL}_n(F) \backslash \mathrm{GL}_n(\mathbb{A}) / K^\natural$: for $V \in \mathsf{VectBun}(X)$, there exists a $U \subseteq X$ with $V|_U \cong {\mathcal{O}_U}^{\oplus n}$. Take $\Lambda_x = V|_{\operatorname{Spec}\mathcal{O}_x} = g_x {\mathcal{O}_x}^{\oplus n}$.

Exercise 15.0.7 (?)
Check that this gives an equivalence of groupoids.

Remark 15.0.8: This equivalence holds for more general split G. For $G = \mathsf{Sp}_{2n}$, a G-bundle is the same as a pair (V, ω) where V is a vector space of rank $2n$ and $\omega V \otimes_{\mathcal{O}_x} V \to \mathcal{O}_x$ is symplectic.

Remark 15.0.9: So far, this is a pointwise story, so we'll geometrize. It's a fact that Bun_G is a moduli stack, and its k-points and $R_{/k}$ points are

$$\mathsf{Bun}_G(k) = \{G\text{-bundles on } X\}$$
$$\mathsf{Bun}_G(R) = \{G\text{-bundles on } X \otimes_k R\}.$$

It's a theorem that these moduli functors are representable by Artin stacks.

Example 15.0.10*(?):* Take $X = \mathbb{P}^1$, then $\mathsf{Bun}_G(k)_{/\sim}$ can be described in terms of group-theoretic data. G-bundles for $G = \mathrm{GL}_n$ are classified by Grothendieck:

$$\begin{aligned} \mathsf{vectBun}(\mathbb{P}^1) &\rightleftharpoons \left\{d_1 \geq d_2 \geq \cdots \geq d_n \;\middle|\; d_i \in \mathbb{Z}\right\} \\ \bigoplus_i \mathcal{O}(d_i) &\leftarrow\!\shortmid \{d_i\}. \end{aligned}$$

In general, fixing $T \leq G$ a torus and W the Weyl group yields

$$\mathsf{Bun}_{G/\mathbb{P}^1}(k)_{/\sim} \rightleftharpoons X_*(T)/W,$$

i.e. bundles are parameterized by the cocharacter lattice, modulo the Weyl group action.

Question 15.0.11
We can regard $\mathcal{A}_{K^\natural}$ as functions on $\mathsf{Bun}_G(k)$, so what is the $\mathcal{H}_K$ action?

Example 15.0.12*(?):* Let $G = \mathrm{GL}_n$, let t_x be a uniformizer at x and take

$$_x := \chi_S, S = K_x \operatorname{diag}(t_x, 1, 1 \cdots, 1)K_x.$$

For $f : \mathsf{Bun}_G(k) \to \mathbb{C}$, we get the **elementary upper modifier of** f:

$$\begin{aligned} f * h_x : \mathsf{Bun}_G(k) &\to \mathbb{C} \\ V &\mapsto \sum_{0 \to V \hookrightarrow V' \twoheadrightarrow k_x \to 0} f(V'). \end{aligned}$$

where k_x is the skyscraper sheaf at x. This is analogous to summing over elliptic curves that are p-isogenous to a given curve.

One could alternatively define a Hecke operator defined by

$$h_x = \chi_S, S := K_-x \operatorname{diag}(t_x^{\lambda_1}, t_x^{\lambda_2}, \cdots, t_x^{\lambda_n}),$$

where λ is a collection of integers, and

$$(f * h_x)(V) = \sum_{\substack{V \to V' \\ \lambda, x}}.$$

15.1 Level Structures

Remark 15.1.1: For interesting automorphic forms, we need to use more general things than $K^\natural$ – many interesting examples come from **parahoric** subgroups of $G(F_x)$. First we define the **Iwahori** as a total preimage of a Borel under a reduction:

$$\begin{array}{ccc} I_x & \hookrightarrow & G(\mathcal{O}_x) \\ \downarrow & \lrcorner & \downarrow \text{reduction} \\ B(k_x) & \hookrightarrow & G(k_x) \end{array}$$

Link to Diagram

For $G = \mathrm{GL}_2$, one gets

$$I_x = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \;\middle|\; a, b, c, d \in \mathcal{O}_X, c \in \mathfrak{m}_x \right\}.$$

Now parahorics are groups that contain the Iwahori, so there is an analogy:

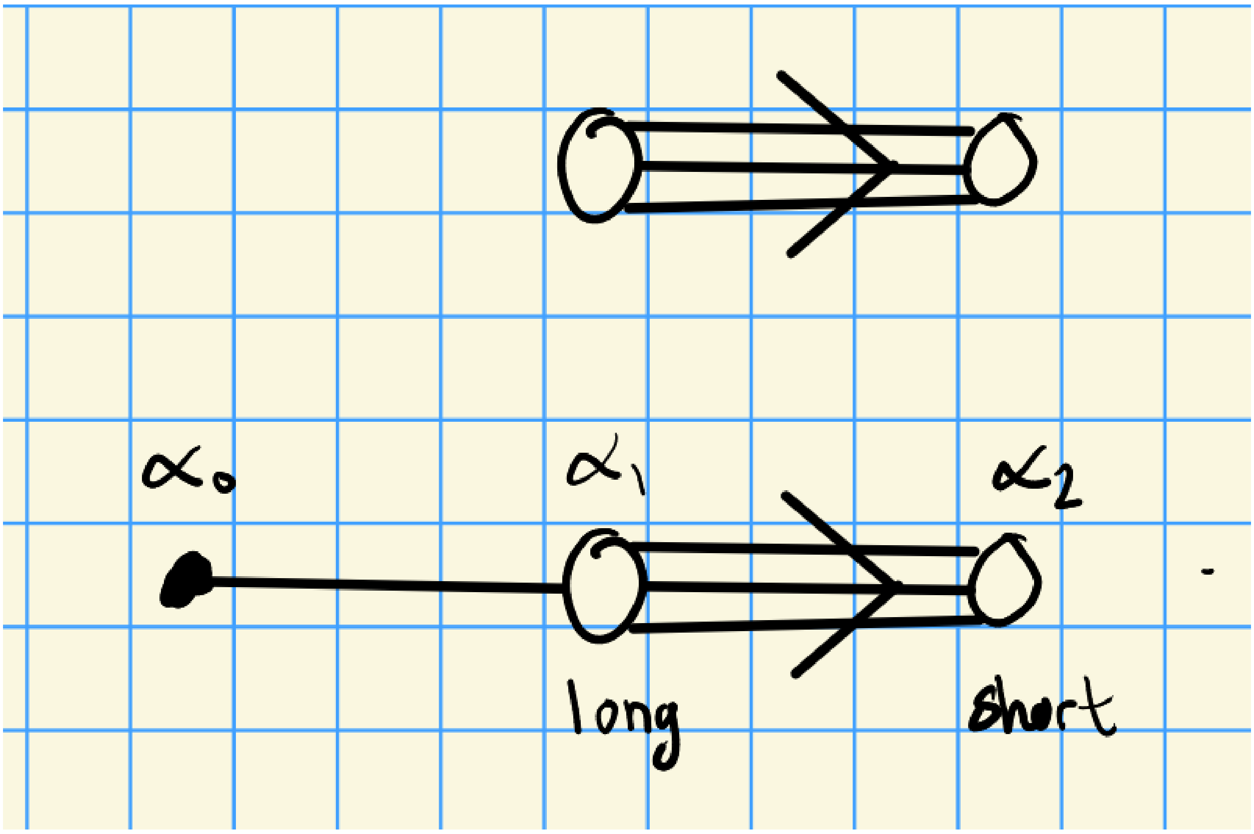
G/k	G/F_x
Borel	Iwahori
Parabolic	Parahoric

Remark 15.1.2: For $G = \mathrm{GL}_n$ and $\Lambda_x \subseteq {F_x}^{\oplus^n}$, we can consider

$$\mathrm{Stab}_{G(F_x)}(\Lambda_x) = \left\{ g \in \mathrm{GL}_n(F_x) \;\middle|\; g\Lambda_x = \Lambda_x \right\}.$$

One can ask for simultaneous stabilizers to get parahorics. In fact, all parahorics occur as stabilizers of chains of lattices where each stage differs by dividing by a uniformizer.

Remark 15.1.3: To visualize these, one needs **affine Dynkin diagrams** – these are generally obtained by adding a new point connected only to the long root. In G_2, the diagram is:



Here taking

- $\emptyset$ yields the Iwahori,
- $\{\alpha_1, \alpha_2\}$ yields $G(\mathcal{O}_x)$
- $\{\alpha_0, \alpha_1\}$ yields $P \twoheadrightarrow \mathrm{SL}_3$
- $\{\alpha_0, \alpha_2\} : Q \twoheadrightarrow \mathrm{SO}_4 = (\mathrm{SL}_2 \times \mathrm{SL}_2)/\Delta(\pm I)$.

16 | Zhiwei Yun, Lecture 2

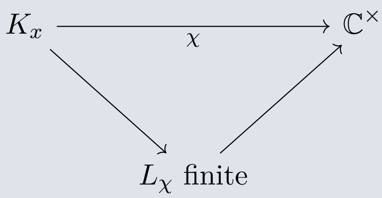
Remark 16.0.1: Today: what is rigidity?

16.1 Automorphic Data

Definition 16.1.1 (Automorphic data)
Given the following:

- $k = \mathbb{F}_q$
- $S \subseteq |X|$ a finite set, e.g. $X = \mathbb{P}^1$ and $S = \{0, \infty\}$ or $S = \{0, 1, \infty\}$.

- For each $x \in S$, a compact open $K_x \subseteq G(F_x)$
- For each $x \in S$, a character:



Link to Diagram

The pair (K_S, χ_S) is **automorphic data**.

Remark 16.1.2: Note that $\chi_S = 1$ recovers $f \in \mathcal{A}_K$ where $K = K_S \times \prod_{x \notin S} G(\mathcal{O}_x)$.

Definition 16.1.3 (Typical data)
A map

$$f \in C^0(G(F) \backslash G(\mathbb{A}) / \textstyle\prod_{x \notin S} G(\mathcal{O}_x) \to \mathbb{C})$$

is (K_S, χ_X)**-typical** iff

$$f(gk_x) = \chi_x(k_x) f(g) \qquad \forall x \in S, k_x \in K_x, g \in G(\mathbb{A}).$$

Remark 16.1.4: We want to make $\dim \mathcal{A}_c(K_S, \chi_S) = 1$. In this case, the Hecke algebra $\mathcal{H}_{K_y} \curvearrowright f \in \mathcal{A}_c(K_S, \chi_S)$ by a character, making f a Hecke eigenform.

16.2 Examples of naive ridigidy

Example 16.2.1 ***(?):*** Let

- $X = \mathbb{P}^1$,
- $G = \mathrm{SL}_2$,
- $S = \{0, 1, \infty\}$,
- $K_x = I_x$ Iwahori for all $x \in S$, where

$$I_x = \left\{ A := \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{Mat}_2(\mathcal{O}_X) \;\middle|\; c \in \mathfrak{m}_x \right\}.$$

Then choosing characters $\chi_x : k^\times \to \mathbb{C}^\times$ *generically* will imply $\dim \mathcal{A}_c(K_S, \chi_S) = 1$. Here generic means that $\prod \chi_i^{\pm 1} \neq 1$. By global Langlands for SL_2, any $f \in \mathcal{A}_c(K_S, \chi_S)$ will yield a 2-dimensional local system on $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ ramified at the 3 punctures. These will be solutions to hypergeometric differential equations.

For $G = \mathrm{PGL}_2$ (where the example works similarly), for $\chi_0, \chi_1 = 1$ and χ_∞ quadratic, there is a cover

$$\begin{array}{c} \{E_t\} : y^2 = x(x-1)(x-t) \\ \big\downarrow \\ \mathbb{P}^1 \setminus \{0, 1, \infty\} \end{array}$$

Moreover $\left\{H^1(E_t)\right\}$ will be a rank 2 local system on this base.

Example 16.2.2 ***(?):*** Let

- $X = \mathbb{P}^1$
- $S = \{0, \infty\}$
- $K_0 = I_0$, $\chi_0 = 1$
- τ a uniformizer at ∞
- $K_\infty = I_\infty^+ = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \;\middle|\; a \equiv d \equiv 1 \bmod \mathfrak{m}_x, b \in \mathcal{O}_x, c \in \mathfrak{m}_x \right\}$, the pro p part

There is a map $K_\infty \to k$ where $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \to b + \dfrac{c}{\tau} \bmod \tau$. Any character $k \xrightarrow{\psi} \mathbb{C}^\times$ can be extended to $\chi_\infty : K_\infty \to k \xrightarrow{\psi} \mathbb{C}^\times$, and $\dim \mathcal{A}_c(K_S, \chi_S)$. This yields a **Kloosterman local system** on $\mathbb{P}^1 \setminus \{0, \infty\}$, where

$$\mathrm{Kl}(a) = \sum_{x \in k^\times} \psi\left(\frac{x+a}{x}\right)$$

recovers the classical Kloosterman sum by taking trace of Frobenius.

16.3 Naive Rigidity

Definition 16.3.1 (Rigidity (Naive Definition))

$$(K_S, \chi_S) \text{ is rigid} \iff \dim \mathcal{A}_c(K_S, \chi_S) = 1.$$

⚠ **Warning 16.3.2**
If $\pi_1 G \neq 1$, then $\pi_0 \mathsf{Bun}_G \geq 2$, yielding multiple components. It's also not clear if this type of dimension bound will hold after a base change $k \to k'$.

16.4 Base Change

Remark 16.4.1: For k'/k finite, write $X' := X \otimes_k k'$ for the base change. Let $S \to S'$ be the preimage of S in S', and consider $k'_x := K_x \otimes_k k'$ How can we base change a character? We need a norm map to fill in the following diagram:

$$K_x \otimes_k k' \xrightarrow{\mathrm{Nm}(-)} K_x \xrightarrow{\chi_x} \mathbb{C}^\times$$

Link to Diagram

Example 16.4.2 ***(Base-changing a character):***

$$I_x \longrightarrow k^\times \xrightarrow{\chi} \mathbb{C}^\times$$

$$I'_x = I_x(k') \longrightarrow (k')^\times \xrightarrow{\mathrm{Nm}_{k'/k}(-)} k^\times \xrightarrow{\chi} \mathbb{C}^\times$$

Link to Diagram

Here $I_x(k') = \{a, b, c, d \in \mathcal{O}_x \widehat{\otimes} k' \cong k'[[t]]\}$.

$$I_x \longrightarrow k^\times \xrightarrow{\chi} \mathbb{C}^\times$$

Example 16.4.3 ***(?):*** $I'_x = I_x(k') \longrightarrow (k')^\times \xrightarrow{\mathrm{Trace}_{k'/k}(-)} k^\times \xrightarrow{\chi} \mathbb{C}^\times$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \longmapsto b + \frac{c}{\tau} \bmod \tau$$

Link to Diagram

Remark 16.4.4: We now geometrize this process to send characters to character sheaves, i.e. rank one local systems on K_x. We have a way of taking (K_S, χ_S) to (K'_S, χ'_S) for extensions $K \to K'$, so we can form $\mathcal{A}(k'; K'_S, \chi'_S)$.

Definition 16.4.5 (Weakly rigid)
Automorphic data (K_S, χ_S) is **weakly rigid** iff $\dim \mathcal{A}_c(k'; K'_S, \chi'_S)$ is uniformly bounded for all extensions $k \to k'$.

16.5 Relevant Points

Remark 16.5.1: Recall that there is a bijection

$$G(F) \backslash G(\mathbb{A}) / K \rightleftharpoons \mathsf{Bun}_G(K)(k),$$

so functions $f \in \mathcal{A}_c(K_S, \chi_S)$ are functions on $\mathsf{Bun}_G(K_S^+)(k)$ where for $x \in S$, $K_x^+ \trianglelefteq K_x$ with $\chi_x|_{K_x^+} = 1$ and K_x/K_x^+ are the k-points of a finite dimensional group L_x.

Example 16.5.2*(?):* $I_x^+ \trianglelefteq I_x = K_x \to \mathbb{G}_m(k)$.

Example 16.5.3*(?):* $I_\infty^{++} = K_\infty^+ \trianglelefteq I_\infty^+ = K_\infty \to k^{\oplus^2}$ where $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto \left(b, \frac{c}{\tau}\right) \bmod \tau$.

Remark 16.5.4: There is a right action

$$C^0(\mathsf{Bun}_G(K_S^+)(k) \to \mathbb{C}?) \curvearrowleft \prod_{x\in S} L_x(k),$$

and the eigenfunctions with eigenvalues $(\chi_x)_{x\in S}$ are in $\mathcal{A}_c(K_S, \chi_S)$. As a set, $\mathsf{Bun}_G(K_S^+)$ has commuting left and right actions, where quotienting by the right action yields a principal homogeneous space. The left action is by $\mathrm{Aut}(\mathcal{E})$ for $\mathcal{E} \in \mathsf{Bun}_G(K_S)$, and permutes points in the fiber in $\tilde{\mathcal{E}} \in \mathsf{Bun}_G(K_S^+)$. So there is an evaluation map which is well-defined up to conjugacy

$$\mathrm{ev}_{\mathcal{E}} : \mathrm{Aut}(\mathcal{E}) \to \prod_{x\in S} L_x(k).$$

Definition 16.5.5 (Relevant points)
A k-point $\mathcal{E} \in \mathsf{Bun}_G(K_s)(k)$ is (K_S, χ_S)-**relevant** iff

$$\mathrm{ev}_{\mathcal{E}}^* \left(\prod_{x\in S} \chi_x \right) \Bigg|_{\mathrm{Aut}(\mathcal{E})^0(k)} = 1.$$

Similarly one can define relevant k'-points for k'/k a finite extension.

Fact 16.5.6

$$\dim \mathcal{A}_c(k'; K_S, \chi_S) \le \sharp\,\mathrm{Rel}(K_S), \quad \mathrm{Rel}(K_S) := \left\{ (K_S', \chi_S')\text{-relevant } k' \text{ points of } \mathsf{Bun}_G(K_S) \right\}.$$

Note that taking connected components in the definition is needed to make this stable under base change.

Corollary 16.5.7 *(?).*
(K_S, χ_S) is weakly rigid $\iff \sharp \mathrm{Rel}(K_S) < \infty$.

Example 16.5.8 *(?):* Let

- $G = \mathrm{SL}_2$
- $K_x = I_x$
- $x \in S := \{0, 1, \infty\}$

Note that

$$\mathsf{Bun}_G(K_S)(k) = \left\{ V \in \mathsf{VectBundle}^{\mathrm{rank}=2}, \iota : \bigwedge^2 V \cong \mathcal{O}_X, \{\ell_x \subseteq V_x\}_{x \in S} \right\},$$

where the ℓ are lines. So these are bundles with extra structure at fixed places, and are parameterized by 5-tuples $\mathcal{E} = (V, \iota, \ell_0, \ell_1, \ell_\infty)$. For all $x \in S$ we have

$$I_x \to \mathbb{G}_m = L_x$$
$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto a \bmod \tau,$$

and the evaluation map is $\mathrm{ev} : \mathrm{Aut}(\mathcal{E}) \to \prod_{x \in S} \mathbb{G}_m$. Each $\gamma \in \mathrm{Aut}(\mathcal{E})$ is a map $V \to V$ where $\gamma_0 \curvearrowright V_0$ preserving ℓ_0. Is it the case that

$$\prod_{x \in S} \chi_x \bigg|_{\mathrm{Aut}(\mathcal{E})^0} \overset{?}{=} 1.$$

For $V = \mathcal{O}^2$ and $\ell_x \subseteq k^2$ in generic position, $\mathrm{Aut}(\mathcal{E}) = \{\pm 1\}$ so they are relevant. Other points are irrelevant: if $V = L \oplus L'$ with $\ell_x \in L_x$ or L'_x, $\mathrm{Aut}(\mathcal{E})$ will contain a copy of $\mathbb{G}_m$ that acts by scaling each L which will map nontrivially to $\prod_{x \in S} \mathbb{G}_m$. Since $\prod \chi_i^{\pm 1} \neq 1$, we get $\mathrm{ev}_{\mathcal{E}}^* \prod \chi_x \big|_{\mathbb{G}_m} \neq 1$.

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Remark 17.0.1: The Langlands correspondence:

Automorphic	Galois
G	$G^\vee$
Eigenforms	Local systems, $\pi_1(X \setminus S) \to G^\vee(\overline{\mathbb{Q}_\ell})$
$f \in \mathcal{A}_c(K_S, \chi_S)$	
Rigid automorphic data	Rigid local systems

Today we'll discuss going from the automorphic side to the Galois side by designing rigid automorphic data.

17.1 Numerical Rigidity

Remark 17.1.1: Automorphic functions will be functions on the algebraic stack $\mathsf{Bun}_G(K_S)$, so we want to consider its k-points. This stack should have (possibly negative) dimension at most zero, what does this tell us about the level groups? For a curve C, there is a formula:

$$\dim \mathsf{Bun}_G(K_S) = 0 \iff \sum_{x \in S} [G(\mathcal{O}_x) : K_x] = (1 - g(C)) \dim G,$$

where the brackets indicate **relative dimension**, which is always non-negative. Recall that I_x is a preimage of a Borel under reduction, and for $K_x = I_x$ we have

$$[G(\mathcal{O}_x) : I_x] = \dim G(\mathcal{O}_x)/I_x = \dim G/B = \sharp \Phi^+.$$

If K_x is not contained in $G(\mathcal{O}_x)$, then

$$[G(\mathcal{O}_x) : K_x] = \dim G(\mathcal{O}_x)/G(\mathcal{O}_x) \cap K_x - \dim K_x/G(\mathcal{O}_x) \cap K_x.$$

The RHS in the formula is non-negative only when $g = 0, 1$, so we expect most rigid data to come from $\mathbb{P}^1$. Genus 1 is a very special case, we get $K_x \sim G(\mathcal{O}_x)$.

Example 17.1.2*(?):* Consider

- G a fixed group,
- $X = \mathbb{P}^1$,
- $S = \{0, 1, \infty\}$,
- K_x a parahoric.

Then

$$\dim G = \sum_{x=0,1,\infty} [G(\mathcal{O}_x) : K_x].$$

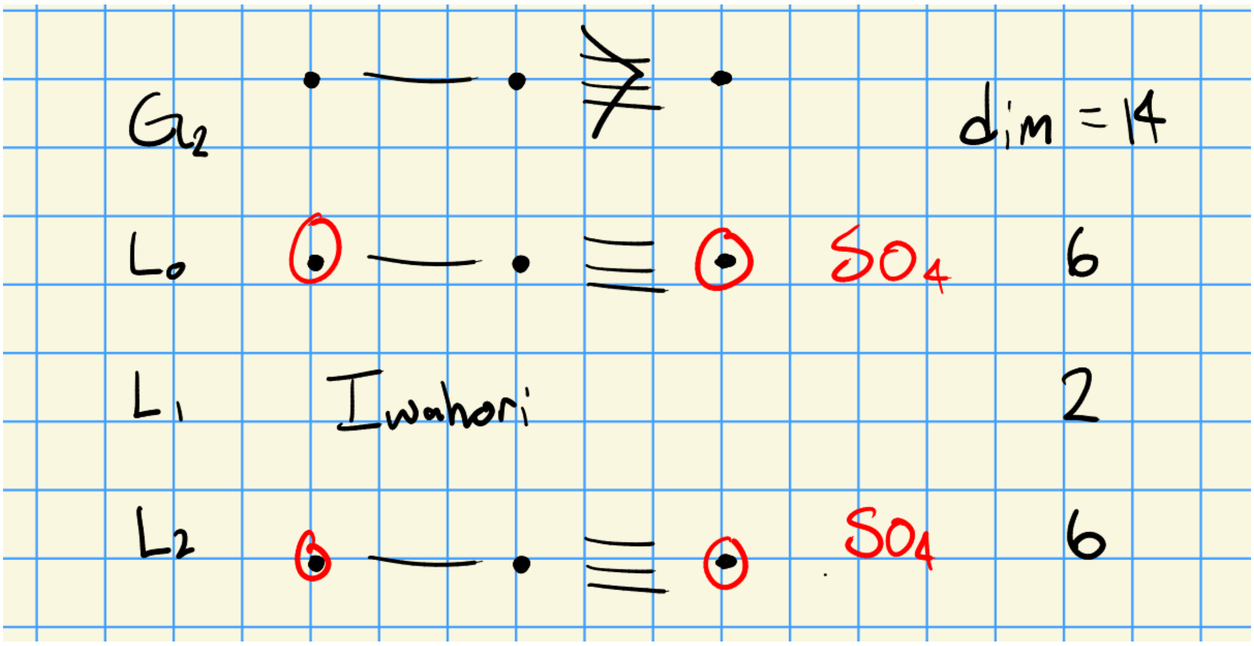
If K_x corresponds to a subdiagram of a Dynkin diagram, we can read off the reductive quotient L_x to get a surjective quotient map $K_x \twoheadrightarrow L_x$. In this case,

$$[G(\mathcal{O}_x) : K_x] = \frac{1}{2} (\dim G - \dim L_x).$$

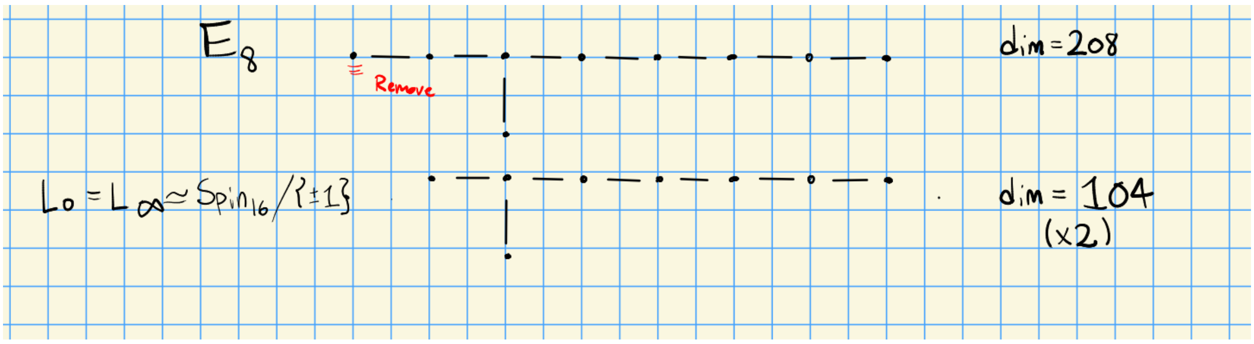
The condition then becomes

$$\dim G = \sum_{x=0,1,\infty} \dim L_x.$$

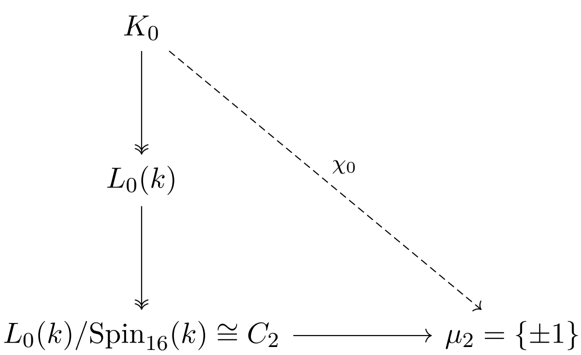
Example 17.1.3*(?):* Consider the same setup for $G = \mathsf{G}_2$.



For $G = \mathbf{E}_8$, take L_1 to be the Iwahori and:



Idea: delete a node to try to get a group of roughly half-dimension. Cook up an order 2 character on the reductive quotient for $x = 0$:



Link to Diagram

Setting $\chi_\infty = \chi_1 = 1$, this yields automorphic datum which turns out to be rigid.

17.2 Matching with local monodromy

Remark 17.2.1: Given a local system, restrict to a formal neighborhood of a puncture to get a representation of the local Galois group, which we can restrict to inertia:

$$
\begin{array}{ccc}
\mathsf{Gal}(\overline{F}_x/F_x) & \xrightarrow{\rho_x \in \mathsf{LocSys}} & G^\vee(\overline{\mathbb{Q}_\ell}) \\
\uparrow & \overset{\rho_x}{\nearrow} & \\
\mathrm{In}_x & &
\end{array}
$$

Link to Diagram

Example 17.2.2*(?)*: If $K_x = I_x$ and one forms the character $K_x \to T(k) \to \overline{\mathbb{Q}_\ell}^\times$, the representation ρ_x is tamely ramified. There is a canonical map to the residue field $\mathrm{In}_x \to k_x^\times$, and χ can be turned into a morphism $k_x^\times \to T^\vee(\overline{\mathbb{Q}_\ell}^\times)$ to the dual torus. The composite dual character $\mathrm{In}_x \to k_x^\times \to T^\vee(\overline{\mathbb{Q}_\ell}^\times)$ has finite order and yields the semisimplification $(\rho_x|_{\mathrm{In}_x})^{\mathrm{ss}}$. This yields unipotent monodromy, usually "maximally" nontrivial.

Example 17.2.3*(?)*: For $K_x = I_x^+ = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \;\middle|\; a-1, d-1, c \equiv 0 \bmod t \right\}$, one gets a character:

$$
\begin{array}{ccccc}
K_x = I_x^+ & \longrightarrow & k & \xrightarrow{\psi} & \mathbb{C}^\times \\
\begin{bmatrix} a & b \\ c & d \end{bmatrix} & & b + \frac{c}{t} \bmod t & &
\end{array}
$$

Link to Diagram

One gets a wildly ramified representation:

$$\rho_x : \mathsf{Gal}(\overline{F}_x/F_x) \to \mathrm{GL}_2(\overline{\mathbb{Q}_\ell}).$$

The Swan conductor is $\mathrm{Sw}(\rho_x) = 1 = \frac{1}{2} + \frac{1}{2}$, where the first factor comes from the shape of the level group I_x^+. There is a **Moy-Prasad filtration** on I_x indexed by $\frac{1}{h}\mathbb{Z}$ for h the Coxeter number of G, which for $G = \mathrm{SL}_2$ yields $h = 2$. The filtration is

$$
\begin{array}{c}
I_x = I_x(0) \\
\big\uparrow \\
I_x^+ = I_x\left(\frac{1}{2}\right) \\
\big\uparrow \\
I_x(1) \\
\big\uparrow \\
I_x\left(\frac{3}{2}\right) \\
\big\uparrow \\
\vdots
\end{array}
$$

Link to Diagram

If $K_x \subseteq P_x(r)$ for $r \in \mathbb{Q}$, then all slopes of ρ_x are at most r, bounding the ramification. In this case depth matches up with slopes.

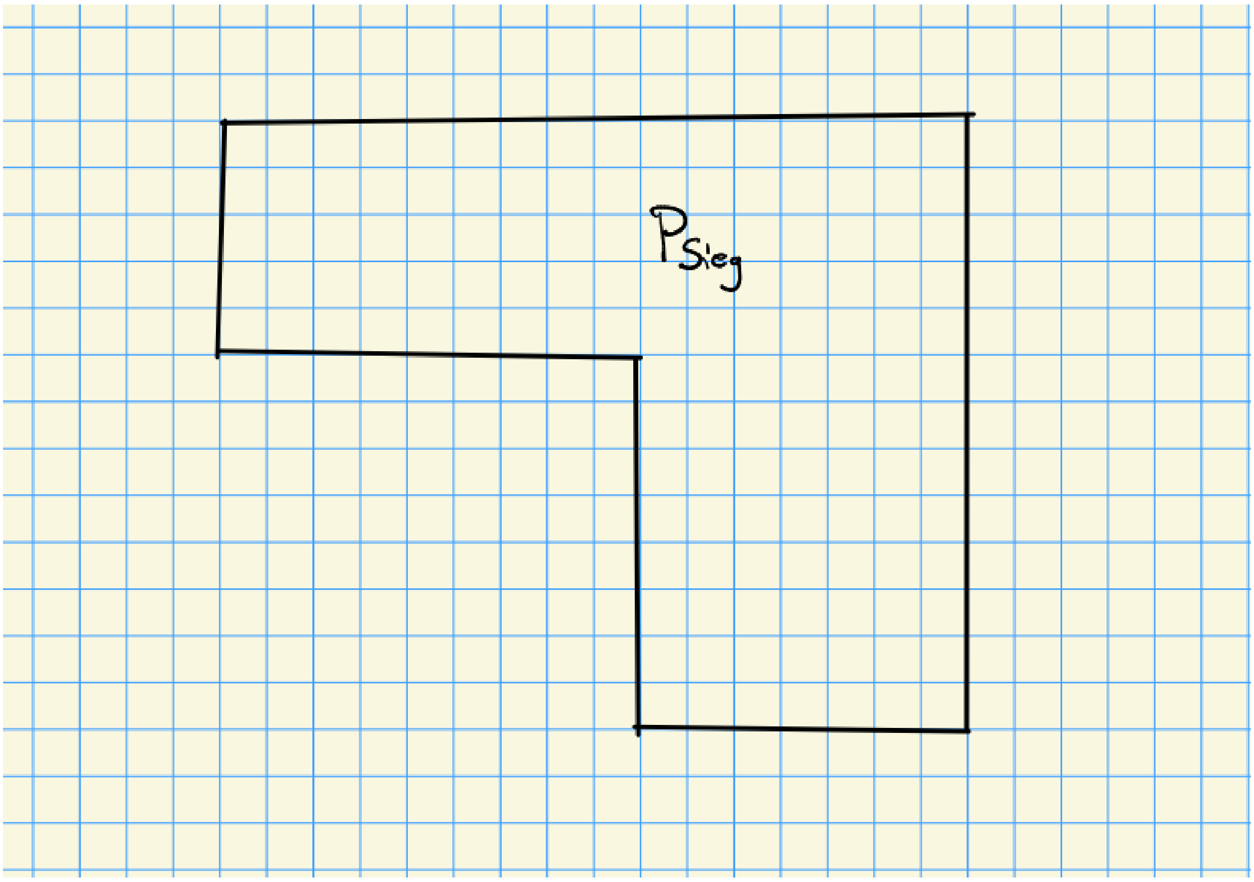
Remark 17.2.4: Ansatz for finding correct automorphic data: if (K_S, χ_S) is rigid and $\rho : \pi_1 \to G^\vee$, there should be an equality involving a the Artin conductor:

$$[G(\mathcal{O}_x) : K_x] = \frac{1}{2} a(\mathrm{Ad}\rho_x).$$

Example 17.2.5 ***(Epipelagic automorphic data):*** Let

- $S = \{0, \infty\}$,
- $K_0 = P_0$ a parahoric,
- $K_\infty = P_\infty^+$ the pro-unipotent of some parahoric
- $\chi : k \to \mathbb{C}^\times$ be an additive character

Compose to get a character $P_\infty^+ \xrightarrow{?} k \xrightarrow{\psi} \mathbb{C}^\times$, where the missing morphism is the interesting bit. For $G = \mathsf{Sp}_{2n} = \mathsf{Sp}(V)$, a Siegel parabolic is the stabilizer of a Lagrangian subspace in V and has the following shape:



Write this as P_{Sieg} preserving a Lagrangian L and take its transpose to get $P^{\text{op}}_{\text{Sieg}}$ which preserves a complementary Lagrangian subspace $L^c \cong L^\vee$. Let $P_0 \subseteq G(\mathcal{O}_0), P_\infty \subseteq G(\mathcal{O}_\infty)$ be the associated parahorics. Define a map

$$P_\infty^+ = \left\{ \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in G(\mathcal{O}_\infty) \;\middle|\; A - I, D - I, C \equiv 0 \bmod \tau \right\} \to W = \text{Sym}^2(L) \oplus \text{Sym}^2(L^\vee)$$
$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \mapsto \left(B \bmod \tau, \frac{C}{\tau} \bmod \tau \right).$$

We then have

- $\overline{A} \in \text{GL}(L)$
- $\overline{D} \in \text{GL}(L^\vee)$
- $\overline{B} : L^\vee \to L$ self-dual, so $\overline{B}^\vee = \overline{B}$,
- $C/\tau : L \to L^\vee$ a self-adjoint operator

We can further apply the trace pairing, fixing $S \in \text{Sym}^2(L)$ and $T \in \text{Sym}^2(L^\vee)$:

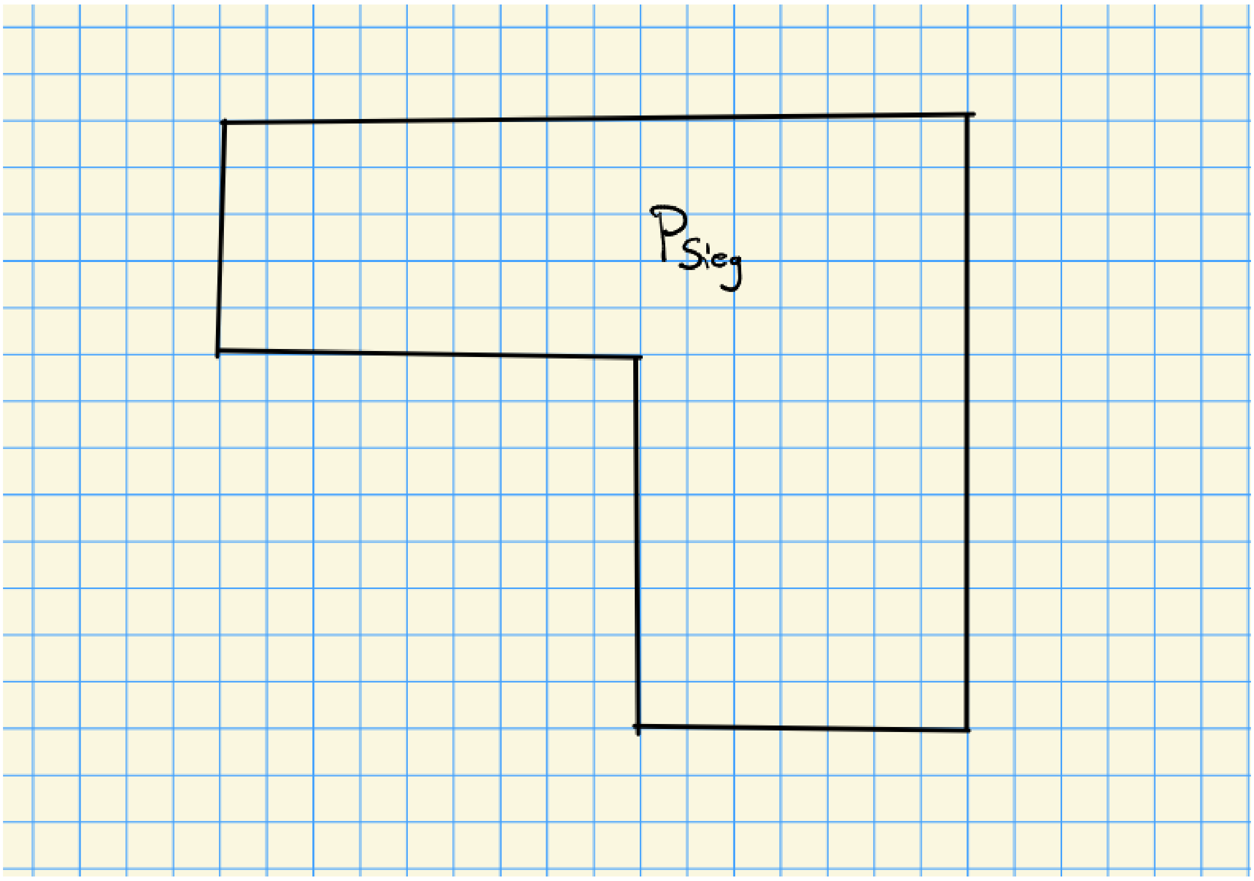
$$\text{Trace} : W \to k$$
$$(X, Y) \mapsto \text{Trace}(XT) + \text{Trace}(YS).$$

Choosing a pair (S,T) yields a character:

$$P_\infty^+ \to W \xrightarrow{(S,T)} k \xrightarrow{\psi} \mathbb{C}^\times.$$

Stable pairs (S,T) will yield rigid data, where stable is the open condition that $ST \in \mathrm{End}(L)$ has distinct nonzero eigenvalues in $\bar{k}$, so regular semisimple and invertible.

Remark 17.2.6: Epipelagic reps of $G(F_\infty)$ due to Reeder-J-K. Yu: for Sp_{2n} this amounts to choosing a matrix of the following shape with equally sized blocks:



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18.1 Kloosterman Automorphic Systems

Remark 18.1.1: We've just been on the automorphic side: today we harvest on the Galois side! For (K_S, χ_S) automorphic data, there is a Hecke action

$$f \in \mathcal{A}_c(K_S, \chi_S) \curvearrowleft \mathcal{H}_{K_x}, \quad x \in |X| \setminus S.$$

The Satake isomorphism yields a correspondence

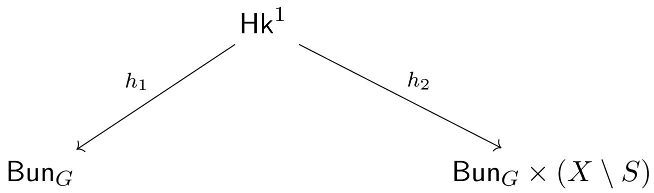
$$\left\{\text{Functions } \mathcal{H}_{K_x} \to \overline{\mathbb{Q}_\ell}\right\} \rightleftharpoons \left\{\text{Semisimple conjugacy classes in } \widehat{G}(\overline{\mathbb{Q}_\ell})\right\}.$$

So for all places $x \notin S$, one gets a Satake parameter $\sigma_x \in \widehat{G}_{\mathrm{ss}}(\overline{\mathbb{Q}_\ell})$. By Langlands, there exists a representation $\rho : \pi_1(X \setminus S) \to \widehat{G}(\overline{\mathbb{Q}_\ell})$ such that $\rho(\mathrm{Frob}_x)^{\mathrm{ss}} \sim \sigma_x$. How do you construct ρ from f?

Remark 18.1.2: We'll geometrize this along the lines of Drinfeld, Laumon, etc. Set $G = \mathrm{GL}_n$ and let $T_x \in \mathcal{H}_{K_x}$ be the characteristic function on $K_x \operatorname{diag}(t_x, 1, \cdots, 1)K_x$. For $f : \mathsf{Bun}_G(k) \to \overline{\mathbb{Q}_\ell}$, define an operator

$$(T_x f)(\mathcal{E}) := \sum_{\mathcal{E}' \hookrightarrow \mathcal{E} \text{ length 1 at } x} f(\mathcal{E}').$$

Note that the index set is isomorphic to $\mathbb{P}(\mathcal{E}_x)$. This translates functions to sheaves: summing corresponds to taking cohomology, characters become character sheaves. Let $\mathsf{Hk}^1 = \{\mathcal{E}' \to \mathcal{E} \text{ of length } 1\}$, then there is a span:



Link to Diagram

The operator T_x geometrizes in the following way:

$$T_1\mathcal{F} := (h_2)_!({h_1}^*\mathcal{F}) \in \mathsf{Sh}(\mathsf{Bun}_G \times (X \setminus S)),$$

and being an *eigensheaf* translates $T_x f = \lambda_x f$ for $\lambda_x \in \overline{\mathbb{Q}_\ell}$ to the condition

$$T_1\mathcal{F} = \mathcal{F} \boxtimes E.$$

The goal is to compute E; this will yield

$$\mathrm{Trace}(\mathrm{Frob}_x; E) = \lambda_x \qquad \forall x \in X \setminus S.$$

Example 18.1.3 ***(Kloosterman automorphic datum):*** Benedict Gross constructed Kl automorphic datum, showed rigidity using a trace formula, and conjectured some properties of ρ related to a Kloosterman local system. Heinloth-Ngo-Y. constructed this ρ, uncovering the story of rigidity here.

Let

- $G = \mathrm{PGL}_n$

- The 0-level $K_0 = I_0$ is the Iwahori and $\chi_0 = 1$
- The infinity level $K_\infty = I_\infty^+$, and the character is given in the following way: add superdiagonal and lower-left corner mod τ (the uniformizer at ∞), so

$$(a_{ij}) \mapsto \sum a_{i,i+1} + \frac{a_{n,1}}{\tau} \bmod \tau \rightsquigarrow I_\infty^+ \to k \xrightarrow{\psi} \overline{\mathbb{Q}_\ell}^\times.$$

In this case,

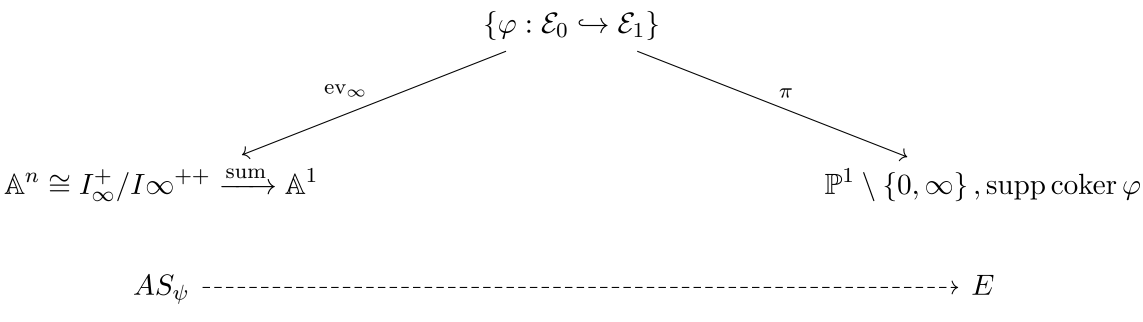
$$\mathsf{Bun}_G(K_0, K_\infty) = \left\{ \begin{array}{l} V \in \mathsf{Bun}\,(\mathrm{GL}_r)^{\mathrm{rank}=n}(\mathbb{P}^1), \\ F^n \supseteq F^{n-1} \supseteq \cdots \supseteq F^1 \text{ a full flag on } V_0, \\ F_0 \subseteq F_1 \subseteq \cdots \subseteq F^n \text{a full flag on } V_\infty, \\ \{e_i\} \text{ a basis of } \mathsf{gr}_i(F_\bullet) \end{array} \right\} /\mathrm{Pic}.$$

There is a unique relevant point on each component of $\mathsf{Bun}_G(K_0, K_\infty)$, where $\deg V \bmod n$ is well-defined. It's given by $\mathcal{E}_0$ where $\mathcal{O}^{\oplus^n} = \bigoplus_{i \le n} \mathcal{O}_i$, with a flag $\{e_n\}, \{e_n, e_{n-1}\}, \cdots$. One can show that $\mathrm{Aut}(\mathcal{E}_0) = 1$ making it automatically relevant.

A point $\mathcal{E}_1$ yields $\bigoplus_{k \le n-1} \mathcal{O} e_k \oplus \mathcal{O}(1) e_n$, with flags

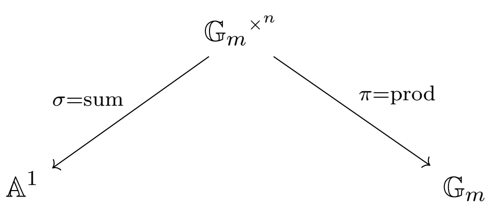
- $F^\bullet : \{e_{n-1}\}, \{e_{n-1}, e_{n-2}\}, \cdots$.
- $F_\bullet : \{e_1\}, \{e_1, e_2\}, \cdots$

There is a Hecke stack Hk containing $\{\varphi : \mathcal{E}_0 \hookrightarrow \mathcal{E}_1\}$, and a span:



Link to Diagram

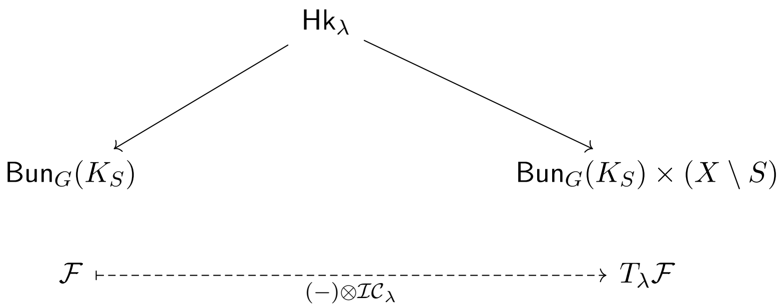
Pull-push yields a local system. Similarly:



Link to Diagram

Defining $E := \mathbb{R}^{n-1}\pi_! \sigma^* \mathrm{AS}_\psi \in \mathsf{LocSys}^{\mathrm{rank}=n}$ exactly recovers Deligne's Kloosterman sheaf.

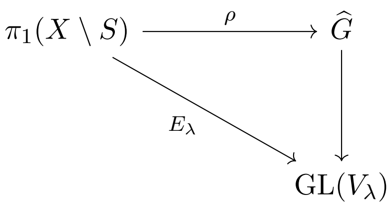
Remark 18.1.4: For more general G, $\mathcal{H}_{K_x}$ has a Kazhdan basis C_λ, where dominant weights $\lambda \in X_*(T)$ correspond to irreducible reps of $G^\vee$. Taking T_x for GL_n recovers the standard representation of $G^\vee = \mathrm{GL}_n$. The geometric incarnation of the Hecke operator is $T_\lambda F$:



Link to Diagram

Here $\mathcal{F}$ is an eigensheaf, so $T_\lambda \mathcal{F} = \mathcal{F} \boxtimes E_\lambda$. Note that the $\mathcal{IC}$ sheaf is not always constant.

Fact 18.1.5
$\lambda \mapsto E_\lambda$ comes from a $\widehat{G}$-local system on $X \setminus S$:



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18.2 Applications

Remark 18.2.1: If (K_S, χ_S) is "tame", where K_x is parahoric, this data will make sense over any base field k which χ_S is replaced by a character sheaf. Note that this only works for multiplicative characters, since additive characters depend on characteristic. One can construct these Hecke eigensheaves and $G^\vee$ local systems for arbitrary fields, e.g. for $\mathbb{P}^1_{/\mathbb{Q}} \setminus S$ where there may not even be a theory of automorphic forms. A first example constructs an $\mathbf{E}_8$-local system on $\mathbb{P}^1_{/\mathbb{Q}} \setminus \{0, 1, \infty\}$, yielding a motive whose motivic Galois group is $\mathbf{E}_8$. One can then apply this to the inverse Galois problem, arguing that there exists a number field K such that

$$\mathsf{Gal}(K/\mathbb{Q}) \cong \mathbf{E}_8(\mathbb{F}_\ell), \qquad \ell \gg 0.$$

See the notes for relations to "rigidity methods" in inverse Galois theory.

18.3 Open Problems

Question 18.3.1
Classification: say $G = \mathrm{GL}_n$, can one classify all rigid automorphic data?

Remark 18.3.2: These should correspond under Langlands to rigid local systems, where there is an algorithmic classification due to Katz in the tame case and Arinkin in general. One can start with rank 1 local systems and apply one of three simple procedures to get local systems of higher rank. Note that hypergeometric local systems occur.

Question 18.3.3
Is there an algorithmic way of producing automorphic data?

Question 18.3.4
Is there a uniform way to check rigidity?

Remark 18.3.5: Checking rigidity requires knowing the specific geometry of Bun_G and some tricky linear algebra. There are some results that provide the uniform bound on dimensions $\mathcal{A}_c$ needed to prove weak rigidity.

Question 18.3.6
Can $\mathcal{A}_c(K_S, \chi_S)$ be further decomposed into Hecke modules when the dimension is bigger than 1?

Remark 18.3.7: This dimension can grow exponentially.

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Goal: proving rigidity. Start with example 2.1.5 in the notes on Kloosterman automorphic data.

ToDos

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Definitions

Theorems

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