Hartshorne, Chapter 1 Answers to exercises. REB 1994

1.1a $k[x,y]/(y-x^2)$ is identical with its subring $k[x]$.

1.1b $A(\mathbf{Z}) = k[x, 1/x]$ which contains an invertible element not in k and is therefore not a polynomial ring over k.

1.1c Any nonsingular conic in P^2 can be reduced to the form $xy + yz + zx = 0$ and this curve is isomorphic to P^1. (Proof: choose any 3 points on the conic, and choose coordinates so that these points are $(1:0:0), (0:1:0), (0,0,1)$; this means the conic must have the equation $cxy + ayz + bzx = 0$, with a, b, c all nonzero (otherwise the conic is singular). Then multiplying x, y, z by a, b, c shows that the conic has equation $xy + yz + zx = 0$. Hence all nonsingular conics are isomorphic to this one, and as it is easy to find one isomorphic to P^1 they all are.) Therefore (regular function on a conic) = (regular functions on the conic $xy + yz + zx = 0$ - some hyperplane) = (regular functions on P^1 - 1 or 2 points) = $A(Y)$ or $A(z)$. The ring is $A(Y)$ if and only if the conic $ax^2 + bxy + cy^2+$(terms of degree < 2) intersects the line at infinity in exactly one point, which happens if and only if $b^2 = 4ac$.

1.2 Y is isomorphic to A^1 and is therefore an affine variety of dimension 1, and $A(Y) = k[x]$. $I(Y)$ is generated by $Y - X^2, Z - X^3$.

1.3 $xy = x$, so $x = 0$ or $z = 1$. $x^2 = yz$, so $x = 0, y = 0$ or $x = 0, z = 0$, or $z = 1, x^2 = y$. Therefore Y is the union of 2 lines and a parabola. The prime ideals are generated by x, y or x, z or $z - 1, x^2 - y$.

1.4 The line $x = y$ is closed in A^2 but not in $A^1 \times A^1$ (at least if k is infinite).

1.5 B is a finitely generated algebra over k and has no nilpotents. If $x_1, ...x_n$ is a set of generators for B then $B = k[x_1, \dots, x_n]/I$ for some ideal I, and $\sqrt{I} = I$ as B has no nilpotents. Hence $I(V(I)) = I$ by the nullstellensatz, so that B is the coordinate ring of $V(I)$ in A^n.

1.6 Put $U \subset X$, U open, X irreducible. Then $X = (X - U) \cup \bar{U}$, so $\bar{U} = X$, so U is dense in X. If $U \subset C_1 \cup C_2$, then $X = \bar{U} = \bar{C}_1 \cup \bar{C}_2 = C_1 \cup C_2$, so C_1 or C_2 contains U, so U is irreducible.

1.7a (i) is equivalent to (iii) by taking complements. (ii) implies (iv) is trivial. (i) implies (ii) because if some set contains no smallest closed subset then we can choose an infinite descending chain $C_1 \supset C_2 \supset \cdots$ using Zorn's lemma. The proof that (iii) is equivalent to (iv) is similar.

1.7b If U is any open cover of X, apply (a)(iv) to the unions of the finite subsets of U.

1.7c Follows from (a)(iv).

1.7d X Noetherian and Hausdorff implies X Hausdorff and every subset Noetherian implies X Hausdorff and every subset compact implies X compact and every subset closed implies X compact and discrete implies X finite and discrete.

1.8 Let H have ideal (f). As Y is not contained in H, f is neither a unit nor a zero divisor in the coordinate ring B of Y. Therefore by 1.11A every minimal prime P containing f has height 1. By 1.8A $\dim(B/P) = r - 1$. If X is an irreducible component of $Y \cup H$ then the ideal of X is a minimal prime ideal P of B containing f and the coordinate ring of X is B/P.

1.9 The dimension of any component of $Z(a)$ = transcendence degree of its function field. This function field contains $x_1, \dots, x_n$ and the algebraic relations between these are a consequence of the r generators of a. Therefore the dimension of any component is at least $n-$number of generators of $a \geq n - r$.

1.10a If $Y_0 \subset Y_1 \subset \cdots \subset Y_n$ is a chain of irreducible closed subsets of Y, then $\bar{Y}_0 \subset \bar{Y}_1 \subset \cdots \subset \bar{Y}_n$ is a chain of irreducible closed subsets of X.

1.10b By (a), $\dim(X) \geq \sup \dim(U_i)$. If $X_0 \subset \cdots X_n$ is a sequence of irreducible closed subsets of X with X_0 a point, choose some set U in the cover with $X_0 \in U$. Then by 1.6 $X_i \cap U$ is irreducible and dense in X_i and therefore not contained in X_{i-1}. Hence $X_0 \cap U \subset X_1 \cap U \subset \cdots \subset X_n \cap U$ is a sequence of closed strictly increasing irreducible subsets of U, so $\dim(X) \leq \dim U \leq \sup \dim U_i$.

1.10c $X = \{u, v\}$ (a 2 point set) with open sets $\emptyset, \{u\} = U, X$.

1.10d If $Y_0 \subset \cdots \subset Y_n$ is a chain of closed irreducible subsets of Y and $Y \neq X$, then we can add X to the end of this chain to see that $\dim(X) \geq \dim(Y) + 1$ so either $\dim(X) = \infty$ or $\dim(X) > \dim(Y)$.

1.10e The set of positive integers, closed sets those of the form $\{1, 2, 3, \dots, n\}$.

1.11 $t \to (t^3, t^4, t^5)$ is a homeomorphism from A^1 to Y, so $\dim(Y) = 1$, so P has height 2. No element of the ideal of P has homogeneous components of degree 0 or 1, and the possible homogeneous components of degree 2 form a vector space of dimension 3, so P needs at least 3 generators. (P is generated by $x^2y - z^2, zx - y^2, x^3 - zy$.)

1.12 $f(x, y) = y^4 + y^2 + x^2(x - 1)^2$.

2.1 a is homogeneous and so defines a cone in A^{n+1}. f vanishes on all the elements of this cone (including 0 as f has positive degree) so $f^q \in a$ for some $q > 0$ by the usual Nullstellensatz.

2.2 (iii) implies (i) is trivial as $x_i^d \in S_d$. Proof that (i) implies (ii): If $Z(a)$ is empty, then in A^{n+1}, $Z(a)$ must be empty or $(0, \ldots, 0)$, so $\sqrt{a}$ must be S or $\oplus_{d>0} S_d$. Proof that (ii) implies (iii): $\sqrt{a}$ contains x_i, so there is some m with $x_i^m \in a$ for all i, so a contains $S_{m(n+1)}$ as any monomial of degree $m(n+1)$ must have x_i^m as a factor for some i.

2.3 (a),(b),(c),(e) are trivial. For (d), clearly $I(Z(a))$ contains $\sqrt{a}$. As $Z(a)$ is nonempty, any nonzero homogeneous polynomial vanishing on it must have positive degree. By 2.1, this implies that $f^q \in a$. Therefore $I(Z(a))$ is contained in $\sqrt{a}$ as it is a homogeneous ideal.

2.4a Follows from 2.3d,e, and 2.2.

2.4b If $Y = Y_1 \cup Y_2$, then $I(Y) = I(Y_1) \cap I(Y_2) \supset I(Y_1)I(Y_2)$. Therefore if $I(Y)$ is prime, $I(Y)$ must be either $I(Y_1)$ or $I(Y_2)$, so Y is Y_1 or Y_2. On the other hand if Y is not prime, then $ab \in I(Y)$, with $a \notin I(Y)$, $b \notin I(Y)$. Therefore Y is the union of the proper subsets $Y \cap Z(a)$, $Y \cap Z(b)$ and is therefore not irreducible.

2.4c $I(P^n) = 0$ which is a prime ideal.

2.5a P^n can be covered by $n+1$ copies of A^n which is Noetherian.

2.5b See proposition 1.5 and part (a) of this question.

2.6 S(Y) is the coordinate ring of the cone in A^{n+1} corresponding to Y (assuming Y is nonempty). $S(Y)_{x_i}$ is the coordinate ring of the cone $Y - (x_i = 0)$ if x_i is not identically 0 on Y, i.e., Y_i is nonempty. Therefore the homogeneous part of degree 0 of $S(Y)_{x_i}$ is the coordinate ring of the cone with $x_i = 0$, which is isomorphic to Y_i, and therefore $S(Y)_{x_i} = A(Y_i)[x_i, 1/x_i]$ as every element of $S(Y)_{x_i}$ is the sum of monomials of the form ($x_i^{\pm n} \times$ element of degree 0). Therefore $Tr.deg.(S(Y)_{x_i}) = Tr.deg.(A(Y_i) + 1) = Tr.deg.S(Y)$. Therefore $\dim(S(Y)) = 1 + \dim(Y_i)$ whenever $Y_i \neq 0$. The Y_i's cover Y, so $\dim(Y) = \sup(\dim(Y_i))$, so $\dim(S(Y) = 1 + \dim(Y)$.

2.7a P^n is covered by $n+1$ open copies of A^n, so $\dim(P^n) = \sup(\dim(A^n)) = n$.

2.7b Y is contained in P^n, and therefore covered by $n+1$ copies of A^n. In each copy A_i of A^n, $\overline{Y \cup A_i} = \overline{Y} \cup A_i$ as A_i is open. Hence $\dim(Y \cap A_i) = \dim(\overline{Y \cap A_i}) = \dim(\overline{Y} \cap A_i)$, and therefore $\dim(Y) = \sup(\dim(Y \cap A_i) = \sup \dim(\overline{Y} \cap A_i) = \dim(\overline{Y})$.

2.8 If f is any homogeneous polynomial of positive degree then the zero set of f has dimension $n-1$ as it has this dimension on some affine subsets and is a proper closed subset of P^n. Also f is irreducible, so the homogeneous ideal generated by it is prime (as rings of polynomials are U.F.D.'s so irreducibles are primes) so its variety is irreducible. Conversely if Y is any proper closed subset of P^n then there is some homogeneous polynomial f vanishing on Y which we can assume to be irreducible because Y is irreducible (so some factor of f must also vanish on Y if f is not irreducible). Then the zero set of f is an irreducible $n-1$ dimensional closed subset of P^n containing the $n-1$ dimensional closed subset Y, and so must be equal to Y (because any proper closed subset of an irreducible topological space has smaller dimension).

2.9a $\beta g(x_0, \ldots, x_n) = x_0^d g(x_1/x_0, \ldots, x_n/x_0)$ if g is of degree d. If g vanishes on Y then βg vanishes on $\bar{Y}$, so $I(\bar{Y}) \supseteq \beta(I(Y))$. If h vanishes on $\bar{Y}$ then we can assume h is homogeneous. If $g(x_1, \ldots, x_n) = h(1, x_1, \ldots, x_n)$, then $h = \beta g$, so $I(\bar{Y})$ is generated by $\beta(I(Y))$

2.9b $\{(t, t^2, t^3)\} = Y$, and $I(Y) = (x_2 - x_1^2, x_3 - x_1^3)$. $\beta(x_2 - x_1^2) = x_0 x_2 - x_1^2$ and $\beta(x_3 - x_1^3) = x_0^2 x_3 - x_1^3$. But $I(\bar{Y})$ contains $x_1 x_3 - x_2^2$ which is not contained in $(\beta(x_2 - x_1^2), \beta(x_3 - x_1^3))$.

2.10a Obvious.

2.10b They have the same ideal, which is prime if and only if they are irreducible.

2.10c See 2.6.

2.11a $I(Y)$ is generated by linear polynomials $\{p_i\}$ if and only if Y is the intersections of the hyperplanes $\{p_i = 0\}$.

2.11b Any hyperplane in P^n is a copy of P^{n-1}, and the intersection of any other hyperplane of P^n with this P^{n-1} is a hyperplane of the P^{n-1}. Therefore any r-dimensional linear variety in P^n is the intersection of $n-r$ hyperplanes and not the intersection of $n-r-1$ hyperplanes. Therefore its ideal is minimally generated by $n-r$ linear polynomials.

2.11c Y is the intersection of $n-r$ hyperplanes and Z is the intersection of $n-s$ hyperplanes, so $Y \cap Z$ is the intersection of $2n-r-s$ hyperplanes, which has dimension at least $n-(2n-r-s)=r+s-n$. In particular it is nonempty if $r+s \geq n$.

2.12a θ maps $k[y_0,\ldots,y_N]$ to an integral domain, so its kernel is a prime ideal. If $f \in k[y_0,\ldots,y_N]$, $f = f_0+f_i+\cdots$ with f_i of degree i, then $\theta(f_i)$ has degree di, so $\theta(f)=0$ if and only if $\theta(f_i)=0$ for all i, and therefore the kernel is also a homogeneous ideal.

2.12b If $f \in Ker(\theta)$ then $f(M_0,\ldots,M_n)=0$. Hence f vanishes on any point $(M_0(a),\ldots,M_n(a))$, so $Im(\rho_d) \subseteq Z(a)$. This proves the easy half. Any monomial raised to the power of d is a product of monomials of the form x_i^d. Choose any point $(m_0,\ldots,m_N) \in Z(a)$. Some m_i is nonzero and $m_i^d = \prod_N m_{j_n}$ where each m_{j_N} corresponds to some monomial x_i^d, hence some m_i corresponding to a monomial x_i^d is nonzero; say $i=0$. If $m_{i_1},\ldots,m_{i_n}$ correspond to $x_0^{d-1}x_1,\ldots,x_0^{d-1}x_n$ then put $x_0=1$, $x_k=m_{i_k}/m_0$, and try to use this to define a map to P^n on the set with $m_0 \neq 0$. We have to show that $m_0M_i(1,x_1,\ldots,x_n)=m_i$ (where m_0 corresponds to x_0^n), i.e., that $m_0M_i(1,m_{i_1}/m_0,\ldots,m_{i_n}/m_0)=m_i$. But this is true because $x_0^dM_i(1,x_1/x_0,\ldots,x_n/x_0)=M_i(x_0,\ldots,x_n)$, and therefore $(m_0,\ldots,m_N)$ is the image of $(x_0,\ldots,x_n)$. Hence $Im(\rho_d) \supseteq Z(a)$.

2.12c ρ_d is continuous and bijective from P^n to $Z(a)$. To show that it is a homeomorphism it is sufficient to show that its inverse is continuous on any open set of $Z(a)$ of the form $m_i \neq 0$ (notation as above) because these open sets cover $Z(a)$. But this follows from the construction of this inverse above.

2.12d The 3-tuple embedding of P^1 into P^3 maps $(x_0:x_1)$ to $(x_0^3:x_0^2x_1:x_0x_1^2:x_1^3)$ which is the projective closure of $\{(x_1,x_1^2,x_1^3)\}$ in P^3, i.e., the twisted cubic curve.

2.13 The map is given by $(x_0:x_1:x_2) \to (x_0^2:x_1^2:x_2^2:x_0x_1:x_1x_2:x_2x_0)$. Any curve in P^2 is defined by some polynomial $f(x_0,x_1,x_2)=0$, f homogeneous, and therefore also by the polynomial $f(x_0,x_1,x_2)^2 = g(x_0^2,x_1^2,x_2^2,x_0x_1,x_1x_2,x_2x_0)$ for some polynomial g. Then some factor of this polynomial g defines a suitable hypersurface containing the image of the curve Z. (This assumes that P^2 is isomorphic to its image which is easy to check (see 2.14 below) once one has defined isomorphisms of varieties, so that curves in the image of P^2 correspond to curves in P^2.)

2.14 The image of ψ is the set Y defined by the equations of the form $x_{ab}x_{cd}=x_{ac}x_{bd}$. Proof: the image is clearly contained in Y. Conversely if $(x_{00}:x_{10}:\cdots:x_{rs}) \in Y$ the we may assume that x_{00} (say) is nonzero. But then the point is the image of $(x_{00}:x_{10}:\cdots:x_{r0}) \times (x_{00}:x_{01}:\cdots:x_{0s}) \in P^r \times P^s$.

2.15a $(a_0:a_1)\times(b_0:b_1)=(a_0b_0:a_0b_1:a_1b_0:a_1b_1)=(w:x:y:z)$, and the image of $P^1 \times P^1$ is then the subvariety $xt-zw=0$ as in 2.14.

2.15b Q is isomorphic to $P^1 \times P^1$, so we can take the two families of lines to correspond to point×line and line×point. (It is easy to check that these are lines in $Q \subset P^3$; for example the image of $(a_0:a_1)\times P^1$ is the set of points $(w:x:y:z) \in P^3$ with $a_1w=a_0y$, $a_1x=a_0z$.)

2.15c The closed subset $x=y$ of Q is not one of these lines.

2.16a $x^2=yw$, $xy=zw$, so $y^2w=xzw$, so $w=0$ or $y^2=xz$. Hence $Q_1 \cap Q_2$ is the intersection of the line $w=x=0$ and the twisted cubic $x^2=yw, xy=zw, y^2=xz$.

2.16b $L \cap C$ is the point $P=(0:0:1)$, so $I(P)=(x,y)$, but $I(L)+I(C)=(x^2,y) \neq (x,y)$.

2.17a By problem 1.8, the intersection of q hypersurfaces has dimension at least $n-q$. If a can be generated by q elements then $Z(y)$ is the intersection of q hypersurfaces and therefore has dimension at least $n-q$ (using problem 2.8).

2.17b If $I(Y)$ can be generated by r elements then Y is the intersection of their hypersurfaces.

2.17c Y is the intersection of $H_1=Z(x^2-wy)$ and $H_2=Z(y^3+wz^2-2xyz)$ as $(xy-wz)^2=w(y^3+wz^2-2xyz)+y^2(x^2-wy)$ and $(y^2-xz)^2=y(y^3+wz^2-2xyz)+z^2(x^2-wy)$, and $y^3=wz^2-2xyz=y(y^2-xz)+z(wz-xy)$. On the other hand $I(Y)$ has no homogeneous elements of degree 0 or 1 and the space of homogeneous elements of degree 2 is 3 dimensional, so any set of generators must have at least 3 elements.

2.17d Still an unsolved problem (as far as I know).

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3.1a Follows from exercise 1.1 as 2 affine varieties are isomorphic if and only if their coordinate rings are.

3.1b The coordinate ring of any proper subset of A^1 has invertible elements not in k and o is not isomorphic to the coordinate ring of A^1.

3.1c The aut group of P^2 acts transitively on sets of 3 points not on a line, so we can assume the conic contains $(0:0:1)$, $(0:1:0)$, and $(1:0:0)$, i.e., it is of the form $axy+byz+czx=0$ for some a,b,c, which are nonzero as otherwise the conic would be a union of two lines. We can multiply x, y, and z by constants to make a, b, and c all equal to 1, so we can assume the conic is $xy+yz+zx=0$, and in particular all conics are isomorphic. Hence we only have to show 1 conic is isomorphic to P^1, e.g., the image of P^1 under the 2-uple embedding.

3.1d Any 2 1-dimensional closed subsets of P^2 intersect (see ex. 3.7a), but A^2 does not have this property.

3.1e By theorem 3.4 the regular functions on a projective variety is the ring k, which is only possible for an affine variety if it is a point.

3.2a If ϕ had an inverse, this would give a polynomial $f(x,y)$ such that $f(t^2,t^3)=t$, which is impossible.

3.2b ϕ is 1:1 because if $x^p=y^p$ in characteristic p then $(x-y)^p=0$ so $x=y$. It has no inverse because there is no polynomial f with $f(t^p)=t$.

3.3a If f is a regular function defined on a neighborhood V of $\phi(p)$ then $f\circ\phi$ is a regular function on the neighborhood $\phi^{-1}(V)$ of p, This gives a map from $O_{\phi(p),Y}$ to $O_{p,X}$ which is a homomorphism.

3.3b We have to show that if V is an open set in X, and f is regular on V, then $f\circ\phi^{-1}$ is regular on $\phi(V)$. If $\phi(p)\in\phi(V)$ then $f\in O_{p,X}$, so ϕ_p^{-1*} maps f to an element of $O_{\phi(p),y}$, so $f\circ\phi^{-1}$ is regular near $\phi(p)$, so it is regular on $\phi(V)$.

3.3c If $\phi_p^*(f)=0$ then f vanishes on $\phi(X)\cap V$ which is a dense subset of V. As f is continuous and vanishes on a dense subset, it must be 0. Therefore ϕ_p^* is injective.

3.4 It is enough to show that ϕ^{-1} is regular near $\phi(1:x_1:\cdots:x_n)$, where ϕ is the d-uple embedding. But near this point ϕ^{-1} takes $(m_0:\cdots:m_N)$ to $(m_{i_0}:\cdots:m_{i_n})$ where m_{i_k} is the coordinate corresponding to the monomial $x_0^{d-1}x_k$, and this is a regular map.

3.5 Identify P^n with its image under the d-uple embedding. Then H is the intersection of a hyperplane in P^N with P^n, so P^n-H is a closed subset of $P^N-H=A^N$ and is therefore an affine variety.

3.6 Any regular function on X has the form $f(x,y)/g(x,y)$ where f and g are coprime. The curves of f and g only intersect in a finite number of points and g can only vanish at $(0,0)$ or where $f=0$, so g has only a finite number of zeros and must therefore be constant. Hence $O(X)=k[x,y]$. Therefore the map from X to A^2 is an isomorphism of their coordinate rings, so if X was affine it would be an isomorphism of varieties, which it obviously is not as is is not surjective on points.

3.7b Suppose $Y\cap H=\phi$. Then Y is a closed subset of an affine variety P^n-H and therefore a finite set of points, as any projective subset of an affine variety is finite.

3.8 Any regular function on P^n-H_i is of the form $f_i(x_0,\ldots,x_n)/x_i^{d_i}$ where d_i is the degree of the homogeneous polynomial f_i. Hence for a function to be regular except on $H_i\cup H_j$ we would have $f_ix_j^{d_j}=f_jx_i^{d_i}$ for some f_i, f_j. But this implies $f_i=x_i^{d_i}$, so the function must be constant.

3.9 $S(X)$ is the polynomial ring $k[X_0,X_1]$, but $S(Y)$ is the subring $k[X_0^2,X_0,X_1,X_1^2]$ of $k[X_0,X_1,X_2]$, which is not a graded polynomial ring in 2 variables (as the space of elements of the smallest nonzero degree is 3 dimensional).

3.10 For any point $x\in X'$ there is an affine neighborhood U of x in X and a regular function f from U to Y with $\phi|U=f$. Therefore f is a regular function from the neighborhood $U\cap X'$ of x to Y and therefore to Y'. Hence ϕ is regular near each point of X' and is therefore regular.

3.11 We can assume that X is affine as the irreducible varieties of X containing P are just the closures of the irreducible varieties containing P of any affine neighborhood of P. But then the varieties containing P just correspond to the prime ideals of $A(X)$ contained in the maximal ideal M of P, which correspond to the prime ideals of the ring $A(X)$ localized at M, which are the prime ideals of the local ring O_P.

3.12 By exercise 2.6 there is an affine neighborhood Y of P with $\dim(Y)=\dim(X)$. But $O_{P,X}=O_{P,Y}$ so $\dim(X)=\dim(Y)=\dim(O_{P,Y})$ (by 3.2c) $=\dim(O_{P,X})$.

3.13 $O_{Y,X}$ is clearly a ring. Put $I=$ image of set of pairs $\{U,f\}$, f regular on U, with $f=0$ on $U\cap Y$. Then I is the unique maximal ideal, because if $\{V,g\}$ is not in I then it has an inverse $\{W,1/g\}$ where $W=V\cap$(set where $g\neq 0$), as $W\cap Y\neq 0$. The residue field is obviously $K(Y)$. To prove the result about dimensions, we can assume X affine. Put $B = A(X)$, p=functions on X vanishing on Y. Then by 1.8A, height$(p) + \dim(B/p) = \dim(B)$. But $\dim(B) = \dim(X)$ and $\dim(B/p) = \dim(Y)$ and height of p in B = height of maximal ideal of $O_{Y,X}$ = dimension of $O_{X,Y}$. Hence $\dim(O_{X,Y} + \dim(Y) = \dim(X)$.

3.14a We can assume that P^n is the set where $x_0 \neq 0$, and p is the point $(1 : 0 : \cdots : 0)$. If $x = (x_0 : \cdots : x_n) \in P^{n+1} - P$, then $x_i \neq 0$ for some $i > 0$. Therefore the line containing P and x meets P^n in $(0 : x_1 : \cdots : x_n)$, which is a morphism in the neighborhood $x_i \neq 0$ of x. Therefore ϕ is a morphism.

3.14b The projection maps (t^3, t^2u, tu^2, u^3) to $(t^3, t^2u, u^3) \in P^2$. It is easy to check that the image is the whole of the variety given by the equation $x_1^3 = x_2x_0^2$. For $x_2 \neq 0$ this is the same as the variety given by $y^3 = x^2$ which has a cusp at $(0, 0)$, i.e., the image has a cusp at $(0, 0, 1)$.

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4.1 If $f = g$ on $U \cap V$, then the function which is f on U and g on V is clearly regular. Therefore the union of all open sets on which f is represented by a regular function is the largest open set on which f is regular.

4.2 A map is regular if and only if it is regular in a neighborhood of each point, so the conclusion follows as in 4.1.

4.3a $f = x_1/x_0$ is defined in the set where $x_0 \neq 0$. This set is isomorphic to A^2, and f is just projection to the first coordinate.

4.3b ϕ is defined everywhere except the point $(1 : 0 : 0)$.

4.4a See exercise 3.1c.

4.4b The maps taking $t \in A^1$ to (t^3, t^2) and (x, y) to y/x (for $x \neq 0$) are inverse birational isomorphisms from the cuspidal curve to A^1.

4.4c The projection maps $(x : y : z)$ to $(x : y)$ if $(x : y : z) \neq (0 : 0 : 1)$. The inverse map from P^1 to Y takes $(x : y)$ to $((y^2 - x^2)x : (y^2 - x^2)y : x^3)$ for $(x : y) \neq (1 : \pm 1)$.

4.5 The subvariety of Q given by $w \neq 0$ is isomorphic to A^2 by $(w : x : y : z) \to (x/w, y/w)$, $(x, y) \to (1 : x : y : xy)$. Thererfore Q is birational to $Q - \{w = 0\}$, which is isomorphic to A^2, which is birational to P^2. Q is isomorphic to $P^1 \times P^1$, which is not isomorphic to P^2 as it contains 2 closed 1-dimensional subvarieties that do not intersect.

4.6ab Put $U = V = \{(x : y : z) | xyz \neq 0\}$. Φ clearly maps U to V, and ϕ^2 maps $(x : y : z)$ to $(ax : xy : az) = (x : y : z)$ where $a = xyz$, so ϕ^2 is the identity map.

4.6c $\phi = \phi^{-1}$ is defined everywhere on P^2 except at the points $(1 : 0 : 0)$, $(0 : 1 : 0)$, $(0 : 0 : 1)$. Remark: The group of birational transformations of P^2 is generated by quadratic transformations (or by one quadratic transformation and $PGL_3(k)$) and very little about it seems to be known beyond the fact that it is very large.

4.7 We can assume that X and Y are closed subsets of A^n, and $P = Q = 0$. If f is a homomorphism from $O_{Q,Y}$ to $O_{P,X}$ then define a map g from an open subset of X to Y by

$$g(x_1, \ldots, x_n) = (f(y_1)(x_1, \ldots, x_n), f(y_2)(x_1, \ldots), \ldots)$$

where y_i is the i'th coordinate function on A^n. This is defined on the open set where all the $f(y_i)$'s are defined. Likewise we can define a similar map from an open set of Y to X, and the composition of these two maps is the identity wherever it is defined. Therefore there is an isomorphism from an open set of X to an open set of Y taking P to Q.

4.8a Clearly the cardinality of P^n is at most $(n + 1)card(k)^n$ which is the cardinality of k. To prove the other inequality we can assume that X is contained in A^n. If the possible values of any coordinate $x_1, \ldots, x_n$ are finite, the X consists of a finite number of points, so we can assume that one coordinate, say x_1, takes on an infinite number of values. By elimination theory the condition for a point with a given value of x_1 to exist on X is given by a finite number of equalities and inequalities in x_1. Therefore the possible values of x_1 are either a finite set or the complement of a finite set in k. But we know the number of possible values of x_1 is infinite, so the number of values is the cardinality of k minus a finite number, which is the cardinality of k.

4.8b Any two curves have the same cardinality and the finite complement topology, and so are homeomorphic.

4.9 We can assume that X is affine and is contained in A^n, the set of points in P^n with first $x_0 \neq 0$. The field of fractions $k(X)$ is generated by $x_1, \ldots, x_n$, so we can assume that $x_1, \ldots, x_r$ is a separating transcendence basis for $k(X)/k$ by 4.7A and 4.8A, and $k(X)$ is generated by $a_{r+1}x_{r+1} + \cdots + a_n x_n$ for some a_i's in k, by 4.6A. As $r \leq n - 2$ we can find a form $b_{r+1}x_{r+1} + \cdots + b_n x_n$ not proportional to $a_{r+1}x_{r+1} + \cdots + a_n x_n$. Choose any point at infinity not in this plane or in $\bar{X}$. Then the projection from this point to the plane maps $k(hyperplane)$ onto $k(X)$, so it is an isomorphism from the function field of the image of X to $k(X)$, and therefore a birational isomorphism.

4.10 If $(x, y, w : z) \in A^2 \times P^2$ is in $\phi^{-1}(Y)-$(exceptional curve) then $y^2 = x^3$, $xz = yw$, so $x^2(z^2 - xw^2) = 0$, so $z^2 - xw^2 = 0$. Therefore the only possibility for this point to lie on the exceptional curve $x = y = 0$ is $(0, 0, 1 : 0)$. If $w = 0$ then $x = 0$ which is not possible, so we can define the map f from $\bar{Y}$ to A^1 by $f(x, y, w : z) = z/w$. The inverse takes t to $(t^2, t^3, 1 : t)$, so $\bar{Y}$ is isomorphic to A^1.

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5.1a This is the tacnode. The singular points are the points with $x^2 = x^4 + y^4$, $2x = 4x^3$, and $4y^3 = 0$, so (at least in characteristic 0) the only singular point is $(0,0)$.

5.1b This is the node; singular point is $(0,0)$.

5.1c This is the cusp; singular point is $(0,0)$.

5.1d This is the triple point; singular point is $(0,0)$.

5.2 The singular points of $f(x,y,z) = 0$ are given by $f = 0$, $\frac{\partial f}{\partial x} = 0$, $\frac{\partial f}{\partial y} = 0$, and $\frac{\partial f}{\partial z} = 0$.

5.2a This is the pinch point; singular points are where $xy^2 = z^2$, $y^2 = 0$, $2xy = 0$, and $2z = 0$, which is the line $y = z = 0$.

5.2b This is the conical double point; singular points are where $x^2 + y^2 = z^2$, $2x = 0$, $2y = 0$, and $2z = 0$, which is the point $(0,0,0)$.

5.2c This is the double line; singular points are where $xy + x^3 + y^3 = 0$, $y + 3x^2 = 0$, $x + 3y^2 = 0$, and $0 = 0$, which is the line $x = y = 0$.

5.3a If P is a point on Y then P is a nonsingular point of Y is equivalent to saying that one of $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ are nonzero at P, which is equivalent to saying that f has a term of degree 1 in x and y, which is equivalent to saying that $\mu_P(Y) = 1$.

5.3b The singularities in 1a, 1b, and 1c have multiplicity 2, and 1d has multiplicity 3.

5.4a f and g both vanish at only a finite number of points, so we can find a polynomial $h(y)$ which vanishes whenever f and g both vanish, so $h^n \in (f,g)$ for some n, so we can assume $n = 1$. The submodules of $O_P/(f,g)$ correspond to ideals of O_P containing f and g, so it is sufficient to show that $k[x,y]/(f,g)$ is finite dimensional (as its dimension is at least the length of $O_P/(f,g)$). But if we have polynomials $h_1(x)$ and $h_2(y)$ of degrees m and n in (f,g) then $k[x,y]/(f,g)$ has dimension at most that of $k[x,y]/(h_1,h_2)$ which is mn which is finite.

5.4b Put $P = (0,0)$ and take any line L not in the tangent cone of Y. We can assume that L is the line $y = 0$, so the terms of lowest degree in f contain x^m (where m is the multiplicity of Y at P). Then $O_P/(f,g) = O_P/(y, x^m + \cdots) = O_Q/(x^m + \cdots)$ which has length m (where O_Q is the local ring of $Q = 0 \in A^1$).

5.4c We can assume that L is $y = 0$. If $z \neq 0$, the equation of the curve Y is $f(x) + y(*) = 0$ where f if a polynomial in x of some degree n. Then if x is a root of f of multiplicity m, we have $(L.Y)_(x,0) = m$, so the sums of the intersection multiplicities along the x axis is the number of roots of f which is n. On the other hand, at the point $(0:1:0)$ the intersection multiplicity is $d - n$ as the equation for f is locally $z^{d-n} + \cdots + x(*) = 0$. So the sum of all intersection multiplicities is $n + d - n = d$.

5.5 If the characteristic p does not divide d we can use $x^d + y^d + z^d = 0$ Otherwise we can use $xy^{d-1} + yz^{d-1} + zx^{d-1} = 0$.

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6.1a By 6.7, Y is isomorphic to an open subset of some projective space, and therefore to a proper open subset of P^1, and therefore to some open subset of A^1.

6.1b We can assume $Y = A^1 \backslash \{a_1, \ldots, a_n\}$. Then Y is isomorphic to the subset $y(x-a_1)\cdots(x-a_n) = 0$ of A^1.

6.1c Any element of $A(Y)$ can be written uniquely in the form $a(x-b_1)^{c_1}\cdots(x-b_n)^{c_n}$ with c_i some integer, and c_i positive if b_i is not one of $a_1, \ldots, a_n$. Hence $A(Y)$ is a U.F.D. with primes $x-b_i$ for $b_i \neq a_1, \ldots, a_n$.

6.2a Singular points must satisfy $y^2-x^3+x = 0$, $2y = 0$, $-3x^2+1 = 0$, and if k does not have characteristic 2 this implies $y = 0$, $x = 0, \pm 1$ which contradicts $-3x^2+1 = 0$. The polynomial y^2-x^3+x is irreducible, because if it were reducible then its two factors would intersect somewhere (possibly at infinity) and this point of intersection would be singular. (We should really also check that the curve has no singular points at infinity!) As Y is nonsingular, all points of Y are normal, so Y is normal, so $A(Y)$ is integrally closed.

6.2b $k[x]$ is clearly a polynomial ring, and $y^2 \in k[x]$ so y is in the integral closure of $k[x]$. So A is contained in the integral closure of $k[x]$, and is therefore equal to the integral closure because A is integrally closed.

6.2c The automorphism $x \mapsto x$, $y \mapsto -y$ is an automorphism of $k[x,y]$ which maps the ideal (y^2-x^3+x) to itself and therefore induces an automorphism of A (fixing x). Any element of A can be written as $yf(x)+g(x)$, so its norm is $(g(x)+yf(x))(g(x)-yf(x)) = g(x)^2-f(x)^2(x^3-x) \in k[x]$. The remaining properties of $N(a)$ are trivial to check.

6.2d If a is a unit then $N(a)$ is also a unit (with inverse $N(1/a)$) so must be an element of k as these are the only units in $k[x]$. But if $a = yf(x)+g(x)$, then its norm is $g(x)^2-f(x)^2(x^3-x)$ and if f is nonzero then the second term has odd degree while the first has even degree so their sum cannot be a constant. Hence $f = 0$, and g^2 is a constant, so a is a constant. To show that A is not a UFD, note that x and y are irreducible (this follows easily by looking at their norms x^2 and x^3-x and noting that there are no elements whose norms is a degree 1 polynomial). But $x|y^2$ and y is not a unit times x, so A cannot be a UFD.

6.2e Y is clearly not P^1, and by exercise 6.1c it is not A^1 minus a finite number of points, so Y is not rational.

6.3a Map $A^2\backslash(0,0)$ to P^1 by $(x,y) \mapsto (x:y)$.

6.3b Map $P^1\backslash\infty$ to A^1 in the obvious way.

6.4 Any nonconstant rational map from Y to P^1 induces ϕ^* from $k(x)$ to $k(Y)$, which is injective. Then every valuation ring of $k(x)$ can be extended to one of $k(Y)$, so every point of P^1 is the image of a point of Y. For every $p \in P^1$, $\phi^{-1}(P)$ is closed. If it was infinite it would have to be all of Y as the closure of any infinite subset of Y is Y, so the map ϕ would have to be constant.

6.5 We know that $\bar{X}$ is a curve. If $x \in \bar{X} - X$ then by 6.8 the map from X to X can be extended to a map from $x \cup X$ to X which is impossible. (Alternatively this problem follows from the fact that the image of any projective variety under a regular map is closed.)

6.6a The inverse of $x \mapsto (ax+b)/(cx+d)$ is $x \mapsto (dx-b)/(a-cx)$ if $ad-bc \neq 0$.

6.6b Follows from corollary 6.12 (i) and (iii).

6.6c Any automorphism of $k(x)$ maps x to $f(x)/g(x)$ for some coprime polynomials f and g, and $x = h(f(x)/g(x))$ for some rational function h. Therefore $f(x)/g(x)$ is not equal to $f(y)/g(y)$ if $x \neq y$. But if f or g have degree greater than 1 then $g(y)a = f(y)$ will usually have more than one solution for y. Hence f and g have degrees at most 1, and the result follows from part (a).

6.7 Any map from one curve to the other can be extended to a map from P^1 to P^1, so the points P_i must be mapped to the points Q_j, so $r = s$. The converse is true if and only if $r \leq 3$, because any set of at most 3 distinct points in P^1 can be mapped to any other set of the same size under $Aut(P^1)$, but this is not true for sets of 4 or more points.

7.1a The polynomials of degree m in $N+1$ variables restricted to the image of P^n in P^N give the polynomials of degree md in $n+1$ variables. Hence the Hilbert polynomial of the embedding of P^n in P^N is $f(dk)$ where $f(k) = \binom{k+n}{n}$ is the Hilbert polynomial of P^n (embedded in itself). So the Hilbert polynomial of the d-tuple embedding is $\binom{dk+n}{n} = (dk)^n/n! + \cdots$ so the degree of the embedding is d^n.

7.1b Similarly we find that the Hilbert polynomial of the Segre embedding of $P^r \times P^s$ is the product of the Hilbert polynomials of P^r and P^s, which is

$$\binom{k+r}{r}\binom{k+s}{s} = (k^r/r! + \cdots)(k^s/s! + \cdots) = \binom{r+s}{r}k^{r+s}/(r+s)! + \cdots$$

so the degree of the Segre embedding is $\binom{r+s}{r}$.

7.2a This follows from the fact that the Hilbert polynomial $P_{P^n}(k) = \binom{n+k}{n}$ has constant term 1.

7.2bc By 7.6c, $P_H(k) = \binom{k+n}{n} - \binom{k-d+n}{n}$, whose value at 0 is $1 - \binom{n-d}{n} = 1 - (-1)^n\binom{d-1}{n}$.

7.2d The Hilbert polynomial of this complete intersection is

$$\binom{k+3}{3} - \binom{k-a+3}{3} - \binom{k-b+3}{3} + \binom{k-a-b+3}{3}$$

whose constant term is $1 - \binom{3-a}{3} - \binom{3-b}{3} + \binom{3-a-b}{3}$ which is $1 - (ab(a+b-4)/2 + 1)$.

7.2e The Hilbert polynomial of $Y \times Z$ is the product of the Hilbert polynomials of Y and Z, from which the result follows easily.

7.3 We can assume that P is $(0,0) \in A^2$, and we can assume that if f if the function defining Y then $f(x,y) = y+$(terms of degree at least 2). By Ex. 5.4 the only line whose intersection multiplicity with Y at P is the line $y = 0$. In general the mapping takes $(x_0 : x_1 : x_2) \in Y$ to the point $(f_0(x_0,x_1,x_2) : f_1(x_0,x_1,x_2) : f_2(x_0,x_1,x_2))$ where $f_i = \frac{\partial f}{\partial x_i}$, which is well defined as long as one of the 3 numbers $f_i(x_0,x_1,x_2)$ is nonzero, i.e., the point P is nonsingular.

7.4 By Ex. 5.4, any line not tangent to Y and not passing through a singular point meets Y in exactly d distinct points. As Y has only a finite number of singular points, the lines intersection at least one of these form a proper closed subset of P^{2*} (in fact a union of lines). By 7.3 the lines tangent to Y are also contained in a proper closed subset of P^{2*}, so there is a nonempty open subset U of lines in P^{2*} intersecting Y in exactly d points.

7.5a We can assume that any point P of multiplicity at least d is $(0,0)$. But then the equation $f(x,y)$ defining Y has all terms of degree exactly d, so it is a product of linear factors, which is not possible if Y is irreducible of degree greater than 1.

7.5b As in 7.5a we can assume that the equation defining Y is of the form $f(x,y) + g(x,y) = 0$ where f is homogeneous of degree $d-1$ and g is homogeneous of degree d. If we make the substitution $t = y/x$ we find that $y = -f(t,1)/g(t,1)$, $x = yt$ gives an inverse rational map so that Y is birational to A^1.

7.6 Any linear variety obviously has degree 1 (by calculating its Hilbert polynomial). Assume that Y has degree 1. Then by 7.6b, Y is irreducible (as all components of Y have the same dimension). By theorem 7.7 if H is any hyperplane then $Y \cap H$ also has degree 1 (or $Y \subset H$, in which case Y is linear by induction on n). Therefore $Y \cap H$ is linear for every hyperplane H, and therefore for every linear variety H. In particular if $p, q \in Y$, then the intersection of Y with the line pq is linear and therefore is the line joining p and q. Hence Y contains any line joining two of its points, and is therefore linear.

7.7a We show that X is birational to the cone on Y which will show that X is irreducible and of dimension $r+1$. We choose a hyperplane "at infinity" in P^n not containing P or Y, and map X to the cone on Y by taking any line $PQ\backslash$(point at infinity) to the affine line on the cone over Y by taking Q to Q, P to the vertex of the cone. We can define a rational inverse in the obvious way.

7.7b We prove this when Y is any closed algebraic set, not necessarily reducible. If Y has dimension 0 then it is a union of d points and X is a union of at most $d-1$ lines, so the result is true in this case. If Y has dimension > 0 choose a generic hyperplane H containing P, which can be chosen to intersect Y transversely at generic points of the intersection as Y is nonsingular at P. The intersection of Y and H has degree at most $d \times \deg(H) = d$ by 7.7. The intersection of X with H is the union of the set of lines joining P and $H \cap Y$ which has degree less than d by induction on $\dim(Y)$. Again by 7.7, the degree of X is equal to the degree of $X \cap H$ as all components of $X \cap H$ have multiplicity 1 in the intersection (as H is generic). Hence the degree of X is less than that of Y. (Note that the intersection $X \cap H$ of an irreducible algebraic set X with a generic hyperplane H need not be irreducible! But see remark 7.9.1 on p. 245 of Hartshorne.)

7.8 Applying 7.7 to Y^r shows that Y is contained in a degree 1 variety H of dimension $r+1$ in P^n, which by 7.6 is a linear variety and therefore isomorphic to P^{r+1}.

Solution to Math256a section IV.1

1.1) Choose a positive integer n larger than $deg(K) = 2g - 2$, and g. By Riemann-Roch theorem, $l(nP) = n+1-g > 1$. Thus there exists a non-constant rational function f over X which has a pole at P of order $n > 0$, and regular everywhere else.

1.2) Induction on r. The case $r = 1$ follows from the previous exercise. Now assume there is a rational function f having poles at each of $P_1, \cdots, P_{r-1}$ of positive orders and regular everywhere else. Since f has no pole at P_r, we may let $n_r \geq 0$ be the coefficient of P_r in (f) (as a divisor). We may choose a rational function g which has a pole at P_r of order $> n_r$ and regular everywhere else (Cf. 1.1). Then $f \cdot g$ has poles precisely at $P_1, \cdots, P_r$ of positive orders.

1.5) Since D is effective, $|K - D| \subseteq |K|$. Therefore $l(K - D) \leq l(K)$. By Riemann-Roch theorem, $l(D) = l(K-D)-g+deg(D)+1 \leq l(K)-g+deg(D)+1 = deg(D)+1$, since $l(K) = g$. It follows that $dim(|D|) = l(D) - 1 \leq deg(D)$. Proof showes that the equality holds iff $l(K - D) = l(K) = g$. If $D = 0$, it is trivially true. If $g = 0$, $deg(K) = -2$, so $deg(K - D) < 0$. Hence $l(K - D) = 0$. It follows that $l(K - D) = l(K) = 0$.

Conversely, suppose $l(K - D) = l(K) = g$. Suppose $D \neq 0$. Let $P \in Supp(D)$. Then $|K - D| \subseteq |K - P| \subseteq |K|$, thus $l(K - D) \leq l(K - P) \leq l(K)$, and hence they are all equal. By Riemann-Roch, $l(P) = l(K - P) + 2 - g = 2$. Therefore there is a rational function f with one pole at P of order 1 and regualr everywhere else. This function defines an isomorphism from X to $\mathbb{P}^1$, thus $g(X) = g(\mathbb{P}^1) = 0$.

1.6) Let P be a point on X. By Riemann-Roch,

$$l((g + 1)P) = l(K - (g + 1)P) + (g + 1) + 1 - g \geq 2.$$

Thus there exists a rational function f with a pole at P of order $g + 1$ and regular everywhere else. This ration function induces a morphism $f : X \to \mathbb{P}^1$ by sending $(g+1)P$ to $\infty \in \mathbb{P}^1$. By II Prop. 6.9, $deg(f) = deg((g+1)P) = g+1$.

1.7) (a) It is clear that $deg(K) = 2g - 2 = 2$ and $dim(|K|) = l(K) - 1 = 1$. Suppose P is a base point of $|K|$, then $l(K - P) = l(K) = 2$ by definition. By Riemann-Roch, $l(P) = 2 + 2 - 2 = 2$. Thus there exist a non-constant rational function f with a pole at P of order 1 and regular everywhere else. As we did before, f defines an isomorphism from X to $\mathbb{P}^1$, contradiction since X has genus 2 not 0. Therefore $|K|$ has no base point. Alternatively, one may apply directly Prop 3.1 on page 307. By II, 7.8.1, there is a finite morphism $f : X \to \mathbb{P}^1$ with degree equal to $deg(K) = 2$. Therefore X must be a hyperelliptic curve.

Solution to Math256a section IV.3 (H.Zhu, 1994)

3.1) One direction follows easily from 3.3.4. We show that D is not very ample when $deg(D) < 5$. Now suppose D is very ample, then $l(D) = l(D - P - Q) + 2 \geq 2$. Futhermore, if $l(D) = 2$, $dim|D| = 1$, thus $|D|$ defines an isomorphism from X to $\mathbb{P}^1$, which is obsurd. Thus we have $l(D) > 2$.

If $deg(D) \leq 1$, Since $l(D) \neq 0$, we may apply Ex. 1.5., $l(D) \leq deg(D) + 1 \leq 2$, thus D is not very ample.

If $deg(D) = 2$, $l(D) = l(K - D) + 1$. Since $D \neq 0$, $l(K - D) < l(K) = 2$. Thus $l(D) \leq 2$. Contradiction.

If $deg(D) = 3$, then $l(K - D) = 0$. So $l(D) = 2$. Contradiction.

If $deg(D) = 4$, then $l(D) = 3$. By 3.2, we know that D is base point free. Thus $|D|$ defines a morphism from X to $\mathbb{P}^2$. But this is impossible since any plane curve has genus $(d-1)(d-2)/2$, which is never 2. Contradiction.

We conclude that $deg(D) \geq 5$.

3.2) (a) From I, Ex.7.2, $g(X) = 3$. It results in $l(K) = 3$ and $deg(K) = 4$. Denote $D =: X.L$. Recall Bezout's theorem from I, 7. so $deg(D) = 4$. Now claim that $l(D) \geq 3$. Since the line L on X is determined exactly by two points (not necessary distinct) so $dim|L| = 2$, i.e. $l(D) = 3$. (This may be rigorously proved by considering the possible linearly independent sets.) Then $l(K-D) = l(D) + g - deg(D) - 1 = 1$. But $deg(K-D) = 0$ and $l(K-D) = deg(K-D)+1$, thus $K = D$ by Ex.1.5.

(b) Since D is an effective divisor of degree 2, $D = P_1 + P_2$ for some two points on X (not necessary distinct). Suppose there is an effective divisor $Q_1 + Q_2$ such that $P_1 + P_2 \sim Q_1 + Q_2$. Since the line passing thru P_1 and P_2 intersects X at two other points P_3 and P_4. By (a) we have $K = P_1 + P_2 + P_3 + P_4$, so Q_1, Q_2, P_3, P_4 is collinear. Hence Q_1, Q_2 coincide with P_1, P_2. Thus $dim|D| = 0$.

(c) From Ex. 1.7.(a), $dim|K| = 1$. But we may pick an effective canonical divisor K such that $dim|K| = 0$ by (b). Thus X can not be a hyperelliptic curve.

3.3) It is clear that the second statement follows from the first one since K is not very ample on a hyperelliptic curve. (Cf. 5.2.) By II.Ex.8.4, $\omega_X \cong \mathcal{O}(\sum d_i - n - 1)$. Since the dimension of the global section of this invertible sheaf equals $g \geq 2$, ω_X has to be very ample. (Otherwise it has no global sections.) This is equivalent to saying that the canonical divisor K is very ample.

We showed in Ex.1.7 that any curve of genus 2 has to be a hyperelliptic curve, and its canonical divisor is not very ample. Thus it can not be a complete intersection in $\mathbb{P}^n$.

3.4) (a) Denote θ as the corresponding ring homomorphism. $deg(\theta) = d$. We know that the image of the d-uple embedding is $Z(ker(\theta))$. We may check that $ker(\theta)$ is generated by $x_{i+1}^2 - x_i x_{i+2}$, for $i = 0, \cdots, d-2$, and $x_0 x_d - x_1 x_{d-1}$.

(b) If i is the close immersion, denote $i^*(\mathcal{O}(1))$ as D. Because $dim|D| = n$ and $deg(D) = d$, we have $l(D) = n + 1 \leq deg(D) + 1 = d + 1 \leq n + 1$. Therefore $n = d$ and $g = 0$ by Ex..1.5. Consequently, $X \cong \mathbb{P}^1$. Since X does not lie in $\mathbb{P}^{n-1}$, the natural map $\Gamma(\mathbb{P}^n, \mathcal{O}(1)) \to \Gamma(X, i^*(\mathcal{O}(1))$ is injective. Thinking X

as $\mathbb{P}^1$, D corresponds to a $(n+1)$-dimension subspace $V \in \Gamma(\mathbb{P}^1, \mathcal{O}(n))$ hence they are equal since the later has dimension $n+1$. Therefore, X is indeed a rational normal curve. (see II. 7.8.1).

(c) It is clear this curve X can not be in $\mathbb{P}^1$. From (b), X is a plane curve of degree 2.

(d) Suppose X is not a plane cubic curve, we apply (b), have $X \subseteq \mathbb{P}^3 \backslash \mathbb{P}^2$, thus it is a rational normal curve of degree 3, which is indeed a twisted cubic.

3.6) (a) When $n \geq 4$, Ex.3.4 (b) implies that X is a rational normal curve. If X is a plane curve, $g(X) = (d-1)(d-2)/2 = 3$. Otherwise, $X \subseteq \mathbb{P}^3 \backslash \mathbb{P}^2$, we claim that $g = 0, 1$. Suppose $g = 2$, then Ex.3.1 shows that any divisor of degree 4 is not very ample, that is X can not be embedded to $\mathbb{P}^3$, which is absurd. If $g = 0$, then it is a rational quartic curve by II,7.8.6. g can not be 3 since X is not a plane curve. Thus g has to be 1.

3.7) Suppose C is a nonsingular curve which projects to the given curve X. We prove that $deg(C) = 4$ which will soon lead a contradiction with assertions in Ex.3.6. To prove our first claim, we carefully choose a suitable hyperplane H passing the projection point to cut C which intersects with $\mathbb{P}^2$ by a line L such that there is a 1-1 map from $C.H$ to $X.L$. We conclude that $deg(C) = 4$ by recalling Bezout's Theorem.

Since C has a node, it can not lie in case (1) or (2) in Ex.3.6. By Hurwitz's theorem, $g(C) \geq g(\tilde{X}) = 3 - 1$ from 3.11.1, thus $g(X) \neq 1$. Contradiction with Ex.3.6. Thus such C does not exist.

3.8) (a) By a simple calculation, the tangent vector is $(1, 0, 0)$ at each point. Pick an point $P = (x_0, y_0, z_0)$ on X, its tangent line is given by the intersection of two hyperplanes: $y = y_0$ and $z = z_0$. Writen in homogeneous polynomial, $y = y_0 w$ and $z = z_0 w$. Thus all tangent lines pass through the point at infinity $(1:0:0:0)$. There is one strange point on this curve.

(b) Note that when $char(k) = 0$, X has finitely many singular points. By choosing a proper projection, we may still project X in $\mathbb{P}^3$. Suppose P is a strange point on X. Choose an affine cover such that P is the infinity point on x-axis, and other relevant conditions in the proof of Theorem 3.9. The resulted morphism is ramified at all but finitely many points on X. The image is thus a point otherwise the map is inseparable which is not the case over a field of *char* 0. Hence X is the line $\mathbb{P}^1$.

3.9) Three points are collinear iff there is a multisecant line passing through them. A hyperpalne in $\mathbb{P}^3$ intersects X at exactly d points iff the hyperplane does not pass any tangent lines of X. Prop 3.5 showed that $dim(Tan(X)) \leq 2$. Also it is not hard to show that the dimension of the space of multisecant lines of X has dimension ≤ 1. Hence the union of these two spaces is a proper closed subspace of $\mathbb{P}^{3*}$ which is of dimension 3. Therefore almost all hyperplanes intersect X in exactly d points.