Problem Sets: Moduli

Problem Set 2

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1.1 1

Problem 1.1.1 (1)
Consider the SL_2 action on $X = (\mathbb{P}^1)^n$ with a linearized invertible sheaf $L = \mathcal{O}_X(d_1, \ldots, d_n), d_i \in \mathbb{N}$. Define $w_i := \frac{2d_i}{\sum d_j}$, so that $\sum w_i = 2$. Prove that a point

$(P_1, \dots, P_n) \in X^{ss}(L)$ (resp. $X^s(L)$) $\iff$ whenever some points P_i, $i \in I, I \subset \{1, \dots, n\}$, coincide, one has $\sum_{i\in I} w_i \leq 1$ (resp. < 1).

Solution:
Write points in this product as

$$X \coloneqq (\mathbf{P}^1)^n = \left\{ \mathbf{p} \coloneqq \begin{bmatrix} x_0 & \cdots & x_n \\ y_0 & \cdots & y_n \end{bmatrix} \right\},$$

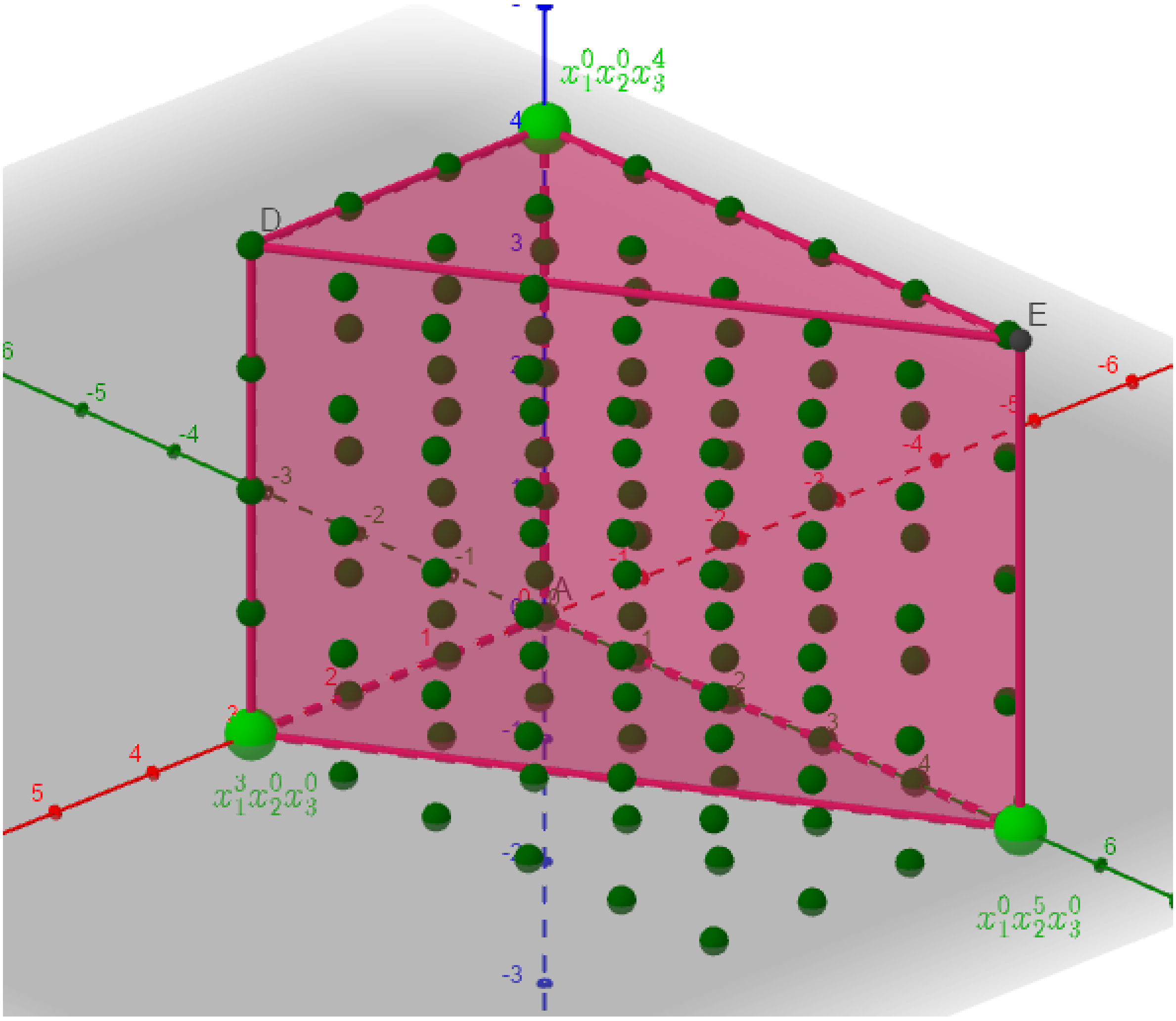
corresponding to the n-tuple $([x_0 : y_0], \cdots, [x_n : y_n])$, with SL_2 action given by

$$\begin{gathered} \mathrm{SL}_2 \curvearrowright X \\ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \mathbf{p} \coloneqq \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_0 & \cdots & x_n \\ y_0 & \cdots & y_n \end{bmatrix} = \begin{bmatrix} ax_0 + by_0 & \cdots & ax_n + by_n \\ cx_0 + dy_0 & \cdots & cx_n + dy_n \end{bmatrix}. \end{gathered}$$

We note that the maximal torus acts as

$$\begin{gathered} T_{\mathrm{SL}_2} \curvearrowright X \\ \begin{bmatrix} t & \cdot \\ \cdot & t^{-1} \end{bmatrix} \cdot \mathbf{p} \coloneqq \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_0 & \cdots & x_n \\ y_0 & \cdots & y_n \end{bmatrix} = \begin{bmatrix} tx_0 & \cdots & tx_n \\ t^{-1}y_0 & \cdots & t^{-1}y_n \end{bmatrix}. \end{gathered}$$

We identify X with its image (which we'll also denote X) under the Veronese embedding $X \to \mathbf{P}^N$ associated to the ample line bundle $\mathcal{L} \coloneqq \mathcal{O}(\mathbf{d})$ where $\mathbf{d} \coloneqq [d_1, \cdots, d_n] \subseteq \mathbf{Z}^n$ viewed as an integer vector. Writing D for the convex hull of the d_i in $\mathbf{Z}^n$, note that every lattice point in $\mathbf{Z}^n \cap D$ defines a monomial, and every point $\mathbf{p} \in X$ corresponds to a a collection of lattice points $P_{\mathbf{p}} = \{\mathbf{k} = [k_1, \cdots, k_n]\} \subseteq D \cap \mathbf{Z}^n$ along with a choice of coefficient $\alpha_{\mathbf{k}}$ for each $\mathbf{k} \in P_{\mathbf{p}}$.
The following is an example D and $P_{\mathbf{p}}$ when $n = 3$ and $\mathbf{d} = [3, 5, 4]$:



The three highlighted lattice points are $\mathbf{k}_1 = [3,0,0], \mathbf{k}_2 = [0,5,0], \mathbf{k}_3 = [0,0,4]$, $P_{\mathbf{p}} \coloneqq \{\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3\}$ corresponds to a polynomial

$$F(x_1, x_2, x_3) = \alpha_1 x_1^3 x_2^0 x_3^0 + \alpha_2 x_1^0 x_2^5 x_3^0 + \alpha_3 x_1^0 x_2^0 x_3^4.$$

In our situation, lattice points will correspond to monomials

$$\mathbf{k}_{IJ} = x^I y^J \coloneqq x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n} \cdot y_1^{j_1} y_2^{j_2} \cdots y_n^{j_n},$$

and so each point in X will correspond to a polynomial

$$F(x_1, \cdots, x_n, y_1, \cdots, y_n) = \sum_{(I,J) \subseteq D} \alpha_{IJ} x^I y^J.$$

where $\sum_{i \in I} i + \sum_{j \in J} j = d_i$.

Todo: this is not quite right. If α_j is associated to the embedding along the d_j direction, then the monomial degrees should just sum up to d_j.

Indexing these monomials systematically, we can write

$$F(x_1, \cdots, y_n) = \sum \alpha_j \prod_{i=1}^{n} x_i^{d_i - k_j} y_i^{k_j}.$$

When points collide, without loss of generality (using the transitive SL_2-action) we can assume that the collision point in $\mathbf{P}^1$ is $[0:1]$, so $p \in X$ is of the form

$$p = \begin{bmatrix} 0 & \cdots & 0 & p_{m+1} & \cdots & p_n \\ 1 & \cdots & 1 & q_{m+1} & \cdots & q_n \end{bmatrix},$$

where we've written m for the number of colliding points. We can now compute the weights of the torus action over such colliding points

$$\begin{aligned} \lambda(t).F(x_1, \cdots, y_n) &= \sum \alpha_j \prod t^{d_i - 2k_j} x_i^{d_i - k_j} y^{k_j} \\ &= \sum t^{w_{ij}} \alpha_j x_i^{d_i - k_j} y^{k_j}, \qquad w_{ij} := \sum_i d_i - 2k_j. \end{aligned}$$

We now need $\mu(x, \lambda) \geq 0$ for semistability, i.e. $\min(w_{ij}) \geq 0$, so $\min(\sum d_i - 2k_j) \geq 0$. We can maximally destabilize such a quantity by taking $k_j = d_i$ for each i, j, and so if the collision set is $\mathcal{S}$, we require

$$\sum_{i=1}^n d_i - \sum_{i \in \mathcal{S}} 2d_i \geq 0 \iff \sum_{i=1}^n d_i \geq \sum_{i \in \mathcal{S}} d_i \iff \frac{\sum_{i \in \mathcal{S}} 2d_i}{\sum_{i=1}^n d_i} \leq 1 \iff \sum_{i \in \mathcal{S}} w_i \leq 1.$$

1.2 2

Problem 1.2.1 (2)

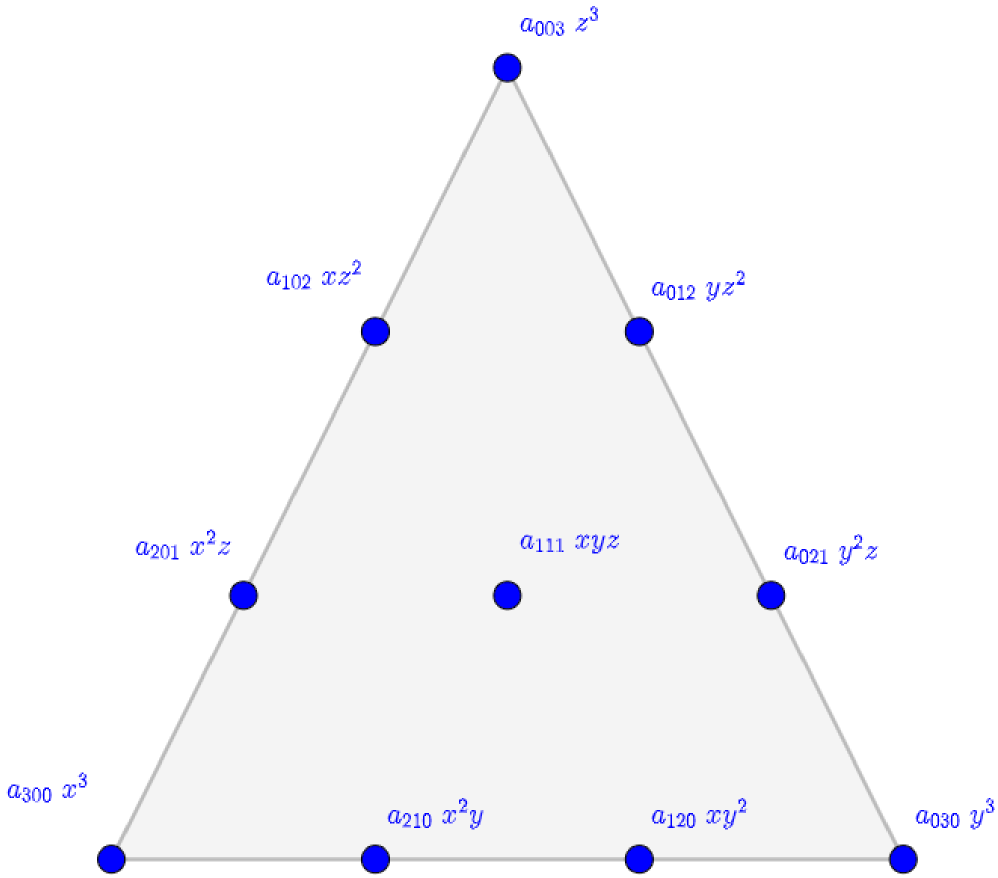
Consider the SL_3 action on the set $X = \mathbb{P}^N, N = \binom{3+2}{2} - 1 = 9$, parameterizing cubic curves $C \subset \mathbb{P}^2$, with a linearized invertible sheaf $L = \mathcal{O}_X(1)$. Prove that C is semistable $\iff C$ has only ordinary double points.

Solution:

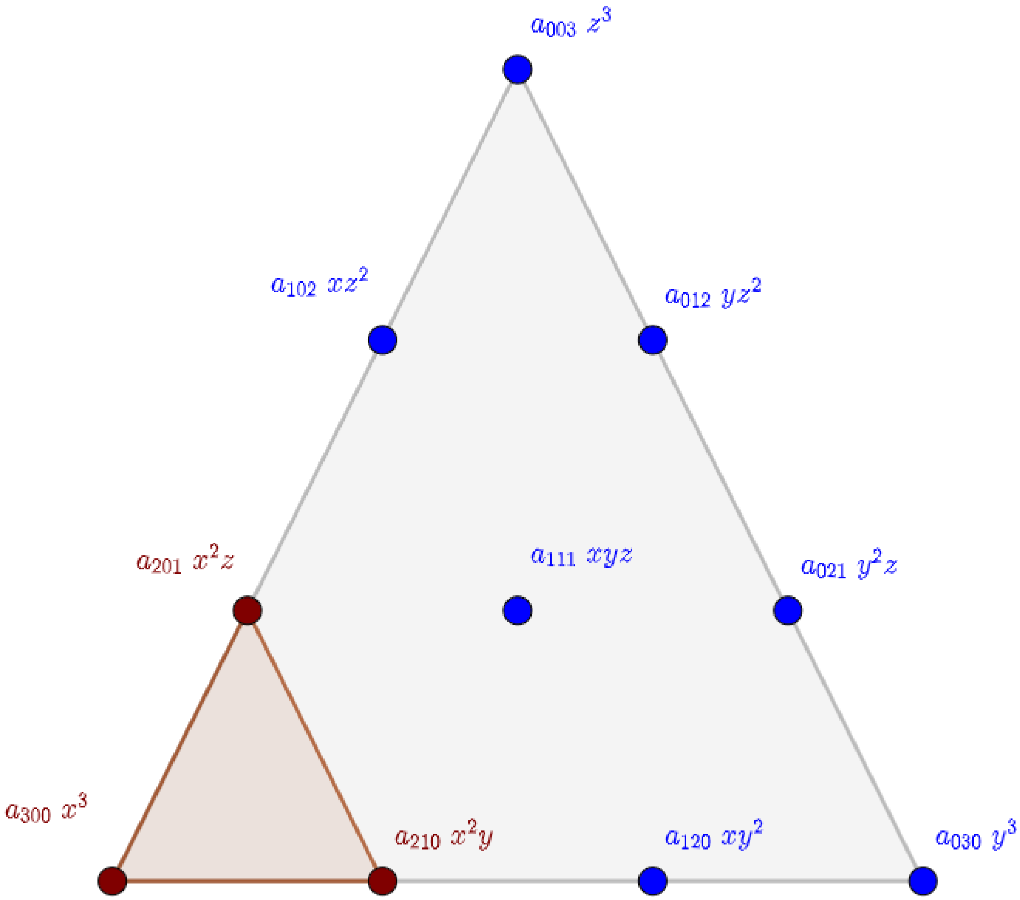
We first note that every choice of cubic curve $C \in Y_{3,2}$ can be represented (after choosing coordinates) by a polynomial

$$F(x, y, z) = \sum_{i+j+k=3} a_{ijk} \, x^i y^j z^k = \sum_{i+j+k=3} a_{\mathbf{i}} x^i y^j z^k \qquad \mathbf{i} := [i, j, k]$$

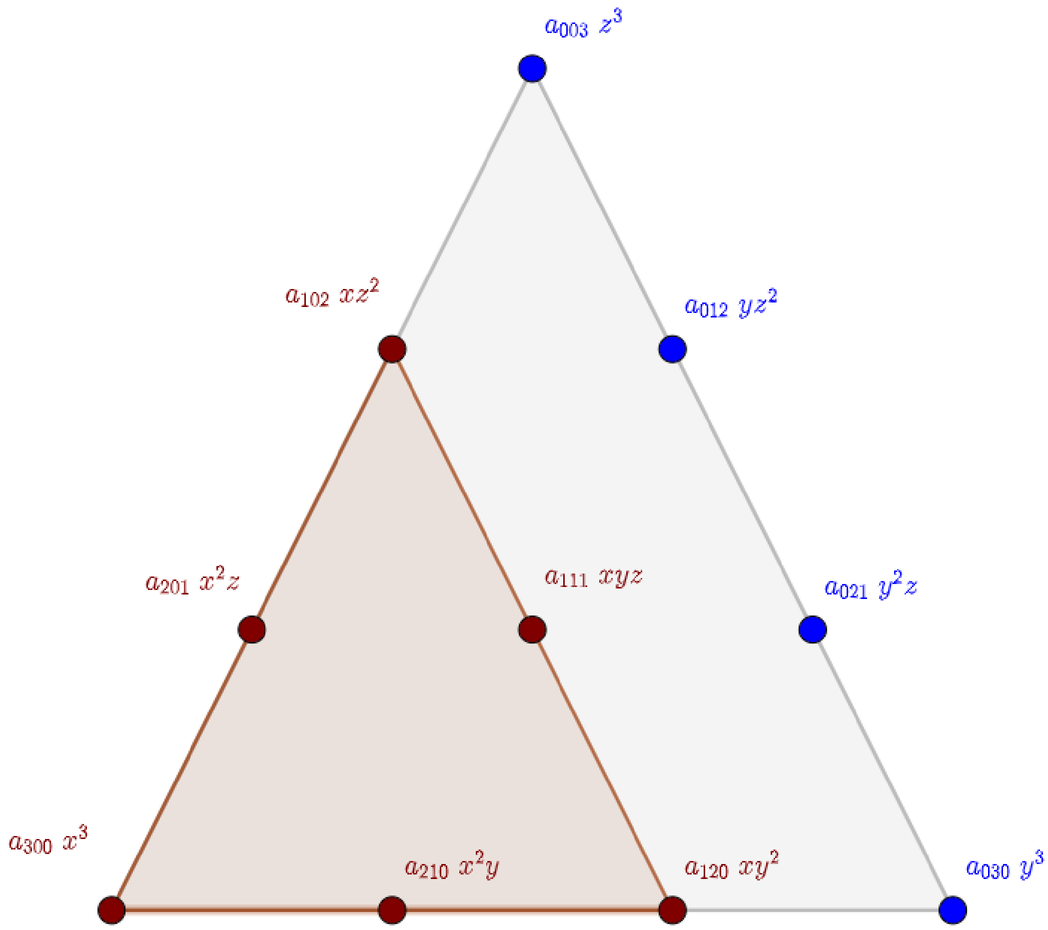
and thus a choice of lattice points C_P in the corresponding weight polytope where each point is labeled with the corresponding coefficient of F:



We record the fact that the point $p := [1:0:0]$ is singular iff $a_{300} = a_{201} = a_{210} = 0$:



Moreover, p is a triple point iff additionally $a_{102} = a_{111} = a_{120} = 0$:



Moreover, all of above holds except a_{102} (the coefficient of xz^2) is nonzero, then p is a double point with only a single tangent, and thus not an ordinary double point. These facts follow from computing the gradients and Hessians which characterize these types of singularities. We also note that if $\lambda : \mathbf{G}_m \to \mathrm{SL}_3$ is a 1-parameter subgroup, then $\lambda(t)$ is conjugate to

$$\tilde{\lambda}(t) = \begin{bmatrix} t^{r_1} & \cdot & \cdot \\ \cdot & t^{r_2} & \cdot \\ \cdot & \cdot & t^{r_3} \end{bmatrix}, \qquad \sum_{i=1}^{3} r_i = 0,$$

and thus determines a vector $\mathbf{r} := [r_1, r_2, r_3] \in \mathbf{Z}^3$. The action can then be written

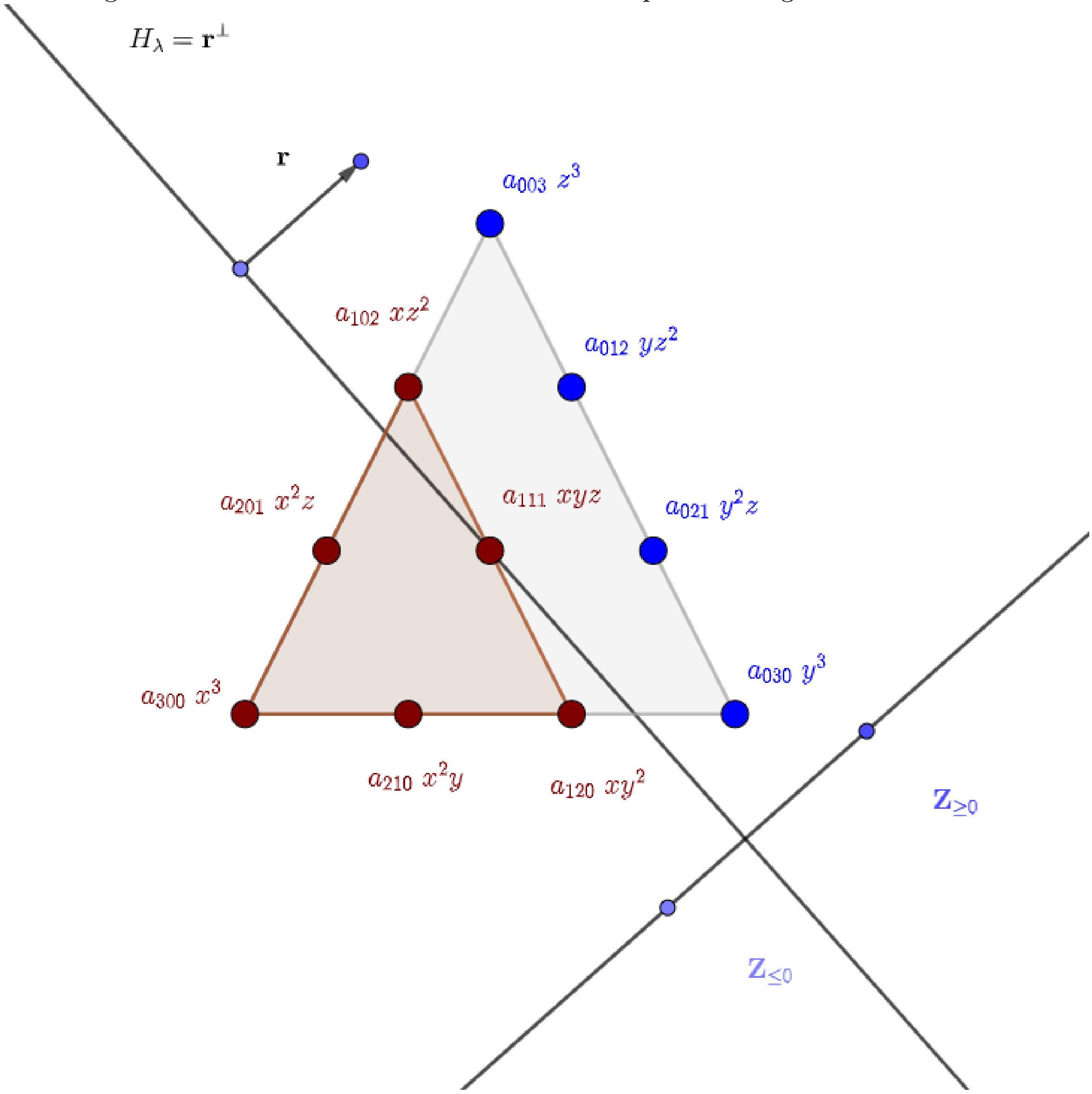
$$\lambda(t) \cdot F(x,y,z) = \sum_{i+j+k=3} a_{\mathbf{i}}\, t^{\langle \mathbf{r},\ \mathbf{i} \rangle} x^i y^j z^k,$$

and so all weights are of the form $w_{\mathbf{i}} = \langle \mathbf{r},\ \mathbf{i} \rangle \in \mathbf{Z}$. We note that $C \in Y_{3,2}$ is unstable iff for every λ, every weight is negative or every weight is positive, so $w_{\mathbf{i}} < 0$ or $w_{\mathbf{i}} > 0$ for all $\mathbf{i} \in C_P$. We'll focus on the strictly positive case, since the positive case follows similarly.

$\implies$: Suppose C is unstable, we will show that p is either a non-ordinary double point, a triple point, or worse. Pick λ and its corresponding $\mathbf{r}$ such that all weights $w_{\mathbf{i}}$ are positive. Then in particular

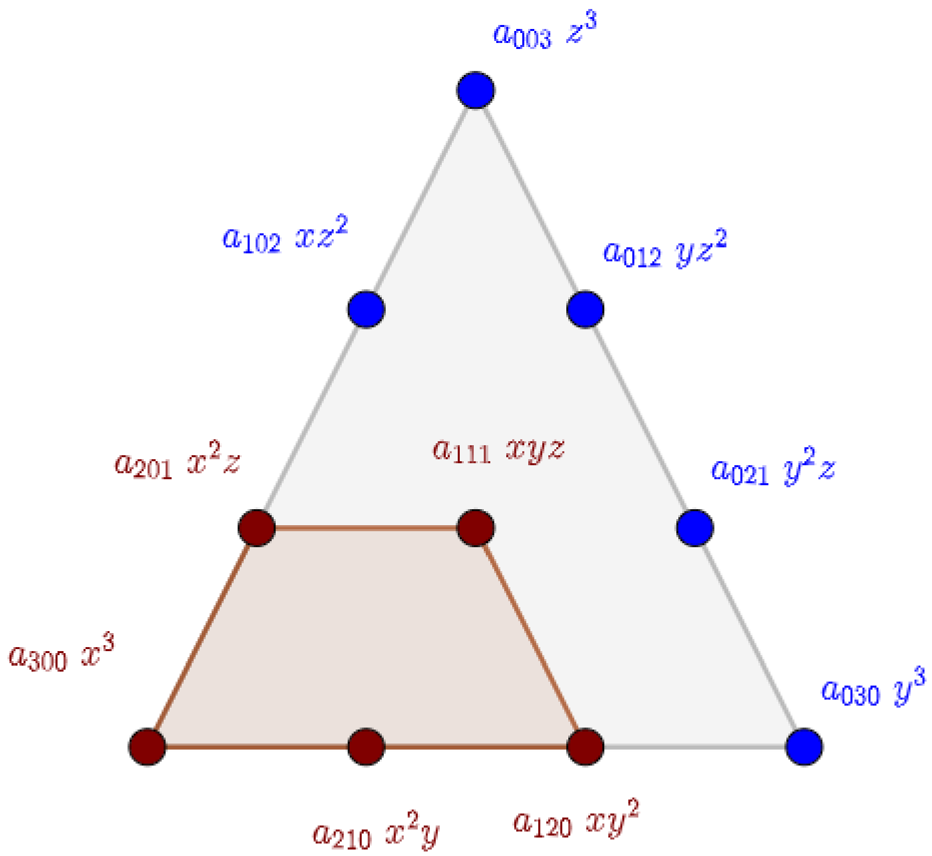
$$\min \left\{ w_{\mathbf{i}} := \langle \mathbf{r}, \ \mathbf{i} \rangle \;\middle|\; \mathbf{i} \in C_P \right\} > 0.$$

Having strictly positive weights can be phrased geometrically as $\left\{ \mathbf{i} \;\middle|\; \mathbf{i} \in C_P, \, a_{\mathbf{i}} \neq 0 \right\}$ being contained in the positive half-space corresponding to the hyperplane $H_C := \mathbf{r}^{\perp}$. Picking a maximally destabilizing λ, without loss of generality (changing coordinates if necessary) we can arrange for the lower-left 5 monomials receive non-positive weights:

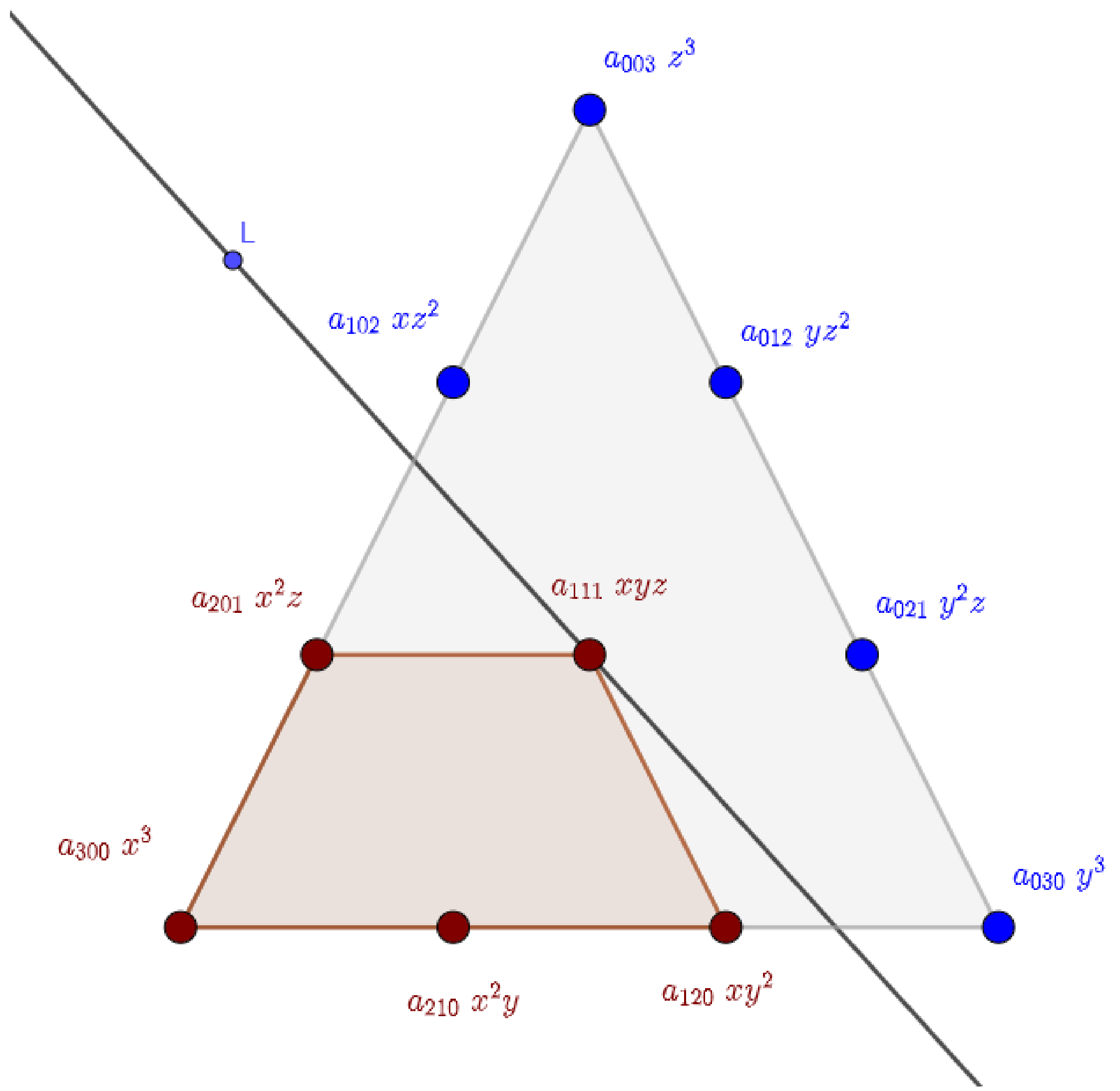


This forces all of the shaded coefficients except for potentially a_{102} to be zero. By the earlier remarks, this forces $p = [1 : 0 : 0]$ to be singular, and if $a_{102} = 0$ this is a triple point. Otherwise, if $a_{102} \neq 0$, this yields a double point which only has a single tangent, and is thus not ordinary. So if C is *not* an unstable curve (i.e. it is semistable), it must have an ordinary

double point at worst.
$\Longleftarrow$: Suppose conversely that C has a triple point or a non-ordinary double point q. Using the transitivity of the SL_3 action, we can move q to $p = [1 : 0 : 0]$ and conclude using the singularity criterion above that the following coefficients vanish:



We can now make a specific choice of λ that yields the following H_λ and gives the remaining coefficients strictly positive weights, allowing us to conclude that C is unstable:



1.3 3

Problem 1.3.1 (3)
Give an example showing that Hilbert-Mumford's criterion of (semi)stability for $G \curvearrowright X$ does not hold in general if X is not assumed to be projective. (In other words, produce a counterexample with a non-projective X.)

Solution:

Consider the following action:

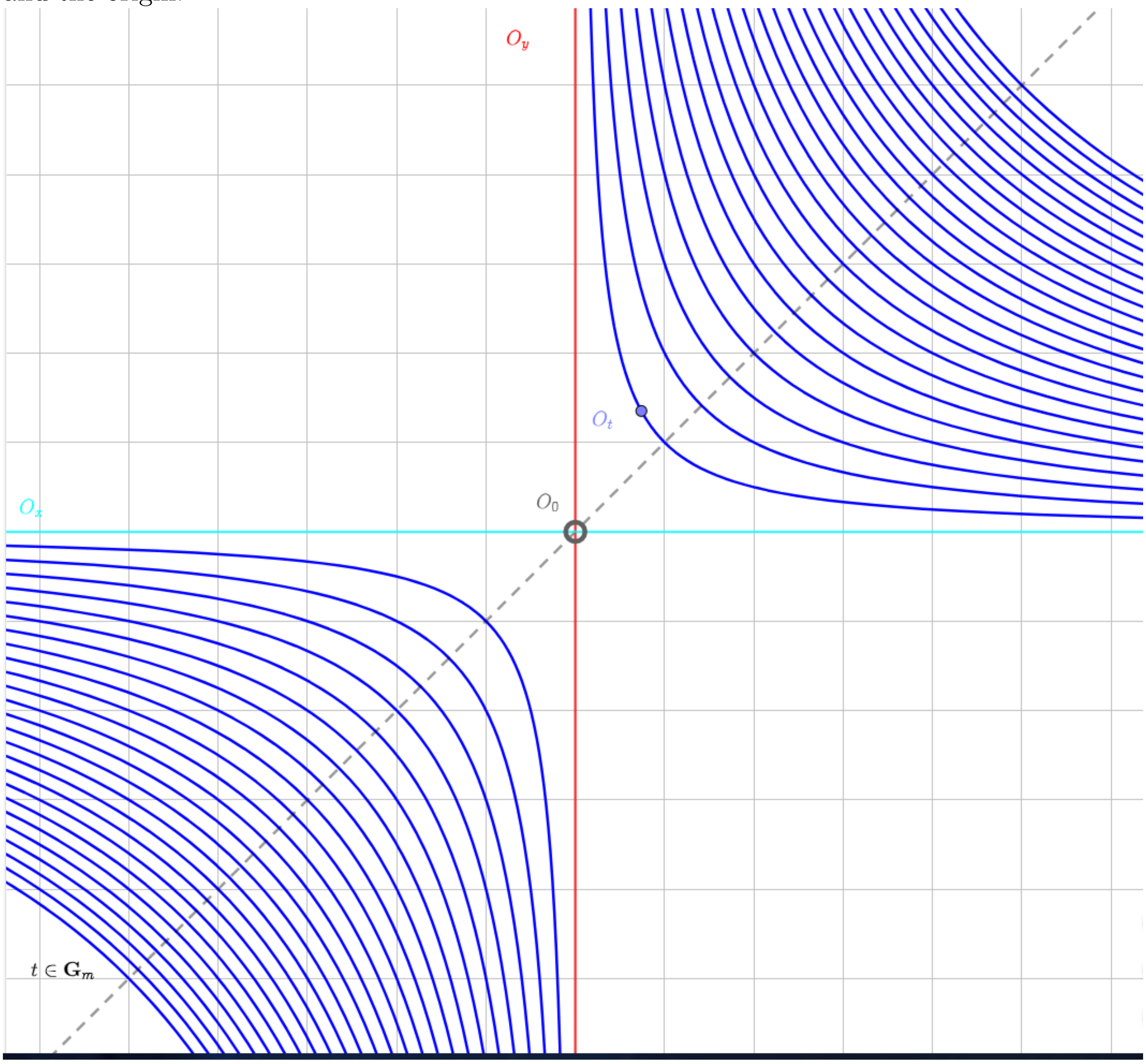
$$\mathbf{G}_m \curvearrowright X := \mathbf{A}^2$$
$$t.[x, y] := [tx, ty].$$

Thus yields a set theoretic orbit space

$$\mathbf{A}^2/\mathbf{G}_m = \left\{ O_t \,\middle|\, t \in \mathbf{G}_m \right\} \cup \{O_x, O_y, O_0\}$$

$$\begin{aligned} O_t &:= \left\{ xy = t \,\middle|\, t \in \mathbf{G}_m \right\} \\ O_x &:= \left\{ [t, 0] \,\middle|\, t \in \mathbf{G}_m \right\} = \mathbf{G}_m.[1, 0] \\ O_y &:= \left\{ [0, t] \,\middle|\, t \in \mathbf{G}_m \right\} = \mathbf{G}_m.[0, 1] \\ O_0 &:= \{0\}, \end{aligned}$$

i.e. there is an orbit for each hyperbola $xy = t$, the punctured x-axis, the punctured y-axis, and the origin:



We record that the following facts:

- The orbits O_t are all closed with 0-dimensional stabilizers,
- The orbits O_x, O_y are not closed but still have 0-dimensional stabilizers, and
- The orbit O_0 is closed but has a 1-dimensional stabilizer $\mathbf{G}_m$.

Thus $X^s = \mathbf{A}^2 \setminus V(xy)$ is the plane with the axes deleted, and for example $0 \in X \setminus X^s$ is an unstable point and $[1,0],[0,1] \in X \setminus X^s$ are not stable points (and may thus either be unstable or semistable).
Noting that $O_x \sim O_y \sim O_0$ are all orbit-closure equivalent since 0 is in the closure of O_x and O_y, we can separate these orbits by redefining our total space to be $X \coloneqq \mathbf{A}^2 \setminus \{0\}$; then O_x, O_y are closed in X' and have 0-dimensional stabilizer and thus points in those orbits become stable for the restricted action $\mathbf{G}_m \curvearrowright X'$.
For example, pick $p \coloneqq [1,0] \in O_x \subseteq X'$, then p is stable by construction. However, we can now check the Hilbert-Mumford numerical criterion and note that every 1-parameter subgroup λ acting with weights r_1, r_2 satisfies

$$\lambda(t).p = [t^{r_1}1, t^{r_2}0] = [t^{r_1}1, 0],$$

and in particular always has strictly positive or strictly negative weights, which would otherwise characterize p as an *unstable* point, yielding the desired counterexample.

1.4 4

Problem 1.4.1 (4)
Provide a complete VGIT (variation of GIT) analysis for the quotients $(\mathbb{P}^1)^3 // \mathbb{G}_m$. The line bundle is $L = \mathcal{O}(1,1,1)$. The $\mathbb{G}_m$-action is defined as

$$t.(x_0 : x_1) = (x_0 : tx_1), \quad t.(y_0 : y_1) = (y_0 : ty_1), \quad t.(z_0 : z_1) = (z_0 : tz_1)$$

The linearization is a lift of this action to the action on the coordinates $w_{ijk} = x_i y_j z_k$ on $(\mathbb{P}^1)^3$ embedded into $\mathbb{P}^7$ with the 8 homogeneous coordinates w_{ijk}. The above equations give an action on the point $(w_{ijk}) \in \mathbb{P}^7$. The linearization is a lift of this action to the point $(w_{ijk}) \in \mathbb{A}^8$.
Determine the following:

(1) The choices for $\mathbb{Q}$-linearizations of L (i.e. linearizations of some $L^d, d \in \mathbb{N}$).

(2) Chamber decomposition.

(3) For each chamber, the quotient.

(4) For neighboring chambers, the induced morphisms between the quotients.

(5) For each chamber, the sets of unstable and strictly semistable points.

Solution:
Todo.

1.5 5

Problem 1.5.1 (5)
Let $X \subset \mathbb{P}^N$ be a singular projective curve. Suppose that X has n irreducible components X_i and that $\deg \mathcal{O}_X(1)|_{X_i} = \lambda_i \in \mathbb{N}$. Let F be a coherent sheaf on X. Then on an open subset $U_i \subset X_i$ of each irreducible component it is a locally free sheaf of rank r_i.
The Seshadri slope of an invertible sheaf F is defined to be

$$\mu(F) = \frac{\chi(F)}{\sum \lambda_i r_i}, \qquad \text{where } r_i = \operatorname{rk} F|_{U_i}.$$

By replacing $\mathcal{O}_X(1)$ by a rational multiple, one can assume that $\lambda_i > 0, \sum \lambda_i = 1$.

1. Let F be a pure-dimensional coherent sheaf on X. Prove that F is Hilbertstable (resp. semistable) $\iff$ for any subsheaf $E \subset F$ one has $\mu(E) < \mu(F)$ (resp. $\leq$). (Note in particular, that this definition depends on the polarization (λ_i), and there is a Variation of GIT here.)

2. Prove, however, that if $\chi(F) = 0$ then the (semi)stability condition does not depend on a polarization (λ_i).

Remark 1.5.1: You can use the following simple observation. If $\pi : \widetilde{X} \to X$ is a normalization then $\tilde{X}$ is a smooth curve, so Riemann-Roch is applicable:

$$\chi(E) = \deg(E) + \operatorname{rank}(E)(1-g),$$

and the difference of Hilbert polynomials

$$\chi(X, F(m)) - \chi\left(\tilde{X}, (\pi^* F)(m)\right)$$

is a constant.

Solution:
We first recall that a sheaf $\mathcal{F} \in \mathsf{Coh}(X)$ is Hilbert stable if for every subsheaf $E \leq F$, we have an inequality of reduced Hilbert polynomials $\tilde{p}_E(n) < \tilde{p}_F(n)$, and semistability is characterized

by replacing $<$ with $\leq$. Noting that

$$p_F(n) \coloneqq \chi(X; F(n)) = c_0 n^{\dim X} = c_0 n + c_1$$

since X is a curve and consequently $\dim X = 1$. We have $\tilde{p}_F(n) = n + \frac{c_0}{c_1}$ and thus $\tilde{p}_E(n) = n + \frac{d_0}{d_1}$ for some constants c_i depending on F and d_i depending on E, and so

$$\tilde{p}_E(n) < \tilde{p}_F(n) \iff \frac{d_0}{d_1} < \frac{c_0}{c_1}.$$

Thus it suffices to show that $\frac{d_0}{d_1} = \mu(E)$ and $\frac{c_0}{c_1} = \mu(F)$. We'll proceed by computing $p_F(n)$ in order to identify what c_0, c_1 are in general.
Noting that X may be singular and thus Riemann-Roch won't apply directly, take the normalization $\pi : \tilde{X} \to X$. Let $X = \cup_i X_i$ be the decomposition of X into irreducible components and let $\tilde{X}_i$ be their lifts in the normalization, which are all curves with some genera g_i. We now have

$$\begin{aligned}
p_F(n) &\coloneqq \chi(X; F(n)) \\
&= \chi(\tilde{X}, (\pi^* F)(n)) + c \qquad \text{for some constant } c \\
&= \sum_{1 \leq i \leq n} \chi(\tilde{X}_i, (\pi^* F)(n)|_{\tilde{X}_i}) + c \\
&= \sum_{1 \leq i \leq n} \Big(\deg\, (\pi^* F)(n)|_{\tilde{X}_i} + (1 - g_i) \Big) + c.
\end{aligned}$$

As an aside, we can compute the degrees inside of the sum as follows:

$$\begin{aligned}
\deg\, (\pi^* F)(n)|_{X_i} &= \deg\, F(n)|_{X_i} \\
&= \deg\, F|_{X_i} \otimes \bigoplus_{1 \leq j \leq r_i} \mathcal{O}_{X_i}(n) \\
&= \deg\, F|_{X_i} + n r_i \lambda_i.
\end{aligned}$$

Continuing the above calculation, we have

$$\begin{aligned}
p_F(n) &= \sum_{1 \leq i \leq n} \Big(\deg\, F|_{X_i} + n r_i \lambda_i + (1 - g_i) \Big) + c \\
&= n \left(\sum_{1 \leq i \leq n} r_i \lambda_i \right) + \left(\sum_{1 \leq i \leq n} \deg\, F|_{X_i} + (1 - g_i) + c \right) \\
&= n \left(\sum_{1 \leq i \leq n} r_i \lambda_i \right) + \left(\sum_{1 \leq i \leq n} \chi(X_i; F|_{X_i}) + c \right) \\
&= n \left(\sum_{1 \leq i \leq n} r_i \lambda_i \right) + \chi(X; F).
\end{aligned}$$

Thus $c_0 = \sum r_i \lambda_i, c_1 = \chi(F)$, and $\frac{c_1}{c_0} = \frac{\chi(F)}{\sum r_i \lambda_i} = \mu(F)$.