Algebraic Geometry Problems

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Contents

Source: Section 1 of Gathmann

1 Problem Set 1

Exercise 1.0.1 ***(Gathmann 1.19):*** Prove that every affine variety $X \subset \mathbb{A}^n/k$ consisting of only finitely many points can be written as the zero locus of n polynomials.

Hint: Use interpolation. It is useful to assume at first that all points in X have different x_1-coordinates.

Solution:
Let $X = \{\mathbf{p}_1, \cdots, \mathbf{p}_d\} = \{\mathbf{p}_j\}_{j=1}^d$, where each $\mathbf{p}_j \in \mathbb{A}^n$ can be written in coordinates

$$\mathbf{p}_j \coloneqq \left[p_j^1, p_j^2, \cdots, p_j^n\right].$$

Remark 1.0.2: Proof idea: for some fixed k with $2 \leq k \leq n$, consider the pairs $(p_j^1, p_j^k) \in \mathbb{A}^2$. Letting j range over $1 \leq j \leq d$ yields d points of the form $(x, y) \in \mathbb{A}^2$, so construct an interpolating polynomial such that $f(x) = y$ for each tuple. Then $f(x) - y$ vanishes at every such tuple.

Doing this for each k (keeping the first coordinate always of the form p_j^1 and letting the second coordinate vary) yields $n-1$ polynomials in $k[x_1, x_k] \subseteq k[x_1, \cdots, x_n]$, then adding in the polynomial $p(x) = \prod_j (x - p_j^1)$ yields a system the vanishes precisely on $\{\mathbf{p}_j\}$.

Claim: Without loss of generality, we can assume all of the first components $\left\{p_j^1\right\}_{j=1}^d$ are distinct.

Todo: follows from "rotation of axes"?

We will use the following fact:

Theorem 1.0.3 ***(Lagrange).***
Given a set of d points $\{(x_i, y_i)\}_{i=1}^d$ with all x_i distinct, there exists a unique polynomial of degree d in $f \in k[x]$ such that $\tilde{f}(x_i) = y_i$ for every i.
This can be explicitly given by

$$\tilde{f}(x) = \sum_{i=1}^d y_i \left(\prod_{\substack{0 \leq m \leq d \\ m \neq i}} \left(\frac{x - x_m}{x_i - x_m} \right) \right).$$

Equivalently, there is a polynomial f defined by $f(x_i) = \tilde{f}(x_i) - y_i$ of degree d whose roots are precisely the x_i.

Using this theorem, we define a system of n polynomials in the following way:

- Define $f_1 \in k[x_1] \subseteq k[x_1, \cdots, x_n]$ by

$$f_1(x) = \prod_{i=1}^{d} \left(x - p_i^1\right).$$

Then the roots of f_1 are precisely the first components of the points p.

- Define $f_2 \in k[x_1, x_2] \subseteq k[x_1, \cdots, x_n]$ by considering the ordered pairs

$$\left\{(x_1, x_2) = (p_j^1, p_j^2)\right\},$$

then taking the unique Lagrange interpolating polynomial $\tilde{f}_2$ satisfying $\tilde{f}_2(p_j^1) = p_j^2$ for all $1 \leq j \leq d$. Then set $f_2 \coloneqq \tilde{f}_2(x_1) - x_2 \in k[x_1, x_2]$.

- Define $f_3 \in k[x_1, x_3] \subseteq k[x_1, \cdots, x_n]$ by considering the ordered pairs

$$\left\{(x_1, x_3) = (p_j^1, p_j^3)\right\},$$

then taking the unique Lagrange interpolating polynomial $\tilde{f}_3$ satisfying $\tilde{f}_2(p_j^1) = p_j^3$ for all $1 \leq j \leq d$. Then set $f_3 \coloneqq \tilde{f}_3(x_1) - x_3 \in k[x_1, x_3]$.

- $\cdots$

Continuing in this way up to $f_n \in k[x_1, x_n]$ yields a system of n polynomials.

Proposition 1.0.4.
$V(f_1, \cdots, f_n) = X$.

Proof .

Claim: $X \subseteq V(f_i)$:
This is essentially by construction. Letting $p_j \in X$ be arbitrary, we find that

$$f_1(p_j) = \prod_{i=1}^{d} \left(p_j^1 - p_i^1\right) = (p_j^1 - p_j^1) \prod_{\substack{i \leq d \\ i \neq j}} \left(p_j^1 - p_i^1\right) = 0.$$

Similarly, for $2 \leq k \leq n$,

$$f_k(p_j) = \tilde{f}_k(p_j^1) - p_j^k = 0,$$

which follows from the fact that $\tilde{f}_k(p_j^1) = p_j^k$ for every k and every j by the construction of $\tilde{f}_k$.

Claim: $X^c \subseteq V(f_i)^c$:
This follows from the fact the polynomials f given by Lagrange interpolation are unique, and thus the roots of $\tilde{f}$ are unique. But if some other point was in $V(f_i)$, then one of its coordinates would be another root of some $\tilde{f}$.

■

Exercise 1.0.5 ***(Gathmann 1.21):*** Determine $\sqrt{I}$ for

$$I := \left\langle x_1^3 - x_2^6,\, x_1 x_2 - x_2^3 \right\rangle \trianglelefteq \mathbb{C}[x_1, x_2].$$

Solution:
For notational purposes, let $\mathcal{I}, \mathcal{V}$ denote the maps in Hilbert's Nullstellensatz, we then have

$$(\mathcal{I} \circ \mathcal{V})(I) = \sqrt{I}.$$

So we consider $\mathcal{V}(I) \subseteq \mathbb{A}^2/\mathbb{C}$, the vanishing locus of these two polynomials, which yields the system

$$\begin{cases} x^3 - y^6 & = 0 \\ xy - y^3 & = 0. \end{cases}$$

In the second equation, we have $(x - y^2)y = 0$, and since $\mathbb{C}[x, y]$ is an integral domain, one term must be zero.

1. If $y = 0$, then $x^3 = 0 \implies x = 0$, and thus $(0, 0) \in \mathcal{V}(I)$, i.e. the origin is contained in this vanishing locus.
2. Otherwise, if $x - y^2 = 0$, then $x = y^2$, with no further conditions coming from the first equation.

Combining these conditions,

$$P := \left\{ (t^2, t) \;\middle|\; t \in \mathbb{C} \right\} \subset \mathcal{V}(I).$$

where $I = \left\langle x^3 - y^6, xy - y^3 \right\rangle$.
We have $P = \mathcal{V}(I)$, and so taking the ideal generated by P yields

$$(\mathcal{I} \circ \mathcal{V})(I) = \mathcal{I}(P) = \left\langle y - x^2 \right\rangle \in \mathbb{C}[x, y]$$

and thus $\sqrt{I} = \left\langle y - x^2 \right\rangle$.

Exercise 1.0.6 ***(Gathmann 1.22):*** Let $X \subset \mathbb{A}^3/k$ be the union of the three coordinate axes. Compute generators for the ideal $I(X)$ and show that it can not be generated by fewer than 3 elements.

Solution:
Claim:

$$I(X) = \langle x_2 x_3,\ x_1 x_3,\ x_1 x_2 \rangle .$$

We can write $X = X_1 \cup X_2 \cup X_3$, where

- The x_1-axis is given by $X_1 := V(x_2 x_3) \implies I(X_1) = \langle x_2 x_3 \rangle$,
- The x_2-axis is given by $X_2 := V(x_1 x_3) \implies I(X_2) = \langle x_1 x_3 \rangle$,
- The x_3-axis is given by $X_3 := V(x_1 x_2) \implies I(X_3) = \langle x_1 x_2 \rangle$.

Here we've used, for example, that

$$I(V(x_2 x_3)) = \sqrt{\langle x_2 x_3 \rangle} = \langle x_2 x_3 \rangle$$

by applying the Nullstellensatz and noting that $\langle x_2 x_3 \rangle$ is radical since it is generated by a squarefree monomial.
We then have

$$\begin{aligned}
I(X) &= I(X_1 \cup X_2 \cup X_3) \\
&= I(X_1) \cap I(X_2) \cap I(X_3) \\
&= \sqrt{I(X_1) + I(X_2) + I(X_3)} \\
&= \sqrt{\langle x_2, x_3 \rangle + \langle x_1 x_3 \rangle + \langle x_1 x_2 \rangle} \\
&= \sqrt{\langle x_2 x_3,\ x_1 x_3,\ x_1 x_2 \rangle} && \text{since } \langle a \rangle + \langle b \rangle = \langle a, b \rangle \\
&= \langle x_2 x_3,\ x_1 x_3,\ x_1 x_2 \rangle,
\end{aligned}$$

where in the last equality we've again used the fact that an ideal generated by squarefree monomials is radical.

Claim: $I(X)$ can not be generated by 2 or fewer elements.
Let $J := I(X)$ and $R := k[x_1, x_2, x_3]$, and toward a contradiction, suppose $J = \langle r, s \rangle$. Define $\mathfrak{m} := \langle x, y, z \rangle$ and a quotient map

$$\pi : J \to J/\mathfrak{m}J$$

and consider the images $\pi(r), \pi(s)$.

Note that $J/\mathfrak{m}J$ is an $R/\mathfrak{m}$-module, and since $R/\mathfrak{m} \cong k$, $J/\mathfrak{m}J$ is in fact a k-vector space. Since $\pi(r), \pi(s)$ generate $J/\mathfrak{m}J$ as a k-module,

$$\dim_k J/\mathfrak{m}J \leq 2.$$

But this is a contradiction, since we can produce 3 k-linearly independent elements in $J/\mathfrak{m}J$: namely $\pi(x_1x_2), \pi(x_1x_3), \pi(x_2x_3)$. Suppose there exist α_i such that

$$\alpha_1\pi(x_1x_2) + \alpha_2\pi(x_1x_3) + \alpha_3\pi(x_2x_3) = 0 \in J/\mathfrak{m}J \iff \alpha_1x_1x_2 + \alpha_2x_1x_3 + \alpha_3x_2x_3 \in \mathfrak{m}J,$$

But we can then note that

$$\mathfrak{m}J = \langle x_1, x_2.x_3\rangle \langle x_1x_2, x_1x_3, x_2x_3\rangle = \left\langle x_1^2x_2,\ x_1^2x_3,\ x_1x_2x_3, \cdots \right\rangle.$$

can't contain any nonzero elements of degree $d < 3$, so no such α_i can exist and these elements are k-linearly independent.

Exercise 1.0.7 ***(Gathmann 1.23: Relative Nullstellensatz):*** Let $Y \subset \mathbb{A}^n/k$ be an affine variety and define $A(Y)$ by the quotient

$$\pi : k[x_1, \cdots, x_n] \to A(Y) := k[x_1, \cdots, x_n]/I(Y).$$

a. Show that $V_Y(J) = V(\pi^{-1}(J))$ for every $J \trianglelefteq A(Y)$.

b. Show that $\pi^{-1}(I_Y(X)) = I(X)$ for every affine subvariety $X \subseteq Y$.

c. Using the fact that $I(V(J)) \subset \sqrt{J}$ for every $J \trianglelefteq k[x_1, \cdots, x_n]$, deduce that $I_Y(V_Y(J)) \subset \sqrt{J}$ for every $J \trianglelefteq A(Y)$.

Conclude that there is an inclusion-reversing bijection

$$\left\{\substack{\text{Affine subvarieties}\\ \text{of } Y}\right\} \iff \left\{\substack{\text{Radical ideals}\\ \text{in } A(Y)}\right\}.$$

Exercise 1.0.8 ***(Extra):*** Let $J \trianglelefteq k[x_1, \cdots, x_n]$ be an ideal, and find a counterexample to $I(V(J)) = \sqrt{J}$ when k is not algebraically closed.

Solution:
Take $J = \left\langle x^2+1 \right\rangle \trianglelefteq \mathbb{R}[x]$, noting that J is nontrivial and proper but $\mathbb{R}$ is not algebraically closed. Then $V(J) \subseteq \mathbb{R}$ is empty, and thus $I(V(J)) = I(\emptyset)$.

Claim: $I(V(J)) = \mathbb{R}[x]$.
Checking definitions, for any set $X \subset \mathbb{A}^n/k$ we have

$$I(X) = \left\{ f \in \mathbb{R}[x] \;\middle|\; \forall x \in X,\ f(x) = 0 \right\}$$

and so we vacuously have

$$I(\emptyset) = \left\{ f \in \mathbb{R}[x] \;\middle|\; \forall x \in \emptyset,\ f(x) = 0 \right\} = \{ f \in \mathbb{R}[x] \} = \mathbb{R}[x].$$

Claim: $\sqrt{J} \neq \mathbb{R}[x]$.
This follows from the fact that maximal ideals are radical, and $\mathbb{R}[x]/J \cong \mathbb{C}$ being a field implies that J is maximal. In this case $\sqrt{J} = J \neq \mathbb{R}[x]$.

That maximal ideals are radical follows from the fact that if $J \trianglelefteq R$ is maximal, we have $J \subset \sqrt{J} \subset R$ which forces $\sqrt{J} = J$ or $\sqrt{J} = R$.

But if $\sqrt{J} = R$, then

$$1 \in \sqrt{J} \implies 1^n \in J \text{ for some } n \implies 1 \in J \implies J = R,$$

contradicting the assumption that J is maximal and thus proper by definition.

2 | Problem Set 2

Exercise 2.0.1 ***(Gathmann 2.17):*** Find the irreducible components of

$$X = V(x - yz, xz - y^2) \subset \mathbb{A}^3/\mathbb{C}.$$

Solution:
Since $x = yz$ for all points in X, we have

$$\begin{aligned} X &= V(x - yz, yz^2 - y^2) \\ &= V\left(x - yz, y(z^2 - y)\right) \\ &= V(x - yz, y) \cup V(x - yz, z^2 - y) \\ &:= X_1 \cup X_2. \end{aligned}$$

Claim: These two subvarieties are irreducible.
It suffices to show that the $A(X_i)$ are integral domains. We have

$$A(X_1) \coloneqq \mathbb{C}[x,y,z]/\langle x-yz, y\rangle \cong \mathbb{C}[y,z]/\langle y\rangle \cong \mathbb{C}[z],$$

which is an integral domain since $\mathbb{C}$ is a field and thus an integral domain, and

$$A(X_2) \coloneqq \mathbb{C}[x,y,z]/\left\langle x-yz, z^2-y\right\rangle \cong \mathbb{C}[y,z]/\left\langle z^2-y\right\rangle \cong \mathbb{C}[y],$$

which is an integral domain for the same reason.

Exercise 2.0.2 ***(Gathmann 2.18):*** Let $X \subset \mathbb{A}^n$ be an arbitrary subset and show that

$$V(I(X)) = \overline{X}.$$

Solution:

$\overline{X} \subseteq V(I(X))$:
We have $X \subseteq V(I(X))$ and since $V(J)$ is closed in the Zariski topology for any ideal $J \trianglelefteq k[x_1, \cdots, x_n]$ by definition, $V(I(X))$ is closed. Thus

$$X \subseteq V(I(X)) \text{ and } V(I(X)) \text{ closed} \implies \overline{X} \subseteq V(I(X)),$$

since $\overline{X}$ is the intersection of all closed sets containing X.

$V(I(X)) \subseteq \overline{X}$:
Noting that $V(\,\cdot\,), I(\,\cdot\,)$ are individually order-reversing, we find that $V(I(\,\cdot\,))$ is order-*preserving* and thus

$$X \subseteq \overline{X} \implies V(I(X)) \subseteq V(I(\overline{X})) = \overline{X},$$

where in the last equality we've used part (i) of the Nullstellensatz: if X is an affine variety, then $V(I(X)) = X$. This applies here because $\overline{X}$ is always closed, and the closed sets in the Zariski topology are precisely the affine varieties.

Exercise 2.0.3 ***(Gathmann 2.21):*** Let $\{U_i\}_{i\in I} \rightrightarrows X$ be an open cover of a topological space with $U_i \cap U_j \neq \emptyset$ for every i, j.

a. Show that if U_i is connected for every i then X is connected.

b. Show that if U_i is irreducible for every i then X is irreducible.

Solution ***(a):***
Suppose toward a contradiction that $X = X_1 \coprod X_2$ with X_i proper, disjoint, and open. Since $\{U_i\} \rightrightarrows X$, for each $j \in I$ this would force one of $U_j \subseteq X_1$ or $U_j \subseteq X_2$, since otherwise $U_j \cap X_1 \cap X_2$ would be nonempty.

So without loss of generality (relabeling if necessary), assume $U_j \in X_1$ for some fixed j. But then for every $i \neq j$, we have $U_i \cap U_j$ nonempty by assumption, and so in fact $U_i \subseteq X_1$ for every $i \in I$. But then $\cup_{i \in I} U_i \subseteq X_1$, and since $\{U_i\}$ was a cover, this forces $X \subseteq X_1$ and thus $X_2 = \emptyset$.

Solution ***(b):***

Claim: X is irreducible $\iff$ any two open subsets intersect.
This follows because otherwise, if $U, V \subset X$ are open and disjoint then $X \setminus U$, $X \setminus V$ are proper and closed. But then we can write $X = (X \setminus U) \coprod (X \setminus V)$ as a union of proper closed subsets, forcing X to not be irreducible.

So it suffices to show that if $U, V \subset X$ then $U \cap V$ is nonempty. Since $\{U_i\} \rightrightarrows X$, we can find a pair i, j such that there is at least one point in $U \cap U_i$ and one point in $V \cap U_j$.

But by assumption $U_i \cap U_j$ is nonempty, so both $U \cap U_i$ and $U_j \cap U_i$ are open nonempty subsets of U_i. Since U_i was assumed irreducible, they must intersect, so there exists a point

$$x_0 \in (U \cap U_i) \cap (U_j \cap U_i) = U \cap (U_i \cap U_j) := \tilde{U}.$$

We can now similarly note that $\tilde{U} \cap V$ and $U_j \cap V$ are nonempty open subsets of V, and thus intersect. So there is a point

$$\tilde{x}_0 \in \left(\tilde{U} \cap V\right) \cap (U_j \cap V) = \tilde{U} \cap V = U \cap V \cap (U_i \cap U_j),$$

and in particular $\tilde{x}_0 \in U \cap V$ as desired.

Exercise 2.0.4 ***(Gathmann 2.22):*** Let $f : X \to Y$ be a continuous map of topological spaces.

a. Show that if X is connected then $f(X)$ is connected.

b. Show that if X is irreducible then $f(X)$ is irreducible.

Solution ***(a):***
Toward a contradiction, if $f(X) = Y_1 \coprod Y_2$ with Y_1, Y_2 nonempty and open in Y, then

$$f^{-1}(f(X)) \subseteq X$$

on one hand, and

$$f^{-1}(f(X)) = f^{-1}(Y_1) \coprod f^{-1}(Y_2)$$

on the other. If f is continuous, the preimages $f^{-1}(Y_i)$ are open (and nonempty), so X contains a disconnected subset. However, every subset of a connected set must be connected, so this contradicts the connectedness of X.

Solution ***(b):***
Suppose $f(X) = Y_1 \cup Y_2$ with Y_i proper closed subsets of Y. Then $f^{-1}(Y_1) \cup f^{-1}(Y^2) = (f^{-1} \circ f)(X) \subseteq X$ are closed in X, since f is continuous. Since X is irreducible, without loss of generality (by relabeling), this forces $X_1 = \emptyset$. But then $f(X_1) = \emptyset$, forcing $f(X) = Y_2$.

Definition 2.0.5 (Ideal Quotient)
For two ideals $J_1, J_2 \trianglelefteq R$, the *ideal quotient* is defined by

$$J_1 : J_2 := \left\{ f \in R \;\middle|\; fJ_2 \subset J_1 \right\}.$$

Exercise 2.0.6 ***(Gathmann 2.23):*** Let X be an affine variety.

a. Show that if $Y_1, Y_2 \subset X$ are subvarieties then

$$I(\overline{Y_1 \setminus Y_2}) = I(Y_1) : I(Y_2).$$

b. If $J_1, J_2 \trianglelefteq A(X)$ are radical, then

$$\overline{V(J_1) \setminus V(J_2)} = V(J_1 : J_2).$$

Solution:
?

Exercise 2.0.7 ***(Gathmann 2.24):*** Let $X \subset \mathbb{A}^n$, $Y \subset \mathbb{A}^m$ be irreducible affine varieties, and show that $X \times Y \subset \mathbb{A}^{n+m}$ is irreducible.

Solution:
That $X \times Y$ is again an affine variety follows from writing $X = V(I), Y = V(J)$, then $X \times Y = V(I + J)$ where $I + J \trianglelefteq k[x_1, \cdots, x_n, y_1, \cdots, y_m]$. So let

$$X \times Y = U \cup V$$

with U, V proper and closed, and let π_X, π_Y be the projections onto the factors.

Claim: For each $x \in X$, $\pi^{-1}(x) \cong Y$ is contained in only one of U or V.
Note that if this is true, we can write $X = G_U \cup G_V$ where

$$G_U := \left\{ x \in X \;\middle|\; \pi_X^{-1}(x) \subseteq U \right\}$$

are the points for which the entire fiber lies in U, and similarly G_V are those for which the fiber lies in V. If we can then show that G_U, G_V are closed, by irreducibility of X this will force (wlog) $G_V = \emptyset$ and $X = G_U$. But then

$$\pi_X^{-1}(X) = X \times Y \text{ and } \pi_X^{-1}(G_U) = U \implies X \times Y = U.$$

which shows that $X \times Y$ is irreducible.

Proof (Every fiber is contained in one irreducible component).
For any fixed x, we can write

$$\pi_X^{-1}(x) = \left(\pi_X^{-1}(x) \cap U\right) \cup \left(\pi_X^{-1}(x) \cap V\right).$$

Since points are closed in the Zariski topology and π_X is continuous, each $\pi_X^{-1}(x)$ is closed. and thus $\pi_X^{-1}(x) \cap U$ is closed (and similarly for V). Noting that $\pi_X^{-1}(x) \cong \{x\} \times Y \cong Y$, where we've assumed Y to be irreducible, we can conclude wlog that $\pi_X^{-1}(x) \cap V = \emptyset$. ■

Proof (G_U, G_V are closed).
Wlog consider $G_U \subseteq X$. Fixing any point $y_0 \in Y$, we have

$$X \cong X_{y_0} \coloneqq X \times \{y_0\} \subseteq X \times Y,$$

so we can identify $G_U \subset X$ with $G_U \subset X_{y_0}$ inside a Y-fiber the product. But then

$$G_U = X_{y_0} \cap U \subseteq X \times Y,$$

where U is closed in $X \times Y$ and thus closed in X_{y_0}, and X_{y_0} is trivially closed in itself. This exhibits G_U as the intersection of two sets that are closed in $X_{y_0} \cong X$. ■

3 | Problem Set 3

Exercise 3.0.1 ***(Gathmann 2.33):*** Define

$$X \coloneqq \left\{ M \in \mathrm{Mat}(2 \times 3, k) \;\middle|\; \mathrm{rank} M \leq 1 \right\} \subseteq \mathbb{A}^6/k.$$

Show that X is an irreducible variety, and find its dimension.

Solution:
We'll use the following fact from linear algebra:

Definition ***(Matrix Minor)***
For an $m \times n$ matrix, a *minor of order* ℓ is the determinant of a $\ell \times \ell$ submatrix obtained by deleting any $m - \ell$ rows and any $n - \ell$ columns.

Theorem 3.0.3 ***(Rank is a Function of Minors).***
If $A \in \mathrm{Mat}(m \times n, k)$ is a matrix, then the rank of A is equal to the order of largest nonzero minor.

Thus

$$M_{ij} = 0 \text{ for all } \ell \times \ell \text{ minors } M_{ij} \iff \mathrm{rank}(M) < \ell,$$

following from the fact that if one takes $\ell = \min(m, n)$ and all $\ell \times \ell$ minors vanish, then the largest nonzero minor must be of size $j \times j$ for $j \leq \ell - 1$. But $\det M_{ij}$ is a polynomial f_{ij} in its entries, which means that X can be written as

$$X = V(\{f_{ij}\}),$$

which exhibits X as a variety. Thus

$$M = \begin{bmatrix} x & y & z \\ a & b & c \end{bmatrix} \implies X = V(\langle xb - ya, yc - zb, xc - za \rangle) \subset \mathbb{A}^6.$$

Claim: The ideal above is prime, and so the coordinate ring $A(X)$ is a domain and thus X is irreducible.

Claim: $\dim(X) = 4$.

Heuristic: there are three degrees of freedom in choosing the first row x, y, z. To enforce the rank 1 condition, the second row must be a scalar multiple of the first, yielding one degree of freedom for the scalar.

Note: I looked at this for a couple of hours, but I don't know how to prove either of these statements with the tools we have so far!

Exercise 3.0.4 ***(Gathmann 2.34):*** Let X be a topological space, and show

a. If $\{U_i\}_{i \in I} \rightrightarrows X$, then $\dim X = \sup_{i \in I} \dim U_i$.

b. If X is an irreducible affine variety and $U \subset X$ is a nonempty subset, then $\dim X = \dim U$. Does this hold for any irreducible topological space?

Solution:

Strictly for notational convenience, we'll treat $\{U_i\}$ is if it were a countable open cover.

Part a: We first note that if $U \subseteq V$, then $\dim U \leq \dim V$. If this were not the case, one could find a chain $\{I_j\}$ of closed irreducible subsets of V of length $n > \dim U$. But then $I_j' := I_j \cap U$ would again be a closed irreducible set, yielding a chain of length n in U. Thus $\dim X \geq \dim U_i$, and it remains true that $\dim X \geq \sup \dim U_i$, so it suffices to show that $\dim X \leq \sup \dim U_i$.

Set $s := \sup_i \dim U_i$ and $n := \dim X$, we want to show that $s \geq n$. Let $\{I_j\}_{j \leq n}$ be a maximal chain of length n of closed irreducible subsets of X, so we have

$$\emptyset \subsetneq I_0 \subsetneq I_1 \subsetneq \cdots \subsetneq I_n \subseteq X.$$

Since $I_0 \subset X$ and $\{U_i\}$ covers X, we can find some $U_0 \in \{U_i\}$ such that $I_0 \cap U_0$ is nonempty, since otherwise there would be a point in $I_0 \cap (X \setminus \cup_{i \in J} U_i) = \emptyset$. We can do this for every I_j, so define $A_j := I_j \cap U_0$.

Each A_j is now closed in U_0, and must remain irreducible, since any decomposition of A_j would lift to a decomposition of I_0. To see that $A_0 \subsetneq A_1$, i.e. that the inclusions are still proper, we can just note that

$$x \in A_{i+1} \setminus A_i \iff x \in (I_{i+1} \cap U_0) \setminus (I_i \cap U_0) = (I_1 \setminus I_2) \cap U_0 = \emptyset.$$

But this exhibits a length n chain in U_0, so $\dim U_0 \geq n$. Taking suprema, we have

$$n \leq \dim U_0 \leq \sup_{i \in J} \dim U_i = s.$$

Part b: The answer is **no**: we can produce a space X with some $\dim X$ and a subset U satisfying $\dim U < \dim X$.
Define a space and a topology by

$$X := \{a, b\} \qquad \tau := \{\emptyset, X, \{1\}\},$$

Here $\{b\}$ is the only proper and closed subset, since its complement is open, so X must be irreducible. We can find an maximal ascending chain of length 1,

$$\emptyset \subsetneq \{b\} \subsetneq X,$$

and so $\dim X = 1$. However, for $U := \{a\}$, there is only one possible maximal chain:

$$\emptyset \subsetneq \{a\} = X,$$

so $\dim U = 0$.

Exercise 3.0.5 ***(Gathmann 2.36):*** Prove the following:

a. Every noetherian topological space is compact. In particular, every open subset of an affine variety is compact in the Zariski topology.

b. A complex affine variety of dimension at least 1 is never compact in the classical topology.

Exercise 3.0.6 ***(Gathmann 2.40):*** Let

$$R = k[x_1, x_2, x_3, x_4] / \langle x_1 x_4 - x_2 x_3 \rangle$$

and show the following:

a. R is an integral domain of dimension 3.

b. $x_1, \cdots, x_4$ are irreducible but not prime in R, and thus R is not a UFD.

c. $x_1 x_4$ and $x_2 x_3$ are two decompositions of the same element in R which are nonassociate.

d. $\langle x_1, x_2 \rangle$ is a prime ideal of codimension 1 in R that is not principal.

Exercise 3.0.7 ***(Problem 5):*** Consider a set U in the complement of $(0,0) \in \mathbb{A}^2$. Prove that any regular function on U extends to a regular function on all of $\mathbb{A}^2$.

4 | Problem Set 4 (Tuesday, October 06)

Problem. (Gathmann 3.20)
Let $X \subset \mathbb{A}^n$be an affine variety and $a \in X$. Show that

$$\mathcal{O}_{X,a} = \mathcal{O}_{\mathbb{A}^n,a}/I(X)\mathcal{O}_{A^n,a},$$

where $I(X)\mathcal{O}_{\mathbb{A}^n,a}$ denotes the ideal in $\mathcal{O}_{\mathbb{A}^n,a}$ generated by all quotients $f/1$ for $f \in I(X)$.

Problem. (Gathmann 3.21)
Let $a \in \mathbb{R}$, and consider sheaves $\mathcal{F}$ on $\mathbb{R}$ with the standard topology:

1. $\mathcal{F} :=$ the sheaf of continuous functions
2. $\mathcal{F} :=$ the sheaf of locally polynomial functions.

For which is the stalk $\mathcal{F}_a$ a local ring?

Recall that a local ring has precisely one maximal ideal.

Problem. (Gathmann 3.22)
Let $\varphi, \psi \in \mathcal{F}(U)$ be two sections of some sheaf $\mathcal{F}$ on an open $U \subseteq X$ and show that

a. If φ, ψ agree on all stalks, so $\overline{(U, \varphi)} = \overline{(U, \psi)} \in \mathcal{F}_a$ for all $a \in U$, then φ and ψ are equal.

b. If $\mathcal{F} := \mathcal{O}_X$ is the sheaf of regular functions on some irreducible affine variety X, then if $\psi = \varphi$ on one stalk $\mathcal{F}_a$, then $\varphi = \psi$ everywhere.

c. For a general sheaf $\mathcal{F}$ on X, (b) is false.

Definition 4.0.1 (Stalk at a subspace)
Let $Y \subset X$ be a nonempty and irreducible subspace of X a topological space with a sheaf $\mathcal{F}$ on X. Then the stalk of $\mathcal{F}$ at Y is defined by the pairs (U, φ) such that $U \subset X$ is open, $U \cap Y$ is nonempty, and $\varphi \in \mathcal{F}(U)$, where we identify $(U, \varphi) \sim (U', \varphi')$ iff there is a small enough open set such that the restrictions agree.

Problem. (Gathmann 3.23: Geometry of a Certain Localization)
Let $Y \subset X$ be a nonempty and irreducible subvariety of an affine variety X, and show that the stalk $\mathcal{O}_{X,Y}$ of $\mathcal{O}_X$ at Y is a k-algebra which is isomorphic to the localization $A(X)_{I(Y)}$.

Problem. (Gathmann 3.24)
Let $\mathcal{F}$ be a sheaf on X a topological space and $a \in X$. Show that the stalk $\mathcal{F}_a$ is a *local object*, i.e. if $U \subset X$ is an open neighborhood of a, then $\mathcal{F}_a$ is isomorphic to the stalk of $\mathcal{F}|_U$ at a on U viewed as a topological space.

5 | Problem Set 5 (Monday, October 26)

Problem. *(Gathmann 4.13)*
Let $f : X \to Y$ be a morphism of affine varieties and $f^* : A(Y) \to A(X)$ the induced map on coordinate rings. Determine if the following statements are true or false:

a. f is surjective $\iff$ f^* is injective.

b. f is injective $\iff$ f^* is surjective.

c. If $f : \mathbb{A}^1 \to \mathbb{A}^1$ is an isomorphism, then f is *affine linear*, i.e. $f(x) = ax + b$ for some $a, b \in k$.

d. If $f : \mathbb{A}^2 \to \mathbb{A}^2$ is an isomorphism, then f is *affine linear*, i.e. $f(x) = Ax + b$ for some $a \in \text{Mat}(2 \times 2, k)$ and $b \in k^2$.

Solution:

a. **True**. This follows because if $p, q \in A(Y)$, then

$$\begin{aligned} f * p &= f^* q \\ &\implies (p \circ f) = (q \circ f) && \text{by definition} \\ &\implies p = q, \end{aligned}$$

where in the last implication we've used the fact that f is surjective iff f admits a right-inverse.

Problem. *(Gathmann 4.19)*
Which of the following are isomorphic as ringed spaces over $\mathbb{C}$?

(a) $\mathbb{A}^1 \setminus \{1\}$

(b) $V\left(x_1^2 + x_2^2\right) \subset \mathbb{A}^2$

(c) $V\left(x_2 - x_1^2, x_3 - x_1^3\right) \setminus \{0\} \subset \mathbb{A}^3$

(d) $V\left(x_1 x_2\right) \subset \mathbb{A}^2$

(e) $V\left(x_2^2 - x_1^3 - x_1^2\right) \subset \mathbb{A}^2$

(f) $V\left(x_1^2 - x_2^2 - 1\right) \subset \mathbb{A}^2$

Problem. *(Gathmann 5.7)*
Show that

a. Every morphism $f : \mathbb{A}^1 \setminus \{0\} \to \mathbb{P}^1$ can be extended to a morphism $\widehat{f} : \mathbb{A}^1 \to \mathbb{P}^1$.

b. Not every morphism $f : \mathbb{A}^2 \setminus \{0\} \to \mathbb{P}^1$ can be extended to a morphism $\widehat{f} : \mathbb{A}^2 \to \mathbb{P}^1$.

c. Every morphism $\mathbb{P}^1 \to \mathbb{A}^1$ is constant.

Problem. *(Gathmann 5.8)*
Show that

a. Every isomorphism $f : \mathbb{P}^1 \to \mathbb{P}^1$ is of the form

$$f(x) = \frac{ax+b}{cx+d} \qquad a, b, c, d \in k.$$

where x is an affine coordinate on $\mathbb{A}^1 \subset \mathbb{P}^1$.

b. Given three distinct points $a_i \in \mathbb{P}^1$ and three distinct points $b_i \in \mathbb{P}^1$, there is a unique isomorphism $f : \mathbb{P}^1 \to \mathbb{P}^1$ such that $f(a_i) = b_i$ for all i.

Proposition 5.0.1 ***(?).***
There is a bijection

$$\{\text{ morphisms } X \to Y\} \stackrel{1:1}{\longleftrightarrow} \{K\text{ -algebra homomorphisms } \mathscr{O}_Y(Y) \to \mathscr{O}_X(X)\}$$
$$f \longmapsto f^*$$

Problem. *(Gathmann 5.9)*
Does the above bijection hold if

a. X is an arbitrary prevariety but Y is still affine?
b. Y is an arbitrary prevariety but X is still affine?