Problem Sets: Lie Algebras

Problem Set 4

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1.1 Section 7

Problem 1.1.1 (Humphreys 7.2)
$M = \mathfrak{sl}(3, \mathbb{F})$ contains a copy of $L := \mathfrak{sl}(2, \mathbb{F})$ in its upper left-hand 2×2 position. Write M as direct sum of irreducible L-submodules (M viewed as L module via the adjoint representation):

$$V(0) \oplus V(1) \oplus V(1) \oplus V(2).$$

Solution:
Noting that

- $\dim V(m) = m+1$
- $\dim \mathfrak{sl}_3(\mathbb{F}) = 8$
- $\dim(V(0) \oplus V(1) \oplus V(1) \oplus V(2)) = 1+2+2+3 = 8$,

it suffices to find distinct highest weight elements of weights $0, 1, 1, 2$ and take the irreducible submodules they generate. As long as the spanning vectors coming from the various $V(n)$ are all distinct, they will span M as a vector space by the above dimension count and individually span the desired submodules.
Taking the standard basis $\{v_1, \cdots, v_8\} := \{x_1, x_2, x_3, h_1, h_2, y_1, y_2, y_3\}$ for $\mathfrak{sl}_3(\mathbb{F})$ with $y_i = x_i^t$, note that the image of the inclusion $\mathfrak{sl}_2(\mathbb{F}) \hookrightarrow \mathfrak{sl}_3(\mathbb{F})$ can be identified with the span of $\{w_1, w_2, w_3\} := \{x_1, h_1, y_1\}$ and it suffices to consider how these 3×3 matrices act.
Since any highest weight vector must be annihilated by the x_1-action, to find potential highest weight vectors one can compute the matrix of ad_{x_1} in the above basis and look for zero columns:

$$\mathrm{ad}_{x_1} = \begin{pmatrix} 0 & 0 & 0 & -2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}.$$

Thus $\{v_1 = x_1, v_2 = x_2, v_8 = y_3\}$ are the only options for highest weight vectors of nonzero weight, since ad_{x_1} acts nontrivially on the remaining basis elements.
Computing the matrix of $\mathrm{ad} h_1$, one can read off the weights of each:

$$\mathrm{ad}_{h_1} = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Thus the candidates for highest-weight vectors are:

- x_1 for $V(2)$,
- x_2 for one copy of $V(1)$,
- y_3 for the other copy of $V(1)$,
- h_1 or h_2 for $V(0)$.

We can now repeatedly apply the y_1-action to obtain the other vectors in each irreducible module.
For $V(2)$:

- $v_0 = x_1$ which has weight 2,
- $v_1 = y_1.v_0 = [y_1, x_1] = -h_1$ which has weight 0,
- $v_2 = \frac{1}{2}y_1^2.v_0 = \frac{1}{2}[y_1, [y_1, x_1]] = -y_1$ which has weight -2.

Since we see h_1 appears in this submodule, we see that we should later take h_2 as the maximal vector for $V(0)$. Continuing with $V(1)$:

- $v_0 = x_2$ which has weight 1,
- $v_1 = y_1.v_0 = [y_1, x_2] = x_3$ which has weight -1.

For the other $V(1)$:

- $v_0 = y_3$ with weight 1,
- $v_1 = -y_2$ with weight -1.

For $V(0)$:

- $v_0 = h_2$.

We see that we get the entire basis of $\mathfrak{sl}_3(\mathbb{F})$ this way with no redundancy, yielding the desired direct product decomposition.

Problem 1.1.2 (Humphreys 7.5)
Suppose char $\mathbb{F} = p > 0, L = \mathfrak{sl}(2, \mathbb{F})$. Prove that the representation $V(m)$ of L constructed as in Exercise 3 or 4 is irreducible so long as the highest weight m is strictly less than p, but reducible when $m = p$.

Note: this corresponds to the formulas in lemma 7.2 parts (a) through (c), or by letting $L \curvearrowright \mathbb{F}^2$ in the usual way and extending $L \curvearrowright \mathbb{F}[x, y]$ by derivations, so $l.(fg) = (l.f)g + f(l.g)$ and taking the subspace of homogeneous degree m polynomials $\langle x^m, x^{m-1}y, \cdots, y^m\rangle$ to get an irreducible module of highest weight m.

Solution:
The representation $V(m)$ in Lemma 7.2 is defined by the following three equations, where $v_0 \in V_m$ is a highest weight vector and $v_k := y^k v_0/k!$:

1. $h \cdot v_i = (m - 2i)v_i$
2. $y \cdot v_i = (i + 1)v_{i+1}$
3. $x \cdot v_i = (m - i + 1)v_{i-1}$.

Supposing $m < p$, the vectors $\{v_0, v_1, \cdots, v_m\}$ still span an irreducible L-module since it contains no nontrivial L-submodules, just as in the characteristic zero case.
However, if $m = p$, then note that $y.v_{m-1} = (m - 1 + 1)v_m = 0v_m = 0$ and consider the set $\{v_0, \cdots, v_{m-1}\}$. This spans an m-dimensional subspace of V, and the equations above show it is invariant under the L-action, so it yields an m-dimensional submodule of $V(m)$. Since

$\dim_{\mathbb{F}} V(m) = m+1$, this is a nontrivial proper submodule, so $V(m)$ is reducible.

Problem 1.1.3 (Humphreys 7.6)
Decompose the tensor product of the two L-modules $V(3), V(7)$ into the sum of irreducible submodules: $V(4) \oplus V(6) \oplus V(8) \oplus V(10)$. Try to develop a general formula for the decomposition of $V(m) \otimes V(n)$.

Solution:
By a theorem from class, we know the weight space decomposition of any $\mathfrak{sl}_2(\mathbb{C})$-module V takes the following form:

$$V = V_{-m} \oplus V_{-m+2} \oplus \cdots \oplus V_{m-2} \oplus V_m,$$

where m is a highest weight vector, and each weight space V_μ is 1-dimensional and occurs with multiplicity one. In particular, since $V(m)$ is a highest-weight module of highest weight m, we can write

$$\begin{aligned} V(3) &= \qquad\qquad\qquad\quad V_{-3} \oplus V_{-1} \oplus V_1 \oplus V_3 \\ V(7) &= V_{-7} \oplus V_{-5} \oplus V_{-3} \oplus V_{-1} \oplus V_1 \oplus V_3 \oplus V_5 \oplus V_7, \end{aligned}$$

and tensoring these together yields modules with weights between $-3-7 = -10$ and $3+7 = 10$:

$$\begin{aligned} V(3) \otimes V(7) = V_{-10} &\oplus {V_{-8}}^{\oplus 2} \oplus {V_{-6}}^{\oplus 3} \oplus {V_{-4}}^{\oplus 4} \oplus {V_{-2}}^{\oplus 4} \\ &\oplus {V_0}^{\oplus 4} \\ &\oplus {V_2}^{\oplus 4} \oplus {V_4}^{\oplus 4} \oplus {V_6}^{\oplus 3} \oplus {V_8}^{\oplus 2} \oplus V_{10}. \end{aligned}$$

This can be more easily parsed by considering formal characters:

$$\begin{aligned} \operatorname{ch}(V(3)) &= e^{-3} + e^{-1} + e^1 + e^3 = \\ \operatorname{ch}(V(7)) &= e^{-7} + e^{-5} + e^{-3} + e^{-1} + e^1 + e^3 + e^5 + e^7 \\ \operatorname{ch}(V(3) \otimes V(7)) &= \operatorname{ch}(V(3)) \cdot \operatorname{ch}(V(7)) \\ \\ &= (e^{-10} + e^{10}) + 2(e^{-8} + e^8) + 3(e^{-6} + e^6) \\ &\quad + 4(e^{-4} + e^4) + 4(e^{-2} + e^2) + 4 \\ \\ &= (e^{-10} + e^{10}) + 2(e^{-8} + e^8) + 3(e^{-6} + e^6) \\ &\quad + 4\operatorname{ch}(V(4)), \end{aligned}$$

noting that $\operatorname{ch}(V(4)) = e^{-4} + e^{-2} + e^2 + e^4$ and collecting terms.
To see that $V(3) \otimes V(7)$ decomposes as $V(4) \oplus V(6) \oplus V(8) \oplus V(10)$ one can check for equality

of characters to see that the various weight spaces and multiplicities match up:

$$
\begin{aligned}
\operatorname{ch}(V(4) \oplus V(6) \oplus V(8) \oplus V(10)) &= \operatorname{ch}(V(4)) + \operatorname{ch}(V(6)) + \operatorname{ch}(V(8)) + \operatorname{ch}(V(10)) \\
&= \left(e^{-4} + \cdots + e^{4}\right) + \left(e^{-6} + \cdots + e^{6}\right) \\
&\quad + \left(e^{-8} + \cdots + e^{8}\right) + \left(e^{-10} + \cdots + e^{10}\right) \\
&= 2\operatorname{ch}(V(4)) + (e^{-6} + e^{6}) \\
&\quad + \operatorname{ch}(V(4)) + (e^{-6} + e^{6}) + (e^{-8} + e^{8}) \\
&\quad + \operatorname{ch}(V(4)) + (e^{-6} + e^{6}) + (e^{-8} + e^{8}) + (e^{-10} + e^{10}) \\
&= 4\operatorname{ch}(V(4)) + 3(e^{-6} + e^{6}) \\
&\quad + 2(e^{-8} + e^{8}) + (e^{-10} + e^{10}),
\end{aligned}
$$

which is equal to $\operatorname{ch}(V(3) \otimes V(7))$ from above.
More generally, for two such modules V, W we can write

$$
V \otimes_{\mathbb{F}} W = \bigoplus_{\lambda \in \mathfrak{h}^\vee} \bigoplus_{\mu_1 + \mu_2 = \lambda} V_{\mu_1} \otimes_{\mathbb{F}} W_{\mu_2},
$$

where we've used the following observation about the weight of $\mathfrak{h}$ acting on a tensor product of weight spaces: supposing $v \in V_{\mu_1}$ and $w \in W_{\mu_2}$,

$$
\begin{aligned}
h.(v \otimes w) &= (hv) \otimes w + v \otimes (hw) \\
&= (\mu_1 v) \otimes w + v \otimes (\mu_2 w) \\
&= (\mu_1 v) \otimes w + (\mu_2 v) \otimes w \\
&= (\mu_1 + \mu_2)(v \otimes w),
\end{aligned}
$$

and so $v \otimes w \in V_{\mu_1 + \mu_2}$.
Taking $V(m_1), V(m_2)$ with $m_1 \geq m_2$ then yields a general formula:

$$
V(m_1) \otimes_{\mathbb{F}} V(m_2) = \bigoplus_{n = -m_1 - m_2}^{m_1 + m_2} \bigoplus_{a + b = n} V_a \otimes_{\mathbb{F}} V_b = \bigoplus_{n = m_1 - m_2}^{m_2 + m_1} V(n).
$$

1.2 Section 8

Problem 1.2.1 (Humphreys 8.9)
Prove that every three dimensional semisimple Lie algebra has the same root system as $\mathfrak{sl}(2, \mathbb{F})$, hence is isomorphic to $\mathfrak{sl}(2, \mathbb{F})$.

Solution:
There is a formula for the dimension of L in terms of the rank of Φ and its cardinality, which is more carefully explained in the solution below for problem 8.10:

$$\dim \mathfrak{g} = \operatorname{rank} \Phi + \sharp \Phi.$$

Thus if $\dim L = 3$ then the only possibility is that $\operatorname{rank} \Phi = 1$ and $\sharp\Phi = 2$, using that $\operatorname{rank} \Phi \leq \sharp\Phi$ and that $\sharp\Phi$ is always even since each $\alpha \in \Phi$ can be paired with $-\alpha \in \Phi$. In particular, the root system Φ of L must have rank 1, and there is a unique root system of rank 1 (up to equivalence) which corresponds to A_1 and $\mathfrak{sl}_2(\mathbb{F})$.
By the remark in Humphreys at the end of 8.5, there is a 1-to-1 correspondence between pairs (L, H) with L a semisimple Lie algebra and H a maximal toral subalgebra and pairs $(\Phi, \mathbb{E})$ with Φ a root system and $\mathbb{E} \supseteq \Phi$ its associated Euclidean space. Using this classification, we conclude that $L \cong \mathfrak{sl}_2(\mathbb{F})$.

Problem 1.2.2 (Humphreys 8.10)
Prove that no four, five or seven dimensional semisimple Lie algebras exist.

Solution:
We can first write

$$\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+, \qquad \mathfrak{n}^+ := \bigoplus_{\alpha\in\Phi^+} \mathfrak{g}_\alpha, \quad \mathfrak{n}^- := \bigoplus_{\alpha\in\Phi^+} \mathfrak{g}_{-\alpha}.$$

Writing $N := \mathfrak{n}^+ \oplus \mathfrak{n}^- = \bigoplus_{\alpha\in\Phi} \mathfrak{g}_\alpha$, we note that $\dim_{\mathbb{F}} \mathfrak{g}_\alpha = 1$ for all $\alpha \in \Phi$. Thus $\dim_{\mathbb{F}} N = \sharp\Phi$ and

$$\dim_{\mathbb{F}} \mathfrak{g} = \dim_{\mathbb{F}} \mathfrak{h} + \sharp\Phi.$$

We can also use the fact that $\dim_{\mathbb{F}} \mathfrak{h} = \operatorname{rank} \Phi := \dim_{\mathbb{R}} \mathbb{R}\Phi$, the dimension of the Euclidean space spanned by Φ, and so we have a general formula

$$\dim_{\mathbb{F}} \mathfrak{g} = \operatorname{rank} \Phi + \sharp\Phi,$$

which we'll write as $d = r + f$.
We can observe that $f \geq 2r$ since if $\mathcal{B} := \{\alpha_1, \cdots, \alpha_r\}$ is a basis for Φ, no $-\alpha_i$ is in $\mathcal{B}$ but $\{\pm\alpha_1, \cdots, \pm\alpha_r\} \subseteq \Phi$ by the axiomatics of a root system. Thus

$$\dim_{\mathbb{F}} \mathfrak{g} = r + f \geq r + 2r = 3r.$$

We can now examine the cases for which $d = r + f = 4, 5, 7$:

- $r = 1$: as shown in class, there is a unique root system A_1 of rank 1 up to equivalence and satisfies $f = 2$ and thus $d = 3$, which is not a case we need to consider.
- $r = 2$: this yields $d \geq 3r = 6$, so this entirely rules out $d = 4, 5$ as possibilities for a semisimple Lie algebra. Using that every $\alpha \in \Phi$ is one of a pair $+\alpha, -\alpha \in \Phi$, we in fact have that f is always even – in other words, $\Phi = \Phi^+ \coprod \Phi^-$ with $\sharp\Phi^+ = \sharp\Phi^-$, so $f := \sharp\Phi = 2 \cdot \sharp\Phi^+$. Thus $d = r + f = 2 + f$ is even in this case, ruling out $d = 7$ when $r = 2$.
- $r \geq 3$: in this case we have $d \geq 3r = 9$, ruling out $d = 7$ once and for all.