Problem Sets: Lie Algebras

Problem Set 3

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1.1 Section 5

Problem 1.1.1 (5.1)
Prove that if L is nilpotent then the Killing form of L is identically zero.

Solution:
Note that if L is nilpotent than every $\ell \in L$ is ad-nilpotent, so letting $x, y \in L$ be arbitrary, their commutator $\ell \coloneqq [xy]$ is ad-nilpotent. Thus $\mathrm{ad}_{[xy]} \in \mathrm{End}(L)$ is a nilpotent endomorphism

of L, which are always traceless.
The claim is the following: for any $x, y \in L$,

$$\operatorname{Trace}(\operatorname{ad}_{[xy]}) = 0 \implies \operatorname{Trace}(\operatorname{ad}_x \circ \operatorname{ad}_y) = 0,$$

from which it follows immediately that β is identically zero.
First we can use the fact that $\operatorname{ad} : L \to \mathfrak{gl}(L)$ preserves brackets, and so

$$\operatorname{ad}_{[xy]_L} = [\operatorname{ad}_x \operatorname{ad}_y]_{\mathfrak{gl}(L)} = \operatorname{ad}_x \circ \operatorname{ad}_y - \operatorname{ad}_y \circ \operatorname{ad}_x,$$

and so

$$0 = \operatorname{Trace}(\operatorname{ad}_{[xy]}) = \operatorname{Trace}(\operatorname{ad}_x \operatorname{ad}_y - \operatorname{ad}_y \operatorname{ad}_x) = \operatorname{Trace}(\operatorname{ad}_x \operatorname{ad}_y) - \operatorname{Trace}(\operatorname{ad}_y \operatorname{ad}_x).$$

where we've used that the trace is an $\mathbb{F}$-linear map $\mathfrak{gl}(L) \to \mathbb{F}$. This forces

$$\operatorname{Trace}(\operatorname{ad}_x \operatorname{ad}_y) = -\operatorname{Trace}(\operatorname{ad}_y \operatorname{ad}_x),$$

but by the cyclic property of traces, we always have

$$\operatorname{Trace}(\operatorname{ad}_x \operatorname{ad}_y) = \operatorname{Trace}(\operatorname{ad}_y \operatorname{ad}_x).$$

Combining these yields $\operatorname{Trace}(\operatorname{ad}_x \operatorname{ad}_y) = 0$.

Problem 1.1.2 (5.7)
Relative to the standard basis of $\mathfrak{sl}_3(\mathbb{F})$, compute $\det \kappa$. What primes divide it?

Hint: use 6.7, which says $\kappa_{\mathfrak{gl}_n}(x, y) = 2n \operatorname{Trace}(xy)$.

Solution:
We have the following standard basis:

$$x_1 = \begin{bmatrix} \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix} \quad x_2 = \begin{bmatrix} \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix} \quad x_3 = \begin{bmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot \end{bmatrix}$$

$$h_1 = \begin{bmatrix} 1 & \cdot & \cdot \\ \cdot & -1 & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix} \quad h_2 = \begin{bmatrix} \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & -1 \end{bmatrix}$$

$$y_1 = \begin{bmatrix} \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix} \quad y_2 = \begin{bmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot \end{bmatrix} \quad y_3 = \begin{bmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot \end{bmatrix}.$$

For notational convenience, let $\{v_1, \cdots, v_8\}$ denote this ordered basis.
Direct computations show

- $[x_1 v_1] = [x_1 x_1] = 0$
- $[x_1 v_2] = [x_1 x_2] = 0$

- $[x_1 v_3] = [x_1 x_3] = e_{13} = x_2 = v_2$
- $[x_1 v_4] = [x_1 h_1] = -2e_{12} = -2x_2 = -2v_2$
- $[x_1 v_5] = [x_1 h_2] = e_{12} = x_1 = v_1$
- $[x_1 v_6] = [x_1 y_1] = e_{11} - e_{22} = h_1 = v_4$
- $[x_1 v_7] = [x_1 y_2] = -e_{31} = -y_2 = v_6$
- $[x_1 v_8] = [x_1 y_3] = 0$

Let E_{ij} denote the elementary 8×8 matrices with a 1 in the (i, j) position. We then have, for example,

$$\begin{aligned} \mathrm{ad}_{x_1} &= 0 + 0 + E_{2,3} - 2E_{2,4} + E_{1,5} + E_{4,6} + E_{6,7} + 0 \\ &= \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & -2 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}. \end{aligned}$$

The remaining computations can be readily automated on a computer, yielding the following matrices for the remaining ad_{v_i}:

- $\mathrm{ad}_{x_1} = 0 + 0 + E_{2,3} - 2E_{1,4} + E_{1,5} + E_{4,6} + E_{8,7} + 0$
- $\mathrm{ad}_{x_2} = 0 + 0 + 0 - E_{2,4} - E_{2,5} - E_{3,6} + (E_{4,7} + E_{5,7}) + E_{1,8}$
- $\mathrm{ad}_{x_3} = -E_{2,1} + 0 + 0 + E_{3,4} - 2E_{3,5} + 0 + E_{6,7} + E_{5,8}$
- $\mathrm{ad}_{h_1} = 2E_{1,1} + E_{2,2} - E_{3,3} + 0 + 0 - 2E_{6,6} - E_{7,7} + E_{8,8}$
- $\mathrm{ad}_{h_2} = -E_{1,1} + E_{2,2} + 2E_{3,3} + 0 + 0 + E_{6,6} - E_{7,7} - 2E_{8,8}$
- $\mathrm{ad}_{y_1} = -E_{4,1} + E_{3,2} + 0 + 2E_{6,4} - E_{6,5} + 0 + 0 - E_{7,8}$
- $\mathrm{ad}_{y_2} = E_{8,1} - (E_{4,2} + E_{5,2}) - E_{6,3} + E_{7,4} + E_{7,5} + 0 + 0 + 0$
- $\mathrm{ad}_{y_3} = 0 - E_{1,2} - E_{5,3} - E_{8,4} + 2E_{8,5} + E_{7,6} + 0 + 0$

Now forming the matrix $(\beta)_{ij} \coloneqq \mathrm{Trace}(\mathrm{ad}_{v_i} \mathrm{ad}_{v_j})$ yields

$$\beta = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & 2 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 6 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 6 \\ \cdot & \cdot & \cdot & 12 & -6 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & -6 & 12 & \cdot & \cdot & \cdot \\ 2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 6 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 6 & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix},$$

whence $\det(\beta) = (2 \cdot 6 \cdot 6)^2 (12^2 - 36) = -2^8 3^7$.

1.2 Section 6

Problem 1.2.1 (6.1)
Using the standard basis for $L = \mathfrak{sl}_2(\mathbb{F})$, write down the Casimir element of the adjoint representation of L *(cf. Exercise 5.5)*. Do the same thing for the usual (3-dimensional) representation of $\mathfrak{sl}_3(\mathbb{F})$, first computing dual bases relative to the trace form.

Solution:
A computation shows that in the basis $\{e_i\} \coloneqq \{x, h, y\}$, the Killing form is represented by

$$\beta = \begin{bmatrix} 0 & 0 & 4 \\ 0 & 8 & 0 \\ 4 & 0 & 0 \end{bmatrix} \implies \beta^{-T} = \begin{bmatrix} 0 & 0 & \frac{1}{4} \\ 0 & \frac{1}{8} & 0 \\ \frac{1}{4} & 0 & 0 \end{bmatrix},$$

yielding the dual basis $\{e_i{}^\vee\}$ read from the columns of β^{-T}:

- $x^\vee = \frac{1}{4} y,$
- $h^\vee = \frac{1}{8} h,$
- $y^\vee = \frac{1}{4} x.$

Thus letting $\varphi = \mathrm{ad}$, we have

$$\begin{aligned} c_\varphi &= \sum \varphi(e_i)\varphi(e_i{}^\vee) \\ &= \mathrm{ad}(x)\mathrm{ad}(x^\vee) + \mathrm{ad}(h)\mathrm{ad}(h^\vee) + \mathrm{ad}(y)\mathrm{ad}(y^\vee) \\ &= \mathrm{ad}(x)\mathrm{ad}(y/4) + \mathrm{ad}(h)\mathrm{ad}(h/8) + \mathrm{ad}(y)\mathrm{ad}(x/4) \\ &= \frac{1}{4}\mathrm{ad}_x\mathrm{ad}_y + \frac{1}{8}\mathrm{ad}_h^2 + \frac{1}{4}\mathrm{ad}_y\mathrm{ad}_x. \end{aligned}$$

For $\mathfrak{sl}_3$, first take the ordered basis $\{v_1, \cdots, v_8\} = \{x_1, x_2, x_3, h_1, h_2, y_1, y_2, y_3\}$ as in the previous problem. So we form the matrix $(\beta)_{ij} \coloneqq \mathrm{Trace}(v_i v_j)$ by computing various products and traces on a computer to obtain

$$\beta = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & 2 & -1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & -1 & 2 & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix} \implies \beta^{-T} = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \frac{2}{3} & \frac{1}{3} & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \frac{1}{3} & \frac{2}{3} & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix},$$

which yields the dual basis

- $x_i{}^\vee = y_i$
- $h_1{}^\vee = \frac{2}{3}h_1 + \frac{1}{3}h_2$
- $h_2{}^\vee = \frac{1}{3}h_1 + \frac{2}{3}h_2$
- $y_i{}^\vee = x_i$

We can thus compute the Casimir element of the standard representation φ on a computer as

$$\begin{aligned} c_\varphi &= \sum_i \varphi(x_i)\varphi(x_i{}^\vee) + \varphi(h_1)\varphi(h_1{}^\vee) + \varphi(h_2)\varphi(h_2{}^\vee) + \sum_i \varphi(y_i)\varphi(y_i{}^\vee) \\ &= \sum_i x_i y_i + h_1 h_1{}^\vee + h_2 h_2{}^\vee + \sum_i y_i x_i \\ &= \sum_i (x_i y_i + y_i x_i) \\ &= \frac{8}{3} I. \end{aligned}$$

Problem 1.2.2 (6.3)
If L is solvable, every irreducible representation of L is one dimensional.

Solution:
Let $\varphi : L \to V$ be an irreducible representation of L. By Lie's theorem. L stabilizes a flag in V, say $F^\bullet = F^1 \subset \cdots F^n = V$ where $F^k = \langle v_1, \cdots, v_k \rangle$ for some basis $\{v_i\}_{i \le n}$. Since φ is irreducible, the only L-invariant subspaces of V are 0 and V itself. However, each F^k is an L-invariant subspace, which forces $n = 1$ and $F^1 = V$. Thus V is 1-dimensional.

Problem 1.2.3 (6.5)
A Lie algebra L for which $\operatorname{Rad} L = Z(L)$ is called reductive.[a]

(a) If L is reductive, then L is a completely reducible ad L-module. [b] In particular, L is the direct sum of $Z(L)$ and $[LL]$, with $[LL]$ semisimple.
(b) If L is a classical linear Lie algebra (1.2), then L is semisimple. *(Cf. Exercise 1.9.)*
(c) If L is a completely reducible $\operatorname{ad}(L)$-module, then L is reductive.
(d) If L is reductive, then all finite dimensional representations of L in which $Z(L)$ is represented by semisimple endomorphisms are completely reducible.

[a] Examples: L abelian, L semisimple, $L = \mathfrak{gl}_n(\mathbb{F})$.
[b] If ad $L \neq 0$, use Weyl's Theorem.

Solution:
Part 1: If $\operatorname{ad}(L) \neq 0$, as hinted, we can attempt to apply Weyl's theorem to the representation $\varphi : \operatorname{ad}(L) \to \mathfrak{gl}(L)$: if we can show $\operatorname{ad}(L)$ is semisimple, then φ (and thus L) will be a completely reducible $\operatorname{ad}(L)$-module. Assume L is reductive, so $\ker(\operatorname{ad}) = Z(L) = \operatorname{Rad}(L)$, and by the first isomorphism theorem $\operatorname{ad}(L) \cong L/\operatorname{Rad}(L)$. We can now use the fact stated in Humphreys on

page 11 that for an arbitrary Lie algebra L, the quotient $L/\operatorname{Rad}(L)$ is semisimple – this follows from the fact that $\operatorname{Rad}(L/\operatorname{Rad}(L)) = 0$, since the maximal solvable ideal in the quotient would need to be a maximal proper ideal in L containing $\operatorname{Rad}(L)$, which won't exist by maximality of $\operatorname{Rad}(L)$. Thus $\operatorname{ad}(L)$ is semisimple, and Weyl's theorem implies it is completely reducible.
To show that $L = Z(L) \oplus [LL]$, we first show that it decomposes as a sum $L = Z(L) + [LL]$, and then that the intersection is empty so the sum is direct. We recall that a Lie algebra is semisimple if and only if it has no nonzero abelian ideals. Since $L/Z(L)$ is semisimple, we have $[L/Z(L), L/Z(L)] = L/Z(L)$ since it would otherwise be a nonzero abelian ideal in $L/Z(L)$. We can separately identify $[L/Z(L), L/Z(L)] \cong [LL]/Z(L)$, since the latter is also semisimple and the former is an abelian ideal in it. Combining these, we have $[LL]/Z(L) \cong L/Z(L) \cong \operatorname{ad}(L)$, and so we have an extension

$$0 \to Z(L) \to L \to [LL] \to 0.$$

Since this sequence splits at the level of vector spaces, $L = Z(L) + [LL]$ as an $\operatorname{ad}(L)$-module, although the sum need not be direct. To show that it is, note that $Z(L) \leq L$ is an $\operatorname{ad}(L)$-invariant submodule, and by complete reducibility has an $\operatorname{ad}(L)$-invariant complement W. We can thus write $L = W \oplus Z(L)$, and moreover $[LL] \subseteq W$, and so we must have $W = [LL]$ and $L = [LL] \oplus Z(L)$.
Finally, to see that $[LL]$ is semisimple, note that the above decomposition allows us to write $L/Z(L) \cong [LL]$, and $\operatorname{Rad}(L/Z(L)) = \operatorname{Rad}(L/\operatorname{Rad}(L)) = 0$ so $\operatorname{Rad}([LL]) = 0$.
Part 2: Omitted for time.
Part 3: Omitted for time.
Part 4: Omitted for time.

Problem 1.2.4 (6.6)
Let L be a simple Lie algebra. Let $\beta(x, y)$ and $\gamma(x, y)$ be two symmetric associative bilinear forms on L. If β, γ are nondegenerate, prove that β and γ are proportional.

Hint: Use Schur's Lemma.

Solution:
The strategy will be to define an irreducible L-module V and use the two bilinear forms to produce an element of $\operatorname{End}_L(V)$, which will be 1-dimensional by Schur's lemma.
The representation we'll take will be $\varphi := \operatorname{ad} : L \to \mathfrak{gl}(L)$, and since L is simple, $\ker \operatorname{ad} = 0$ since otherwise it would yield a nontrivial ideal of L. Since this is a faithful representation, we will identify L with its image $V := \operatorname{ad}(L) \subseteq \mathfrak{gl}(L)$ and regard V as an L-module.
As a matter of notation, let $\beta_x(y) := \beta(x, y)$ and similarly $\gamma_x(y) := \gamma(x, y)$, so that β_x, γ_x can be regarded as linear functionals on V and thus elements of $V^\vee$. This gives an $\mathbb{F}$-linear map

$$\begin{aligned} \Phi_1 : V &\to V^\vee \\ x &\mapsto \beta_x, \end{aligned}$$

which we claim is an L-module morphism.
Assuming this for the moment, note that by the general theory of bilinear forms on vector spaces, since β and γ are nondegenerate, the assignments $x \mapsto \beta_x$ and $x \mapsto \gamma_x$ induce vector space isomorphisms $V \xrightarrow{\sim} V^\vee$. Accordingly, for any linear functional $f \in V^\vee$, there is a unique

element $z(f) \in V$ such that $f(v) = \gamma(z(f), v)$. So define a map using the representing element for γ:

$$\begin{aligned} \Phi_2 : V^\vee &\to V \\ f &\mapsto z(f), \end{aligned}$$

which we claim is also an L-module morphism.
We can now define their composite

$$\begin{aligned} \Phi := \Phi_2 \circ \Phi_1 : V &\to V \\ x &\mapsto z(\beta_x), \end{aligned}$$

which sends an element $x \in V$ to the element $z = z(\beta_x) \in V$ such that $\beta_x(-) = \gamma_z(-)$ as functionals. An additional claim is that Φ commutes with the image $V := \mathrm{ad}(L) \subseteq \mathfrak{gl}(L)$. Given this, by Schur's lemma we have $\Phi \in \mathrm{End}_L(V) = \mathbb{F}$ (where we've used that a compositions of morphisms is again a morphism) and so $\Phi = \lambda \, \mathrm{id}_L$ for some scalar $\lambda \in \mathbb{F}$.
To see why this implies the result, we have equalities of functionals

$$\begin{aligned} \beta(x, -) &= \beta_x(-) \\ &= \gamma_{z(\beta_x)}(-) \\ &= \gamma(z(\beta_x), -) \\ &= \gamma(\Phi(x), -) \\ &= \gamma(\lambda x, -) \\ &= \lambda\gamma(x, -), \end{aligned}$$

and since this holds for all x we have $\beta(-,-) = \lambda\gamma(-,-)$ as desired.

Claim: Φ_1 is an L-module morphism.

Proof (?).
We recall that a morphism of L-modules $\varphi : V \to W$ is an $\mathbb{F}$-linear map satisfying

$$\varphi(\ell.\mathbf{x}) = \ell.\varphi(\mathbf{x}) \qquad \forall \ell \in L, \forall \mathbf{x} \in V.$$

In our case, the left-hand side is

$$\Phi_1(\ell.\mathbf{x}) := \Phi_1(\mathrm{ad}_\ell(\mathbf{x})) = \Phi_1([\ell, \mathbf{x}]) = \beta_{[\ell,\mathbf{x}]} = \beta([\ell, \mathbf{x}], -).$$

and the right-hand side is

$$\ell.\Phi_1(\mathbf{x}) := \ell.\beta_{\mathbf{x}} := (y \mapsto -\beta_{\mathbf{x}}(\ell.y)) := (\mathbf{y} \mapsto -\beta_{\mathbf{x}}([\ell, \mathbf{y}])) = -\beta(\mathbf{x}, [\ell, -]).$$

By anticommutativity of the bracket, along with $\mathbb{F}$-linearity and associativity of β, we have

$$\beta([\ell, \mathbf{x}], \mathbf{y}) = -\beta([\mathbf{x}, \ell], \mathbf{y}) = -\beta(\mathbf{x}, [\ell, \mathbf{y}]) \qquad \forall \mathbf{y} \in V$$

and so the above two sides do indeed coincide.

■

Claim: Φ_2 is an L-module morphism.

Proof (?).
Omitted for time, proceeds similarly. ■

Claim: Φ commutes with $\mathrm{ad}(L)$.

Proof (?).
Letting $x \in L$, we want to show that $\Phi \circ \mathrm{ad}_x = \mathrm{ad}_x \circ \Phi \in \mathfrak{gl}(L)$, i.e. that these two endomorphisms of L commute. Fixing $\ell \in L$, the LHS expands to

$$\Phi(\mathrm{ad}_x(\ell)) = z(\beta_{\mathrm{ad}_x(\ell)}) = z(\beta_{[x\ell]}),$$

while the RHS is

$$\mathrm{ad}_x(\Phi(\ell)) = \mathrm{ad}_x(z(\beta_\ell)) = [x, z(\beta_\ell)].$$

Recalling that $\Phi(t) = z(\beta_t)$ is defined to be the unique element $t \in L$ such that $\beta(t, -) = \gamma(z(\beta_t), -)$, for the above two to be equal it suffices to show that

$$\beta([x,\ell], -) = \gamma([x, z(\beta_\ell)], -)$$

as linear functionals. Starting with the RHS of this expression, we have

$$\begin{aligned}
\gamma([x, z(\beta_\ell)], -) &= -\gamma([z(\beta_\ell), x], -) \quad \text{by antisymmetry} \\
&= -\gamma(z(\beta_\ell), [x, -]) \quad \text{by associativity of } \gamma \\
&= -\beta(\ell, [x, -]) \quad \text{by definition of } z(\beta_\ell) \\
&= -\beta([\ell, x], -) \\
&= \beta([x, \ell], -).
\end{aligned}$$

■

Problem 1.2.5 (6.7)
It will be seen later on that $\mathfrak{sl}_n(\mathbb{F})$ is actually simple. Assuming this and using Exercise 6, prove that the Killing form κ on $\mathfrak{sl}_n(\mathbb{F})$ is related to the ordinary trace form by $\kappa(x, y) = 2n\,\mathrm{Tr}(xy)$.

Solution:
By the previous exercise, the trace pairing $(x, y) \mapsto \mathrm{Trace}(xy)$ is related to the Killing form by $\kappa(x, y) = \lambda\,\mathrm{Trace}(x, y)$ for some λ – here we've used the fact that since $\mathfrak{sl}_n(\mathbb{F})$ is simple, $\mathrm{Rad}(\mathrm{Trace}) = 0$ and thus the trace pairing is nodegenerate. Since the scalar only depends on the bilinear forms and not on any particular inputs, it suffices to compute it for any pair (x, y), and in fact we can take $x = y$. For $\mathfrak{sl}_n$, we can take advantage of the fact that in the standard basis, ad_{h_i} will be diagonal for any standard generator $h_i \in \mathfrak{h}$, making $\mathrm{Trace}(\mathrm{ad}_{h_i}^2)$ easier to compute for general n.
Take the standard $h_1 \coloneqq e_{11} - e_{22}$, and consider the matrix of ad_{h_1} in the ordered basis

$\{x_1, \cdots, x_k, h_1, \cdots, h_{n-1}, y_1, \cdots, y_k\}$ which has $k + (n-1) + k = n^2 - 1$ elements where $k = (n^2 - n)/2$. We'll first compute the Killing form with respect to this basis. In order to compute the various $[h_1, v_i]$, we recall the formula $[e_{ij}, e_{kl}] = \delta_{jk} e_{il} - \delta_{li} e_{kj}$. Applying this to h_1 yields

$$[h_1, e_{ij}] = [e_{11} - e_{22}, e_{ij}] = [e_{11}, e_{ij}] - [e_{22}, e_{ij}] = (\delta_{1i} e_{2j} - \delta_{1j} e_{i1}) - (\delta_{2i} e_{2j} - \delta_{2j} e_{i2}).$$

We proceed to check all of the possibilities for the results as i, j vary with $i \neq j$ using the following schematic:

$$\left[\begin{array}{c|c|c} \cdot & a & R_1 \cdots \\ \hline b & \cdot & R_2 \cdots \\ \hline C_1 & C_2 & \\ \vdots & \vdots & M \end{array}\right].$$

The possible cases are:

- $a : i = 1, j = 2 \implies [h_1, e_{ij}] = 2e_{12}$, covering 1 case.
- $b : i = 2, j = 1 \implies [h_1, e_{ij}] = -2e_{21}$ covering 1 case.
- $R_1 : i = 1, j > 2 \implies [h_1, e_{ij}] = e_{1j}$ covering $n-2$ cases.
- $R_2 : i = 2, j > 2 \implies [h_1, e_{ij}] = e_{2j}$ covering $n-2$ cases.
- $C_1 : j = 1, i > 2 \implies [h_1, e_{ij}] = e_{i1}$ covering $n-2$ cases.
- $C_2 : j = 2, i > 2 \implies [h_1, e_{ij}] = e_{i2}$ covering $n-2$ cases.
- $M : i, j > 2 \implies [h_1, e_{ij}] = 0$ covering the remaining cases.

Thus the matrix of ad_{h_1} has $4(n-2)$ ones and $2, -2$ on the diagonal, and $\mathrm{ad}_{h_1}^2$ as $4(n-2)$ ones and $4, 4$ on the diagonal, yielding

$$\mathrm{Trace}(\mathrm{ad}_{h_1}^2) = 4(n-2) + 2(4) = 4n.$$

On the other hand, computing the standard trace form yields

$$\mathrm{Trace}(h_1^2) = \mathrm{Trace}(\mathrm{diag}(1, 1, 0, 0, \cdots)) = 2,$$

and so

$$\mathrm{Trace}(\mathrm{ad}_{h_1}^2) = 4n = 2n \cdot 2 = 2n \cdot \mathrm{Trace}(h_1^2) \implies \lambda = 2n.$$