HOMEWORK 2

MATH 8330 GIT

Problem 1. Consider the SL_2 action on $X = (\mathbb{P}^1)^n$ with a linearized invertible sheaf $L = \mathcal{O}_X(d_1, \dots, d_n)$, $d_i \in \mathbb{N}$. Define $w_i := \frac{2d_i}{\sum d_j}$, so that $\sum w_i = 2$. Prove that a point $(P_1, \dots, P_n) \in X^{ss}(L)$ (resp. $X^s(L)$) $\iff$ whenever some points P_i, $i \in I$, $I \subset \{1, \dots, n\}$, coincide, one has $\sum_{i \in I} w_i \le 1$ (resp. < 1).

Problem 2. Consider the SL_3 action on the set $X = \mathbb{P}^N$, $N = \binom{3+2}{2} - 1 = 9$, parameterizing cubic curves $C \subset \mathbb{P}^2$, with a linearized invertible sheaf $L = \mathcal{O}_X(1)$. Prove that C is semistable $\iff$ C has only ordinary double points.

Problem 3. Give an example showing that Hilbert-Mumford's criterion of (semi)stability for $G \curvearrowright X$ does *not* hold in general if X is not assumed to be projective. (In other words, produce a counterexample with a non-projective X.)

Problem 4. Provide a complete VGIT (variation of GIT) analysis for the quotients $(\mathbb{P}^1)^3 // \mathbb{G}_m$. The line bundle is $L = \mathcal{O}(1,1,1)$. The $\mathbb{G}_m$-action is defined as

$$t.(x_0 : x_1) = (x_0 : tx_1), \quad t.(y_0 : y_1) = (y_0 : ty_1), \quad t.(z_0 : z_1) = (z_0 : tz_1)$$

The linearization is a lift of this action to the action on the coordinates $w_{ijk} = x_i y_j z_k$ on $(\mathbb{P}^1)^3$ embedded into $\mathbb{P}^7$ with the 8 homogeneous coordinates w_{ijk}. The above equations give an action on the point $(w_{ijk}) \in \mathbb{P}^7$. The linearization is a lift of this action to the point $(w_{ijk}) \in \mathbb{A}^8$.

Determine the following:

1. The choices for $\mathbb{Q}$-linearizations of L (i.e. linearizations of some L^d, $d \in \mathbb{N}$).
2. Chamber decomposition.
3. For each chamber, the quotient.
4. For neighboring chambers, the induced morphisms between the quotients.
5. For each chamber, the sets of unstable and strictly semistable points.

Problem 5. Let $X \subset \mathbb{P}^N$ be a singular projective curve. Suppose that X has n irreducible components X_i and that $\deg \mathcal{O}_X(1)|_{X_i} = \lambda_i \in \mathbb{N}$. Let F be a coherent sheaf on X. Then on an open subset $U_i \subset X_i$ of each irreducible component it is a locally free sheaf of rank r_i.

The *Seshadri slope* of an invertible sheaf F is defined to be

$$\mu(F) = \frac{\chi(F)}{\sum \lambda_i r_i}, \quad \text{where } r_i = \mathrm{rk}\, F|_{U_i}.$$

By replacing $\mathcal{O}_X(1)$ by a rational multiple, one can assume that $\lambda_i > 0$, $\sum \lambda_i = 1$.

1. Let F be a pure-dimensional coherent sheaf on X. Prove that F is Hilbert-stable (resp. semistable) $\iff$ for any subsheaf $E \subset F$ one has $\mu(E) < \mu(F)$ (resp. $\le$). (Note in particular, that this definition depends on the polarization (λ_i), and there is a Variation of GIT here.)

Date: Nov 4; due Thu Nov 18.

(2) Prove, however, that if $\chi(F) = 0$ then the (semi)stability condition does *not* depend on a polarization (λ_i).

You can use the following simple observation. If $\pi : \widetilde{X} \to X$ is a normalization then $\widetilde{X}$ is a smooth curve, so Riemann-Roch is applicable:

$$\chi(E) = \deg(E) + \operatorname{rank}(E)(1 - g),$$

and the difference of Hilbert polynomials

$$\chi\big(X, F(m)\big) - \chi\big(\widetilde{X}, (\pi^* F)(m)\big)$$

is a constant.