Problem Sets: Lie Algebras

Problem Set 1

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1.1 Section 1

Problem 1.1.1 (Humphreys 1.1)
Let L be the real vector space $\mathbf{R}^3$. Define $[xy] = x \times y$ (cross product of vectors) for $x, y \in L$, and verify that L is a Lie algebra. Write down the structure constants relative to the usual basis of $\mathbf{R}^3$.

Solution:
It suffices to check the 3 axioms given in class that define a Lie algebra:

- **L1 (Bilinearity)**: This can be quickly seen from the formula

$$a \times b = \|a\| \cdot \|b\| \sin(\theta_{ab}) \widehat{n}_{ab}$$

where $\widehat{n}_{ab}$ is the vector orthogonal to both a and b given by the right-hand rule. The result follows readily from a direct computation:

$$\begin{aligned}(ra) \times (tb) &= \|ra\| \cdot \|tb\| \sin(\theta_{ra,tb}) \widehat{n}_{ra,tb} \\ &= (rt) \|a\| \cdot \|b\| \sin(\theta_{a,b}) \widehat{n}_{a,b} \\ &= (rt)\,(a \times b)\,,\end{aligned}$$

where we've used the fact that the angle between a and b is the same as the angle between any of their scalar multiples, as is their normal.

- **L2**: that $a \times a = 0$ readily follows from the same formula, since $\sin(\theta_{a,a}) = \sin(0) = 0$.

- **L3 (The Jacobi identity)**: This is most easily seen from the "BAC - CAB" formula,

$$a \times (b \times c) = b\langle a,\ c\rangle - c\langle a,\ b\rangle.$$

We proceed by expanding the Jacobi expression:

$$\begin{aligned}a \times (b \times c) + c \times (a \times b) + b \times (c \times a) &= b\langle a,\ c\rangle - c\langle a,\ b\rangle \\ &\quad + a\langle c,\ b\rangle - b\langle c,\ a\rangle \\ &\quad + c\langle a,\ b\rangle - a\langle b,\ c\rangle \\ &= 0.\end{aligned}$$

For the structure constants, let $\{e_1, e_2, e_3\}$ be the standard Euclidean basis for $\mathbb{R}^3$; we can then write $e_i \times e_j = \sum_{k=1}^{3} c_{ij}^k e_k$ and we would like to determine the c_{ij}^k. One can compute the following multiplication table:

$e_i \times e_j$	e_1	e_2	e_3
e_1	0	e_3	$-e_2$
e_2	$-e_3$	0	e_1
e_3	e_2	$-e_1$	0

Thus the structure constants are given by the antisymmetric Levi-Cevita symbol,

$$c_{ij}^k = \varepsilon^{ijk} := \begin{cases} 0 & \text{if any index } i, j, k \text{ is repeated} \\ \operatorname{sgn} \sigma_{ijk} & \text{otherwise,} \end{cases}$$

where $\sigma_{ijk} \in S_3$ is the permutation associated to (i, j, k) in cycle notation and $\operatorname{sgn} \sigma$ is the sign homomorphism.
Remark 1.1.1: An example to demonstrate how the Levi-Cevita symbol works:

$$e_1 \times e_2 = c_{12}^1 e_1 + c_{12}^2 e_2 + c_{12}^3 e_3 = 0e_1 + 0e_2 + 1e_3$$

since the first two terms have a repeated index and

$$c_{12}^3 = \varepsilon_{1,2,3} = \operatorname{sgn}(123) = \operatorname{sgn}(12)(23) = (-1)^2 = 1$$

using that $\operatorname{sgn} \sigma = (-1)^m$ where m is the number of transpositions in σ.

Problem 1.1.2 (Humphreys 1.6)
Let $x \in \mathfrak{gl}_n(\mathbb{F})$ have n distinct eigenvalues $a_1, \ldots, a_n$ in $\mathbb{F}$. Prove that the eigenvalues of ad_x are precisely the n^2 scalars $a_i - a_j$ for $1 \leq i, j \leq n$, which of course need not be distinct.

Solution:
For a fixed n, let $e_{ij} \in \mathfrak{gl}_n(\mathbb{F})$ be the matrix with a 1 in the (i, j) position and zeros elsewhere. We will use the following fact:

$$e_{ij} e_{kl} = \delta_{jk} e_{il},$$

where $\delta_{jk} = 1 \iff j = k$, which implies that

$$[e_{ij} e_{kl}] = e_{ij} e_{kl} - e_{kl} e_{ij} = \delta_{jk} e_{il} - \delta_{li} e_{kj}.$$

Suppose without loss of generality[a] that x is diagonal and of the form $x = \operatorname{diag}(a_1, a_2, \cdots, a_n)$. Then the eigenvectors of x are precisely the e_{ij}, since a direct check via matrix multiplication shows $x e_{ij} = a_i e_{ij}$.
We claim that every e_{ij} is again an eigenvector of ad_x with eigenvalue $a_i - a_j$. Noting that the e_{ij} are also left eigenvectors satisfying $e_{ij} x = a_j e_{ij}$, one readily computes

$$\operatorname{ad}_x e_{ij} := [x, e_ij] = x e_{ij} - e_{ij} x = a_i e_{ij} - a_j e_{ij} = (a_i - a_j) e_{ij},$$

yielding at least n^2 eigenvalues. Since ad_x expanded in the basis $\{e_{ij}\}_{1 \leq i,j \leq n}$ is an $n \times n$ matrix, this exhausts all possible eigenvalues.

[a] If x is not diagonal, one can use that x is diagonalizable over $\mathbb{F}$ since x has distinct eigenvalues in $\mathbb{F}$. So one can reduce to the diagonal case by a change-of-basis of $\mathbb{F}^n$ that diagonalizes x.

Problem 1.1.3 (Humphreys 1.9, one Lie type only)
When $\operatorname{ch}\mathbb{F} = 0$, show that each classical algebra $L = \mathrm{A}_\ell, \mathrm{B}_\ell, \mathrm{C}_\ell$, or D_ℓ is equal to $[LL]$. (This shows again that each algebra consists of trace 0 matrices.)

Solution:
We will check for this type A_n, corresponding to $L := \mathfrak{sl}_{n+1}$. Since $[LL] \subseteq L$, it suffices to show $L \subseteq [LL]$, and we can further reduce to writing every basis element of L as a commutator in $[LL]$. Note that L has a standard basis given by the matrices

- $\left\{x_i := e_{ij} \;\middle|\; i > j\right\}$ corresponding to $\mathfrak{n}^-$,
- $\left\{h_i := e_{ii} - e_{i+1,i+1} \;\middle|\; 1 \leq i \leq n\right\}$ corresponding to $\mathfrak{h}$, and
- $\left\{y_i := e_{ij} \;\middle|\; i < j\right\}$ corresponding to $\mathfrak{n}^+$.

Considering the equation $[e_{ij}e_{kl}] = \delta_{jk}e_{il} - \delta_{li}e_{kj}$, one can choose $j = k$ to preserve the first term and $l \neq i$ to kill the second term. So letting t, i, j be arbitrary with $i \neq j$, we have

$$[e_{it}e_{tj}] = \delta_{tt}e_{ij} - \delta_{ij}e_{tt} = e_{ij},$$

yielding all of the x_i and y_i. But in fact we are done, using the fact that $h_i = [x_i y_i]$.

Problem 1.1.4 (Humphreys 1.11)
Verify that the commutator of two derivations of an $\mathbb{F}$-algebra is again a derivation, whereas the ordinary product need not be.

Solution:
We want to show that $[\operatorname{Der}(L)\operatorname{Der}(L)] \subseteq \operatorname{Der}(L)$, so let $f, g \in \operatorname{Der}(L)$. The result follows from a direct computation; letting $D := [fg]$, we have

$$\begin{aligned}
D(ab) = [fg](ab) &= (fg - gf)(ab) \\
&= fg(ab) - gf(ab) \\
&= f\left(g(a)b + ag(b)\right) - g\left(f(a)b + af(b)\right) \\
&= f\left(g(a)b\right) + f\left(ag(b)\right) - g\left(f(a)b\right) - g\left(af(b)\right) \\
&= (fg)(a)b + g(a)f(b) \\
&\quad + f(a)g(b) + a(fg)(b) \\
&\quad - (gf)(a)b + f(a)g(b) \\
&\quad - g(a)f(b) - a(gf)(b) \\
&= [fg](a)b - a[fg](b) \\
&= D(a)b - aD(b).
\end{aligned}$$

To see that ordinary products of derivations need not be derivations, consider the operators $D_x := \frac{\partial}{\partial x}, D_y = \frac{\partial}{\partial y}$ acting on a finite-dimensional vector space of multivariate polynomials of some bounded degree, as a sub $\mathbb{R}$-algebra of $\mathbb{R}[x, y]$. Take $f(x, y) = x + y$ and $g(x, y) = xy$, so that $fg = gf = x^2y + xy^2$. Then $[D_x D_y] = 0$ since mixed partial derivatives are equal, but

$$D_x D_y(fg) = D_x \left(x^2 + 2xy\right) = 2x + 2y \neq 0.$$

Problem 1.1.5 (Humphreys 1.12)
Let L be a Lie algebra over an algebraically closed field $\mathbb{F}$ and let $x \in L$. Prove that the subspace of L spanned by the eigenvectors of ad_x is a subalgebra.

Solution:
Let $E_x \subseteq L$ be the subspace spanned by eigenvectors of ad_x; it suffices to show $[E_x E_x] \subseteq E_x$. Letting $y_i \in E_x$, we have $\mathrm{ad}_x(y_i) = \lambda_i y_i$ for some scalars $\lambda_i \in \mathbb{F}$, and we want to show $\mathrm{ad}_x([y_1 y_2]) = \lambda_{12}[y_1 y_2]$ for some scalar λ_{12}. Note that the Jacobi identity is equivalent to ad acting as a derivation with respect to the bracket, i.e.

$$\mathrm{ad}_x([yz]) = [\mathrm{ad}_x(y)z] + [y\mathrm{ad}_x(z)] \implies [x[yz]] = [[xy]z] + [y[xz]].$$

The result then follows from a direct computation:

$$\begin{aligned} \mathrm{ad}_x([y_1 y_2]) &= [[xy_1]y_2] + [y_1[xy_2]] \\ &= [\lambda_1 y_1 y_2] + [y_1 \lambda_2 y_2] \\ &= (\lambda_1 + \lambda_2)[y_1 y_2]. \end{aligned}$$

1.2 Section 2

Problem 1.2.1 (Humphreys 2.1)
Prove that the set of all inner derivations $\mathrm{ad}_x, x \in L$, is an ideal of $\mathrm{Der}\, L$.

Solution:
It suffices to show $[\mathrm{Der}(L)\,\mathrm{Inn}(L)] \subseteq \mathrm{Inn}(L)$, so let $f \in \mathrm{Der}(L)$ and $\mathrm{ad}_x \in \mathrm{Inn}(L)$. The result follows from the following check:

$$\begin{aligned} [f\mathrm{ad}_x](l) &= (f \circ \mathrm{ad}_x)(l) - (\mathrm{ad}_x \circ f)(l) \\ &= f([xl]) - [xf(l)] \\ &= [f(x)l] + [xf(l)] - [xf(l)] \\ &= [f(x)l] \\ &= \mathrm{ad}_{f(x)}(l), \qquad \text{and } \mathrm{ad}_{f(x)} \in \mathrm{Inn}(L). \end{aligned}$$

Problem 1.2.2 (Humphreys 2.2)
Show that $\mathfrak{sl}_n(\mathbb{F})$ is precisely the derived algebra of $\mathfrak{gl}_n(\mathbb{F})$ (cf. Exercise 1.9).

Solution:
We want to show $\mathfrak{gl}_n(\mathbb{F})^{(1)} := [\mathfrak{gl}_n(\mathbb{F})\mathfrak{gl}_n(\mathbb{F})] = \mathfrak{sl}_n(\mathbb{F})$.
$\subseteq$: This immediate from the fact that for any matrices A and B,

$$\operatorname{tr}([AB]) = \operatorname{tr}(AB - BA) = \operatorname{tr}(AB) - \operatorname{tr}(BA) = \operatorname{tr}(AB) - \operatorname{tr}(AB) = 0.$$

$\supseteq$: From a previous exercise, we know that $[\mathfrak{sl}_n(\mathbb{F})\mathfrak{sl}_n(\mathbb{F})] = \mathfrak{sl}_n(\mathbb{F})$, and since $\mathfrak{sl}_n(\mathbb{F}) \subseteq \mathfrak{gl}_n(\mathbb{F})$ we have

$$\mathfrak{sl}_n(\mathbb{F}) = \mathfrak{sl}_n(\mathbb{F})^{(1)} \subseteq \mathfrak{gl}_n(\mathbb{F})^{(1)}.$$

Problem 1.2.3 (Humphreys 2.5)
Suppose $\dim L = 3$ and $L = [LL]$. Prove that L must be simple. Observe first that any homomorphic image of L also equals its derived algebra. Recover the simplicity of $\mathfrak{sl}_2(\mathbb{F})$ when $\operatorname{ch}\mathbb{F} \neq 2$.

Solution:
Let $I \trianglelefteq L$ be a proper ideal, then $\dim L/I < \dim L$ forces $\dim L/I = 1, 2$. Since $L \twoheadrightarrow L/I$, the latter is the homomorphic image of a Lie algebra and thus $(L/I)^{(1)} = L/I$ by the hint. Note that in particular, L/I is not abelian. We proceed by cases:

- $\dim L/I = 1$.
  - In this case, $L/I = \mathbb{F}x$ is generated by a single element x. Since $[xx] = 0$ in any Lie algebra, we have $(\mathbb{F}x)^{(1)} = 0$, contradicting that L/I is not abelian. $\lightning$
- $\dim L/I = 2$: Write $L/I = \mathbb{F}x + \mathbb{F}y$ for distinct generators x, y, and consider the multiplication table for the bracket.
  - If $[xy] = 0$, then L/I is abelian, a contradiction. $\lightning$
  - Otherwise, without loss of generality $[xy] = x$ as described at the end of section 1.4. In this case, $(L/I)^{(1)} \subseteq \mathbb{F}_x \subsetneq L/I$, again a contradiction. $\lightning$

So no such proper ideals I can exist, forcing L to be simple.
Applying this to $L := \mathfrak{sl}_2(\mathbb{F})$, we have $\dim_{\mathbb{F}} \mathfrak{sl}_2(\mathbb{F}) = 2^2 - 1 = 3$, and from a previous exercise we know $\mathfrak{sl}_2(\mathbb{F})^{(1)} = \mathfrak{sl}_2(\mathbb{F})$, so the above argument applies and shows simplicity.

Problem 1.2.4 (Humphreys 2.10)
Let σ be the automorphism of $\mathfrak{sl}_2(\mathbb{F})$ defined in (2.3). Verify that

- $\sigma(x) = -y$,
- $\sigma(y) = -x$,
- $\sigma(h) = -h$.

Note that this automorphism is defined as

$$\sigma = \exp(\mathrm{ad}_x) \circ \exp(\mathrm{ad}_{-y}) \circ \exp(\mathrm{ad}_x).$$

Solution:
We recall that $\exp \mathrm{ad}_x(y) := \sum_{n \geq 0} \frac{1}{n!} \mathrm{ad}_x^n(y)$, where the exponent denotes an n-fold composition of operators. To compute these power series, first note that $\mathrm{ad}_t(t) = 0$ for $t = x, y, h$ by axiom **L2**, so

$$(\exp \mathrm{ad}_t)(t) = 1(t) + \mathrm{ad}_t(t) + \frac{1}{2}\mathrm{ad}_t^2(t) + \cdots = 1(t) = t$$

where 1 denotes the identity operator. It is worth noting that if $\mathrm{ad}_t^n(t') = 0$ for some n and some fixed t, t', then it is also zero for all higher n since each successive term involves bracketing with the previous term:

$$\mathrm{ad}_t^{n+1}(t') = [t\, \mathrm{ad}_t^n(t')] = [t\, 0] = 0.$$

We first compute some individual nontrivial terms that will appear in σ. The first order terms are given by standard formulas, which we collect into a multiplication table for the bracket:

	x	h	y
x	0	$-2x$	h
h	$2x$	0	$-2y$
y	$-h$	$2y$	0

We can thus read off the following:

- $\mathrm{ad}_x(y) = h$
- $\mathrm{ad}_x(h) = -2x$
- $\mathrm{ad}_{-y}(x) = [-yx] = [xy] = h$
- $\mathrm{ad}_{-y}(h) = [-yh] = [hy] = -2y$

For reference, we compute and collect higher order terms:

- $\mathrm{ad}_x^n(y)$:
  - $\mathrm{ad}_x^1(y) = h$ from above,
  - $\mathrm{ad}_x^2(y) = \mathrm{ad}_x([xy]) = \mathrm{ad}_x(h) = [xh] = -[hx] = -2x$,
  - $\mathrm{ad}_x^3(y) = \mathrm{ad}_x(-2x) = 0$, so $\mathrm{ad}_x^{\geq 3}(y) = 0$.
- $\mathrm{ad}_x^n(h)$:
  - $\mathrm{ad}_x^1(h) = -2x$ from above,
  - $\mathrm{ad}_x^2(h) = \mathrm{ad}_x(-2x) = 0$, so $\mathrm{ad}_x^{\geq 2}(h) = 0$.
- $\mathrm{ad}_{-y}^n(x)$:
  - $\mathrm{ad}_{-y}^1(x) = h$ from above,
  - $\mathrm{ad}_{-y}^2(x) = \mathrm{ad}_{-y}(h) = [-yh] = [hy] = -2y$,
  - $\mathrm{ad}_{-y}^2(x) = \mathrm{ad}_{-y}(-2y) = 0$, so $\mathrm{ad}_{-y}^{n \geq 2}(x) = 0$.
- $\mathrm{ad}_{-y}^n(h)$:
  - $\mathrm{ad}_{-y}^1(h) = -2y$ from above, and so $\mathrm{ad}_{-y}^{\geq 2}(h) = 0$.

Finally, we can compute the individual terms of σ:

$$(\exp \operatorname{ad}_x)(x) = x$$

$$\begin{aligned}(\exp \operatorname{ad}_x)(h) &= 1(h) + \operatorname{ad}_x(h)\\ &= h + (-2x)\\ &= h - 2x\end{aligned}$$

$$\begin{aligned}(\exp \operatorname{ad}_x)(y) &= 1(y) + \operatorname{ad}_x(y) + \frac{1}{2}\operatorname{ad}_x^2(y)\\ &= y + h + \frac{1}{2}(-2x)\\ &= y + h - x\end{aligned}$$

$$\begin{aligned}(\exp \operatorname{ad}_{-y})(x) &= 1(x) + \operatorname{ad}_{-y}(x)x + \frac{1}{2}\operatorname{ad}_{-y}^2(x)\\ &= x + h + \frac{1}{2}(-2y)\\ &= x + h - y\end{aligned}$$

$$\begin{aligned}(\exp \operatorname{ad}_{-y})(h) &= 1(h) + \operatorname{ad}_{-y}(h)\\ &= h - 2y\end{aligned}$$

$$(\exp \operatorname{ad}_{-y})(y) = y,$$

and assembling everything together yields

$$
\begin{aligned}
\sigma(x) &= (\exp \operatorname{ad}_x \circ \exp \operatorname{ad}_{-y} \circ \exp \operatorname{ad}_x)(x) \\
&= (\exp \operatorname{ad}_x \circ \exp \operatorname{ad}_{-y})(x) \\
&= (\exp \operatorname{ad}_x)(x + h - y) \\
&= (x) + (h - 2x) - (y + h - x) \\
&= -y
\end{aligned}
$$

$$
\begin{aligned}
\sigma(y) &= (\exp \operatorname{ad}_x \circ \exp \operatorname{ad}_{-y} \circ \exp \operatorname{ad}_x)(y) \\
&= (\exp \operatorname{ad}_x \circ \exp \operatorname{ad}_{-y})(y + h - x) \\
&= \exp \operatorname{ad}_x \left((y) + (h - 2y) - (x + h - y)\right) \\
&= \exp \operatorname{ad}_x (-x) \\
&= -x
\end{aligned}
$$

$$
\begin{aligned}
\sigma(h) &= (\exp \operatorname{ad}_x \circ \exp \operatorname{ad}_{-y} \circ \exp \operatorname{ad}_x)(h) \\
&= (\exp \operatorname{ad}_x \circ \exp \operatorname{ad}_{-y})(h - 2x) \\
&= (\exp \operatorname{ad}_x)((h - 2y) - 2(x + h - y)) \\
&= (\exp \operatorname{ad}_x)(-2x - h) \\
&= -2(x) - (h - 2x) \\
&= -h.
\end{aligned}
$$