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Chapter I Varieties
Section 1. Affine Varieties

1.1. (a) Note that A(Y) = k[x,y]/(y-x^2) i
p
Define phi : k[x] -> A(Y) = k[x,y]/(y-x^2) by the composition k[x] -> k[x,y] -> k[x,y]/(y-x^2)
Claim: phi is injective; Let phi(f)=phi(g) for f,g in k[x]. Then, f(x)-g(x) in (y-x^2) <=> f(x) - g(x)= h(x,y) (y-x^2) for some h in k[x,y]. But, if h is not zero, deg_y h(x,y) (y-x^2) >=1, and deg_y LHS = 0. Hence h ==0, i.e. f(x)=g(x)./
Claim: phi is surjective; Let h(x,y) + (y-x^2) be in A(Y). Then, if h(x,y) = sum_{i=0 to n} f_i (x) y^i, f_i are in k[x]. Note that h(x,y)-h(x,x^2) = sum_{i=0 to n} f_i(x) (y^i - x^2i) = sum_{i=0 to n} f_i(x) (y-x^2)(y^(i-1) + y^(i-2) x^2 + ... + y (x^2)^(i-2) + (x^2)^(i-1)) is in (y-x^2). Hence, h(x,y) + (y-x^2) = h(x,x^2) + (y-x^2). Let g(x)=h(x,x^2), then, phi(g) = h(x,y) + (y-x^2)./

(b) Note that A(Z) = k[x,y]/(xy-1). Assume that phi: A(Y) -> A(Z) is an isomorphism. Since phi is surjective, there are f,g in k[x] s.t. phi(f(x))= x+(xy-1), phi(g(x))=y+(xy-1). => phi(f(x)g(x)) = xy + (xy-1) = 1+(xy-1) = unity of A(Z). Since phi is an isomorphism, f(x)g(x)= unity of A(Y), i.e. f,g are in k. Then for any h(x,y) + (xy-1) in A(Z), h(f,g) is in k, and phi(h(f,g)) = h(x,y) +(xy-1) i.e. phi|k (k) = k[x,y]/(xy-1), but, it is a contradiction.//

(c) Let f(x,y) in k[x,y] be an irreducible conic polynomial. Let's write f(x,y) = ax^2 + 2bxy + cy^2 + dx + ey + g, a,b,c,d,e,g in k, and not all of a,b,c are zero. For this f, define D(f) = b^2-ac. We prove the following claims:
  1) Whether D(f) =0 or D(f) !=0 is stable under the following operations on f:
      i) multiply by a nonzero constant u in k.
      ii) translation x->x+l, y->y+m, l,m in k.
      iii) linear transform (x,y)^t -> A (x,y)^t for any A in GL(2,k).
  2) Any irreducible conic f(x,y) can be transformed into one of the following two cases using only above three operations:
      i) y-x^2 if D(f)=0 (parabolic)
      ii) xy-1 if D(f)!=0 (elliptic)
  3) For irreducible conic f(x,y), its affine coordinate ring k[x,y]/(f) is stable (up to isomorphism) under the operations in i).

proof of 1); i) D(uf)=(bu)^2 - (au)(cu)=D(f)u^2.
ii) Let f' be the transformed conic. Note that, when calculating D(f'), only coefficients of x^2, xy, y^2 matter. But, translation does not change these coefficients. Hence, D(f')=D(f).
iii) linear transforms preserve degrees, so, we may assume that f is homogeneous of degree 2. Note that f=ax^2 +2bxy + cy^2 = (x y)([[a,b],[b,c]])(x,y)^t and D(f)= -det([[a,b],[b,c]]). So by (x,y)^t -> A(x,y)^t, we obtain (x,y)^t A^t ([[a,b],[b,c]])A(x,y)^t, so D(f')=-detA^t D(f) detA. But, A is in GL(2,k), so, D(f')=nonzero constant multiple of D(f)./

proof of 2); i) D(f)=b^2-ac=0. Then, f=ax^2 + 2bxy + cy^2 + dx+ey+g = (sqrt(a)x + sqrt(c)y)^2 + dx+ey+g (k: algebraically closed => sqrt(a), sqrt(c), sqrt(-1) exist).
Take transform x<- sqrt(a)/sqrt(-1) x + sqrt(c)/sqrt(-1) y
y<- dx + ey + g
This transform is a composition of operation ii), iii).
Then, f-> y-x^2.

ii) D(f)=b^2-ac!=0. Then, f(x,y)=ax^2+2bxy+cy^2+dx+ey+g =
a{(x+b/a y)^2 + c/a y^2 - b^2/a^2 y^2} + dx+ey +g = a(x+b/ay)^2 + (ac-b^2)/a
y^2 + dx+ey+g = (sqrt(a)x + b/sqrt(a) y)^2 + (sqrt((ac-b^2)/a) y)^2 +
dx+ey+g = (sqrt(a)x + (b/sqrt(a) + sqrt((ac-b^2)/a) sqrt(-1) y) (sqrt(a)x +
(b/sqrt(a) - sqrt((ac-b^2)/a)sqrt(-1) y) + dx+ey+g.
Take transform x<- sqrt(a)x + (b/sqrt(a) + sqrt((ac-b^2)/a)sqrt(-1) y
               y<- sqrt(a)x + (b/sqrt(a) - sqrt((ac-b^2)/a)sqrt(-1) y
which is operation iii).
=> f -> xy + d'x + e'y + g for some d', e' in k.
        = (x+e')(y+d') + g-e'd'
Take transform x<- x+e'
               y<- y+d'
which is operation ii).
=> f -> xy + g-e'd'
Since f is irreducible, g-e'd' is not zero, so there is u in k s.t
u(g-e'd')=-1.
Then, operation i): f-> (ux)y -1
      operation iii) : x<-ux, y<-y
=> f -> xy-1.

proof of 3); i) multiplication by u in k^* results in,
k[x,y]/(f') = k[x,y]/(uf) = k[x,y]/(f) so stable.
ii) translation x-> x+l , y-> y+m results in,
k[x,y]/(f') = k[x,y]/(f(x+l,y+m)) -> k[x,y]/(f)
                g(x,y) + (f') -> g(x-l,y-m) + (f)
is an isomorphism
iii) linear transform (x,y)^t -> A (x,y)^t results in,
k[x,y]/(f') = k[x,y]/(f(A(x,y)^t)^t) -> k[x,y]/(f)
                g(x,y) + (f') -> g((A^-1 (x,y)^t))^t) + (f)
is an isomorphism./

So, any irreducible conic f can be transformed into
  y-x^2 if D(f)=0,
  xy-1 if D(f)!=0.
Under this transformation, coordinate ring is stable upto isomorphism. Hence
for any irreducible conic f in k[x,y], A(W)=k[x,y]/(f) is isomorphic to
k[x]=A(Y) or k[x,y]/(y-x^2) = A(Z).//

1.2. Clearly, A(Y)=k[x,y,z]/(z-x^3, y-x^2). Note that A(Y)==
k[x,x^2,x^3]=k[x]: ID, so Y is irreducible, and Y is an affine variety. and
dim Y = dim A(Y) = dim k[x] = 1 and I(Y) = (z-x^3,y-x^2).//

1.3. Y= Z(x^2-yz, xz-x)= Z(x^2-yz, x(z-1)). Let x^2-yz=0 be (i), x(z-1)=0 be
(ii). From (ii), if x=0 => y=0 or z=0 in (i). If x!=0 => z=1 and x^2-y=0.
Hence we have three cases: {x^2-y=0 and z=1}, {x=0,y=0},{x=0,z=0}. Hence
Y=Z(x^2-y, z-1) union Z(x,y) union Z(x,z).
Note that A(Z(x^2-y,z-1))=k[x,y,z]/(x^2-y,z-1) == k[x,x^2,1]=k[x] : ID
          A(Z(x,y))==k[z] : ID
          A(Z(x,z))==k[y] : ID
So, Z(x^2-y, z-1), Z(x,y), Z(x,z) are all irreducible and, above expression
of Y is the irreducible decomposition.//

1.4. Consider Z(x-y) in A^2. It is closed in A^2. But, in A^1 x A^1, closed
sets are finite union or arbitrary intersection of V1 x V2, V1,V2 : closed
sets of A^1. Since V1,V2 are empty or A^1 or finite sets, V1xV2 must be
empty or finite set or {finite}xA^1, A^1x{finite} or A^1 x A^1. Z(x-y) is
not any of above form. So, A^2 is not homeomorphic to A^1 x A^1.//

1.5. (=>) Assume that B== k[x1,...,xn]/I(X) for some X. Clearly, B is then
finitely generated. Assume that f+I(X) satisfies (f+I(X))^m = 0 for some m
in N. Then, f^m is in I(X) => f^m (x) = 0 for all x in X => f is in I(X) =>
f(x)+I(X)=0. Hence, there is no nilpotent element./
(<=) Let a1,a2,...,an be generators of B is a k-algebra. Then, phi :
k[x1,...,xn] -> B mapping xi to ai is a surjection. Then,
k[x1,...,xn]/ker(phi) = B. So, we have to prove that ker(phi) is a radical

ideal. Let f be in sqrt(ker(phi)). => f^m is in ker(phi). => phi(f^m) = (phi(f))^m = 0 but, B does not have nilpotent elements, so phi(f)=0. i.e. f is in ker(phi). Hence ker(phi) is a radical ideal, and so, ker(phi)=I(X) for some X. //

1.6. Let U be nonempty open in X, X:irreducible.
Claim: U is dense and irreducible.
If U is not dense, there is a nonempty open set V in X s.t. U doesnot meet V. Then U^c union V^c is X, contradicting the irreducibility of X. Hence U is dense.
If U is not irreducible, there is a nonempty proper closed sets W_1,W_2 in U s.t. W_1 union W_2 = U. Since W_1=U intersection V_1, W_2=U intersection V_2 for some closed sets V_1,V_2 of X. Then, (V_1 union V_2) union U^c =X contradicting the irreducibility of X. Hence is U is irreducible.

Claim: When Y is irreducible, so is Y~.
Lemma: If every nonempty open set U of Z is dense, then Z is irreducible.
pf) If not, V_1 U V_2 = Z, V_1,V_2;nonempty proper closed. Then, (V_1)^c intersection (V_2)^c = empty, contradicting denseness of (V_1)^c./
Hence ETS: Any nonempty open set U in Y~ is dense. Let V be any nonempty open set of Y~. Then, U intersection Y, V intersection Y are nonempty open sets of Y(because Y is dense in Y~). Hence intersection of all U,V,Y = nonempty. => intersection of U,V is = nonempty.//

1.7. (a) (i) => (ii) Let S be a nonempty collection of closed subsets. On elements of S, give a partial order '<=' as F1<=F2 if F1 contains F2. Let {F_i} be a chain in S. Since X is a noetherian space, {F_i} is actually a finite set, so there is a maximal element. Hence by Zorn's lemma, there is a maximal element in S with respect to '<=', i.e. a minimal element in S with respect to the inclusion./

(ii) => (i) Let F1 \contain F2 \contain F3 .... be a descending chain of closed subsets of X. Then, S= {Fi| i in N} has a minimal element by (ii), Let F_N be the minimal element. Then, of course, this chain is stationary beyond Nth./

(iii)=>(iv) and (iv)=>(iii) can be done in the same way.

(i) => (iii) Let G1 \contained_in G2 \contained_in G3 ..... be an ascending chain of open sets. Then, G1^c \contain G2^c \contain G3^c ..... is a descedning chain of closed sets. By the DCC, there is N s.t. Gn^c = GN^c for all n>=N, i.e. Gn = GN. Hence, ACC for open sets hold./

(iii)=>(i) In the same way as in (i)=>(iii).
Hence, (i), (ii), (iii), (iv) are all equivalent.//

(b) Assume not, i.e. let I be a collection of open sets of a noetherian space X s.t. any finite subcollection of I cannot cover X. -------(*)
Choose any nonempty U_1 in I. By (*), U_1 is properly in X. Let X_1 = (U_1)^c;nonempty closed. Since I covers X, there is nonempty U_2 in I such that U_2 meets X_1. By (*) again, U_1 U U_2 is properly in X, so, let X_2 = (U_1 U U_2)^c; nonempty closed, and X_1 properly contain X_2.
Assume U_1,...,U_n in I, X_1,...,X_n are given s.t. X_i = (Union of U_1,...,U_n)^c;nonempty closed and {X_i,i=1,...,n} is strictly decreasing. Since I covers X, there is a nonempty U_n+1 in X, so, let X_n+1 = (Union of U_1,...,U_n+1)^c;nonempty closed, and, then, {X_i,i=1,...,n+1} is strictl decreaing. Hence by induction, we can obtain an infinite properly descending chian of closed subsets of X=> contradiction because X is noetherian.//

(c) Let Y be in X, and X is a noetherian space. If Y is not noetherian, there is a sequence of strictly decreasing closed subsets of Y: {Y_i,i=1,2,.....}. Since Y has induced topology, Y_i = Y intersection F_i for closed subsets F_i of X. Since Y_i contains Y_i+1, Y_i+1 = Y_i intersection Y_i+1 = Y intersection (F_i intersection F_i+1). Hence, by replacing F_i by intersection of F1,...,F_i, WMA {F_i,i=1,2,..} is an decreasing sequence of closed sets of X. Since X is noetherian, F_N=F_N+1=.... => Y_N=Y_N+1=..... (contradiction)//

(d) If V is an irreducible closed set, V is a single point. (If not, two points x,y have disjoint open sets, which are dense.) Since X is noetherian, it is a finite union of irreducible closed sets, i.e. finite union of points. Hence X has only finitely many points.//

1.8. Let Y=Z(p), H=Z(f) where p is a prime ideal of k[x1,...,xn], f: a nonconstant polynomial. Since Y \not_contained_in H, f is not in p, and Y \intersection H = Z(p,f).
Consider A(Y)= k[x1,...,xn]/p. Then, f+p in A(Y) is not zero, not unit, not a zero-divisor. since A(Y) is a noetherian space, there is a primary decomposition (f+p) = \intersection_{i=1 to m} q_i with r(q_i)=p_i; minimal prime ideals of (f+p).
Then, by the Krull's Hauptidealsatz, ht(p_i)=1, and every component of Y \intersection H corresponds to p_i.
Here, ht(p_i) + dim A(Y)/p_i = k[x1,...,xn]/(p,p_i) : coordinate ring of each irreducible component => dim(component) = dim A(Y)/p_i = dim A(Y) - ht(p_i) = r-1.//

1.9. Let a = (f_1,f_2,...,f_r)
=> 0 \contained_in (f_1) \contained_in (f_2) .... \contained_in (f_1,f_2,...,f_r)=a.
=> A^n \contains Z(f_1) \contains Z(f_2) ..... \contains Z(f_1,f_2,...,f_r)=Z(a).
so, note that dim Z(f_1,f_2,...,f_i) >= dim Z(f_1,f_2,...,f_i+1) for each i.
If f_i+1 is not a zero-divisor in A(Z(f_1,f_2,...,f_i)), there is a minimal prime ideal p in A(Z(f_1,f_2,...,f_i)) containing the image of f_i+1 with ht(p)=1. So, ht(p) + dim A(Z(f_1,...,f_i))/p = dim A(Z(f_1,...,f_i)), so, dim Z(f_1,...,f_i+1) = dim A(Z(f_1,...,f_i+1) >=dim A(Z(f_1,...,f_i))/p = dim Z(f_1,...,f_i))-1.
Otherwise, Z(f_1,...,f_i)=Z(f_1,...,f_i+1).
Hence, at each step, dimension decreses by 0 or 1. Because there are r such steps, hence dimension of Z(f_1,...,f_r) >=n-r.//

1.10. (a) Any closed subset of Y is of the form : Y \intersection F, F: closed in X.
For any chain Y \intersection F_0 \strictly_in Y \intersection F_1 .... of irreducible closed sets, cl(Y \intersection F_0) \strictly_in cl(Y \intersection F_1) .... : irreducible closed sets of X. Hence dim Y <= dim X.

(b) dim U_i <= dim X is clear by (a) for all i. Hence sup dim U_i <= dim X. Conversely, for any chain of irreducible closed subsets F_0 \strictly_in F_1 \strictly_in ..... F_n, choose an open set U_0 in {U_i} s.t. F_0 \intersection U_0 !=\empty. Then, since F_0 \strictly_in F_1, F_1 \intersection U_0 !=\empty. Since F_1 is irreducible and F_1 \intersection U0 is a nonempty open subset, it must be dense in F_1. THen, F_1 - F_0 : nonempty open subset of F_1 => (F_1 - F_0) \intersection U_0 != \empty. Hence, F_0 \intersection U_0 \strictly_in F_1 \intersection U_0. But, using same argument, we can construct a strict chain {F_i \intersection U_0} i=0,1,...,n. Hence dim U_0 >=dim X. Hence sup dim U_i = dim X.//

(c) Consider X={0,1} with topology T={\empty, {0}, {0,1}}. Then, {0} is dense open subset, because arbitrary open set intersects it. But, dim{0} = 0, because {0} doesnot contain any nonempty closed subsets other than itself. But, {1} \strictly_in {1,2} is the maximal chain of irreducible closed sets of X, so dim X=1. Hence dim U<dim X.//

(d) Assume not, i.e. Y \strictly_in X. Choose a maximal chain of irreducible closed subsets of Y with the maximal length: Y_0 \strictly_in Y_1 .... \strictly_in Y_n. Then, by adding X, we obtain a chain with length n+1, so dim X>=n+1 > n=dimY. contradiction .//

(e) (From Atiyah-Macdonald)
Let A=k[x_1,...,x_n,....], m_1,m_2, ... : increading sequence of natural numbers, s.t. m_i+1 - m_i > m_i - m_i-1 for all i>1. Let p_i = (x_{m_i +1} , ..., x_{m_{i+1}}), S= (Union of all P_i)^c. Let X = Spec(S^-1 A) => X is noetherian because S^-1 A is a noetherian ring (Chapter 7, Ex.9 of Atiyah-Macdonald) but,, S^-1 p_i has height m_{i+1} - m_i : dim S^-1 A = infinity.//

1.11. Define phi: k[x,y,z] -> k[t^3, t^4, t^5] by phi(f(x,y,z)) = f(t^3,t^4,t^5). Clearly phi is surjective and f is in ker(phi) iff f(t^3,t^4,t^5)=0, i.e. f is in I(Y). Hence ker(phi) = I(Y). Note that k[t^3,t^4,t^4] is in k[t], so it is an ID, hence I(Y) is a prime ideal.
Now, ht(I(Y)) + dim (k[x,y,z]/I(Y)) = dim (k[x,y,z]).
Note dim k[x,y,z]/I(Y) = dim k[t^3,t^4,t^5] = dim k[t] (because k[t] is an integral extension of k[t^3,t^4,t^5]) = 1. Hence, ht(I(Y))= 3-1=2.
Now we show that I(Y) cannot be generated by 2 elements. Let's search for the elements of I(Y) with "minimal degree, next minimal degree, etc."
Let f(x,y,z) be in I(Y), f(x,y,z)= sum_{i,j,k} b(i,j,k) x^i y^j z^k, b(i,j,k) is in k.
Then, f(t^3,t^4,t^5)=0 implies sum_{i,j,k} b(i,j,k) t^(3i+4j+5k) = 0. Let's collect terms with same degree with respect to t. Let n=3i+4j+5k
If n=0, (i,j,k)=(0,0,0) => b(0,0,0)=0.
n=1, (i,j,k) does not exist
n=2, (i,j,k) does not exist
n=3, (i,j,k)=(1,0,0) => b(1,0,0)=0.
n=4, (i,j,k)=(0,1,0) => b(0,1,0)=0.
n=5, (i,j,k)=(0,0,1) => b(0,0,1)=0.
n=6, (i,j,k)=(2,0,0) => b(2,0,0)=0.
n=7, (i,j,k)=(1,1,0) => b(1,1,0)=0.
n=8, (i,j,k)=(1,0,1) and (0,2,0) => b(1,0,1) + b(0,2,0) = 0 i.e. we find xz-y^2 = f_1.
n=9, (i,j,k)=(3,0,0) and (0,1,0) => b(3,0,0) + b(0,1,1) = 0 i.e. we find x^3 - yz = f_2.
n=10, (i,j,k)=(2,1,0) and (0,0,2) => b(2,1,0) + b(0,0,2) = 0 i.e. we find x^2 y - z^2 = f_3.
n=11, (i,j,k)=(1,2,0) and (2,0,1) => b(1,2,0) + b(2,0,1) = 0 i.e. xy^2 - x^2z = x(y^2-xz)=xf_1.
n>=12, if there are solutions (i,j,k), i+j+k>=3. So, polynomials obtained from now must have degree>=3 for every term.
So, we have (f_1, f_2, f_3) \contained_in I(Y).
Note that f_1 is not in (f_2, f_3) (degf_1 = 2, deg of minimal degree term of f_2,f_3 = 2)
f_2 is not in (f_1,f_3) (f_2 has yz but, deg 2 terms of f_1 and f_3 are xz,y^2,z^2)
f_3 is not in (f_1,f_2) (f_3 has z^2 but, deg 2 terms of f_1 and f_2 are xz,y^2,yz)
Note that if we find all other generators f_4,f_5,... of I(Y), each term of f_i (i>=4) must have degree >=3 and f_1, f_2, f_3 are the only generators those who have terms of degree 2. If (g_1 , g_2) = I(Y), i.e. I(Y) can be generated by only two elements, (g_1, g_2) must contain f_1,f_2,f_3. By the minimality of the degrees of f_1,f_2,f_3 in I(Y), g_1, g_2 must be constant multiples of f_1,f_2,f_3. But, it is not possible. Hence I(Y) must have at least 3 generators./

Remark: We did not prove that (xz-y^2, x^3-yz , x^2 y -z^2) = I(Y). It requires more work.

1.12. Let f(x,y) = (x^2 -1)^2 + y^2 = x^4 -2x^2 +1 + y^2. It is irreducible. (f has a factorization in C[x,y]: (x^2 -1 + iy)(x^2-1 -iy) in to irreducibles. If f factors in R[x,y], since both R[x,y] and C[x,y] are UFDs and R[x,y] is in C[x,y], the factorization must be equal to (x^2 -1 + iy)(x^2-1-iy), but it is not possible in R[x,y].)
But, Z(f) = {(1,0),(-1,0)} = Z(x-1,y) \union Z(z+1,y). which is obviously reducible.//

Robin Hartshorne's Algebraic Geometry Solutions

by Jinhyun Park

Chapter II Section 2 Schemes

2.1. Let A be a ring, let $X = \operatorname{Spec}(A)$, let $f \in A$ and let $D(f) \subset X$ be the open complement of $V((f))$. Show that the locally ringed space $(D(f), \mathcal{O}_X|_{D(f)})$ is isomorphic to $\operatorname{Spec}(A_f)$.

Proof. From a basic commutative algebra, we know that prime ideals in A_S, for a multiplicative set S of A, correspond to prime ideals of A which do not intersect S. In particular, $A_f = A_S$ for $S = \{1, f, f^2, \cdots, \}$ so that prime ideals of A_f correspond to prime ideals of A not containing f. This shows that the underlying topological spaces are homeomorphic. For the morphism of structure sheaves, Prop. 2.2 -(b) gives the answer. This proves the assertion. $\square$

2.2. Let $(X, \mathcal{O}_X)$ be a scheme, and let $U \subset X$ be any open subset. Show that $(U, \mathcal{O}_X|_U)$ is a scheme. We call this the *induced scheme structure* on the open set U, and we refer to $(U, \mathcal{O}_X|_U)$ as an *open subscheme* of X.

Proof. By the remark on p. 71 above the Prop. 2.2., affine subschemes of X form a basis for the topology of X. Thus, for any open $U \subset X$ there is an affine open subscheme $Y \subset U$, thus, by definition, $(U, \mathcal{O}_X|_U)$ is a scheme. $\square$

2.3. *Reduced Schemes.* **A scheme $(X, \mathcal{O}_X)$ is *reduced* if for every open set $U \subset X$, the ring $\mathcal{O}_X(U)$ has no nilpotent elements.**

(a) **Show that $(X, \mathcal{O}_X)$ is reduced if and only if for every $P \in X$, the local ring $\mathcal{O}_{X,P}$ has no nilpotent elements.**

Proof. ($\Rightarrow$) Assume not, i.e. there is $P \in X$ and $0 \neq f \in \mathcal{O}_{X,P}$ such that $f^m = 0$ for some $m \in \mathbb{N}$. Then there is an open set $V \ni P$ and $g \in \mathcal{O}_X(V)$ which represents f. But, then $g^m = 0$ which is a contradiction.

($\Leftarrow$) Assume that for some open $V \subset X$, there is nonzero $g \in \mathcal{O}_X(V)$ such that $g^m = 0$. Then, there is $P \in V$ for which the image $f \in \mathcal{O}_{X,P}$ of g is nonzero and $f^m = 0$, which is a contradiction. $\square$

(b) **Let $(X, \mathcal{O}_X)$ be a scheme. Let $(\mathcal{O}_X)_{\text{red}}$ be the sheaf associated to the presheaf $U \mapsto \mathcal{O}_X(U)_{\textbf{red}}$, where for any ring A, we denote by A_{red} the quotient of A by its ideal of nilpotent elements. Show that $(X, (\mathcal{O}_X)_{\text{red}})$ is a scheme. We call it the *reduced scheme* associated to X, and denote it by X_{red}. How that there is a morphism of schemes $X_{\text{red}} \to X$, which is a homeomorphism on the underlying topological spaces.**

Claim. *$(X, (\mathcal{O}_X)_{\text{red}})$ is a scheme.*

Proof. For any affine schemes $V \subset U \subset X$, $(\mathcal{O}_X|_U)_{\text{red}}(V) = (\mathcal{O}_X)_{\text{red}}|_U(V)$, so, the rest is obvious. $\square$

Claim. *There is a morphism of schemes $X_{\text{red}} \to X$ which is a homeomorphism on the underlying spaces.*

Proof. Just define $f : X_{\text{red}} \to X$ to be the identity map on the underlying spaces. We define $f^\sharp : \mathcal{O}_X \to f_*(\mathcal{O}_X)_{\text{red}}$ to be

$$f^\sharp(U) : \mathcal{O}_X(U) \to \mathcal{O}_X(U)/\mathfrak{nilrad}(\mathcal{O}_X(U))$$

for any open subset $U \subset X$. $\square$

(c) **Let $f : X \to Y$ be a morphism of schemes, and assume that X is reduced. Show that there is a unique morphism $g : X \to Y_{\mathrm{red}}$ such that f is obtained by composing g with the natural map $Y_{\mathrm{red}} \to Y$.**

Proof. Define $g : X \to Y_{\mathrm{red}}$ as follows. As a map on underlying spaces, $g = f$. As a morphism of sheaves, $g^\sharp : (\mathcal{O}_Y)_{\mathrm{red}} \to g_*\mathcal{O}_X = g_*(\mathcal{O}_X)_{\mathrm{red}}$ is defined from $f^\sharp : \mathcal{O}_Y \to f_*\mathcal{O}_X$ by taking $g^\sharp = (f^\sharp)_{\mathrm{red}}$. This is possible because a nilpotent is sent to a nilpotent so that a nilradical is sent to a nilradical. □

2.4. Let A be a ring and let $(X, \mathcal{O}_X)$ be a scheme. Given a morphism $f : X \to \mathrm{Spec}(A)$, we have an associated map on sheaves $f^\sharp : \mathcal{O}_{\mathrm{Spec}(A)} \to f_*\mathcal{O}_X$. Taking global sections we obtaion a homomorphism $A \to \Gamma(X, \mathcal{O}_X)$. Thus there is a natural map

$$\alpha : \mathrm{Hom}_{\mathfrak{Sch}}(X, \mathrm{Spec}(A)) \to \mathrm{Hom}_{\mathfrak{Rings}}(A, \Gamma(X, \mathcal{O}_X)).$$

Show that α is bijective (cf. (I, 3.5) for an analogous statement about varieties).

Proof. Let $\phi : A \to \Gamma(X, \mathcal{O}_X)$ be a ring homomorphism. We want to construct a natural morphism of schemes which corresponds to ϕ.

Notice that for any affine open $U \subset X$, we have $A \xrightarrow{\phi} \Gamma(X, \mathcal{O}_X) \xrightarrow{\rho_X^U} \Gamma(U, \mathcal{O}_X)$ from which we can obtain $\phi_U^* = \mathrm{Spec}(\rho_X^U \circ \phi) : U \simeq \mathrm{Spec}(\Gamma(U, \mathcal{O}_X)) \to \mathrm{Spec}(A)$. The question is whether they glue together nicely so that we can we can actually obtain a map from X to $\mathrm{Spec}(A)$. But, this is easy: for two affine open sets U and V and any affine open subset $W \subset U \cap V$, the restiction maps ρ are transitive so that

$$\rho_X^W = \rho_U^W \circ \rho_X^U = \rho_V^W \circ \rho_X^W$$

and Spec is contravariant functorial so that the morphism of schemes

$$\phi_U^*|_W = \mathrm{Spec}(\rho_X^U \circ \phi) \circ \mathrm{Spec}(\rho_U^W) = \mathrm{Spec}(\rho_U^W \circ \rho_X^U \circ \phi) = \mathrm{Spec}(\rho_X^W \circ \phi) = \phi_W^*$$

and by symmetry, $\phi_V^*|_W = \phi_W^*$. Thus, by collecting $\{\phi_U^*\}_{U \subset X}$, we have $\phi^* : X \to \mathrm{Spec}(A)$. That these two procedures are inverse to each other is obvious. □

2.5. Describe $\mathrm{Spec}(\mathbb{Z})$, and show that it is a final object for the category of schemes, i.e., each scheme X admits a unique morphism to $\mathrm{Spec}(\mathbb{Z})$.

Proof. $\mathrm{Spec}(\mathbb{Z}) = \{(0)\} \cup \{(p) | p\text{:prime number}\}$ with (0), not closed and (p) are closed points. This is a dimension 1 scheme. On the other hand, take $A = \mathbb{Z}$ in Ex. 2.4. Then, $\mathrm{Hom}_{\mathfrak{Rings}}(\mathbb{Z}, \Gamma(X, \mathcal{O}_X))$ has only one element, namely, the ring homomorphism sending 1 to 1. This corresponds to a unique morphism of schemes $X \to \mathrm{Spec}(\mathbb{Z})$, thus, it is a final object for the category of schemes. □

2.6. Describe the spectrum of the zero ring, and show that it is an initial object for the category of schemes. (According to our conventions, all ring homomorphisms must take 1 to 1. Since $0 = 1$ in the zero ring, we see that each ring R admits a unique homomorphism to the zero ring, but that there is no homomorphism from the zero ring to R unless $0 = 1$ in R.)

Proof. For $A = 0$, $\mathrm{Spec}(A) = \phi$. On the other hand, for any scheme X, any ring homomorphism $\Gamma(X, \mathcal{O}_X) \to 0$ is 0. Hence, by Ex. 2.4, there is a unique morphism of schemes $\mathrm{Spec}(0) \to X$, namely, the inclusion of empty set to X. Hence, $\mathrm{Spec}(0)$ is an initial object in the category of schemes. □

2.7. Let X be a scheme. For any $x \in X$, let $\mathcal{O}_x$ be the local ring at x, and $\mathfrak{m}_x$ its maximal ideal. We define the *residue field* of x on X to be the field $k(x) = \mathcal{O}_x/\mathfrak{m}_x$. Now let K be any field. Show that to give a morphism of $\mathrm{Spec}(K)$ to X is equivalent to give a point $x \in X$ and an inclusion map $k(x) \to K$.

Proof. ($\Rightarrow$) Let $(\eta, \eta^\sharp) : \mathrm{Spec}(K) \to X$ be a morphism of schemes. As a map on topological spaces, since $\mathrm{Spec}(K)$ consists of a single point $\{*\}$, there is a unique point $x \in X$ with $x := \eta(*)$.

Now, from $\eta^\sharp$, we obtain a local homomorphism $\eta^\sharp_* : \mathcal{O}_{X,x} \to \mathcal{O}_{\mathrm{Spec}(K),*} = K$, thus, the map of their residue fields

$$\overline{\eta^\sharp_*} : k(x) = \frac{\mathcal{O}_{X,x}}{\mathfrak{m}_{X,x}} \to \frac{\mathcal{O}_{\mathrm{Spec}(K),*}}{\mathfrak{m}_{\mathrm{Spec}(K),*}} = \frac{K}{0} = K.$$

This is injective because $k(x)$ is a field.
($\Leftarrow$) Conversely, suppose that $x \in X$ and an embedding $k(x) \hookrightarrow K$ are given. We have the obvious map on topological spaces $\eta : \mathrm{Spec}(K) \to X$ defined to be $* \mapsto x$, thus, we need to construct $\eta^\sharp : \mathcal{O}_X \to \eta_* \mathcal{O}_{\mathrm{Spec}(K)}$. But, this is easy:

If $x \in U \subset X$, then, $\eta^\sharp(U) : \mathcal{O}_X(U) \to \left(\eta_* \mathcal{O}_{\mathrm{Spec}(K)}\right)(U) = K$ is defined to be the composition of maps

$$\mathcal{O}_X(U) \to \mathcal{O}_{X,x} \to \mathcal{O}_{X,x}/\mathfrak{m}_{X,x} = k(x) \hookrightarrow K.$$

If $x \notin U \subset X$, we let $\eta^\sharp(U) = 0$, where the target is the zero ring.

Thus, we constructed the desired morphism of schemes $(\eta, \eta^\sharp) : \mathrm{Spec}(K) \to X$. This finishes the proof. □

2.8. Let X be a scheme. For any point $x \in X$, we define the *Zariski tangent space* T_x to X at x to be the dual of the $k(x)$-vector space $\mathfrak{m}_x/\mathfrak{m}_x^2$. Now assume that X is a scheme over a field k, and let $k[\epsilon]/\epsilon^2$ be the *ring of dual numbers* over k. Show that to give a k-morphism of $\mathrm{Spec}\left(k[\epsilon]/\epsilon^2\right)$ to X is equivalent to giving a point $x \in X$, *rational over* k (i.e., such that $k(x) = k$), and an element of T_x.

Proof. Notice first that as a topological space, $\mathrm{Spec}\left(k[\epsilon]/\epsilon^2\right)$ is a single point $\{*\}$ with residue field $k(*) = k$.
($\Rightarrow$) Let $(\eta, \eta^\sharp) \in \mathfrak{Mor}_{k-\mathfrak{sch}}\left(\mathrm{Spec}\left(k[\epsilon]/\epsilon^2\right), X\right)$ be given. Let $x = \eta(*)$. Since η is a k-morphism, and $k(*) = k$, we must have $k(x) = k$ and x is a rational point.

On the other hand, we have a k-algebra *local* homomorphism $\eta^\sharp_* : \mathcal{O}_{X,x} \to \mathcal{O}_{\mathrm{Spec}(k[\epsilon]/\epsilon^2),*} = k[\epsilon]/\epsilon^2 =: k[\overline{\epsilon}]$, thus, $\eta^\sharp_*(\mathfrak{m}_{X,x}) \subset (\overline{\epsilon})$. But, since $(\overline{\epsilon}^2) = 0$, we have $\eta^\sharp_*(\mathfrak{m}^2_{X,x}) \subset (\overline{\epsilon}^2) = 0$, thus we get a k-vector space homomorphism

$$\eta^\sharp_* : \mathfrak{m}_{X,x}/\mathfrak{m}^2_{X,x} \to (\overline{\epsilon}) \simeq k,$$

where the last map is an isomorphism of k-vector spaces.

Thus, we obtained a k-rational point $x \in X$ and $\eta^\sharp_* \in \mathrm{Hom}_k\left(\mathfrak{m}_x/\mathfrak{m}_x^2, k\right) = T_x$ as desired.
($\Leftarrow$) Conversely, suppose that we have a k-rational point $x \in X$ and a k-linear map $\xi \in \mathrm{Hom}_k\left(\mathfrak{m}_x/\mathfrak{m}_x^2, k\right)$. Out of this data, we will define an element $(\eta, \eta^\sharp) \in \mathfrak{Mor}_{k-\mathfrak{sch}}\left(\mathrm{Spec}\left(k[\epsilon]/\epsilon^2\right), X\right)$.

First, as a map of topological spaces, just define $\eta(*) = x$. Let's define $\eta^\sharp$.

If $x \notin U \subset X$, define $\eta^\sharp(U) : \mathcal{O}_X(U) \to \left(\eta_* \mathcal{O}_{\mathrm{Spec}(k[\epsilon]/\epsilon^2)}\right)(U) = 0$ to be 0.
If $x \in U \subset X$, notice that since x is a k-rational point, we first have a decomposition $\mathcal{O}_{X,x} = k \oplus \mathfrak{m}_{X,x}$. Then, using this, define $\eta^\sharp(U)$ as the composition of maps

$$\mathcal{O}_X(U) \to \mathcal{O}_{X,x} = k \oplus \mathfrak{m}_{X,x} \xrightarrow{\alpha} k[\epsilon]/\epsilon^2 = \left(\eta_* \mathcal{O}_{\mathrm{Spec}(k[\epsilon]/\epsilon^2)}\right)(U)$$

where the second map α sends $(a,b) \mapsto a + \xi(\bar{b})\bar{\epsilon}$, where $\bar{b}$ denotes its image in $\mathfrak{m}_x/\mathfrak{m}_x^2$. This proves the assertion.

□

2.9. If X is a topological space, and Z an irreducible closed subset of X, a *generic point* for Z is a point ζ such that $Z = \{\zeta\}^-$. If X is a scheme, show that every (nonempty) irreducible closed subset has a unique generic point.

Proof. Choose an affine open subset $V \subset X$, $V \simeq \mathrm{Spec}(A)$, with $V \cap Z \neq \phi$.

Claim (1). *$Z = \overline{V \cap Z}$ where the closure is taken in X.*

Set theoretically, $Z = \overline{V \cap Z} \cup (Z \cap (X - V))$. But, since Z is irreducible and $Z \cap (X - V)$ is a proper subset of Z, this claim is true.

Claim (2). *$V \cap Z$ is irreducible.*

If not, there are two proper closed subsets F_1, F_2 of Z such that $V \cap Z = (V \cap F_1) \cup (V \cap F_2)$ so that $Z = (Z \cap F_1) \cup (Z \cap F_2) \cup (Z \cap (X - V))$ which contradicts the irreducibility of Z.

Thus, $V \cap Z$ is an irreducible closed subset of an affine variety V, i.e. there is a point x corresponding to a prime ideal of A such that $V \cap Z = \{x\}^-$, where the closure here is taken in V. Hence, by extending the closure in X, by Claim (1), $Z = \overline{V \cap Z} = \{x\}^-$, which shows the existence of a generic point.

If there are two generic points x_1, x_2, then, $x_1 \in \{x_2\}^-$. Thus, if we choose an affine open subset V containing x_2, x_1 must lie in V as well, and for two prime ideals p_1, p_2 corresponding to x_1, x_2, $p_1 \supset p_2$. But, by interchanging the roles of x_1 and x_2, we also have $p_1 \subset p_2$, which means, $x_1 = x_2$. Hence there is a unique generic point. □

2.10. Describe $\mathrm{Spec}(\mathbb{R}[x])$. How does its topological space compare to the set $\mathbb{R}$? to $\mathbb{C}$?

Proof. See my solutions for Atiyah-MacDonald's *Introduction to commutative algebra* Chapter 1. □

2.11. Let $k = \mathbb{F}_p$ be the finite field with p elements. Describe $k[x]$. What are the residue fields of its points? How many points are there with a given residue field?

Proof. First of all, what are $\mathrm{Spec}(k[x])$? (0) is the generic point and (f) are closed points, when f are nonzero irreducible polynomials. It is in general not very easy to enumerate all irreducible polynomials. But, we can count the number of them, which will be done in the sequel.

When $\xi = (0)$, $k[x]_\xi = k(x)$ and $\mathfrak{m}_\xi = 0$, thus, the residue field is same as the fraction field so that $k(\xi) = k(x)$. When $\xi = (f)$, where f is an irreducible polynomial of degree $n \geq 1$, then, $k[x]_\xi = \left\{\frac{h}{g} | f \not| g\right\}$, $\mathfrak{m}_\xi = \left\{\frac{h}{g} | f \not| g, f | h\right\}$ so that

$$k[x]_\xi/\mathfrak{m}_\xi \simeq (k[x]/(f))_\xi \simeq k[x]/(f) \simeq \mathbb{F}_{p^n}.$$

So, $\mathbb{F}_p[x]$ and $\mathbb{F}_{p^n}, n \geq 1$ are all possible residue fields. Obviously only the generic point can have $k[x]$ as the residue field.

To compute the number of points which have a specific $\mathbb{F}_{p^n}$ as its residue field is equivalent to count the number of monic irreducible polynomials of degree n over $\mathbb{F}_p$. To do so, we will use the collection of all maps from $\mathrm{Spec}\,(F_n := \mathbb{F}_{p^n})$ to $\mathrm{Spec}(k[x])$.

If $\xi = (f) \neq 0$ with $\deg f = m$ is a F_n-rational point, then it means, the image of $f \in k(\xi) = k[x]/(f) \hookrightarrow F_n$ is an element of F_n. In particular, $m|n$ and there are m distinct embeddings coming from various conjugates. Conversely, each nonzero element of F_n is a root of a unique monic irreducible polynomial of degree m dividing n. Hence each irreducible monic polynomial of degree m, $m|n$ determines m elements of F_n and each element of F_n is also determined by an irreducible monic polynomial.

So, let S_n be the number of monic irreducible polynomials. Let $T_n = nS_n$. Then,

$$\sum_{m|n} T_m = p^n.$$

To solve this equation, we use the Möbius inversion formulae: if $g(n) = \sum_{d|n} f(d)$, then, $f(n) = \sum_{d|n} \mu(d) g\left(\frac{n}{d}\right)$ where

$$\mu(n) = \begin{cases} 1 & \text{n = 1} \\ 0 & n \text{ is not square free.} \\ (-1)^k & n = p_1 \cdots p_k\text{: distinct primes} \end{cases}.$$

(See any reasonable number theory book.)

Hence,

$$S_n = \frac{1}{n} \sum_{d|n} \mu(d) \left(p^{\frac{n}{d}}\right)$$

is the number of monic irreducible polynomials of degree n over $\mathbb{F}_p$, which is equal to the number of points of $\mathrm{Spec}(k[x])$ whose residue field is exactly $\mathbb{F}_{p^n}$. □

2.12. *Glueing Lemma.* **Generalize the glueing procedure described in the text (2.3.5) as follows. Let $\{X_i\}$ be a family of schemes (possibly infinite). For each $i \neq j$, suppose given an open subset $U_{ij} \subset X_i$, and let it have the induced scheme structure (Ex. 2.2). Suppose also given for each $i \neq j$ an isomorphism of schemes $\phi_{ij} : U_{ij} \to U_{ji}$ such that (1) for each i, j, $\phi_{ji} = \phi_{ij}^{-1}$, and (2) for each i, j, k, $\phi_{ij}(U_{ij} \cap U_{ik}) = U_{ji} \cap U_{jk}$, and $\phi_{ik} = \phi_{jk} \circ \phi_{ij}$ on $U_{ij} \cap U_{ik}$. Then show that there is a scheme X, together with morphisms $\psi_i : X_i \to X$ for each i, such that (1) ψ_i is an isomorphism of X_i onto an open subscheme of X, (2) the $\psi_i(X_i)$ cover X, (3) $\psi_i(U_{ij}) = \psi_i(X_i) \cap \psi_j(X_j)$ and (4) $\psi_i = \psi_j \circ \phi_{ij}$ on U_{ij}. We say that X is obtained by *glueing* the schemes X_i along the isomorphisms ϕ_{ij}. An interesting special case is when the family X_i is arbitrary, but the U_{ij} and ϕ_{ij} are all empty. Then the scheme X is called the *disjoint union* of the X_i, and is denoted $\coprod X_i$.**

Proof. Obvious. □

2.13. A topological space is *quasi-compact* if every open cover has a finite subcover.

(a) **Show that a topological space is noetherian (I, §1) if and only if every open subset is quasi-compact.**

Proof. (⇒) By Ex. I-1.7-(c), any open subset is noetherian, hence, by Ex. I-1.7-(b), it is quasi-compact.

(⇐) let $U_1 \subset U_2 \cdots$ be an ascending chain of open subsets of X. Let $U = \bigcup_i U_i$. By assumption, U is quasi-compact so that $U = \bigcup_{i=1}^r U_i$ for some r. Then, $U_r = U_{r+1} = \cdots$ so that X is noetherian. □

(b) **If X is an affine scheme, show that $sp(X)$ is quasi-compact, but not in general noetherian. We say that X is *quasi-compact* is $sp(X)$ is.**

Proof. Let $X = \mathrm{Spec}(A)$. We know that for $g \in A$, $D(g) \simeq \mathrm{Spec}(A_g)$ form a basis for X. Hence, $\mathrm{Spec}(A) = \bigcup_{g\in A} D(g)$ which means $V(1) = V\left(\sum_{g\in A}(g)\right)$, which means $1 \in \sum_{g\in A}(g)$, thus, $1 = \sum_{i=1}^{r} c_i g_i$ for some $c_i \in A$ and $g_i \in A$. But, then it means $\mathrm{Spec}(A) = \bigcup_{i=1}^{r} D(g_i)$. Hence $\mathrm{Spec}(A)$ is quasi-compact.

For $A = k[x_1, x_2, \cdots]$, $\mathrm{Spec}(A)$ is not noetherian. □

(c) **If A is a noetherian ring, show that $sp(\mathrm{Spec}(A))$ is a noetherian topological space.**

Proof. Let $V(a_1) \supset V(a_2) \supset \cdots$ be a descending chain of closed subsets of $\mathrm{Spec}(A)$. Then, $\sqrt{a_1} \subset \sqrt{a_2} \subset \cdots$. Since A is noetherian, for all sufficiently large N, $\sqrt{a_N} = \sqrt{a_{N+1}} = \cdots$. Hence, by applying $V(\ )$ again and noting that $V(a_i) = V(\sqrt{a_i})$, $V(a_N) = V(a_{N+1}) = \cdots$. Hence $\mathrm{Spec}(A)$ is noetherian. □

(d) **Give an example to show that $sp(\mathrm{Spec}(A))$ can be noetherian even when A is not.**

Proof. ? □

2.14.

2.15.

2.16.

2.17.

2.18.

2.19.

Robin Hartshorne's Algebraic Geometry Solutions
by Jinhyun Park

Chapter II Section 7, Projective Morphisms

7.1.

7.9. Let $r+1$ be the rank of $\mathcal{E}$.

(a). There are several ways to prove it.

Proof 1 We assume the following result from Chow group theory: (See Appendix A section2 A11 and section 3. The group $A(X)$ is here $CH(X)$.)

$$CH^*(\mathbb{P}(\mathcal{E})) \simeq \left(\mathbb{Z}[\xi] / \sum_{i=0}^{r} (-1)^i c_i(\mathcal{E}) \xi^{r-i} \right) \otimes_{\mathbb{Z}} CH^*(X)$$

as graded rings. If we look at the grade 1 part, as $\mathbb{Z}$-modules,

$$CH^1(\mathbb{P}(\mathcal{E})) \simeq (\mathbb{Z} \otimes_{\mathbb{Z}} CH^0(X)) \oplus (\mathbb{Z} \otimes_{\mathbb{Z}} CH^1(X))$$

and $CH^1(-) = \mathrm{Pic}(-)$ so that $\mathrm{Pic}(\mathbb{P}(\mathcal{E})) \simeq \mathbb{Z} \oplus \mathrm{Pic}(X)$ as desired.

Proof 2 We can use the Grothendieck groups, i.e. K-theory to do so. Note that

$$K(\mathbb{P}(\mathcal{E})) \simeq \left(\mathbb{Z}[\xi] / \sum_{i=0}^{r} (-1)^i c_i(\mathcal{E}) \xi^{r-i} \right) \otimes_{\mathbb{Z}} K(X)$$

as rings. For the detail, see Yuri Manin *Lectures on the K-functor in Algebraic Geometry*, Russian Mathematical Surveys, 24 (1969) 1-90, in particular, p. 44, from Prop (10.2) to Cor. (10.5).

Proof 3 Here we give a direct proof. In fact, it adapts a way from Proof 1. It can also use the method from Proof 2. Totally your choice.

Define a map $\phi : \mathbb{Z} \oplus \mathrm{Pic}(X) \to \mathrm{Pic}(\mathbb{P}(\mathcal{E}))$ by $(n, \mathcal{L}) \mapsto (\pi^*\mathcal{L})(n) := (\pi^*\mathcal{L}) \otimes \mathcal{O}_{\mathbb{P}(\mathcal{E})}(n)$.

Claim. *This map is injective.*

Assume that $\phi(n, \mathcal{L}) = \mathcal{O}_{\mathbb{P}(\mathcal{E})}$, i.e. $\pi^*\mathcal{L} \otimes \mathcal{O}_{\mathbb{P}(\mathcal{E})}(n) \simeq \mathcal{O}_{\mathbb{P}(\mathcal{E})}$. Apply π_* to it. From II (7.11), recall thet

$$\pi_*(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(n)) = \begin{cases} 0 & n < 0 \\ \mathcal{O}_X & n = 0 \\ \mathrm{Sym}^n(\mathcal{E}) & n > 0 \end{cases}.$$

So, by applying the projection formula (Ex. II (5.1)-(d)), we obtain, $\mathcal{L} \otimes \pi_* \mathcal{O}_{\mathbb{P}(\mathcal{E})}(n) \simeq \mathcal{O}_X$, i.e.

$$\pi_* \mathcal{O}_{\mathbb{P}(\mathcal{E})} \simeq \mathcal{L}^{-1}.$$

Note that it is a line bundle and $\mathrm{rk}(\mathrm{Sym}^n(\mathcal{E})) \geq r+1 \geq 2$ if $n > 0$ by the given assumption, so that the only possible choice for n is $n = 0$. Then, it implies that $\mathcal{L} \simeq \mathcal{O}_X$. Hence ϕ is injective.

Claim. *This map is surjective.*

In case $\mathcal{E}$ is a trivial bundle, then $\mathbb{P}(\mathcal{E}) \simeq X \times \mathbb{P}^r$ so that we already know the result.

In general, choose an open subset $U \subset X$ over which $\mathcal{E}$ is trivial and let $Z = X - U$. Then, we have a closed immersion $\mathbb{P}(\mathcal{E}|_Z) \hookrightarrow \mathbb{P}(\mathcal{E})$ and an open immersion $\mathbb{P}(\mathcal{E}_U) \hookrightarrow \mathbb{P}(\mathcal{E}|_U) \simeq U \times \mathbb{P}^r$. Let $m = \dim X$. Then we have

$$\begin{tikzcd} CH_{m+r-1}(\mathbb{P}(\mathcal{E}|_Z)) \arrow[r] & \mathrm{Pic}(\mathbb{P}(\mathcal{E})) \arrow[r] & \mathrm{Pic}(\mathbb{P}(\mathcal{E}|_U)) \arrow[r] & 0 \; . \\ \mathbb{Z} \oplus CH_{m-1}(Z) \arrow[r] \arrow[u, "\phi_Z"] & \mathbb{Z} \oplus \mathrm{Pic}X \arrow[r] \arrow[u, "\phi_X"] & \mathbb{Z} \oplus \mathrm{Pic}U \arrow[r] \arrow[u, "\phi_U"] & 0 \end{tikzcd}$$

By induction on the dimension, ϕ_Z is surjective and we already know that ϕ_U is an isomorphism. Hence, by a simple diagram chasing, we have the surjectivity of ϕ_X.

(b). Let $\pi : \mathbb{P}(\mathcal{E}) \to X$, $\pi' : \mathbb{P}(\mathcal{E}') \to X$ be the structure morphisms and let $\phi : \mathbb{P}(\mathcal{E}) \to \mathbb{P}(\mathcal{E})$ be the given isomorphism over X:

$$\begin{tikzcd} & \mathbb{P}(\mathcal{E}) \; . \arrow[d, "\pi"] \\ \mathbb{P}(\mathcal{E}') \arrow[ur, "\phi"] \arrow[r, "\pi'"] & X \end{tikzcd}$$

$\phi^*\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ is an invertible sheaf on $\mathbb{P}(\mathcal{E})$ so that by part (a), we have

$$(1) : \phi^*\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1) \simeq {\pi'}^*\mathcal{L}' \otimes \mathcal{O}_{\mathbb{P}(\mathcal{E}')}(n')$$

for some $\mathcal{L}' \in \mathrm{Pic}X$ and $n' \in \mathbb{Z}$. Similarly, ϕ^{-1} being a morphism, we have

$$(2) : {\phi^{-1}}^*\mathcal{O}_{\mathbb{P}(\mathcal{E}')}(1) \simeq \pi^*\mathcal{L} \otimes \mathcal{O}_{\mathbb{P}(\mathcal{E})}(n)$$

for some $\mathcal{L} \in \mathrm{Pic}X$ and $n \in \mathbb{Z}$. By applying ϕ^* to (2), we have

$$\begin{aligned} \mathcal{O}_{\mathbb{P}(\mathcal{E}')}(1) &\simeq \phi^*{\phi^{-1}}^*\mathcal{O}_{\mathbb{P}(\mathcal{E}')}(1) \simeq \phi^*\pi^*\mathcal{L} \otimes \phi^*\mathcal{O}_{\mathbb{P}(\mathcal{E})}(n) \simeq {\pi'}^*\mathcal{L} \otimes \left(\phi^*\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)\right)^{\otimes n} \\ &\simeq {\pi'}^*\mathcal{L} \otimes \left({\pi'}^*\mathcal{L}' \otimes \mathcal{O}_{\mathbb{P}(\mathcal{E})}(n')\right)^{\otimes n} \simeq {\pi'}^*\left(\mathcal{L} \otimes {\mathcal{L}'}^{\otimes n}\right) \otimes \mathcal{O}_{\mathbb{P}(\mathcal{E}')}(nn'). \end{aligned}$$

Recall that

$$\pi'_*\left(\mathcal{P}_{\mathbb{P}(\mathcal{E}')}(n)\right) = \begin{cases} 0 & m < 0 \\ \mathcal{O}_X & m = 0 \\ \mathrm{Sym}^m\mathcal{E}' & m > 0 \end{cases}$$

so that if we apply π'_* to the above, then by the projection formula, we will have

$$\mathcal{O}_X \simeq \mathcal{L} \otimes {\mathcal{L}'}^{\otimes n} \otimes \pi'_*\left(\mathcal{O}_{\mathbb{P}(\mathcal{E}')}(nn')\right).$$

Since $\mathcal{O}_X$, $\mathcal{L} \otimes {\mathcal{L}'}^{\otimes n}$ are invertible sheaves, it makes sense only when $nn' = 1$. Hence we have either $(n, n') = (-1, -1)$ or $(n, n') = (1, 1)$.

If $(n, n') = (-1, -1)$, then, we have $\mathcal{L} \simeq \mathcal{L}'$ and (2) becomes ${\phi^{-1}}^*\mathcal{O}_{\mathbb{P}(\mathcal{E}')}(1) \simeq \pi^*\mathcal{L} \otimes \mathcal{O}_{\mathbb{P}(\mathcal{E})}(-1)$. ϕ being an isomorphism, ${\phi^{-1}}^* = \phi_*$, so that $\pi'_* = \pi_*\phi_* = \pi_*{\phi^{-1}}^*$ and the projection formula gives $\mathcal{E}' \simeq \mathcal{L} \otimes 0 \simeq 0$ which is not possible. Hence $(n, n') = (1, 1)$.

Hence, we have (2): ${\phi^{-1}}^*\mathcal{O}_{\mathbb{P}(\mathcal{E}')}(1) \simeq \pi^*\mathcal{L} \otimes \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$, and as above, noting that ${\phi^{-1}}^* = \phi_*$, applying π_* and using the projection formula, we will have $\mathcal{E}' \simeq \mathcal{L} \otimes \mathcal{E}$ as desired.

Robin Hartshorne's Algebraic Geometry Solutions

by Jinhyun Park

Chapter II Section 8, Differentials

8.1.

(a).

(b).

(c).

(d).

8.2.

8.3.

(a).

(b).

(c).

8.4.

(a).

(b).

(c).

(d).

(e).

(f).

(g).

8.5.

(a).

(b).

8.6.

(a). Here, we assume that there exists at least one lefting $g : A \to B'$. We prove all the required propositions.

Claim. *I has a natural structure of B-module.*

Let $b \in B$, $x \in I$. Let $b' \in B'$ be a lifting of b under the given surjection $p : B' \to B$. Define $b \cdot x = b'x \in I$. If $b'' \in B'$ is another lifting of b, then $p(b'' - b') = 0$ implies $b'' - b' \in I$. Hence, $b''x - b'x = (b'' - b')x \in I^2 = 0$, i.e. $b \cdot x$ is well defined. It proves the claim.

Since we have a k-algebra homomorphism $f : A \to B$ and $g : A \to B'$ is a lifting, in fact, $b \cdot x = g(b)x$ by above claim for any lifting g.

If $g' : A \to B'$ is another such lifting, then obviously the image of $\theta = g - g'$ lies in I.

Claim. *$\theta : A \to I$ is a k-derivation.*

Obviously, it is additive because g, g' are. For $a \in k$, since $g(1) = g'(1)$, $\theta(a) = g(a) - g'(a) = ag(1) - ag'(1) = 0$. We now need to prove that for $a, b \in A$, $\theta(ab) = a\theta(b) + b\theta(a)$, i.e.

$$g(ab) - g'(ab) = a(g(b) - g'(b)) + b(g(a) - g'(a)).$$

Recall how the action of A was defined on I. Hence,

$$\begin{aligned} RHS &= g(a)(g(b) - g'(b)) + g'(b)(g(a) - g'(a)) = g(ab) - g(a)g'(b) + g'(b)g(a) - g'(ab) \\ &= g(ab) - g'(ab) = LHS \end{aligned}$$

so that θ is a k-derivation, i.e. $\theta \in Der_k(A, I) = \mathrm{Hom}_A(\Omega_{A/k}, I)$. It proves the claim.

Now, conversely, let $\theta \in \mathrm{Hom}_A(\Omega_{A/k}, I) = Der_k(A, I)$.

Claim. *$g' := g + \theta$ is another lifting of f.*

Since θ is additive, so is g'. Now,

$$\begin{aligned} g'(ab) &= g(ab) + \theta(ab) = g(ab) + a\theta(b) + b\theta(a) \\ &= g(a)g(b) + g(a)\theta(b) + g(b)\theta(a) + \theta(a)\theta(b) \\ &= (g(a) + \theta(a))(g(b) + \theta(b)) = g'(a)g'(b) \end{aligned}$$

so that g' is multiplicative.

If $a \in k$, then θ is a k-derivation so that $\theta(a) = 0$. Hence $g'(a) = g(a) = ag(1) = a$. Hence g' is a k-algebra homomorphism. Now, $(p \circ g')(a) = p(g(a) + \theta(a)) = p \circ g(a) + p(\theta(a)) = f(a)$ because $\theta(a) \in I$ and $p(I) = 0$. Hence g' is another lifting of f.

(b). For each i, choose $b_i \in B'$ such that $p(b_i) = f(\bar{x_i})$. Define $h : P = k[x_1, \cdots, x_n] \to B'$ be the k-algebra homomorphism determined by $h(x_i) := b_i$. Obviously, the diagram commutes by construction.

Let $q : P \to A$ be the given surjection. If $j \in J$, then since $q(j) = 0$, we have $f(q(j)) = p(h(j)) = 0$ i.e. $h(j) \in I$. Hence we have $h|_J : J \to I$. But $I^2 = 0$ implies that we have a k-homomorphism $\bar{h} : J/J^2 \to I$.

Claim. *This map is even A-linear.*

First, we note that J/J^2 has a canonical A-action. Let $a \in A$, $[j] \in J/J^2$. Choose any lifting $a' \in P$ of a and define $a \cdot [j] = [a'j]$. If we have another lifting a'' of a, then $a'' - a' \in J$ so that $(a'' - a')j \in J^2$, i.e. $[a'j] = [a''j]$ so, this action is well-defined.

In part (a), we noted that the action of A on I is well-defined. To show that $\bar{h} : J/J^2 \to I$ is A-equivariant, it is enough to show that the action of A is preserved. This is easy: Let $a \in A$ and choose a lifting $a' \in P$. Then by the commutativity of the diagram, $h(a')$ is a lifting of $f(a)$ so that for $[j] \in J/J^2$,

$$\bar{h}(a \cdot [j]) = \bar{h}([a'j]) = h(a'j) = h(a')h(j) = a \cdot h(j) = a \cdot \bar{j}([j]).$$

It proves the required A-linearity.

(c). By the hypothesis, $\mathrm{Spec}A \hookrightarrow \mathbb{A}_k^n$ is a nonsingular subvariety. Hence by (8.17), we have an exact sequence

$$0 \to J/J^2 \to \Omega_{P/k} \otimes A \to \Omega_{A/k} \to 0.$$

A being nonsingular, $\Omega_{A/k}$ is projective (because the sheaf $\Omega_{\mathrm{Spec}A/k}$ is locally free). Hence, above sequence splits and so by applying $\mathrm{Hom}_A(-I)$, we obtain

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Hom}_A(\Omega_{A/k}, I) \arrow[r] & \mathrm{Hom}_A(\Omega_{P/k} \otimes A, I) \arrow[r] \arrow[d, "\simeq"'] & \mathrm{Hom}_A(J/J^2, I) \arrow[r] & 0 . \\ & & \mathrm{Hom}_P(\Omega_{P/k}, I) \arrow[r, "="] & Der_k(P, I) & \end{tikzcd}$$

Let $\theta \in \mathrm{Hom}_P(\Omega_{P/k}, I)$ be an element mapped to $\bar{h} \in \mathrm{Hom}_A(J/J^2, I)$ defined in part (b). Regard θ as a k-derivation of P to $B' \supset I$. Let $h' = h - \theta$.

Claim. *$h' : P \to B'$ is a k-homomorphism such that $h'(J) = 0$.*

Obviously, θ being a k-derivation, $h'(a) = a$ for $a \in k$ and h' is additive. If $a, b \in P$, then

$$h'(ab) = h(ab) - \theta(ab) = h(ab) - b\theta(a) - a\theta(b) + \theta(a)\theta(b)$$

$$= (h(a) - \theta(a))(h(b) - \theta(b)) = h'(a)h'(b).$$

If $j \in J$, $\theta(j) = \bar{h}(j) = h(j)$ so that $h(j) = h(j) - \theta(j) = 0$. Hence h' gives a rise to a k-homomorphism $g : A \to B'$. Since h was a lifting of f from P to B', obviously, g is indeed a required lifting.

8.7. As an application of the infinitesimal lifting property, we consider the following general problem. Let X be a scheme of finite type over k, and let $\mathcal{F}$ be a coherent sheaf on X. We seek to classify schemes X' over k, which have a sheaf of ideals $\mathcal{I}$ such that $\mathcal{I}^2 = 0$ and $(X', \mathcal{O}_{X'}/\mathcal{I}) \simeq (X, \mathcal{O}_X)$, and such that $\mathcal{I}$ with its resulting structure of $\mathcal{O}_X$-module is isomorphic to the given sheaf $\mathcal{F}$. Such a pair $X', \mathcal{F}$ we call an *infinitesimal extension* of the scheme X bye the sheaf $\mathcal{F}$. One such extension, the *trivial* one, is obtained as follows. Take $\mathcal{O}_{X'} = \mathcal{O}_X \oplus \mathcal{F}$ as sheaves of abelian groups, and define multiplication by $(a \oplus f) \cdot (a' \oplus f') = aa' \oplus (af' + a'f)$. Then the topological space X with the sheaf of rings $\mathcal{O}_{X'}$ is an infinitesimal extension of X by $\mathcal{F}$.

The general problem of classifying extensions of X by $\mathcal{F}$ can be quite complicated. So for now, just prove the following special case: if X is affine and nonsingular, then any extension of X by a coherent sheaf $\mathcal{F}$ is isomorphic to the trivial one. See (III, Ex. 4.10) for another case.

Proof. Suppose that we have an infinitesimal extension

$$0 \to I \to A' \xrightarrow{\alpha} A \to 0$$

defined by a ring A' and its square-zero ideal I with $I^2 = 0$. By the infinitesimal lifting property we have a lifting f, that is a k-algebra homomorphism, of the identity map of A:

$$\begin{tikzcd} & 0 \arrow[d] \\ & I \arrow[d] \\ & A' \arrow[d, "\alpha"] \\ A \arrow[ur, "f"] \arrow[r, "\mathrm{id}"] & A \arrow[d] \\ & 0 \end{tikzcd}$$

and it gives a splitting of $A' \simeq A \oplus I$ as k-modules. We show that it is in fact an isomorphism of k-algebras, where $A \oplus I$ is seen as given the structure of the trivial extension as in the statement of the problem.

For each $x, y \in A'$, we have $x - f(\alpha(x)), y - f(\alpha(y)) \in I$. Since $I^2 = 0$ we have

$$(x - f(\alpha(x)))(y - f(\alpha(y))) = 0$$

that gives $xy = -f(\alpha(x))f(\alpha(y)) + xf(\alpha(y)) + f(\alpha(x))y$. Thus,

$$\begin{aligned} xy - f(\alpha(xy)) &= xy - f(\alpha(x))f(\alpha(y)) = -2f(\alpha(x))f(\alpha(y)) + xf(\alpha(y)) + f(\alpha(x))y \\ &= (x - f(\alpha(x)))f(\alpha(y)) + f(\alpha(x))(x - f(\alpha(y))). \end{aligned}$$

This immediately implies that, when we identify $x \in A'$ with the pair $(f(\alpha(x)), x - f(\alpha(x)))$ of $A \oplus I$, the product structure of A' is identical to that of $A \oplus I$, as desired. Thus there is only one extension up to isomorphism. $\square$

8.8.

Robin Hartshorne's Algebraic Geometry Solutions
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Chapter III Cohomology Section 4 Cohomology of Affine spaces

4.1.

4.2.

4.3.

4.4.

4.5.

4.6.

4.7.

4.8.

4.9.

4.10. Let X be a nonsingular variety over an algebraically closed field k, and let $\mathcal{F}$ be a coherent sheaf on X. Show that there is a one-to-one correspondence between the set of infinitesimal extensions of X by $\mathcal{F}$ (II, Ex. 8.7) up to isomorphism, and the group $H^1(X, \mathcal{F} \otimes \mathcal{T})$, where $\mathcal{T}$ is the tangent sheaf of X (II, §8). [*Hint*: Use (II, Ex. 8.6) and (4.5)]

Proof. Let $\mathcal{U} = \{U_i\}_{i\in I}$ be an affine open cover of X. Note that since X is nonsingular, so is each U_i. Let $(X', \mathcal{O}_{X'})$ be an infinitesimal extension of X by $\mathcal{F}$, that is, there is a scheme X' with an ideal sheaf $\mathcal{I} \simeq \mathcal{F}$ as $\mathcal{O}_X$-modules, such that $\mathcal{I}^2 = 0$, $(X, \mathcal{O}_{X'}/\mathcal{I}) \simeq (X, \mathcal{O}_X)$ as ringed spaces, and a short exact sequence of $\mathcal{O}_X$-modules

$$0 \to \mathcal{I} \to \mathcal{O}_{X'} \to \mathcal{O}_X \to 0.$$

Since each U_i is nonsingular and affine, by Ex.II-8.7, the above short exact sequence restricts to a split exact sequence on U_i, where the splitting is given by a lifting $\alpha_i : \mathcal{O}_X|_{U_i} \to \mathcal{O}_{X'}|_{U_i}$.

On each $U_{ij} = U_i \cap U_j$, that is affine since X is separated, we have two liftings $\alpha_i|_{U_{ij}}$, $\alpha_j|_{U_{ij}} : \mathcal{O}_X|_{U_{ij}} \to \mathcal{O}_{X'}|_{U_{ij}}$, and they differs by a section β_{ij} in $\mathrm{Der}_k(\mathcal{O}_X(U_{ij}), \mathcal{I}(U_{ij}))$ so that on U_{ij} we have

$$\alpha_i - \alpha_j = \beta_{ij}.$$

Notice that β_{ij} can be seen as a section in $(\mathcal{F} \otimes \mathcal{T})(U_{ij})$ via isomorphisms

$$\mathrm{Der}_k(\mathcal{O}_X(U_{ij}), \mathcal{I}(U_{ij})) \simeq \mathrm{Hom}_{\mathcal{O}_{X/k}(U_{ij})}(\Omega_{X/k}(U_{ij}), \mathcal{I}(U_{ij})) \simeq (\mathcal{F} \otimes \Omega^*_{X/k})(U_{ij}) \simeq (\mathcal{F} \otimes \mathcal{T})(U_{ij}).$$

Restricting all the above sections onto $U_{ijk} = U_i \cap U_j \cap U_k$, we thus obtain

$$\beta_{ij} + \beta_{jk} + \beta_{ki} = (\alpha_i - \alpha_j) + (\alpha_j - \alpha_k) + (\alpha_k - \alpha_i) = 0,$$

and $\{\beta_{ij}\}$ gives a cocycle of the Čech complex $\check{C}^\bullet(\mathcal{U}, \mathcal{F} \otimes \mathcal{T})$ in degree 1.

For a different choice of liftings $\mu_i : \mathcal{O}_X|_{U_i} \to \mathcal{O}_{X'}|_{U_i}$ for each U_i, as above we have the corresponding sections β'_{ij} of $\mathrm{Der}_k(\mathcal{O}_X(U_{ij}), \mathcal{I}(U_{ij}))$ with $\mu_i - \mu_j = \beta'_{ij}$ on U_{ij}, and with $\beta'_{ij} + \beta'_{jk} + \beta'_{ki} = 0$ on U_{ijk}.

Applying the Ex. II-8.6- (a) again to the pair of liftings α_i and μ_i on U_i, we have sections ξ_i of $\mathrm{Der}_k(\mathcal{O}_X(U_i), \mathcal{I}(U_i))$ for each U_i with $\alpha_i - \mu = \xi_i$, that can also be seen as a section of $\mathcal{F} \otimes \mathcal{T}$ on U_i. Then, on U_{ij} we have

$$\beta_{ij} - \beta'_{ij} = (\alpha_i - \alpha_j) - (\mu_i - \mu_j) = \xi_i - \xi_j,$$

thus the cocycles $\{\beta_{ij}\}$ and $\{\beta'_{ij}\}$ give the same cohomology class in $\check{H}^1(\mathcal{U}, \mathcal{F} \otimes \mathcal{T})$. This last group is isomorphic to $H^1(X, \mathcal{F} \otimes T)$ by (4.5). The converse is easy. This finishes the proof. $\square$

4.11.

Robin Hartshorne's Algebraic Geometry Solutions
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Chapter III Cohomology Section 5 The Cohomology of Projective Space

5.1.

5.2.

5.3.

5.4.

5.5.

5.6.

(a) Identify $Q = \mathbb{P}^1 \times \mathbb{P}^1$ and let $|Y| = \mathbb{P}^1 \times *$, $|Z| = * \times \mathbb{P}^1$. First, observe that Q is birational to $\mathbb{P}^2$ and $h^1(X, \mathcal{O}_X)$ is a birational invariant, so, $h^1(Q, \mathcal{O}_Q) = h^1(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}) = 0$.

Claim (1)**.** *Let $p > 0$. Then, $H^1(Q, \mathcal{O}_Q(p, 0)) = 0$.*

Proof. Let $Y = \mathbb{P}^1 \times \{p\text{-points}\}$. Then, we have a short exact sequence $0 \to \mathcal{O}_Q(-p, 0) \to \mathcal{O}_Q \to \mathcal{O}_Y \to 0$. Tensor it by $\mathcal{O}_Q(p, 0)$ then, we obtain $0 \to \mathcal{O}_Q \to \mathcal{O}_Q(p, 0) \to \mathcal{O}_Y(p, 0) \to 0$. Then, from the cohomology long exact sequence, we obtain

$$0 = H^1(Q, \mathcal{O}_Q) \to H^1(Q, \mathcal{O}_Q(p, 0)) \to \bigoplus_p H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(p|Y|^2)) \to 0,$$

but, $|Y|^2 = 0$, so, by Serre duality, $H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) \simeq H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-2))^* = 0$. Hence, $H^1(Q, \mathcal{O}_Q(p, 0)) = 0$ for $p > 0$. This finishes the proof of Claim 1. $\square$

By symmetry, we also have $H^1(Q, \mathcal{O}_Q(0, q)) = 0$ for $q > 0$.

Claim (2)**.** *For all $p \geq 0, q \geq 0$, $H^1(Q, \mathcal{O}_Q(p, q)) = 0$.*

Proof. If $(p, q) = (0, 0)$ or $p = 0$ or $q = 0$, then, we already know this result, so, assume that $p, q > 0$. Tensor the sequence $0 \to \mathcal{O}_Q(-p, 0) \to \mathcal{O}_Q \to \mathcal{O}_Y \to 0$ with $\mathcal{O}_Q(p, q)$ to obtain a short exact sequence $0 \to \mathcal{O}_Q(0, q) \to \mathcal{O}_Q(p, q) \to \mathcal{O}_Y(p, q) \to 0$. Then, from the cohomology long exact sequence we have

$$H^1(Q, \mathcal{O}_Q(0, q)) \to H^1(Q, \mathcal{O}_Q(p, q)) \to \bigoplus_p H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(p|Y|^2 + q|Y|.|Z|))$$

but $p|Y|^2 + q|Y|.|Z| = q$ and by Serre duality, $H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(q)) \simeq H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-q - 2))^* = 0$ as $-q - 2 < 0$. By Claim 1, we know that $H^1(Q, \mathcal{O}_Q(0, q)) = 0$ so, $H^1(Q, \mathcal{O}_Q(p, q)) = 0$ consequently. This proves the result. $\square$

Claim (3)**.** *For any $p \in \mathbb{Z}$, $H^1(Q, \mathcal{O}_Q(p, -1)) \simeq H^1(Q, \mathcal{O}_Q(0, -1))$.*

Proof. If $p = 0$, it is obvious. First consider the case when $p > 0$. From the sequence $0 \to \mathcal{O}_Q(-p, 0) \to \mathcal{O}_Q \to \mathcal{O}_Y \to 0$, by tensoring with $\mathcal{O}_Q(p, -1)$, we obtain $0 \to \mathcal{O}_Q(0, -1) \to \mathcal{O}_Q(p, -1) \to \mathcal{O}_Y(p, -1) \to 0$. Hence, the long exact sequence gives

$$\bigoplus_p H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(p|Y|^2 + (-1)|Y|.|Z| = -1) \to H^1(Q, \mathcal{O}_Q(0, -1)) \to H^1(Q, \mathcal{O}_Q(p, -1)) \to \bigoplus_p H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-1)).$$

Then, $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-1)) = 0$ and $H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-1)) \simeq H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-1))^* = 0$, so, $H^1(Q, \mathcal{O}_Q(p,-1)) \simeq H^1(Q, \mathcal{O}_Q(0,-1))$ indeed.

Now consider the case when $p < 0$. let $p' = -p > 0$ and let $Y' = \mathbb{P}^1 \times \{p'\text{-points}\}$. Then we have $0 \to \mathcal{O}_Q(-p', 0) \to \mathcal{O}_Q \to \mathcal{O}_{Y'} \to 0$ and by tensoring with $\mathcal{O}_Q(0,-1)$, we obtain $0 \to \mathcal{O}_Q(-p', -1) \to \mathcal{O}_Q(0,-1) \to \mathcal{O}_{Y'}(0,-1) \to 0$. Hence, the long exact sequence gives us

$$\bigoplus_{p'} H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-1)) \to H^1(Q, \mathcal{O}_Q(-p',-1)) \to H^1(Q, \mathcal{O}_Q(0,-1)) \to \bigoplus_{p'} H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-1))$$

and $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-1)) = H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-1)) = 0$. This shows that $H^1(Q, \mathcal{O}_Q(p,-1)) \simeq H^1(Q, \mathcal{P}_Q(0,-1))$ for $p < 0$. $\square$

Claim (4)**.** (i) $H^1(Q, \mathcal{O}_Q(0,q)) \neq 0$ *if* $q \leq -2$.
(ii) $H^1(Q, \mathcal{O}_Q(0,-1)) = 0$.

Proof. Let $p > 0$. From $0 \to \mathcal{O}_Q(-p,0) \to \mathcal{O}_Q \to \mathcal{O}_Y \to 0$, by tensoring with $\mathcal{O}_Q(0,q)$, we obtain $0 \to \mathcal{O}_Q(-p,q) \to \mathcal{O}_Q(0,q) \to \mathcal{O}_Y(0,q) \to 0$ so that the long exact sequence gives

$$\bigoplus_{p} H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(q)) \to H^1(Q, \mathcal{O}_Q(-p,q)) \to H^1(Q, \mathcal{O}_Q(0,q)) \to \bigoplus_{p} H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(q)) \to 0.$$

When $q \leq -2$, $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-1)) = 0$ and $h^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(q)) = h^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-q-2)) > 0$ so that $H^1(0,q) \neq 0$. This proves (i) and by symmetry we also have $H^1(p,0) \neq 0$ if $p \leq -2$. This proves (3).

When $p = 1, q = 0$, we have

$$k = H^0(Q, \mathcal{O}_Q) \xrightarrow{\simeq} H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) = k \to H^1(Q, \mathcal{O}_Q(-1,0)) \to H^1(Q, \mathcal{O}_Q) = 0$$

so that $H^1(Q, \mathcal{O}_Q(-1,0)) = 0$. This proves (ii) and similarly we have $H^1(Q, \mathcal{O}_Q(0,-1)) = 0$. $\square$

Now, we prove (2). From V. 1.4.4, the canonical line bundle $K \simeq \mathcal{O}_Q(-2,-2)$, so, when $a, b < 0$, by Serre duality,

$$H^1(Q, \mathcal{O}_Q(a,b)) \simeq H^1(Q, \mathcal{O}_Q(-a-2, -b-2))^*.$$

If $a, b \leq -2$, then, by Claim 2, this group vanishes.

In case $(a,b) = (0,-1), (-1,0), (-2,-1), (-1,-2), (-1,-1)$, the previous claims already show it. Hence, it is 0 for any $a, b < 0$. This proves (2). (1) is trivial once we have (2) and the previous claims.

(b) (1) For Y, $0 \to \mathcal{O}_Q(-Y) \to \mathcal{O}_Q \to \mathcal{O}_Y \to 0$ is exact and $\mathcal{O}_Q(-Y) \simeq \mathcal{O}_Q(-a,-b)$. Thus,

$$0 \to H^0(Q, \mathcal{P}_Q(-a,-b)) \to H^0(Q, \mathcal{O}_Q) \to H^0(Y, \mathcal{O}_Y) \to H^1(Q, \mathcal{O}_Q(-a,-b)) \to 0$$

so, $H^0(Y, \mathcal{O}_Y) \simeq H^0(Q, \mathcal{O}_Q) \simeq k$. Hence Y has only 1 connected component, i.e. connected.

(2) Let $\mathcal{L}$ be a line bundle on Q of type (a,b) with $a > 0, b > 0$. Then, by II. 7.6.2, $\mathcal{L}$ is very ample so that it gives an embedding of Q into a projective space $\mathbb{P}^N$. Then, by Bertini's theorem (II, 8.18), there is a hyperplane $H \subset \mathbb{P}^N$ whose intersection with Q is a nonsingular projective curve Y and this $\mathcal{O}_Q(Y)$ is isomorphic to $\mathcal{L}$, i.e. Y is of type (a,b).

(3) By Ex. II 5.14-(d), $X \subset \mathbb{P}^r_A$ is projectively normal if and only if it is normal and for all $n \geq 0$, the natural map $\Gamma(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(n)) \to \Gamma(X, \mathcal{O}_X(n))$ is surjective. We will use this.

Since we have a sequence of closed embeddings $Y \hookrightarrow Q \hookrightarrow \mathbb{P}^3$, it gives a commutative diagram

$$\begin{tikzcd} \Gamma(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(n)) \arrow[rr] \arrow[dr] & & \Gamma(Y, \mathcal{O}_Y(n)) \\ & \Gamma(Q, \mathcal{O}_Q(n)) \arrow[ur] & \end{tikzcd}$$

so, if, $\Gamma(Q, \mathcal{O}_Q(n)) \to \Gamma(Y, \mathcal{O}_Y(n))$ is not surjective, then, $\Gamma(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(n)) \to \Gamma(Y, \mathcal{O}_Y(n))$ cannot be surjective.

On the other hand, since $Q = V(xy - zw) \subset \mathbb{P}^3$, the ideal sheaf of Q $\mathcal{I}_Q \simeq \mathcal{O}_{\mathbb{P}^3}(-2)$ so that the sequence $0 \to \mathcal{O}_{\mathbb{P}^3}(-2) \to \mathcal{O}_{\mathbb{P}^3} \to \mathcal{O}_Q \to 0$ is exact. Hence, by tensoring with $\mathcal{O}_{\mathbb{P}^3}(n)$, we have $0 \to \mathcal{O}_{\mathbb{P}^3}(n-2) \to \mathcal{O}_{\mathbb{P}^3}(n) \to \mathcal{P}_Q(n) \to 0$ whose cohomology long exact sequence gives

$$H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(n)) \to H^0(Q, \mathcal{O}_Q(n)) \to H^1(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(n-2)) = 0.$$

Consequently, the map $\Gamma(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(n)) \to \Gamma(Q, \mathcal{O}_Q(n))$ is always surjective and it implies that $\Gamma(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(n)) \to \Gamma(Y, \mathcal{O}_Y(n))$ is surjective if and only if $\Gamma(Q, \mathcal{O}_Q(n)) \to \Gamma(Y, \mathcal{O}_Y(n)))$ is surjective if and only if $Y \subset \mathbb{P}^3$ is projectively normal, because being nonsingular, Y is already normal.

Hence, it remains to show that $\Gamma(Q, \mathcal{O}_Q(n)) \to \Gamma(Y, \mathcal{O}_Y(n))$ is surjective if and only if $|a - b| \leq 1$.

($\Leftarrow$) Suppose that $|a - b| \leq 1$. Then, from $0 \to \mathcal{O}_Q(-a, -b) \to \mathcal{O}_Q \to \mathcal{O}_Y \to 0$, we obtain $0 \to \mathcal{O}_Q(n-a, n-b) \to \mathcal{O}_Q(n, n) \to \mathcal{O}_Y(n) \to 0$ which gives us

$$\Gamma(Q, \mathcal{O}_Q(n)) \to \Gamma(Y, \mathcal{O}_Y(n)) \to H^1(Q, \mathcal{O}_Q(n-a, n-b).$$

But, $|a-b| \leq 1$ means $|(n-a)-(n-b)| \leq 1$ so, by part (a) - (1), $H^1(Q, \mathcal{O}_Q(n-a, n-b))$ vanishes and the natural map is surjective.

($\Rightarrow$) Conversely, suppose that the natural map is surjective for all $n \geq 0$. Then, the same sequence gives

$$\Gamma(Q, \mathcal{O}_Q(n)) \to \Gamma(Y, \mathcal{O}_Y(n)) \to H^1(Q, \mathcal{O}_Q(n-a, n-b)) \to H^1(Q, \mathcal{O}_Q(n, n))$$

where the last one is 0 by Claim 2 of (a) and the first map is surjective. Hence, we must have $H^1(Q, \mathcal{O}_Q(n-a, n-b)) = 0$ for all $n \geq 0$.

Toward contradiction, so, suppose that $|a - b| \geq 2$, i.e. $a \geq b + 2$ or $b \geq a + 2$. For the first case, when $n = b$, $n - a \leq -2$ so that by (a)- (3), we have $H^1(Q, \mathcal{O}_Q(n-a, n-b)) \neq 0$, which is a contradiction. For the second case, we will have the same contradiction. Hence $|a - b| \leq 1$.

Hence, a nonsingular $Y \subset Q$ of type (a, b) with $a, b > 0$ is projectively normal in $\mathbb{P}^3$ if and only if $|a - b| \leq 1$.

(c) First, we reduce this problem to a nonsingular Y. By part (b)-(2), Y is linearly (hence rationally) equivalent to a nonsingular projective curve lying on Q and this new curve has the same bidegree. Also, since this is a rational equivalence, they belong to the same flat family, so, the arithmetic genera are unchanged (which are defined to be $h^1(Y, \mathcal{O}_Y)$). Hence, we may replace Y by its linearly equivalent nonsingular Y. Then, for this Y, the arithmetic genus $p_a(Y) = p_g(Y)$, the geometric genus, and we can compute it in terms of a, b as follows: $\mathcal{O}_Q(Y) = \mathcal{O}_Q(a, b)$ and the first Chern class $c_1(N_{Q/Y}) = \deg_Y(N_{Q/Y}) = Y.Y = (ah+bk)^2 = ab(h.k)+ba(k.h) = 2ab$ where h, k are generators of Pic$Q \simeq \mathbb{Z}\oplus\mathbb{Z}$ with intersection product $h^2 = k^2 = 0$, $h.k = k.h = 1$.

On the other hand, $T_{\mathbb{P}^1\times\mathbb{P}^1} = \left(\Omega^1_{\mathbb{P}^1\times\mathbb{P}^1}\right)^*$ implies that $c_1\left(T_{\mathbb{P}^1\times\mathbb{P}^1}\right) = c_1\left(\wedge^2 T_{\mathbb{P}^1\times\mathbb{P}^1}\right) = c_1\left(K^*_{\mathbb{P}^1\times\mathbb{P}^1}\right) = -(c_1(K_{\mathbb{P}^1}), c_1(K_{\mathbb{P}^1})) = -(2\cdot 0-2, 2\cdot 0-2) = (2,2) = 2h+2k$ and so, $c_1(T_Q|_Y) = \deg_Y\left(T_Q\otimes_{\mathcal{O}_Q}\mathcal{O}_C\right) = \deg_Y\left(\wedge^2 T_Q\otimes_{\mathcal{O}_Q}\mathcal{O}_C\right) = [K^*_Q]\cdot(ah+bk) = (2h+2k)\cdot(ah+bk) = 2(a+b)$.

Of course, $c_1(T_Y) = -\deg_Y(K_Y) = -(2g-2) = 2-2g$. Hence the short exact sequence

$$0 \to T_Y \to T_Q|_Y \to N_{Q/Y} \to 0$$

gives $c_1(T_Y)+c_1(N_{Q/Y}) = c_1(T_Q|Y)$ and it is equivalent to $2(a+b) = 2-2g+2ab$, i.e. $g = ab-a-b+1 = (a-1)(b-1)$. This proves the result.

5.7.

5.8.

5.9.

5.10.

Robin Hartshorne's Algebraic Geometry Solutions

by Jinhyun Park

CHAPTER III SECTION 9 FLAT MORPHISMS

9.1.

9.2.

9.3.

9.4.

9.5.

9.6.

9.7.

***9.8. Let A be a finitely generated k-algebra. Write A as a quotient of a polynomial ring P over k, and let J be the kernel:**

$$0 \to J \to P \to A \to 0.$$

Consider the exact sequence of (II, 8.4A)

$$J/J^2 \to \Omega_{P/k} \otimes_P A \to \Omega_{A/k} \to 0.$$

Apply the functor $\operatorname{Hom}_A(\cdot, A)$, and let $T^1(A)$ be the cokernel:

$$\operatorname{Hom}_A(\Omega_{P/k} \otimes A, A) \to \operatorname{Hom}_A(J/J^2, A) \to T^1(A) \to 0.$$

Now use the construction of (II, Ex. 8.6) to show that $T^1(A)$ classifies infinitesimal deformations of A, i.e., algebras A' flat over $D = k[t]/t^2$, with $A' \otimes_D k \simeq A$. It follows that $T^1(A)$ is independent of the given representation of A as a quotient of a polynomial ring P. (For more details, see Lichtenbaum and Schlessinger [1].)

Proof. Suppose that $P = k[x_1, \cdots, x_n]$ is a polynomial k-algebra of which A is a quotient with the kernel J. Let $P_2 := k[x_1, \cdots, x_n, y_1, \cdots, y_n]$.

For each infinitesimal deformation A' of A, we can define a k-algebra homomorphism $f : P_2 \to A'$ so that we obtain the following commutative diagram with exact rows and columns:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & J \arrow[r, "t"] \arrow[d] & K \arrow[r] \arrow[d] & J \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & P \arrow[r, "t"] \arrow[d] & P_2 \arrow[r] \arrow[d, "f"] & P \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A \arrow[r, "t"] \arrow[d] & A' \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

where K is an ideal of $P-2$.

Notice that to give a k-algebra A' with the required properties is equivalent to give an ideal K, and the ambiguity is given by the choice of the k-algebra homomorphism f. Thus, the set of equivalence classes of infinitesimal deformations A' of A is equal to

$$\frac{\{\text{ choices of an ideal } K\}}{\{\text{ choices of } f\}}.$$

We will identify the numerator and the denominator.

Claim. $\{$ *choices of an ideal* $K\} \simeq \operatorname{Hom}_P(J, A)$ *as sets.*

Notice that the middle row splits via the natural inclusion $P \to P_2$ of the right hand side P. So that as modules, $P_2 = P \oplus tP$.

Suppose an ideal K was chosen. For each $x \in J$, lift it to $\tilde{x} \in K$. Since $P_2 = P \oplus tP \supset K$, $\tilde{x} = x + t(y)$ for some $y \in P$. Two liftings of x differ by an image of tz for some $z \in I$, thus, $y \in P$ is not uniquely determined by x, but $\bar{y} \in A$ is uniquely determined. Thus, it defines a map in $\operatorname{Hom}_P(J, A)$ that sends $x \mapsto \bar{y}$.

Conversely, suppose that $\phi \in \operatorname{Hom}_P(J, A)$. Define an ideal K of P_2 by

$$K = \{x + ty | x \in J, y \in P \text{ such that } \bar{y} = \phi(x) \text{ in } A\}.$$

It is easy to see that K is an ideal of P_2, and the image of K in P is J so that

$$0 \to J \to K \to J \to 0$$

is exact. It defines $A' := P_2/K$, and here f is the canonical quotient map. Thus, it shows the claim.

Claim. $\{$ *choices of* $f\} \simeq \operatorname{Der}_k(P, A)$ *as sets.*

A choice of $f : P_2 \to A'$ gives after composing with $t : P \to P_2$, a lifting of $P \to A$ to $P \to A'$. Thus, Ex. II-8.6-(a) shows the assertion. This proves the claim.

Hence, the obvious identities

$$\operatorname{Hom}_P(J, A) \simeq \operatorname{Hom}_A(J/J^2, A), \text{ and}$$

$$\operatorname{Der}_k(P, A) \simeq \operatorname{Hom}_P(\Omega_{P/k}, A)$$

show that the set of isomorphism classes of infinitesimal deformations are in one-to-one correspondence with the coker $\big(\operatorname{Hom}_P(\Omega_{P/k}, A) \to \operatorname{Hom}_A(J/J^2, A)\big)$, which is by definition $T^1(A)$. This finishes the proof. $\square$

Remark. In fact, via a natural map

$$T_1(A) \supset \operatorname{Ext}^1_A(\Omega_{A/k}, A),$$

where the natural map will be apparent from the following discussion.

For the exact sequence

$$J/J^2 \to \Omega_{P/k} \otimes_P A \to \Omega_{A/k} \to 0,$$

let L be the kernel of the second map so that we have a natural projection $J/J^2 \to L$ and a commutative diagram

$$\begin{tikzcd}
 & J/J^2 \arrow[d] \arrow[dr] & & & \\
0 \arrow[r] & L \arrow[r] & \Omega_{P/k} \otimes_P A \arrow[r] & \Omega_{A/k} \arrow[r] & 0.
\end{tikzcd}$$

Then, it induces a commutative diagram with exact rows

$$\begin{tikzcd}
\mathrm{Hom}_P(\Omega_{P/k}, A) \arrow[r] & \mathrm{Hom}_P(J/J^2, A) \arrow[r] & T^1(A) \arrow[r] & 0 \\
\mathrm{Hom}_P(\Omega_{P/k}, A) \arrow[r] \arrow[u, "="] & \mathrm{Hom}_P(L, A) \arrow[r] \arrow[u] & \mathrm{Ext}^1_A(\Omega_{A/k}, A) \arrow[r] & \mathrm{Ext}^1_P(\Omega_{P/k}, A)
\end{tikzcd}$$

First of all, since P is smooth over k, $\mathrm{Ext}^1_P(\Omega_{P/k}, A) \simeq 0$, $\Omega_{P/k}$ being projective. Hence, by diagram chasing we can define a map

$$\mathrm{Ext}^1_A(\Omega_{A/k}, A) \to T^1(A)$$

and furthermore, the diagram implies that it must be injective.

It is known that this map becomes an isomorphism when

(1) k is a perfect field, and

(2) A is a reduced k-algebra of finite type,

according to Lichtenbaum and Schlessinger.

9.9.

9.10.

9.11.

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by Jinhyun Park

Chapter III Section 10 Smooth morphisms

10.1. Over a nonperfect field, smooth and regular are not equivalent. For example, let k_0 be a field of characteristic $p > 0$, let $k = k_0(t)$, and let $X \subset \mathbb{A}_k^2$ be the curve defined by $y^2 = x^p - t$. Show that every local ring of X is a regular local ring, but X is not smooth over k.

Proof. We need to suppose that char(k$_0$) = p > 2. Let $f = y^2 - x^p + t \in k[x, y]$. Then $\frac{\partial f}{\partial x} = 0$, $\frac{\partial f}{\partial y} = 2y$ so that $\mathrm{rk}\begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{pmatrix} = \mathrm{rk}\begin{pmatrix} 0 \\ 2y \end{pmatrix} = 1$ everywhere on X because on X $y \neq 0$. Indeed, if $y = 0$, then $x^p = t$ over $k = k_0(t)$, which is not possible. Hence X is regular everywhere and every local ring of X is a regular local ring.

Let's now prove that $X \to \mathrm{Spec}(k)$ is not smooth. Toward contradiction, suppose that it is smooth. Then by base change via $\mathrm{Spec}\left(\overline{k}\right) \to \mathrm{Spec}\left(k\right)$, the morphism $X_{\overline{k}} \to \mathrm{Spec}\left(\overline{k}\right)$ is smooth. But, this is not true: $X_{\overline{k}} \subset \mathbb{A}_{\overline{k}}^2$ is defined by the equation $y^2 = x^p - t = \left(x - t^{\frac{1}{p}}\right)^p$ over $\overline{k}$ and the point $(x, y) = \left(t^{\frac{1}{p}}, 0\right)$ on $X_{\overline{k}}$ has multiplicity 2 so that it is not regular at this point. Contradiction. Hence $X \to \mathrm{Spec}(k)$ is not smooth. □

10.2. Let $f : X \to Y$ be a proper, flat morphism of varieties over k. Suppose for some point $y \in Y$ that the fibre X_y is smooth over $k(y)$. Then show that there is an open neighborhood U of y in Y such that $f : f^{-1}(U) \to U$ is smooth.

Proof. Let n be the relative dimension of the flat morphism $f : X \to Y$. Since $X_y \to \mathrm{Spec}\left(k(y)\right)$ is smooth, $\Omega_{X_y/k(y)}$ is a locally free coherent sheaf of rank n on X_y. That is, for each $x \in X_y$, $\dim_{k(x)}\left(\Omega_{X/Y} \otimes k(x)\right) = \dim_{k(x)}\left(\Omega_{X_y/k(y)} \otimes k(x)\right) = n$. But, $\Omega_{X/Y} \otimes k(x) = \left(\Omega_{X/Y}\right)_x / \mathfrak{m}_x \left(\Omega_{X/Y}\right)_x$ so that by Nakayama's lemma, there exist sections $s_1, \cdots, s_n$ of $\Omega_{X/Y}$ over a neighborhood U_x of x whose images in $\Omega_{X/Y} \otimes k(x) = \left(\Omega_{X/Y}\right)_x / \mathfrak{m}_x \left(\Omega_{X/Y}\right)_x$ form a $k(x)$-basis and they generate $\Omega_{X/Y}$ over U_x. This implies that for all $z \in U_x$, $\dim_{k(z)}\left(\Omega_{X/Y} \otimes k(z)\right) \leq n$. But, by Theorem II-8.6A, $\dim_{k(z)}\left(\Omega_{X/Y} \otimes k(z)\right) \geq n$ so that $\dim_{k(z)}\left(\Omega_{X/Y} \otimes k(z)\right) = n$ for all $z \in U_x$, i.e. $\Omega_{X/Y}|_{U_x}$ is locally free of rank n. Since $x \in X_y$ was arbitrary, by collecting all such U_x, we see that $\{U_x\}_{x \in X_y}$ is a cover of X_y such that $\Omega_{X/Y}$ is locally free on $\bigcup_{x \in X_y} U_x$.

The remaining point is to find a kind of tubular neighborhood of X_y. Since $f : X \to Y$ is proper, by base change $\mathrm{Spec}\left(k(y)\right) \to Y$, $X_y \to \mathrm{Spec}\left(k(y)\right)$ is also proper. Thus, in particular, X_y is quasi-compact and there are finitely many points $x_1, \cdots, x_m \in X_y$ such that $U_{x_1}, \cdots, U_{x_m}$ cover X_y. Since f is flat, it is an open map so that $f(U_{x_i} \subset Y$ is open containing y. Let $U = \bigcap_{i=1}^m f(U_{x_i})$. This is then an open subset of Y containing y. Obviously, $f^{-1}(U) \subset \bigcup_{x \in X_y} U_x$ and thus $\Omega_{X/Y}|_{f^{-1}(U)}$ is locally free on $f^{-1}(U)$. Because flatness is stable under base change, that $\Omega_{X/Y}|_{f^{-1}(U)}$ is locally free of rank n on $f^{-1}(U)$ is equivalent to that $f^{-1}(U) \to U$ is smooth. This finishes the proof. □

10.3. A morphism $f : X \to Y$ of schemes of finite type over k is *étale* if it is smooth of relative dimension 0. It is *unramified* if for every $x \in X$, letting $y = f(x)$, we have $\mathfrak{m}_y \cdot \mathcal{O}_x = \mathfrak{m}_x$, and $k(x)$ is a separable algebraic extension of $k(y)$. Show that the following conditions are equivalent:

(i) f is étale;

(ii) f **is flat, and** $\Omega_{X/Y} = 0$**;**
(iii) f **is flat and unramified.**

Proof. (i) $\Leftrightarrow$ (ii) is obvious by definition. (ii) $\Leftrightarrow$(iii) is a direct consequence of Theorem II-8.6A. $\square$

10.4. Show that a morphism $f : X \to Y$ **of schemes of finite type over** k **is étale if and only if the following condition is satisfied: for each** $x \in X$**, let** $y = f(x)$**. Let** $\widehat{\mathcal{O}}_x$ **and** $\widehat{\mathcal{O}}_y$ **be the completions of the local rings at** x **and** y**. Choose fields of representatives (II, 8.25A)** $k(x) \subset \widehat{\mathcal{O}}_x$ **and** $k(y) \subset \widehat{\mathcal{O}}_y$ **so that** $k(y) \subset k(x)$ **via the natural map** $\widehat{\mathcal{O}}_y \to \widehat{\mathcal{O}}_x$**. Then our condition is that for every** $x \in X$**,** $k(x)$ **is a separable algebraic extension of** $k(y)$**, and the natural map**

$$\widehat{\mathcal{O}}_y \otimes_{k(y)} k(x) \to \widehat{\mathcal{O}}_x$$

is an isomorphism.

Proof. By definition, $f : X \to Y$ is unramified if and only if for all $x \in X$ with $y = f(x)$, $k(x)$ is separable over $k(y)$ and $\mathfrak{m}_y \cdot \mathcal{O}_x = \mathfrak{m}_x$. Also, Ex III-10.3 shows that f is étale if and only if f is flat and unramified. Thus, it is enough to show that f is flat if and only if for all $x \in X$ with $y = f(x)$, $\widehat{\mathcal{O}}_y \otimes_{k(y)} k(x) \xrightarrow{\simeq} \widehat{\mathcal{O}}_x$.

Since flatness is a local condition, it follows from the following three statements. (All rings are supposed to be noetherian.)

Claim (1)**.** *A is a ring and $I \subset A$ is an ideal. Then, the I-adic completion $A \to \widehat{A}$ is faithfully flat if and only if $I \subset \sqrt{A}$. (It is always flat.)*

Proof. See SGA 1, IV-Cor 3.2 $\square$

Claim (2)**.** *Let $(A, \mathfrak{m})$, $(B, \mathfrak{n})$ be local rings with a local homomorphism $A \to B$. Then,*

$$grB \simeq grA \otimes_A B \Leftrightarrow \widehat{B} \simeq \widehat{A} \otimes_A B.$$

Proof. Easy. $\square$

Claim (3)**.** *Let $(A, \mathfrak{m}), (B, \mathfrak{n})$, $A \to B$ be as above. Then, $A \to B$ is flat if and only if $\widehat{B} \simeq \widehat{A} \otimes_A B$.*

Proof. ($\Rightarrow$) Since $0 \to \mathfrak{m}^{r+1} \to \mathfrak{m}^r \to \mathfrak{m}^r/\mathfrak{m}^{r+1} \to 0$ is exact and $A \to B$ is flat, $0 \to \mathfrak{m}^{r+1} \otimes_A B \to \mathfrak{m}^r \otimes_A B \to \mathfrak{m}^r/\mathfrak{m}^{r+1} \otimes_A B \to 0$ is exact. Since $\mathfrak{m}B = \mathfrak{n}$ and $\mathfrak{m}^r/\mathfrak{m}^{r+1} \otimes_A B = \mathfrak{m}^r B/\mathfrak{m}^{r+1}B = (\mathfrak{m}B)^r/(\mathfrak{m}B)^{r+1}$, we immediately obtain that $grB \simeq grA \otimes_A B$ which implies that $\widehat{B} \simeq \widehat{A} \otimes_A B$ by Claim (2).

($\Leftarrow$) Let $M \to N$ be an injective A-module homomorphism. We want to show that $M \otimes_A B \to N \otimes_A B$ is an injection. Since $A \to \widehat{A}$ is flat, $B \otimes_A \widehat{A}$ is also a flat A-module. Thus, $M \otimes_A \left(B \otimes_A \widehat{A}\right) \to N \otimes_A \left(B \otimes_A \widehat{A}\right)$ is an injection But, $- \otimes_A \left(B \otimes_A \widehat{A}\right) \simeq (- \otimes_A B) \otimes_A \widehat{A}$ is an injection and since $A \to \widehat{A}$ is faithfully flat by Claim (1), $M \otimes_A B \to N \otimes_A B$ is injective as desired. This finishes the proof. $\square$

Thus, taking $A = \mathcal{O}_y$, $B = \mathcal{O}_x$ gives the desired result because, when $L = k(y)$, $k = k(x)$, we have $L = B \otimes_A k$ so taht $- \otimes_k L = - \otimes_k k \otimes_A B = - \otimes_A B$. $\square$

10.5. If x is a point of a scheme X, we define an *étale neighborhood* of x to be an étale morphism $f : U \to X$, together with a point $x' \in U$ such that $f(x') = x$. As an example of the use of étale neighborhoods, prove the following: if $\mathcal{F}$ is a coherent sheaf on X, and if every point of X has an étale neighborhood $f : U \to X$ for which $f^*\mathcal{F}$ is a free $\mathcal{O}_U$-module, then $\mathcal{F}$ is locally free on X.

Proof. The question being local, we may suppose that both U and X are affine. Furthermore, by localizing them at x and x', we reduce the problem to show the following:

$A \to B$ is an étale local homomorphism of local rings and M is an A-module such that $M \otimes_A B \simeq B^n$. Then, $M \simeq A^n$.

But, this is easy: $M \otimes_A B \simeq B^n \simeq A^n \otimes_A B$ and since $B \neq 0$ and B is A-flat, B is a faithfully flat A-module (SGA1 IV-cor 2.2) so that $M \simeq A^n$. This finishes the proof. $\square$

Algebraic Geometry by Robin Hartshorne

Exercises solutions by Jinhyun Park

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Chapter 4.Curves, Section 1.Riemann-Roch Theorem.

1. Choose $Q \in C$. Choose n big enough so that deg $n(2P - Q) > 2g - 2,\ g,\ 1$. $\Rightarrow h^0(n(2P-Q)) = 1 - g + n(2P-Q) > 1 \Rightarrow \exists$ effective divisor $D \in |n(2P-Q)| \Rightarrow \exists f \in K(C)$ such that $D + nQ - 2nP = (f)$. Since deg $D = n$, so D cannot cancel $-2nP$ i.e. f has a pole only at P.//

2. Let $F = \{P_1, \cdots, P_r\}$. Multiplying functions of Ex.IV.1.1 might give cancellation of poles and zeros, so we need slightly different approach.
Let $Q \in C - F$. Consider $D' = n(P_1 + \cdots P_r - (r-1)Q)$. Choose $n > 2g-2,\ g$. Then, $\exists D \in |D'|$ i.e. $\exists f \in K(c)$ such that $D + (r-1)Q - nP_1 - \cdots - nP_r = (f)$. Note that deg $D = n$. Each P_i occurs with order $-n$ so, if $P_i \in \mathrm{Supp}D$, then either (i) $\mathrm{ord}_{P_i} D < n$ or (ii) $D = nP_i$ for some i, in this case, WMA $i = 1$ WLOG.

For (i) there is no problem.
For (ii) f has ples only at $P_2, \cdots, P_r$ not at P_1. By Ex.IV.1.1, $\exists g \in K(C)$ which has a pole only at P_1. Let $\mathrm{ord}_{P_i} g = n_i$ $2 \le i \le r$. Then, if we choose $m > 1, n_2, \cdots, n_r$, then, $f^m g$ has poles at and only at F.//

3. Proof 1) By I-(6.10), there is a projective nonsingular curve $\bar{X}$ over k such that X is an open subset of $\bar{X}$, i.e. $\bar{X} - X$ is a finite set, say, $\{P_1, \cdots, P_r\} \neq \phi$ because X is not proper.
Then, by Ex.IV.1.2, $\exists f \in k(\bar{X}) = k(X)$ such that f has poles only at $P_1, \cdots, P_r$.
We can consider $f \in k(\bar{X})$ as a morphism $f : \bar{X} \to \mathbb{P}^1$.Then, $f^{-1}(\mathbb{A}^n) = X$, so, $g = f|_X : X \to \mathbb{A}^1$ is a morphism.
$f(\bar(X))$ is proper over k(because $\bar{X}$ is proper) and irreducible and, $f(\bar{X}) \neq$a point. Hence, $f(\bar{X}) = \mathbb{P}^1$. And by (II-6.8), f is a finite morphism, in particular, affine morphism. Hence, $f^{-1}(\mathbb{A}^1) = X$ is affine.

Proof 2) As above, let $F = \{P_1, \cdots, P_r\} = \bar{X} - X$. Choose m such that $mr > 2g$ Then, $D = m(P_1 + \cdots P_r$ has a degree $> 2g$, so by (3), it is very ample. Then, it gives an embedding of $\bar{X}$ into a projective space $\mathbb{P}^N$ for some N, and $D = \bar{X}.H$ for a hyper plane H of $\mathbb{P}^N$. Then, $\bar{X} - F =$ a closed subscheme of $\mathbb{A}^N = \mathbb{P}^N - H$ which is affine, so, $X = \bar{X} - F$ is also affine.//

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CHAPTER V SECTION 5 BIRATIONAL TRANSFORMATIONS

5.8. *A surface singularity.* **Let k be an algebraically closed field, and let X be the surface in $\mathbb{A}^3_k$ defined by the equation $x^2+y^3+z^5=0$. It has an isolated singularity at the origin $P=(0,0,0)$.**

(a). **Show that the affine ring $A = k[x,y,z]/(x^2+y^3+z^5)$ of X is a unique factorization domain, as follows. Let $t = z^{-1}$; $u = t^3x$, and $v = t^2y$. Show that z is irreducible in A; $t \in k[u,v]$, and $A[z^{-1}] = k[u,v,t^{-1}]$. Conclude that A is a UFD.**

Claim. *z is irreducible in A.*

Proof. Notice that

$$\begin{aligned} &z \text{ is irreducible in } A. \\ &\Leftrightarrow (z) \text{ is a prime ideal in } A. \\ &\Leftrightarrow A/(z), \text{ which is } k[x,y]/(x^2+y^3), \text{ is an integral domain.} \\ &\Leftrightarrow x^2+y^3 \text{ is irreducible in } k[x,y]. \end{aligned}$$

We prove the last statement. Suppose that for some f, g in $k[x,y]$, we have

$$fg = x^2+y^3.$$

(1) (case 1) Assume that $\deg_x(f)$, the degree of f in x, is zero. Then f is a polynomial in y, and we can write $g = cx^2+ax+b$ for some c in $k^\times$ and a, b in $k[y]$. Thus,
$$x^2+y^3 = fg = cfx^2+fax+fb.$$
This implies that $f = 1/c$, which is a unit in $k[x,y]$.

(2) (case 2) Assume that $\deg_x f = 1$. Then, by multiplying a suitable constant in $k^\times$, we may assume that $f = x+a$ and $g = x+b$ for some a, b in $k[y]$. Then,
$$x^2+y^3 = fg = x^2+(a+b)x+ab$$
so that $a+b=0$ and $ab=y^3$. Then, $b^2=-y^3$ and since y is irreducible in $k[y]$, $b = yb'$ for some b' in $k[y]$. Hence $y^2(b')^2 = -y^3$, thus $(b')^2 = -y$, which is impossible because we then have $2\deg_y(b') = 1$.

(3) (case 3) Assume that $\deg_x(f) = 2$. Then, by symmetry, (case 1) shows that g must be a unit.

Hence x^2+y^3 is irreducible in $k[x,y]$, and thus z is irreducible in A. □

Claim. *$t \in k[u,v]$.*

Proof. The equality $x^2+y^3+z^5=0$ implies that in the fraction field we have $-x^2/z^6 - t^3/z^6 = 1/z$. This is equivalent to $t = -u^2-v^3$. □

Claim. *$A[z^{-1}] = k[u,v,t^{-1}]$.*

Proof. The equalities

$$\begin{cases} x = (t^{-1})^3t^3x = (t^{-1})^3u, \\ y = (t^{-1})^2t^2y = (t^{-1})^2v, \\ z = t^{-1} \end{cases}$$

show that $A \subset k[u,v,t^{-1}]$. By the previous claim $z^{-1} = t \in k[u,v]$, thus $A[z^{-1}] \subset k[u,v,t^{-1}]$.

Conversely, we have
$$\begin{cases} u = (z^{-1})^3 x \in A[z^{-1}], \\ v = (z^{-1})^2 y \in A[z^{-1}], \\ t^{-1} = z \in A \end{cases}$$
so that $A[z^{-1}] \supset k[u, v, t^{-1}]$. This finishes the proof. $\square$

Claim. *A is a UFD.*

Notice that, being a polynomial ring, $A[z^{-1}] = k[u, v, t^{-1}]$ is a UFD.

Lemma (1)**.** *Let f be an irreducible element in A, that is not in (z). Then, f is irreducible as an element in $A[z^{-1}]$.*

Proof. Suppose that $(g/z^m)(h/z^n) = f/1$ for some integers $m, n \geq 0$ and g, h in A, so that $gh = z^{m+n} f$. If $m > 0$ or $n > 0$, then since the quantity $gh = z^{m+n} f$ is in the ideal (z), either $g \in (z)$ or $h \in (z)$. By canceling a suitable number of z's if necessary, we may assume that $m = n = 0$. Thus, $gh = f$ in A. But, since f is irreducible in A, g or h must be a unit in A. Hence $g/1$ or $h/1$ is a unit in $A[z^{-1}]$, thus, $f/1$ is irreducible in $A[z^{-1}]$. $\square$

Lemma (2)**.** *If a nonzero element $f/1 \in A[z^{-1}]$ is irreducible for some f in A, then $f = z^m g$ for some integer $m \geq 0$ and an irreducible element g in A, where $g \notin (z)$.*

Proof. Since f is nonzero, we can write $f = z^m g$ for some integer $m \geq 0$ and an element $g \in A$ that is not in the ideal (z). We need to check that this g is irreducible in A.

If not, then for some nonunits p, q in A, the equality $g = pq$ holds. Thus, $f/1 = (z^m/1)(g/1) = (z^m/1)(p/1)(q/1)$. Since $f/1$ is irreducible in $A[z^{-1}]$ and z^m is a unit, one of $p/1$ and $q/1$ must be a unit element in $A[z^{-1}]$, say $p/1$, without loss of generality. Thus, for some r in A and an integer $n \geq 0$, we have $(p/1)(r/z^n) = 1$, that is, $pr = z^n$. Thus, $pr \in (z)$, and z being irreducible either $p \in (z)$ or $r \in (z)$. But since $g \notin (z)$ and $g = pq$, the element p must not be in (z). Hence $r \in (z)$. Thus, by repeating this argument, we may assume that $n = 0$. Then, we have the equality $pr = 1$ in A, contradicting the assumption that p is not a unit in A. $\square$

We now prove that A is a UFD. For any nonzero $f \in A$, since the ring $A[z^{-1}]$ is a UFD, we have a factorization of $f/1$
$$\frac{f}{1} = \frac{u}{z^m} \frac{f_1}{z^{m_1}} \cdots \frac{f_n}{z^{m_n}}$$
for some nonnegative integers $m, m_1, \cdots, m_n$, a unit u in A, and $f_1, \cdots, f_n$ in A, where f_i/z^{m_i} are irreducible in $A[z^{-1}]$. Since each z^{m_i} is a unit, by replacing $m + m_1 + \cdots + m_n$ by m, we may simplify the above equation as
$$\frac{f}{1} = \frac{u}{z^m} \frac{f_1}{1} \cdots \frac{f_n}{1}$$
where f_i are irreducible in $A[z^{-1}]$. Thus, $z^m f = u f_1 \cdots f_n$ in A. By the Lemma (2), each $f_i = z^{r_i} g_i$ for some integer $r_i \geq 0$ and an irreducible element $g_i \in A$, where $g_i \notin (z)$, so that
$$z^m f = u z^{r_1 + \cdots + r_n} g_1 \cdots g_n.$$
Note that we must have $m \geq r_1 \cdots r_n$ since all g_i is not in (z). Thus, $f = u z^s g_1 \cdots g_n$, where $s = r_1 + \cdots + r_n - m \geq 0$, gives a factorization of f into a product of irreducible elements of A.

To show that this factorization is unique, suppose that we have two such factorizations
$$g = u g_1 \cdots g_n = v h_1 \cdots h_m,$$
where u, v in A are units, and g_i, h_j in A, $1 \leq i \leq n$, $1 \leq j \leq m$, are irreducible. Since $f = u g_1 \cdots g_n$ is in the ideal (h_1), for some i, the element g_i must be in (h_1). We may

assume that $g_1 \in (h_1)$ so that $g_1 = h_1 k$ for some k in A. Since g_1 is irreducible and h_1 is not a unit (being irreducible), k must be a unit in A. Hence, continuing in this way, by suitably renumbering them if necessary, we must have $m = n$ and each irreducible element g_i is a unit multiple of h_i. This finishes the proof.

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APPENDIX C; THE WEIL CONJECTURES

Exercise 5.1. Let $X = \coprod_i X_i$. Obviously, then, $N_r(X) = \sum_i N_r(X_i)$ so that

$$Z(X,t) = \exp(\sum_{r=1}^{\infty} N_r(X)\frac{t^r}{r}) = \exp(\sum_{r=1}^{\infty}\sum_i N_r(X_i)\frac{t^r}{r})$$

$$= \exp(\sum_i \sum_{r=1}^{\infty} N_r(X_i)\frac{t^r}{r}) = \prod_i \exp(\sum_{r=1}^{\infty} N_r(X_i)\frac{t^r}{r}) = \prod_i Z(X_i,t).$$

Exercise 5.2. The point is to compute the number of $k_r = \mathbb{F}_{q^r}$-rational points, i.e. to compute N_r in $\bar{X} = \mathbb{P}^n_{\bar{k}}$. We can consider the following *stratification* of $\mathbb{P}^n$:

$$\mathbb{P}^n = \mathbb{A}^n \cup \mathbb{P}^{n-1} = \cdots = \mathbb{A}^n \cup \mathbb{A}^{n-1} \cup \cdots \cup \mathbb{A}^1 \cup \{*\}.$$

Hence for a field k_r of q^r elements, $\mathbb{P}^n_{k_r}$ has $N_r = 1 + q^r + q^{2r} + \cdots + q^{nr}$ points. Hence,

$$Z(\mathbb{P}^n,t) = \exp(\sum_{r=1}^{\infty}\{1+q^r+\cdots+q^{nr}\}\frac{t^r}{r}) = \prod_{i=0}^{n}\exp(\sum_{r=1}^{\infty}\frac{(q^it)^r}{r})$$

$$= \prod_{i=0}^{n}\exp(-\log(1-q^it)) = \prod_{i=0}^{n}\frac{1}{1-q^it} = \frac{1}{(1-t)(1-qt)\cdots(1-q^nt)}.$$

Obviously, this is a rational function so that (1.1) is true. Also, we have $P_{\text{odd}}(t) = 1$, $P_{2i}(t) = 1 - q^it$ so that (1.3) is true, because $|q^i| = q^{\frac{2i}{2}}$, indeed.

Let's find the self intersection number E of Δ in $\mathbb{P}^n \times \mathbb{P}^n$. Note that $CH^*(\mathbb{P}^n \times \mathbb{P}^n) \simeq CH^*(\mathbb{P}^n) \otimes CH^*(\mathbb{P}^n)$ so that in particular, we have

$$CH^n(\mathbb{P}^n \times \mathbb{P}^n) \simeq \bigoplus_{i+j=n, 0\le i,j\le n} \mathbb{Z}s^it^j,$$

where s^it^j corresponds to an n-cycle $\mathbb{P}^i \times \mathbb{P}^j$ in $\mathbb{P}^n \times \mathbb{P}^n$. Hence if $\Delta = \sum_{i,j} a_{ij}s^it^j$, then if we look at the intersection product, $(\Delta.(\mathbb{P}^i \times \mathbb{P}^j)) = 1$. Also, $(s^it^j).(s^{i'}t^{j'}) = 1$ iff $i + i' = n$ and $j + j' = n$ and otherwise it is 0. Hence each $a_{ij} = 1$. That is,

$$E = (\Delta.\Delta) = (\sum_{i,j} s^it^j)(\sum_{i',j'} s^{i'}t^{j'}) = n+1.$$

Now,

$$Z(\mathbb{P}^n, \frac{1}{q^nt}) = \frac{1}{(1-\frac{1}{q^nt})(1-\frac{1}{q^{n-1}t})\cdots(1-\frac{1}{t})} = \frac{(q^nt)(q^{n-1}t)\cdots(t)}{(q^nt-1)(q^{n-1}t-1)\cdots(t-1)}$$

$$= (-1)^{n+1}\frac{q^{\frac{n(n+1)}{2}}t^{n+1}}{(1-t)(1-qt)\cdots(1-q^nt)} = (-1)^{n+1}q^{\frac{nE}{2}}t^E Z(\mathbb{P}^n,t),$$

so that we have (1.2).

Now, obviously, from $Z(\mathbb{P}^n,t)$, we see that $B_i = 0$ if i is odd and $B_i = 1$ if i is even. Hence $E = n+1 = \sum_{i=0}^{2n}(-1)^iB_i$, indeed. Also, for $\mathbb{P}^n_{\mathbb{C}}$, we had $H^i(\mathbb{P}^n_{\mathbb{C}},\mathbb{Z}) = \begin{cases} \mathbb{Z} & i: \text{ even} \\ 0 & i: \text{ odd} \end{cases}$, so that B_i indeed is $\mathrm{rk}H^i(\mathbb{P}^n_{\mathbb{C}},\mathbb{Z})$. This one shows (1.4). Hence $\mathbb{P}^n$ satisfies the Weil conjectures.

Exercise 5.3. Obviously, $N_r(X \times \mathbb{A}^1) = q^r N_r(X)$. Hence,

$$Z(X \times \mathbb{A}^1, t) = \exp(\sum_{r=1}^{\infty} N_r(X \times \mathbb{A}^1)\frac{t^r}{r}) = \exp(\sum_{r=1}^{\infty} q^r N_r(X)\frac{t^r}{r})$$
$$= \exp(\sum_{r=1}^{\infty} N_r(X)\frac{(qt)^r}{r}) = Z(X, qt).$$

Exercise 5.4. Let $N_r(x)$ be the contribution of the closed point $x \in X$ to N_r in X. A $\mathbb{F}_{q^r}$-rational point is determined by the number of morphisms $\mathrm{Spec}\mathbb{F}_{q^r} \to X$, which is the same as to give a $\mathbb{F}_q$-homomorphism $k(x) \to \mathbb{F}_{q^r}$ and $N_r(x)$ is the number of all such embeddings. This is possible iff $\deg x = [k(x) : \mathbb{F}_q] | r$, and, the number of all such embeddings is just $\deg x$. (If you are confused, consider, say $\deg x = 1$. How many $\mathbb{F}_q$-linear maps are there? Just 1!.)

Thus, in fact, $N(x) = q^{\deg x}$ so that

$$\zeta_X(x) = \prod_{x\in X} \frac{1}{1 - N(x)^{-s}} = \prod_{x\in X} \frac{1}{1 - (q^{\deg x})^{-s}} = \prod_{x\in X} \frac{1}{1 - (q^{-s})^{\deg x}} = \exp(\sum_{x\in X}\sum_{r=1}^{\infty} \frac{(q^{-s\deg x})^r}{r})$$
$$= \exp(\sum_{x\in X}\sum_{r=1}^{\infty} \frac{(q^{-s})^{r\deg x}}{r}) = \exp(\sum_{x\in X}\sum_{r=1}^{\infty} \frac{(\deg x)(q^{-s})^{r\deg x}}{r\deg x}) = \exp(\sum_{x\in X}\sum_{n=1}^{\infty} \frac{N_n(x)(q^{-s})^n}{n})$$
$$= \exp(\sum_{n=1}^{\infty} N_n\frac{(q^{-s})^n}{n} = Z(X, q^{-s}).$$

Exercise 5.5. By the Weil conjectures, since $\dim H^1(X, \mathbb{Q}_l) = B_1 = \deg P_1(t) = 2g$, for some α_i, we have

$$Z(X,t) = \frac{P_1(t)}{(1-t)(1-qt)} = \frac{\prod_{i=1}^{2g}(1-\alpha_i t)}{(1-t)(1-qt)}$$
$$= \exp(\sum_{r=1}^{\infty}(\frac{t^r}{r} + \frac{q^r t^r}{r} - \sum_{i=1}^{2g}\alpha_i^r\frac{t^r}{r})) = \exp(\sum_{r=1}^{\infty} N_r\frac{t^r}{r}).$$

Hence, $N_r = 1 + q^r - \sum_{i=1}^{2g}\alpha_i^r$ for all $r \geq 1$.

Now, from the functional equation, we have

$$Z(\frac{1}{qt}) = \pm q^{1-g}t^{2-2g}Z(t).$$

The left hand side of the equation is,

$$\frac{\prod_{i=1}^{2g}(1-\frac{\alpha_i}{qt})}{(1-\frac{1}{qt})(1-\frac{1}{t})} = \frac{t^{2-2g}q^{1-g}\prod_{i=1}^{2g}(\sqrt{q}t - \frac{\alpha_i}{\sqrt{q}})}{(1-t)(1-qt)}.$$

Hence, by comparing terms, we have

$$P_1(t) = \prod_{i=1}^{2g}(1-\alpha_i t) = \pm\prod_{i=1}^{2g}(\sqrt{q}t - \frac{\alpha_i}{\sqrt{q}}).$$

Recall that $\frac{1}{-a+bt} = \frac{-1}{a}(1 + \frac{b}{a}t + (\frac{b}{a})^2t^2 + \cdots)$ so that $\log(-a+bt) = -(\frac{b}{a}t + (\frac{b}{a})^2t^2 + \cdots)$. Hence, if we replace $P_1(t)$ be the right hand side of above,

$$Z(X,t) = \exp(\sum_{r=1}^{\infty}(\frac{t^r}{r} + \frac{q^r t^r}{r} + \sum_{i=1}^{2g}\frac{q^r}{\alpha_i^r}\frac{t^r}{r})) = \exp(\sum_{r=1}^{\infty}(1 + q^r + q^r\sum_{i=1}^{2g}\frac{1}{\alpha_i^r}\frac{t^r}{r})).$$

Hence, $N_r = 1+q^r+q^r\sum_{i=1}^{2g}\frac{1}{\alpha_i^r}$ as well, which was also $1+q^r-\sum_{i=1}^{2g}\alpha_i^r$. Since we know $N_1, N_2, \cdots, N_g$, we hence know all of $\sum_{i=1}^{2g}\alpha_i^r$ for $-g \le r \le g$. Using some combinatorial argument and Nowton's identity on symmetric polynomials, above information is enough to determine all $\sum\alpha_i^r$ for $r > g$ as well. Hence $N_r = 1+q^r-\sum\alpha_i^r$ is determined as well.

Exercise 5.6. From IV, Exercise 4.16, $N_r = q^r - (f^r + \breve{f}^r) + 1$. Hence,

$$Z(,t) = \exp(\sum_{r=1}^{\infty} N_r\frac{t^r}{r}) = \exp(\sum_{r=1}^{\infty}(q^r - (f^r+\breve{f}^r)+1)\frac{t^r}{r})$$

$$= \frac{1}{1-qt}\frac{1}{1-t}(1-ft)(1-\breve{f}t) = \frac{1-at+qt^2}{(1-t)(1-qt)},$$

since $f\breve{f} = q$ and $a = f+\breve{f} \in \mathbb{Z}$.

Now from the functional equation, we will have

$$P_1(t) = (1-ft)(1-\breve{f}t) = \pm(\sqrt(q)t - \frac{f}{\sqrt{q}})(\sqrt{q}t - \frac{\breve{f}}{\sqrt{q}})$$

so that

$$|a| = |f+\breve{f}| \le 2g \Leftrightarrow |f| = |\breve{f}| = \sqrt{q}.$$

(See Exercise 5.7, (b) and (c).)

Exercise 5.7.

(a). We have

$$Z(X,t) = \frac{P_1(t)}{(1-t)(1-qt)} = \frac{\prod_{i=1}^{2g}(1-\alpha_i t)}{(1-t)(1-qt)}$$

$$= \exp(\sum_{r=1}^{\infty}\frac{(1-\sum_{i=1}^{2g}\alpha_i^r + q^r)t^r}{r}).$$

Hence, $N_r = 1-\sum_{i=1}^{2g}\alpha_i^r + q^r = 1 - a_r + q^r$ so that $a_r = \sum_{i=1}^{2g}(\alpha_i)^r$.

(b). ($\Leftarrow$) If $|\alpha_i| \le \sqrt{q}$ for all i, then,

$$|\alpha_i| \le \sum_{i=1}^{2g}|\alpha_i|^r \le \sum_{i=1}^{2g}\sqrt{q^r} = 2g\sqrt{q^r}.$$

($\Rightarrow$) Consider the following easy power series expansion:

$$\sum_{i=1}^{2g}\frac{\alpha_i t}{1-\alpha_i t} = \sum_{r=1}^{\infty}a_r t^r.$$

Since $|a_r| \le 2g\sqrt{q^r}$, the RHS is holomorphic in $|t| < \frac{1}{\sqrt{q}}$. If $|\alpha_i| > \sqrt{q}$ for some i, then, the LHS has a pole of order 1 in $|t| < \frac{1}{\sqrt{q}}$ at $t = \frac{1}{\alpha_i}$ which is hence a contradiction.

(c). By using the functional equations, as in Ex (5.5), we have

$$P_1(t) = \prod_{i=1}^{2g}(1 - \alpha_i t) = \pm \prod_{i=1}^{2g}(\sqrt{q}t - \frac{\alpha_i}{\sqrt{q}}).$$

Hence,

$$\{\alpha_1^{-1}, \cdots, \alpha_{2g}^{-1}\} = \{\frac{\alpha_1}{q}, \cdots, \frac{\alpha_{2g}}{q}\}$$

so that for all j, there is a unique number $i(j)$ with $\alpha_j^{-1} = \frac{\alpha_{i(j)}}{q}$, so that $|\alpha_i| \leq \sqrt{q}$ then implies $|\alpha_j^{-1}| = \frac{|\alpha_i|}{q} \leq \frac{1}{\sqrt{q}}$ i.e. $|\alpha_j| \geq \sqrt{q}$ for all j. This proves that $|\alpha_i| = \sqrt{q}$.