Problem Sets: Moduli

Problem Set 1

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Problem 1.0.1 (1)

Denote by $\mu_n \leq \mathrm{SL}_n(\mathbb{C})$ the subgroup generated by $M := \begin{bmatrix} \varepsilon & 0 \\ 0 & \varepsilon^{-1} \end{bmatrix}$ for $\varepsilon^n = 1$ a primitive nth root of unity, and consider its action $\mu_n \curvearrowright \mathbb{C}[x, y]$ restricted from the standard action $\mathrm{SL}_2(\mathbb{C}) \curvearrowright \mathbb{C}[x, y]$. Explicitly, this can be written geometrically as

$$\mu_n \curvearrowright \mathbb{A}^2$$
$$M.(x, y) = (\varepsilon x, \varepsilon^{-1} y).$$

Write a general polynomial in $\mathbb{C}[x, y]$ as $f(x, y) = \sum_{i,j \geq 0} c_{ij} x^i y^j$, then under the action of μ_n we have

$$M.f(x, y) = \sum_{i,j \geq 0} c_{ij} (\varepsilon x)^i (\varepsilon^{-1} y)^j = \sum_{i,j \geq 0} c_{ij} \varepsilon^{i-j} x^i y^j.$$

The polynomial f will be in the invariant subring $\mathbb{C}[x,y]^{\mu_n}$ if and only if $M.f = f$, and equating coefficients in the above expression imposes the condition that for a fixed i, j,

- For $i-j=0$, so $i=j$, no extra condition is enforced. Such a middle coefficient occurs if and only if n is even.
- For $i \neq j$ with $4 \not| i-j$, since $\varepsilon^{i-j} \neq 1$ we must have $c_{ij}=0$.

Inspecting such polynomials, if n is even one can find

$$a(x,y) \coloneqq (xy)^{\frac{n}{2}}, \qquad b(x,y) = xy,$$

from which the relation $a^2 = b^n$ is readily seen to hold. If n is odd, no such invariants exist – this follows from writing

$$a(x,y) = a_{n,0}x^n + a_{0,n}y^n, \qquad b(x,y) = b_{2,0}x^2 + b_{1,1}xy + b_{0,2}y^2$$

and setting $a^2 - b^n = 0$, which yields

$$\begin{aligned} 0 &= 2\,a_{0n}a_{n0}x^n y^n + a_{n0}^2 x^{2\,n} + a_{0n}^2 y^{2\,n} - \left(b_{20}x^2 + b_{11}xy + b_{02}y^2\right)^n \\ &= 2\,a_{0n}a_{n0}x^n y^n + a_{n0}^2 x^{2\,n} + a_{0n}^2 y^{2\,n} - \sum_{i+j+k=n} b_{20}^i b_{11}^j b_{02}^k \; x^{2i}(xy)^j y^{2k} \\ &= 2\,a_{0n}a_{n0}x^n y^n + a_{n0}^2 x^{2\,n} + a_{0n}^2 y^{2\,n} - \sum_{i+j+k=n} \binom{n}{i,j,k} b_{20}^i b_{11}^j b_{02}^k \; x^{2i+j} y^{2k+j}, \end{aligned}$$

where we've taken a general trinomial expansion. Setting $(i,j,k) = (1,0,n-1)$ shows $b_{20}=0$, and similarly setting $(0,1,n-1)$ forces $b_{11}=0$ and $(n-1,0,1)$ forces $b_{02}=0$.

Problem 1.0.2 (2)
The isomorphism with D_{2n}: Let $BD_{4n} \coloneqq \langle R, S\rangle$ where

$$R \coloneqq \begin{bmatrix} \varepsilon & 0 \\ 0 & \varepsilon^{-1} \end{bmatrix}, \quad \varepsilon^{2n} = 1, \qquad S = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

To see that BD_{4n} has order exactly $4n$, we can start listing elements.

- The subset $\{R, R^2, R^3, \cdots, R^{2n-1}, R^{2n} = I\}$ contributes $2n$ distinct elements, and
- The subset $\{SR, SR^2, SR^3, \cdots, SR^{2n-1}, SR^{2n} = S\}$ contributes $2n$ more distinct elements. That these are distinct from each other and the previous set is clear from computing the products directly:

$$SR^k = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \varepsilon^k & 0 \\ 0 & \varepsilon^{-k} \end{bmatrix} = \begin{bmatrix} 0 & \varepsilon^{-k} \\ -\varepsilon^k & 0 \end{bmatrix}.$$

We can also note that $S^2 = -I = R^n$, so the sets $\left\{S^2R^k \;\middle|\; k \geq 0\right\}, \left\{S^3R^k \;\middle|\; k \geq 0\right\}$ are redundant and exhaust all possibilities for elements in this group, since S, R commute up to multiplication by -1 and $R^n = -R$ occurs in the first subset.

To see that the image of BD_{4n} in $\mathrm{SO}_3(\mathbb{R})$ is isomorphic to D_{2n}, note that the subgroup BD_{4n} already lies in SU_2, viewed as a subgroup of $\mathrm{SL}_2(\mathbb{C})$, and so we look for a map $\mathrm{SU}_2 \to \mathrm{SO}_3(\mathbb{R})$. For this, we can use the following isomorphism to the unit quaternions $Q^\times$:

$$F_1 : \mathrm{SU}_2 \to Q^\times$$
$$\begin{bmatrix} a+bi & -c+di \\ c-di & a-bi \end{bmatrix} \mapsto a\mathbf{1} + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}.$$

Unit quaternions can be mapped to rotation matrices using the following well-known formula:

$$F_2 : Q^\times \to \mathrm{SO}_3(\mathbb{R})$$
$$a\mathbf{1} + b\mathbf{i} + c\mathbf{j} + d\mathbf{k} \mapsto \begin{bmatrix} 1-2\left(c^2+d^2\right) & 2\left(bc-da\right) & 2\left(bd+ca\right) \\ 2\left(bc+da\right) & 1-2\left(b^2+d^2\right) & 2\left(cd-ba\right) \\ 2\left(bd-ca\right) & 2\left(cd+ba\right) & 1-2\left(b^2+c^2\right) \end{bmatrix}.$$

So we can use $\Phi := F_2 \circ F_1 : \mathrm{SU}_2 \to \mathrm{SO}_3(\mathbb{R})$ and investigate the image. A computation shows that

$$\Phi(S) = F_2(-1\mathbf{j}) = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \implies \Phi(S)^2 = I,$$

and

$$\Phi(R) = \Phi\left(\begin{bmatrix} a_n + ib_n & 0 \\ 0 & a_n - ib_n \end{bmatrix}\right), \qquad a_n = \cos(2\pi/n),\ b_n = \sin(2\pi/n)$$

$$= F_2(a_n\mathbf{1} + b_n\mathbf{i})$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1-2b_n^2 & -2a_nb_n \\ 0 & 2a_nb_n & 1-2b_n^2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\pi/n) & -\sin(\pi/n) \\ 0 & \sin(\pi/n) & \cos(\pi/n) \end{bmatrix}$$

$$= \left[\begin{array}{c|c} I & 0 \\ \hline 0 & R_{\pi/n} \end{array}\right],$$

where $R_\theta \in \mathrm{SO}_2(\mathbb{R})$ is the rotation by θ matrix and we have applied several double angle formulas. In this form, we can easily check

$$\Phi(R)^n = \left[\begin{array}{c|c} I^n & 0 \\ \hline 0 & R_{\pi/n}^n \end{array}\right] = I,$$

and so the image of $\Phi(R)$ is order n. Finally, we note the presentation

$$D_{2n} = \left\langle r, s \;\middle|\; r^n = s^2 = 1, sr = r^{-1}s \right\rangle,$$

and so in order to verify that the image is isomorphic to D_{2n}, it suffices to check that $r := \Phi(R)$ and $s := \Phi(S)$ satisfy the same relations, since (by the same argument as in $\mathrm{SL}_2(\mathbb{C})$) they already generate a finite subgroup of $\mathrm{SO}_3(\mathbb{R})$ of order $2n$. That this relation holds in the image follows from the fact that it holds for the original two matrices and group homomorphisms preserve relations:

$$R^{-1}S = \begin{bmatrix} \varepsilon^{-1} & 0 \\ 0 & \varepsilon \end{bmatrix} \cdot \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \varepsilon^{-1} \\ -\varepsilon & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} \varepsilon & 0 \\ 0 & \varepsilon^{-1} \end{bmatrix} = SR.$$

Finding invariant polynomials: We can first check which polynomials are invariant under the M-action:

$$M.f(x,y) = f(x,y) \implies \sum c_{ij}x^iy^j = \sum c_{ij}\varepsilon^{i-j}x^iy^j,$$

which implies that $c_{ij} = 0$ unless $i = j$ or $2n \mid i - j$. Thus the general polynomials of degrees $2n, 4$, and $2n+2$ respectively satisfying these conditions are of the form

$$a(x,y) = a_{2n,0}x^{2n} + a_{n,n}x^ny^n + a_{0,2n}y^{2n}$$

$$b(x,y) = \begin{cases} b_{4,0}x^4 + b_{2,2}x^2y^2 + b_{0,4}y^4, & n = 2 \\ b_{2,2}x^2y^2, & n > 2, \end{cases}$$

$$c(x,y) = c_{2n+1,1}x^{2n+1}y + c_{n+1,n+1}x^{n+1}y^{n+1} + c_{1,2n+1}xy^{2n+1}.$$

We can then further check which polynomials are invariant under the i-action:

$$i.f(x,y) = f(x,y) \implies \sum c_{ij}x^iy^j = \sum c_{ij}(-1)^jx^jy^i,$$

which implies that $c_{ij} = c_{ji}$ when j is even and $c_{ij} = -c_{ji}$ when j is odd. Incorporating these new restrictions, the general such invariant polynomials will be of the following forms:

$$a(x,y) = \alpha_0x^{2n} + \alpha_1x^ny^n + \alpha_0y^{2n}$$

$$b(x,y) = \begin{cases} \beta_0x^4 + \beta_1x^2y^2 + \beta_0y^4, & n = 2 \\ \beta_1x^2y^2, & n > 2, \end{cases}$$

$$c(x,y) = \gamma_0x^{2n+1}y + \gamma_1x^{n+1}y^{n+1} - \gamma_0xy^{2n+1}.$$

Since we have freedom to change coordinates, we can assume these polynomials are monic, potentially at the cost of getting a slightly different relation than $ba^2 = 4b^{n+1}$. Setting

$\alpha_0 = \beta_0 = \gamma_0 = 1$, we're left considering polynomials of the form

$$a(x,y) = x^{2n} + \alpha_1 x^n y^n + y^{2n}$$

$$b(x,y) = \begin{cases} x^4 + \beta_1 x^2 y^2 + y^4, & n = 2 \\ \beta_1 x^2 y^2, & n > 2, \end{cases}$$

$$c(x,y) = x^{2n+1} y + \gamma_1 x^{n+1} y^{n+1} - x y^{2n+1}.$$

Generalizing example 1.13 in Mukai suggests that invariants of the following forms may work, corresponding to setting $\alpha_1 = \gamma_1 = 0$ and $\beta_1 = 1$:

$$\begin{aligned} a(x,y) &\coloneqq x^{2n} + y^{2n} \\ b(x,y) &\coloneqq x^2 y^2 \\ c(x,y) &\coloneqq xy(x^{2n} - y^{2n}). \end{aligned}$$

One can then check directly that the desired relation holds:

$$\begin{aligned} b(x,y)a(x,y)^2 - 4b(x,y)^{n+1} &= (xy)^2(x^{4n} + y^{4n} + 2(xy)^{2n}) - 4(xy)^2(xy)^{2n} \\ &= (xy)^2(x^{4n} + y^{4n} - 2(xy)^{2n}) \\ &= c(x,y)^2. \end{aligned}$$

Problem 1.0.3 (3)
Let $\varepsilon^n = 1$ and $\varepsilon.(x,y) \coloneqq (\varepsilon x, \varepsilon y)$, and let $f(x,y) = \sum c_{ij} x^i y^j \in \mathbb{C}[x,y]$. Then f is invariant iff

$$\varepsilon.f(x,y) = f(x,y) \iff \sum c_{ij} x^i y^j = \sum c_{ij} \varepsilon^{i+j} x^i y^j \iff n \mid i+j,$$

and so the invariant ring is

$$\mathbb{C}[x,y]^{\mu_n} = \bigoplus_{k \geq 0} \mathbb{C}[x,y]_{kn},$$

the nth graded piece of $\mathbb{C}[x,y]$ along with the pieces corresponding to all higher multiples kn of n. This is generated as a graded ring by the degree n monomials $\langle x^n, x^{n-1}y, \cdots, xy^{n-1}, y^n \rangle$, so

$$\mathbb{C}[x,y]^{\mu_n} = \mathbb{C}[x^n, x^{n-1}y, \cdots, xy^{n-1}, y^n].$$

- For $n = 3$, this recovers $\mathbb{C}[x,y]^{\mu_3} = \mathbb{C}[x^3, x^2y, xy^2, y^3]$.
- For $n = 4$, it is $\mathbb{C}[x,y]^{\mu_4} = \mathbb{C}[x^4, x^3y, x^2y^2, xy^3, y^4]$.

Problem 1.0.4 (4)

Part 1: To fix notation, let $R = k[M]$ and $G = \operatorname{Spec} R$, and write the given maps as

$$\begin{aligned} m^* : R &\to R \otimes_k R \\ x^m &\mapsto x^m \otimes x^m \end{aligned}$$

$$\begin{aligned} i^* : R &\to R \\ x^m &\mapsto x^{-m} \end{aligned}$$

$$\begin{aligned} \varepsilon^* : R &\to k \\ x^m &\mapsto 1. \end{aligned}$$

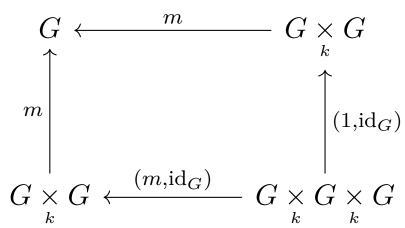
Equipping G with the structure of a group scheme requires producing the following maps:

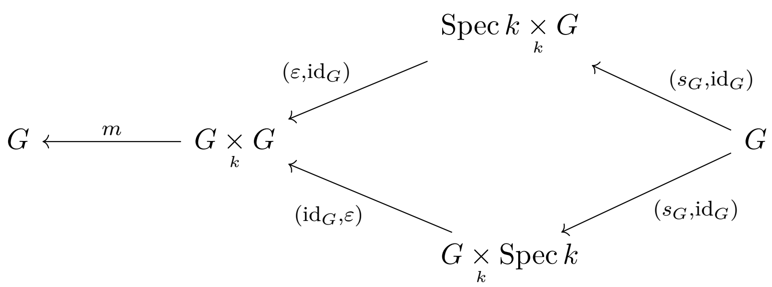
$$m : G \to G \underset{k}{\times} G$$

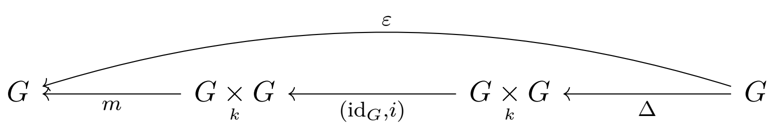
$$i : G \to G$$

$$\varepsilon : \operatorname{Spec} k \to G,$$

which are required to fit into commutative diagrams of k-schemes, where $s_G : G \to \operatorname{Spec} k$ is the structure morphisms of G and $\Delta : G \to G \underset{k}{\times} G$ is the diagonal morphism:

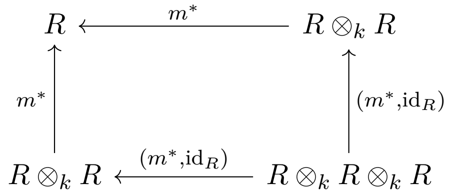


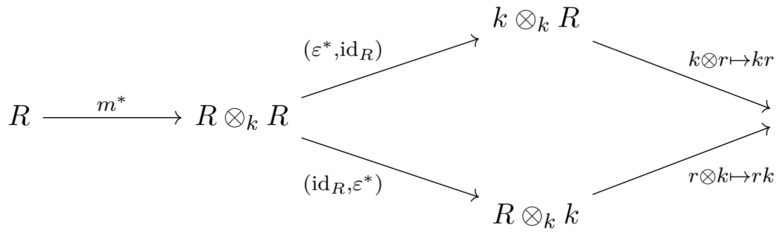


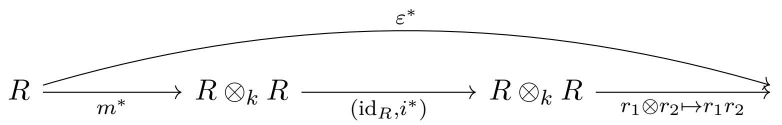


Link to Diagram

Since morphisms of affine schemes correspond bijectively to k-algebra morphisms between their global sections, if we set m, i, ε to be the morphisms corresponding to m^*, i^*, ε^* induced by the Spec functor, it suffices to show the following diagrams of k-algebras commute:







Link to Diagram

- The first diagram commutes:
  - The bottom path is $x^a \otimes x^b \otimes x^c \mapsto x^{a+b} \otimes x^c \mapsto x^{a+b+c}$,
  - The top path is $x^a \otimes x^b \otimes x^c \mapsto x^a \otimes x^{b+c} \mapsto x^{a+b+c}$.
- The second diagram commutes:
  - The bottom path is $x^m \mapsto x^m \otimes x^m \mapsto x^m \otimes 1 \mapsto x^m \cdot 1 = x^m$,
  - The top path is $x^m \mapsto x^m \otimes x^m \mapsto 1 \otimes x^m \mapsto 1 \cdot x^m = x^m$.
- The third diagram commutes:
  - The bottom path is $x^m \mapsto x^m \otimes x^m \mapsto x^m \otimes x^{-m} \mapsto x^{m+(-m)} = x^0 = 1$,
  - The top path is $x^m \mapsto 1$.

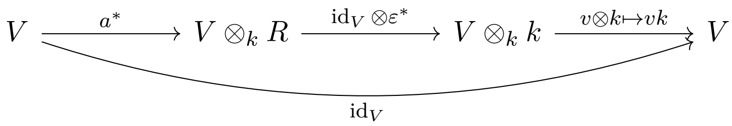
Part 2: Write $M \cong \mathbb{Z}^r \oplus \bigoplus_{i=1}^{\ell} \mathbb{Z}/n_i\mathbb{Z}$, then

$$\begin{aligned}\operatorname{Spec} k[M] &\cong \operatorname{Spec} k\left[\mathbb{Z}^r \oplus \bigoplus_{i=0}^{\ell} \mathbb{Z}/n_i\mathbb{Z}\right] \\ &\cong \operatorname{Spec}\left(k[\mathbb{Z}^r] \otimes_k k[\mathbb{Z}/n_1\mathbb{Z}] \otimes_k \cdots \otimes_k k[\mathbb{Z}/n_\ell\mathbb{Z}]\right) \\ &\cong \operatorname{Spec} k[\mathbb{Z}^r] \underset{k}{\times} \operatorname{Spec} k[\mathbb{Z}/n_0\mathbb{Z}] \underset{k}{\times} \cdots \underset{k}{\times} \operatorname{Spec} k[\mathbb{Z}/n_\ell\mathbb{Z}] \\ &\cong \mathbb{G}_m \underset{k}{\times} \mu_{n_0} \underset{k}{\times} \cdots \underset{k}{\times} \mu_{n_\ell},\end{aligned}$$

where we've used that $k[A \times B] = k[A] \otimes_k k[B]$ and $\operatorname{Spec}(R \otimes_k S) = \operatorname{Spec}(R) \underset{k}{\times} \operatorname{Spec}(S)$.

Part 3: $\Longrightarrow$: suppose one is given such a linear coaction, we will show that it induces a direct sum decomposition of vector spaces.

Definition 3.54 in Mukai describes a coaction of R on V as a morphism $a^* : V \to V \otimes_k R$ such that the following diagrams commute:



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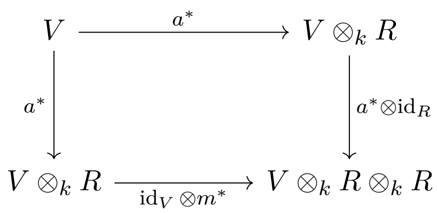

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As in class, we can note that for any $v \in V$, we have $a^*(v) = \sum_{m \in M} v_m \otimes x^m$ for some components v_m, and by the commutativity of the above diagram, the composition

$$v \mapsto \sum_{m \in M} v_m \otimes x^m \mapsto \sum_{m \in M} v_m \otimes 1 \mapsto \sum_{m \in M} v_m$$

is equal to the identity and so $v = \sum_{m \in M} v_m$. This yields $V = \sum_{m \in M} V_m$ for some subsets V_m, which can be defined as all of those $w \in V$ such that the term $v_m \otimes x^n$ occurs in the expansion of the image $a^*(w) = \sum_{m \in M} v_m \otimes x^m$. These are linear subspaces, because for example if $m_1, m_2 \in V_m$, then

$$a^*(v_{m_1} + v_{m_2}) = a^*(v_{m_1}) + a^*(v_{m_2}) = (v_{m_1} \otimes x^m) + (v_{m_2} \otimes x^m) = (v_{m_1} + v_{m_2}) \otimes x^m,$$

and so setting $w := v_{m_1} + v_{m_2}$ shows that their sum is again in V_m. It remains to show that this sum of subspaces is direct.

It suffices to show that if any $v_m \in V_m$ can be expressed as $v_m = \sum_{n\neq m} v_n$ with $v_n \in V_n$ then $v_m = 0$. This shows that $V_m \cap V_n = 0$ for all m and n, making the sum direct. To this end, note that $a^*(v_m) = v_m \otimes x^m$ is an elementary tensor. If $v_m = \sum_{n\neq m} v_n$, then $a^*(v) = \sum_{n\neq m} v_n \otimes x^n$. Since a^* is a well-defined map, it must be the case that

$$v_m \otimes x^m = \sum_{n\neq m} v_n \otimes x^n.$$

Equating components of these tensors forces $v_n = 0$ for all $n \neq m$, so $v_m = 0$.
$\Longleftarrow$: suppose now that one has a decomposition $V = \bigoplus_{m\in M} V_m$; then the naturally associated map $v \mapsto \sum m \in M v_m \otimes x^m$ yields the desired coaction.
Part 4: This follows from the same proof as in part 3 – the only new aspect is that the coaction map $a^* : A \to A \otimes k[M]$ is now a map of k-algebras which preserves the grading on A. If $a_i \in A_i$ and $a_j \in A_j$ with $A = \bigoplus_{m\in M} A_m$, then $a_i a_j \in A_{i+j}$, and

$$a^*(a_i a_j) = a^*(a_i)a^*(a_j) = (a_i \otimes x^i)(a_j \otimes x^j) = (a_i a_j) \otimes x^{i+j}.$$

Problem 1.0.5 (5)
Throughout this problem, we work over a fixed field k write $\mathbb{A}^2 := \operatorname{Spec} k[x, y]$. All tensor products are implicitly over k.

1. First noting that we can write $\mathbb{G}_a = \operatorname{Spec} k[\xi]$ for an indeterminate a, we can use the isomorphism $R \otimes V := k[x, y] \otimes k[\xi] \cong k[\xi][x, y]$ to regard elements in polynomials in the variables x, y with coefficients in $k[\xi]$. The coaction

$$\mathbb{G}_a \curvearrowright \mathbb{A}^2$$
$$\xi.(x, y) := (x, \xi x + y)$$

can then be written as

$$a^* : k[x, y] \to k[\xi] \otimes k[x, y] \cong k[\xi][x, y]$$
$$x \mapsto x$$
$$y \mapsto \xi x + y.$$

2. Write $\mathbb{G}_m = \operatorname{Spec} k[\lambda, \lambda^{-1}]$ and use the isomorphism $k[\lambda, \lambda^{-1}] \cong k[z, w]/(zw-1)$ to write

$$R \otimes V = k[x, y] \otimes k[\lambda, \lambda^{-1}] \cong k[x, y] \otimes \frac{k[z, w]}{(zw-1)} \cong \frac{k[z, w]}{(zw-1)}[x, y],$$

so z corresponds to λ and w to λ^{-1}. Then the action

$$\mathbb{G}_m \curvearrowright \mathbb{A}^2$$
$$\lambda.(x, y) := (\lambda x, \lambda^{-1} y)$$

has the following corresponding coaction:

$$a^* : k[x, y] \mapsto \frac{k[z, w]}{(zw-1)}[x, y]$$
$$x \mapsto zx$$
$$y \mapsto wy.$$

3. Write $\mu_n = \operatorname{Spec} k[\xi]/(\xi^n - 1)$, then

$$R \otimes V = k[x,y] \otimes \frac{k[\xi]}{(\xi^n - 1)} \cong \frac{k[\xi]}{(\xi^n - 1)}[x,y]$$

and the coaction is

$$\begin{aligned} a^* : k[x,y] &\to \frac{k[\xi]}{(\xi^n - 1)}[x,y] \\ x &\mapsto \xi x \\ y &\mapsto \xi^{-1} y = \xi^{n-1} y. \end{aligned}$$

4. We first write the geometric action as

$$\begin{aligned} S_3 \curvearrowright \mathbb{A}^3 &= \operatorname{Spec} k[x_1, x_2, x_3] \\ \sigma.[x_1, x_2, x_3] &\coloneqq \left[x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}\right]. \end{aligned}$$

We can then write

$$R \otimes_k V = \left(\bigoplus_{\sigma \in S_3} k \right) \otimes_k k[x_1, x_2, x_3] \cong \bigoplus_{\sigma \in S_3} k[x_1, x_2, x_3].$$

Thus the coaction is

$$\begin{aligned} k[x_1, x_2, x_3] &\to \bigoplus_{\sigma \in S_3} k[x_1, x_2, x_3] \\ x_i &\mapsto \bigoplus_{\sigma \in S_3} x_{\sigma(i)}. \end{aligned}$$

For example, writing $S_3 = \{(), (12), (23), (13), (123), (132)\}$, the map on the first coordinate is the following:

$$x_1 \mapsto [x_1, x_2, x_1, x_3, x_2, x_3].$$