Notes: Todo

2022 Cantrell Lectures: Akshay Vankatesh

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1 Talk 1: Number Theory and 3-Dimensional Geometry

Abstract: There is a wonderful analogy between the theory of numbers, and 3-dimensional geometry. For example, prime numbers behave like knots!
I will explain some of the history of this analogy and how it is evolving.

Remark 1.0.1:
- 60s, Mazur's *Remarks on the Alexander polynomial*: primes are analogous to knots.
- Impetus: Thomson 1867, *On vortex atoms.*
- Idea: topological distinction of knots may explain distinct atoms, and links for molecules.
- P. G. Tait (and Kirkman? Around 1880) tabulates many knots.
- Rigorous tools to distinguish: Wirtinger, Dehn, Alexander in late 1800s/early 1990s.
- Dehn (1910), Wirtinger (1905): a presentation of the knot group $\pi_1(\mathbb{R}^3 \setminus K)$ in terms of generators and relations.
- Alexander (1927): *Topological invariants of knots and links*, observes difficulties distinguishing group presentations, introduces the simpler Alexander polynomial.
  - E.g. for the trefoil, $p(x) = 1 - x + x^2$.
  - He gives a simple algorithm computing it as the determinant of a matrix, and shows how it can be derived from the knot group.
  - Succeeded in distinguishing many tabulated knots (but not all).
  - See SnapPy! You can fly around the hyperbolic space of a knot complement, and compute a large number of invariants.

Remark 1.0.2:
- NT: consider solving $x^2 + y^2 = z^2$.
  - Factor $x^2+y^2 = (x-iy)(x+iy)$ and write $x \pm iy = (m \pm in)^2$ to obtain $x^2 = m^2 - n^2, y = 2mn, z = m^2 + n^2$.
  - This yields *all* solutions, using unique factorization in $\mathbb{Z}[i]$.
- Consider $x^p + y^p = z^p$ (Fermat)
  - To solve: factor $z^p = \prod_k x + \zeta_p^k y$.
- Lame (1847): an incorrect proof that falsely assumes $L_p := \mathbb{Z}[\zeta_p]$ is a UFD, since $\mathrm{cl}(L_p) \neq 1$ for $p = 23$ (shown by Kummer) and factorization fails.
- Kummer fixes this proof assuming a weaker condition: $p \nmid \mathrm{cl}(L_p)$, i.e. this is **regular**.
  - This fails for $p = 37$.
  - Kummer proves FLT fails for regular primes.
- Kummer (1850) shows p is irregular iff p divides on of the first $p - 3$ coefficients of $\frac{x}{e^x - 1}$.
- Iwasawa (59, 62) gives a polynomial P_p whose degree quantifies the irregularity of p; p is irregularity iff $P_p = 1$ is constant.
  - This definition uses a Galois group G_p.

Remark 1.0.3: Mazur (63) observes similarity between the Alexander polynomial and Iwasawa's polynomial. Both are pure group theory, and the construction arises from action of $G^{\mathrm{ab}} \curvearrowright [G, G]^{\mathrm{ab}}$ and taking a characteristic polynomial. Analogies:

- G_p is similar to $\pi_1(S^3 \setminus K)$,
- the linking number is similar to the quadratic residue $\left(\frac{p}{q}\right)$.
- Symmetry of linking numbers is analogous to quadratic reciprocity.
- $\operatorname{Spec} \mathbb{Z}/p\mathbb{Z} \hookrightarrow \operatorname{Spec} \mathbb{Z}$ is like a knot in a 3-manifold.
- $\operatorname{Spec} \mathbb{Z}$ is like S^3, with primes corresponding to embedded knots.

Analogies (partially) due to Mazur, guided by results of Artin and Tate.

Remark 1.0.4: Defining G_p: take all rings over k with some finite rank r which admit *exactly* r embeddings in $\bar{k}$, take their union, take ring automorphisms. Constructing such rings: find polynomials p with $\operatorname{disc}_p = p^k$ for some k. Problem: G_p is very mysterious! Probably finitely generated but infinite, the only element we can produce is complex conjugation, $z \mapsto \bar{z}$. Tate (62) shows that G_p cohomologically looks like a knot group, currently the strongest formal analogy.

Warning 1.0.5
Fixing a specific prime p, it is not analogous to any specific not, and $\mathbb{Z}$ is not analogous to any specific manifold. Instead, there is some unknown refinement which recovers properties of both.

Remark 1.0.6: Idea: compare statistical properties of random knots vs random primes.

Question 1.0.7
If $\mathbb{Z} \sim M^3$ and $p \sim K_p \hookrightarrow M^3$, what object is analogous to K^c?

2 Talk 2: Symplectic L-functions and their topological analogues

Abstract: The topology of the symplectic group enters into many different areas of mathematics. After discussing a couple of "classical" manifestations of this, I will explain a new one, in the theory of L-functions, as well as a purely topological analogue of the statement. I am not going to assume any familiarity with the theory of L-functions for the talk. Joint work with Amina Abdurrahman.

2.1 Symplectic Spaces

Remark 2.1.1: Note: for L a field,

$$L_{\square} := \left\{ a^2 \;\middle|\; a \in L \right\}.$$

Remark 2.1.2: Recall that any symplectic vector space is isomorphic to $\left(\mathbb{R}^{2g}, J := \begin{bmatrix} 0 & \mathrm{id}_g \\ -\mathrm{id}_g & 0 \end{bmatrix}\right)$.
Write

$$\mathrm{SP}(V) = \left\{ g : V \to V \;\middle|\; \langle gx,\ gy \rangle = \langle x,\ y \rangle \right\} = \left\{ A \in \mathrm{GL}_{2g}(\mathbb{R}) \;\middle|\; A^t J A = J \right\}.$$

Note

$$\mathrm{SP}_2(\mathbb{R}) = \left\{ M \in \mathrm{GL}_2(\mathbb{R}) \;\middle|\; \det M = 1 \right\},$$

and $\mathrm{SP}_{2g}(\mathbb{R})$ is connected but $\mathrm{SP}_{2g}(\mathbb{R}) \simeq \mathbb{R}^2 \setminus \{0\}$. Let $p : G \to \mathrm{SP}_{2g}(\mathbb{R})$ be the universal cover; since the base is a topological group, picking any $p^{-1}(1)$ yields an essentially unique group structure on G by analytically continuing the group law. In fact G is a Lie group and $\ker p \cong \mathbb{Z} \in Z(G)$ is central, so there is a SES

$$\mathbb{Z} \to G \to \mathrm{SP}_{2g}(\mathbb{R}).$$

Note that there are not faithful finite-dimensional reps of G, but there are infinite-dimensional reps important to quantization in physics.

2.2 Surfaces

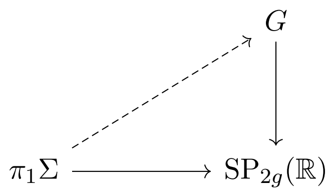
Remark 2.2.1: A theorem of Meyer on surfaces: let Σ be a (compact closed oriented smooth) surface and $\rho : \pi_1\Sigma \to \mathrm{SP}_{2g}\mathbb{R}$; this corresponds to a local system, so form the twisted cohomology $H^1(\Sigma; \rho)$. Using the cup product and symplectic pairing, one can produce a *symmetric* pairing

$$L : H^1(\Sigma; \rho)^{\otimes^2_{\mathbb{R}}} \to \mathbb{R}.$$

There is an isomorphism $(H^1, L) \cong (\mathbb{R}^{p+q}, \mathrm{diag}_p(-1) \oplus \mathrm{diag}_q(1))$, so $\mathrm{sig}(\Sigma, \rho) := \mathrm{sig}\, L := p - q$ is an interesting invariant.

Remark 2.2.2: Recall that 4-manifolds M also admit a signature on $H^2(M; \mathbb{R})$. Let $M \to \Sigma$ be surface bundle over a surface – if M is a product, $\mathrm{sig}\, M = 0$. Chern-Hirzebruch-Serre: there is a monodromy morphism $\pi_1\Sigma \to \mathrm{SP}_{2g}\mathbb{R}$ which is trivial if $\mathrm{sig}\, M = 0$. In fact $\mathrm{sig}(M) = \mathrm{sig}(\Sigma, \rho)$, so this situation naturally arises for fibered 4-manifolds.

Remark 2.2.3: Recall $\pi_1\Sigma_g = \left\langle a_i, b_i \;\middle|\; \prod_i [a_i, b_i] = e \right\rangle$. Consider trying to form a lift:



Link to Diagram

Note $m := \prod[\tilde\rho a_i, \tilde\rho b_i] \in \ker p \cong \mathbb{Z}$, and it turns out that

$$\mathrm{sig}(\Sigma, \rho) = p - q = 4m.$$

The fact that $\ker p \in Z(G)$ was essential in making sure this doesn't depend on choices. Note that this only determines m up to sign!

Remark 2.2.4: Toward generalizing, the above central extension determines a class $b \in H^2(\mathrm{SP}_{2g}(\mathbb{R}); \mathbb{Z})$ and $p^* b \in H^2(\Sigma; \mathbb{Z})$. Using the pairing against the fundamental class $[\Sigma]$ yields

$$\mathrm{sig}(\Sigma, \rho) = 4 \int_\Sigma p^* b.$$

When replacing $\mathbb{R}$ with L, replace $\mathbb{Z}$ by W_L, the Witt group of quadratic forms over L.

3 | 3-manifolds

Remark 3.0.1: Consider now varying the situation in a 1-dimensional family – take $\Sigma \to M \to S^1$ a surface bundle over a circle with fibers Σ_t, each yielding $\rho_t : \pi_1 \Sigma_t \to \mathrm{SP}_{2g}(\mathbb{R})$ for $t \in S^1$. Each t yields a quadratic vector space $H^1(\Sigma_t, \rho_t)$ of signature (p, q). Monodromy yields an element $m \in \mathrm{Aut}\, H^1(\Sigma_0, \rho_0) \subseteq \mathrm{O}_{p,q}$. If $p, q > 0$ then $\sharp \pi_0 \mathrm{O}_{p,q} = 4$ – which connected component does this land in? How to separate the components: there is a determinant map

$$\det : \mathrm{O}_{p,q} \to \{\pm 1\}.$$

There is a *spinor norm*

$$\begin{aligned} \mathrm{spinornorm} : \mathrm{O}_{p,q} &\to \{\pm 1\} = \mathbb{R}^\times / {\mathbb{R}_\square}^\times \\ \mathrm{reflection}_v &\mapsto \langle v,\ v \rangle. \end{aligned}$$

This works with $\mathbb{R}$ replaced by L, using the fact that $\mathrm{O}_{p,q}$ is generated by reflections. For (V, L) a symmetric space, one gets

$$\det : \mathrm{O}(V) \to \{\pm 1\}$$

and

$$\mathrm{spinornorm} : \mathrm{O}(V) \to L^\times / {L_\square}^\times.$$

Theorem 3.0.2 *(?)*.
It turns out that

- $m \in \mathrm{O}(H^1(\Sigma_0, \rho_0))$,
- $\det m = 1$, and
- $\operatorname{spinornorm}(m) = \int_M p^* c \in L^\times / L_\square{}^\times$.

Note that there is a Soulé etale Chern class

$$m \in H^3(\mathrm{SP}_{2g}(L), L^\times / L_\square{}^\times).$$

Pulling this back and integrating yields

$$p^* c \in H^3(M; L^\times / L_\square{}^\times) \xrightarrow{\int_M} L^\times / L_\square{}^\times.$$

Proof (?).
Re-interplay $\operatorname{spinornorm}(m)$ as a quantity $\mathrm{RT}(M, \rho)$ where RT is the *Reidemeister torsion*, which makes sense for any (M, ρ) (not just those fibered over S^1). This is a bordism invariant, i.e. if $M \sim N$ are bordant then $\mathrm{RT}(M, \rho) = \mathrm{RT}(N, \rho')$. Thus there exists some formula of the desired type, where c is unknown – the trick is to compute enough examples to determine c. ■

3.1 Symplectic L-functions

Remark 3.1.1: For $X \in \mathsf{sm}\,\mathrm{proj}\,\mathsf{Var}_{/\mathbb{F}_q}$ and $\rho : \pi_1 X \to \mathrm{SP}_{2g} k$ for k finite containing $\sqrt{q}$, there is an associated L-function $L(X, \rho; T) \in k(T)$ where $T \approx q^{-s}$. There is a functional equation relating $T \rightleftharpoons \frac{1}{qT}$. Evaluate at the center of symmetry $T = \frac{1}{\sqrt{q}}$ to define $L(X, \rho) := L(X, \rho; 1/\sqrt{q})$.

Theorem 3.1.2 *(?)*.
Suppose $L(X, \rho) \neq 0$; then modulo squares

$$L(X, \rho) = \int_X \rho^* c \qquad \in k^\times / k_\square{}^\times,$$

which is a spinor norm of Frobenius (see Zassenhaus). This requires some conditions, e.g. $\rho \Big|_{\pi_1 X}$ is surjective, congruences on $\sharp k$ and $q \bmod 8$, and $\gcd(q, \sharp \mathrm{SP}_{2g}(k)) = 1$ (which may not be necessary).

Remark 3.1.3: Interpretation: $L(X, \rho) = ab^2$ where a is simple and b is complicated. BSD gives a conjectural formula for this which includes a lot of squares. So there is a cohomological obstruction to the existence of a square root of $L(X, \rho)$.

Remark 3.1.4: On the proof: try to pass validity from one known example (X, ρ) to other examples (X', ρ') with $X_{/\mathbb{F}_q}, X'_{/\mathbb{F}'_q}$ and images in $\mathrm{SP}_{2g}(k), \mathrm{SP}_{2g}(k')$ respectively.

- Pass from q to q' using a moduli space of pairs (X, ρ) where a topological theorem controls the fiber over $\mathbb{C}$.
- Pass from k to k' using *compatible local systems* from NT.

Remark 3.1.5: Final comparison:

- For 2-manifolds, $\mathrm{sig}(\Sigma, \rho) = \int_\Sigma p^* b$.
- For 3-manifolds, $\mathrm{RT}(M, \rho) = \int_M p^* c$.
- Symplectic L-functions: ?

4 Talk 3: Relative Langlands Duality

Abstract: If we are given a compact Lie group G acting on a space X, a powerful tool in "approximately" decomposing the G-action on functions on X is the orbit method. I will describe this method and how it sometimes refines to an exact algebraic statement which involves a "dual" group G^ and dual space X^. This is part of a joint work with David Ben-Zvi and Yiannis Sakellaridis about duality in the relative Langlands program. I will do my best to make the talk comprehensible without any familiarity with the framework of the Langlands program.

ToDos

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