Problem Sets: Lie Algebras

Problem Set 2

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1.1 Section 3

Problem 1.1.1 (Humphreys 3.1)
Let I be an ideal of L. Then each member of the derived series or descending central series of I is also an ideal of L.

Solution:
To recall definitions:

- The derived series of L is $L \supseteq L^{(0)} := [LL] \supseteq L^{(1)} := [[LL][LL]] \supseteq \cdots$ and termination implies solvability.
- The descending central series of L is $L \supseteq L^1 := [LL] \supseteq L^2 := [L[LL]] \supseteq \cdots$, and termination implies nilpotency (and hence solvability since $[LL] \subseteq L \implies L^{(i)} \subseteq L^i$).
- $I \trianglelefteq L \iff [L, I] \subseteq I$.

For the derived series, inductively suppose $I := I^{(i)}$ is an ideal, so $[LI] \subseteq I$. We then want to show $I^{(i+1)} := [I, I]$ is an ideal, so $[L, [I, I]] \subseteq [I, I]$. Letting $l \in L$, and $i, j \in I$, one can use the Jacobi identity, antisymmetry of the bracket, and the fact that $[I, I] := L^{(i+1)} \subseteq I$ to write

$$\begin{aligned} [L, [I, I]] &\ni [l[ij]] \\ &= [[li]j] - [[lj]i] \\ &\in [[L, I], I] - [[L, I], I] \\ &\subseteq [[L, I], I] \subseteq [I, I]. \end{aligned}$$

Similarly, for the lower central series, inductively suppose $I := I^i$ is an ideal, so $[L, I] \subseteq I$; we want to show $[L, [L, I]] \subseteq [L, I]$. Again using the Jacobi identity and antisymmetry, we have

$$\begin{aligned} [L, [L, I]] &\ni [l_1, [l_2, i]] \\ &= [[i, l_1], l_2] + [[l_2, l_1], i] \\ &\subseteq [[I, L], L] + [[L, L], I] \\ &\subseteq [I, L] + [L, I] \subseteq [L, I]. \end{aligned}$$

Problem 1.1.2 (Humphreys 3.4)
Prove that L is solvable (resp. nilpotent) if and only $\text{ad}(L)$ is solvable (resp. nilpotent).

Solution:
$\implies$: By the propositions in Section 3.1 (resp. 3.2), the homomorphic image of any solvable (resp. nilpotent) Lie algebra is again solvable (resp. nilpotent).
$\impliedby$: There is an exact sequence

$$0 \to Z(L) \to L \xrightarrow{\text{ad}} \text{ad}(L) \to 0,$$

exhibiting $\text{ad}(L) \cong L/Z(L)$. Thus if $\text{ad}(L)$ is solvable, noting that centers are always solvable, we can use the fact that the 2-out-of-3 property for short exact sequences holds for solvability. Moreover, by the proposition in Section 3.2, if $L/Z(L)$ is nilpotent then L is nilpotent.

Problem 1.1.3 (Humphreys 3.6)
Prove that the sum of two nilpotent ideals of a Lie algebra L is again a nilpotent ideal. Therefore, L possesses a unique maximal nilpotent ideal. Determine this ideal for the nonabelian 2-dimensional algebra $\mathbb{F}x + \mathbb{F}y$ where $[xy] = x$, and the 3-dimensional algebra $\mathbb{F}x + \mathbb{F}y + \mathbb{F}z$ where

- $[xy] = z$

- $[xz] = y$
- $[yz] = 0$

Solution:
To see that sums of nilpotent ideals are nilpotent, suppose $I^N = J^M = 0$ are nilpotent ideals. Then $(I+J)^{M+N} \subseteq I^M + J^N$ by collecting terms and using the absorbing property of ideals. One can now construct a maximal nilpotent ideal in L by defining M as the sum of all nilpotent ideals in L. That this is unique is clear, since M is nilpotent, so if M' is another maximal nilpotent ideal then $M \subseteq M'$ and $M' \subseteq M$.
Consider the 2-dimensional algebra $L := \mathbb{F}x + \mathbb{F}y$ where $[xy] = x$ and let I be the maximal nilpotent ideal. Note that L is not nilpotent since $L^k = \mathbb{F}x$ for all $k \geq 0$, since $L^1 = \mathbb{F}x$ and $[L, \mathbb{F}x] = \mathbb{F}x$ (since all brackets are either zero or $\pm x$). However, this also shows that the subalgebra $\mathbb{F}x$ is an ideal, and is in fact a nilpotent ideal since $[\mathbb{F}x, \mathbb{F}x] = 0$. Although $\mathbb{F}y$ is a nilpotent subalgebra, it is not an ideal since $[L, \mathbb{F}y] = \mathbb{F}x$. So I is at least 1-dimensional, since it contains $\mathbb{F}x$, and at most 1-dimensional, since it is not all of L, forcing $I = \mathbb{F}x$.
Consider now the 3-dimensional algebra $L := \mathbb{F}x + \mathbb{F}y + \mathbb{F}z$ with the multiplication table given in the problem statement above. Note that L is not nilpotent, since $L^1 = \mathbb{F}y + \mathbb{F}z = L^k$ for all $k \geq 2$. This follows from consider $[L, \mathbb{F}y + \mathbb{F}z]$, where choosing $x \in L$ is always a valid choice and choosing y or z in the second slot hits all generators; however, no element brackets to x. So similar to the previous algebra, the ideal $J := \mathbb{F}x + \mathbb{F}y$ is an ideal, and it is nilpotent since all brackets between y and z vanish. By similar dimensional considerations, J must equal the maximal nilpotent ideal.

Problem 1.1.4 (Humphreys 3.10)
Let L be a Lie algebra, K an ideal of L such that L/K is nilpotent and such that $\operatorname{ad}_x|_K$ is nilpotent for all $x \in L$. Prove that L is nilpotent.

Solution:
Suppose that $M := L/K$ is nilpotent, so the lower central series terminates and $M^n = 0$ for some n. Then $L^n \subseteq K$ for the same n, and the claim is that L^n is nilpotent. This follows from applying Engel's theorem: let $x \in L^n \subseteq K$, then $\operatorname{ad}_x|_{L^n} = 0$ by assumption. So every element of L^n is ad-nilpotent, making it nilpotent. Since $0 = (L^n)^k = L^{n+k}$ for some k, this forces L to be nilpotent as well.

1.2 Section 4

Problem 1.2.1 (Humphreys 4.1)
Let $L = \mathfrak{sl}(V)$. Use Lie's Theorem to prove that $\operatorname{Rad} L = Z(L)$; conclude that L is semisimple.

Hint: observe that Rad L *lies in each maximal solvable subalgebra* B *of* L. *Select a basis of* V *so that* $B = L \cap \mathfrak{t}(n, \mathrm{F})$, *and notice that* B^t *is also a maximal solvable subalgebra of* L. *Conclude that* Rad $L \subset L \cap \mathfrak{d}(n, \mathrm{F})$ *(diagonal matrices), then that* Rad $L = Z(L)$.]

Solution:
Let $R = \mathrm{Rad}(L)$ be the radical (maximal solvable ideal) of L. Using the hint, if $S \leq L$ is a maximal solvable subalgebra then it must contain R. By (a corollary of) Lie's theorem, S stabilizes a flag and thus there is a basis with respect to which all elements of S (and thus R) are upper triangular. Thus $S \subseteq \mathfrak{b}$; however, taking the transpose of every element in S again yields a maximal solvable ideal which is lower triangular and thus contained in $\mathfrak{b}^-$. Thus $R \subseteq S \subseteq \mathfrak{b} \cap \mathfrak{b}^- = \mathfrak{h}$, which consists of just diagonal matrices.
We have $Z(L) \subseteq R$ since centers are solvable, and the claim is that $R \subseteq \mathfrak{h} \implies R \subseteq Z(L)$. It suffices to show that R consists of scalar matrices, since it is well-known that $Z(\mathfrak{gl}_n(\mathbb{F}))$ consists of precisely scalar matrices, and this contains $Z(L)$ since $L \leq \mathfrak{gl}_n(\mathbb{F})$ is a subalgebra. This follows by letting $\ell = \sum a_i e_{i,i}$ be an element of $\mathrm{Rad}(L)$ and considering bracketing elements of $\mathfrak{sl}_n(\mathbb{F})$ against it. Bracketing elementary matrices $e_{i,j}$ with $i \neq j$ yields

$$[e_{i,j}, \ell] = a_j e_{i,j} - a_i e_{i,j},$$

which must be an element of $\mathrm{Rad}(L)$ and thus diagonal, which forces $a_j = a_i$ for all i, j.
To conclude that L is semisimple, note that a scalar traceless matrix is necessarily zero, and so $Z(\mathfrak{sl}(V)) = 0$. This suffices since $\mathrm{Rad}(L) = 0 \iff L$ is semisimple.

Problem 1.2.2 (Humphreys 4.3, Failure of Lie's theorem in positive characteristic)
Consider the $p \times p$ matrices:

$$x = \begin{bmatrix} 0 & 1 & 0 & . & . & 0 \\ 0 & 0 & 1 & 0 & . & 0 \\ . & . & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot & \cdot & 1 \\ 1 & . & \cdot & \cdot & \cdots & 0 \end{bmatrix}, \qquad y = \mathrm{diag}(0, 1, 2, 3, \cdots, p-1).$$

Check that $[x, y] = x$, hence that x and y span a two dimensional solvable subalgebra L of $\mathfrak{gl}(p, F)$. Verify that x, y have no common eigenvector.

Solution:
Note that x acts on the left on matrices y by cycling all rows of y up by one position, and

similar yacts on the right by cycling the columns to the right. Thus

$$
\begin{aligned}
xy - yx &= \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & p-1 \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \vdots & \cdots & 0 & 2 & 0 \\ 0 & 0 & \cdots & 0 & 3 \\ p-1 & 0 & \cdots & 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -(p-1) & 0 & 0 & 0 & 0 \end{bmatrix} \\
&\equiv \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \in \mathrm{GL}_n(\mathbb{F}_p) \\
&= x.
\end{aligned}
$$

Thus $L := \mathbb{F}x + \mathbb{F}y$ span a solvable subalgebra, since $L^{(1)} = \mathbb{F}x$ and so $L^{(2)} = 0$.
Moreover, note that every basis vector e_i is an eigenvector for y since $y(e_i) = ie_i$, while no basis vector is an eigenvector for x since $x(e_i) = e_{i+1}$ for $1 \leq i \leq p-1$ and $x(e_p) = e_1$, so x cycles the various basis vectors.

Problem 1.2.3 (Humphreys 4.4)
For arbitrary p, construct a counterexample to Corollary C[a] as follows: Start with $L \subset \mathfrak{gl}_p(\mathbb{F})$ as in Exercise 3. Form the vector space direct sum $M = L \oplus \mathbb{F}^p$, and make M a Lie algebra by decreeing that $\mathbb{F}^p$ is abelian, while L has its usual product and acts on $\mathbb{F}^p$ in the given way. Verify that M is solvable, but that its derived algebra ($= \mathbb{F}x + \mathbb{F}^p$) fails to be nilpotent.

[a] Corollary C states that if L is solvable then every $x \in L^{(1)}$ is ad-nilpotent, and thus $L^{(1)}$ is nilpotent.

Solution:
For pairs $A_1 \oplus v_1$ and $A_2 \oplus v_2$ in M, we'll interpret the given definition of the bracket as

$$[A_1 \oplus v_1, A_2 \oplus v_2] := [A_1, A_2] \oplus (A_1(v_2) - A_2(v_1)),$$

where $A_i(v_j)$ denotes evaluating an endomorphism $A \in \mathfrak{gl}_p(\mathbb{F})$ on a vector $v \in \mathbb{F}^p$. We also define $L = \mathbb{F}x + \mathbb{F}y$ with x and y the given matrices in the previous problem, and note that L is solvable with derived series

$$L = \mathbb{F}x \oplus \mathbb{F}y \supseteq L^{(1)} = \mathbb{F}x \supseteq L^{(2)} = 0.$$

Consider the derived series of M – by inspecting the above definition, we have

$$M^{(1)} \subseteq L^{(1)} \oplus \mathbb{F}^p = \mathbb{F}x \oplus \mathbb{F}^p.$$

Moreover, we have

$$M^{(2)} \subseteq L^{(2)} \oplus \mathbb{F}^p = 0 \oplus \mathbb{F}^p,$$

which follows from considering considering bracketing two elements in $M^{(1)}$: set $w_{ij} := A_i(v_j) - A_j(v_i)$, then

$$\begin{aligned}&[[A_1, A_2] \oplus w_{1,2},\ [A_3, A_4] \oplus w_{3,4}] \\ &\qquad = [[A_1, A_2], [A_3, A_4]] \oplus [A_1, A_2](w_{3,4}) - [A_3, A_4](w_{1,2}).\end{aligned}$$

We can then see that $M^{(3)} = 0$, since for any $w_i \in \mathbb{F}^p$,

$$[0 \oplus w_1,\ 0 \oplus w_2] = 0 \oplus 0(w_2) - 0(w_1) = 0 \oplus 0,$$

and so M is solvable.
Now consider its derived subalgebra $M^{(1)} = \mathbb{F}x \oplus \mathbb{F}^p$. If this were nilpotent, every element would be ad-nilpotent, but let $v = [1, 1, \cdots, 1]$ and consider $\mathrm{ad}_{x\oplus 0}$. We have

$$\mathrm{ad}_{x\oplus 0}(0 \oplus v) = [x \oplus 0, 0 \oplus v] = 0 \oplus xv = 0 \oplus v,$$

where we've used that x acts on the left on vectors by cycling the entries. Thus $\mathrm{ad}^n_{x\oplus 0}(0 \oplus v) = 0 \oplus v$ for all $n \geq 1$ and $x \oplus 0 \in M^{(1)}$ is not ad-nilpotent.