Solutions to *Algebraic Geometry*[*]

Steven V Sam
ssam@math.mit.edu

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Contents

1 Affine Varieties

1. **Exercise.**

   (a) Let Y be the plane curve $y = x^2$. Show that $A(Y)$ is isomorphic to a polynomial ring in one variable over k.

   (b) Let Z be the plane curve $xy = 1$. Show that $A(Z)$ is not isomorphic to a polynomial ring in one variable over k.

   (c) Let f be any irreducible quadratic polynomial in $k[x, y]$, and let W be the conic defined by f. Show that $A(W)$ is isomorphic to $A(Y)$ or $A(Z)$. Which one is it when?

   Solution.

   (a) The coordinate ring $A(Y)$ is $k[x, y]/(y - x^2)$. We have a map $k[z] \to A(Y)$ given by $z \mapsto x$. This is clearly injective and surjective, so $k[z] \cong A(Y)$.

   (b) The coordinate ring $A(Z)$ is $k[x, y]/(xy - 1)$. If $A(Z) \to k[z]$ is a map, then x and y are both mapped to units, and hence elements of k. Thus no such map can be surjective.

   (c) Given an irreducible quadratic polynomial $P = ax^2 + bxy + cy^2 + dx + ey + f$, we may assume without loss of generality that either $a \neq 0$ or $b \neq 0$. Note that performing invertible changes of coordinates preserve the property that P is irreducible.

   In the first case, we complete the square for $x^2 + (by + d)x$ and do a change of coordinates so that we may write the polynomial as $x^2 + cy^2 + ey + f$ for some new values of c, e, f. If $c = 0$, then we must have $e \neq 0$ because of the irreducibility, so doing a linear change of coordinates gives $x^2 - y$. If $c \neq 0$, then again completing the square and doing a change

[*] by Robin Hartshorne

of coordinates gives $x^2 + y^2 + f$ for some new (nonzero) value of f. We may use a linear change $x \mapsto \alpha x + \beta y$ and $y \mapsto \gamma x + \delta y$ in order to cancel the resulting x^2 and y^2 terms, so we are left with something of the form $xy + f$ where $f \neq 0$.

In the second case, we can assume that $a = c = 0$ or else we are in the previous case, so we have $xy + dx + ey + f$. Then writing it as $x(y + d) + ey + f$ and doing $y \mapsto y - d$ gives $y(x + e) + (f - de)$, and again doing $x \mapsto x - e$ gives $xy + f - de$ and $f - de \neq 0$ by irreducibility, so we can scale our coordinates to get $xy - 1$. $\square$

2. **Exercise (The twisted cubic curve).** Let $Y \subseteq \mathbf{A}^3$ be the set $Y = \{(t, t^2, t^3) \mid t \in k\}$. Show that Y is an affine variety of dimension 1. Find generators for the ideal $I(Y)$. Show that $A(Y)$ is isomorphic to a polynomial ring in one variable over k. We say that Y is given by the *parametric representation* $x = t$, $y = t^2$, $z = t^3$.

   Solution. The ideal $I(Y)$ is generated by the polynomials $z - x^3$ and $y - x^2$. We have a map $k[t] \to A(Y)$ defined by $t \mapsto x$ which is injective and surjective. Since this is an isomorphism, $A(Y)$ has dimension 1, so Y has dimension 1. Also, $k[t]$ is a domain, so $I(Y)$ is a prime ideal, and hence Y is an affine variety. $\square$

3. **Exercise.** Let Y be the algebraic set in $\mathbf{A}^3$ defined by the two polynomials $x^2 - yz$ and $xz - x$. Show that Y is a union of three irreducible components. Describe them and find their prime ideals.

   Solution. The intersection of Y with the plane $z = 1$ is defined by $x^2 = y$. The intersection of Y with the plane $z = 0$ is defined by $x = 0$. On the rest of Y, the equation $xz = x$ has no solution except $x = 0$, and the equation $x^2 - yz$ becomes $yz = 0$, so we also have $y = 0$. Hence we see that Y is the union of the affine varieties defined by the prime ideals $(z - 1, x^2 - y)$, (x, y), and (x, z). $\square$

4. **Exercise.** If we identify $\mathbf{A}^2$ with $\mathbf{A}^1 \times \mathbf{A}^1$ in the natural way, show that the Zariski topology on $\mathbf{A}^2$ is not the product topology of the Zariski topologies on the two copies of $\mathbf{A}^1$.

   Solution. The zero locus of $y = x^2$ is a closed subset in the Zariski topology of $\mathbf{A}^2$, but cannot be a closed subset in the product topology on $\mathbf{A}^1 \times \mathbf{A}^1$ because a basis for the topology is $U \times V$ where U and V are both complements of finite sets of points in $\mathbf{A}^1$, and there is no way to write the complement of $y - x^2$ as a union of such sets. $\square$

5. **Exercise.** Show that a k-algebra B is isomorphic to the affine coordinate ring of some algebraic set in $\mathbf{A}^n$, for some n, if and only if B is a finitely generated k-algebra with no nilpotent elements.

   Solution. Since B is finitely generated, we can write $B \cong k[x_1, \ldots, x_n]/\mathfrak{a}$ for some n and some ideal $\mathfrak{a}$. The condition that B have no nilpotents is equivalent to $\mathfrak{a}$ being a radical ideal. So B is the coordinate ring of the algebraic set $Z(\mathfrak{a})$ (Corollary 1.4). $\square$

6. **Exercise.** Any nonempty open subset of an irreducible topological space is dense and irreducible. If Y is a subset of a topological space X, which is irreducible in its induced topology, then the closure $\overline{Y}$ is also irreducible.

   Solution. Let X be an irreducible space, and let $U \subseteq$ be a nonempty open subset. Then $X = \overline{U} \cup (X \setminus U)$, where $\overline{U}$ denotes the closure of U, so either $X = \overline{U}$ or $X = X \setminus U$. The latter is ruled out since U is nonempty, so U is dense in X. Also, if U is the union of two closed subsets $U_1 \cup U_2$, then there exist X_1 and X_2 such that $X_1 \cap U = U_1$ and $X_2 \cap U = U_2$. Hence $X = X_1' \cup X_2'$ where $X_i' = X_i \cup (X \setminus U)$. If $U_1 \neq U_2$, then $X_1' \neq X_2'$, which contradicts that X is irreducible, so U is irreducible.

Now let $Y \subseteq X$ be some subset which is irreducible with respect to its subspace topology. Suppose we can write its closure $\overline{Y}$ as a union of proper closed subsets $\overline{Y} = Y_1 \cup Y_2$. Then Y_1 and Y_2 are both closed in X, so $Y_1 \cap Y$ and $Y_2 \cap Y$ are closed in Y. Then either $Y = Y_1 \cap Y$ or $Y = Y_2 \cap Y$, or equivalently, $Y \subseteq Y_1$ or $Y \subseteq Y_2$. In the first case, this implies that $Y_1 = \overline{Y}$, which is a contradiction, and similarly in the second case. Thus $\overline{Y}$ is also irreducible. □

7. **Exercise.**

(a) Show that the following conditions are equivalent for a topological space X: (i) X is Noetherian; (ii) every nonempty family of closed subsets has a minimal element; (iii) X satisfies the ascending chain condition for open subsets; (iv) every nonempty family of open subsets has a maximal element.

(b) A Noetherian topological space is *quasi-compact*, i.e., every open cover has a finite subcover.

(c) Any subset of a Noetherian topological space is Noetherian in its induced topology.

(d) A Noetherian space which is also Hausdorff must be a finite set with the discrete topology.

Solution.

(a) The equivalence of (i) and (iii) is trivial, as is the equivalence of (ii) and (iv). We show that (i) is equivalent to (ii). Assuming (i), any nonempty family of closed subsets must have a minimal element. For instance, pick any closed subset Z_0. If it is not minimal, pick Z_1 contained in it. If that is not minimal, pick Z_2 contained in it. Eventually, this must terminate because we get a descending chain of closed subsets $Z_0 \supseteq Z_1 \supseteq Z_2 \supseteq \cdots$. Assuming (ii), let $Z_0 \supseteq Z_1 \supset Z_2 \supseteq \cdots$ be a descending chain of closed subsets. Then this is a family of closed subsets which has a minimal element, say Z_r. So $Z_r = Z_t$ for $t \geq r$, which means X is Noetherian.

(b) Let $\{U_\alpha\}$ be an open cover of a Noetherian space X. Pick any open set U_1. Inductively if $U_1 \cup \cdots U_{i-1} \neq X$, we pick U_i to be any open subset not contained in $U_1 \cup \cdots \cup U_{i-1}$, otherwise, let $U_i = U_{i-1}$. Then setting $Z_i = X \setminus (U_1 \cup \cdots \cup U_i)$, we get a descending chain of closed subsets $Z_1 \supseteq Z_2 \supseteq \cdots$, which must terminate, so there exists r such that $Z_r = Z_t$ for $t \geq r$. This implies that at the rth step, $U_1 \cup \cdots \cup U_r = X$, so X is quasi-compact.

(c) This is an immediate consequence of characterization (iv) of (a).

(d) Let X be a Noetherian Hausdorff space, and suppose X has infinitely many points. Then pick $x_1, y_1 \in X$, we can find disjoint open sets $U_1 \ni x_1$ and $V_1 \ni y_1$. Then either $X \setminus U_1$ is infinite or $X \setminus V_1$ is infinite. Without loss of generality, $X \setminus U_1$ is infinite. Since $X \setminus U_1$ is Hausdorff, we can find $U_2 \subseteq X \setminus U_1$ which is open relative to $X \setminus U_1$ and such that $X \setminus (U_1 \cup U_2)$ is also infinite. In this way, we can continue to find U_i for all $i > 0$ such that $U_i \subseteq X \setminus (U_1 \cup \cdots \cup U_{i-1})$ is open relative to $X \setminus (U_1 \cup \cdots \cup U_{i-1})$ and $X \setminus (U_1 \cup \cdots \cup U_i)$ is infinite. For each U_i, there is an open set $U_i' \subseteq X$ such that $U_i' \cap (X \setminus (U_1 \cup \cdots \cup U_{i-1}))$ is open relative to $X \setminus (U_1 \cup \cdots \cup U_{i-1})$. Setting $Z_i = U_1' \cup \cdots \cup U_i'$, we get an ascending chain of open sets $Z_1 \subseteq Z_2 \subseteq \cdots$, which must terminate by (a), and which contradicts the infinitude of X. Hence X must be a finite set. Since the points of a Hausdorff space are closed, we conclude that X has the discrete topology. □

8. **Exercise.** Let Y be an affine variety of dimension r in $\mathbf{A}^n$. Let H be a hypersurface in $\mathbf{A}^n$, and assume $Y \not\subseteq H$. Then every irreducible component of $Y \cap H$ has dimension $r - 1$.

Solution. Let f be the irreducible polynomial defining H. Since $Y \not\subseteq H$, the image of f under the quotient map $k[x_1, \ldots, x_n] \to A(Y)$ is nonzero. The irreducible components of $H \cap Y$ are precisely the affine varieties defined by the minimal primes of $A(Y)$ which contain f. Hence they have dimension $\dim Y - 1$ by Theorem 1.11A. $\square$

9. **Exercise.** Let $\mathfrak{a} \subseteq A = k[x_1, \ldots, x_n]$ be an ideal which can be generated by r elements. Then every irreducible component of $Z(\mathfrak{a})$ has dimension $\geq n - r$.

   Solution. The irreducible components Z_i of $Z(\mathfrak{a})$ correspond to prime ideals $\mathfrak{p}_i$ of $A/\mathfrak{a}$. Hence by Theorem 1.8A(b), $\dim Z_i = \dim A/\mathfrak{p}_i = n - \operatorname{height} \mathfrak{p}_i$. Using Theorem 1.11A, we see that the height of a prime ideal is bounded above by its minimal number of generators, so $\dim Z_i \geq n - r$. $\square$

10. **Exercise.**

    (a) If Y is any subset of a topological space X, then $\dim Y \leq \dim X$.

    (b) If X is a topological space which is covered by a family of open subsets $\{U_i\}$, then $\dim X = \sup \dim U_i$.

    (c) Give an example of a topological space X and a dense open subset U with $\dim U < \dim X$.

    (d) If Y is a closed subset of an irreducible finite-dimensional topological space X, and if $\dim Y = \dim X$, then $Y = X$.

    (e) Give an example of a Noetherian topological space of infinite dimension.

    Solution.

    (a) Let Y be a subset of topological space X. If $Y_0 \subset Y_1 \subset Y_2 \subset \cdots$ is an ascending chain of closed irreducible subsets of Y with strict inclusions, then $\overline{Y}_0 \subset \overline{Y}_1 \subset \overline{Y}_2 \subset \cdots$ is an ascending chain of closed irreducible subsets of X with strict inclusions, so $\dim Y \leq \dim X$.

    (b) From (a), we get that $\dim X \geq \sup \dim U_i$. Let $X_0 \subset X_1 \subset \cdots$ be a chain of closed irreducible subsets of X with maximal length. We can find some U_i such that $X_0' = X_0 \cap U_i \neq \varnothing$. Then X_0' is open relative to X_0 and hence is dense (Ex. 1.6). Since $X_1 \setminus X_0$ is a nonempty open subset of X_1, it intersects X_0', so intersects U_i. Continuing this reasoning, $U_i \cap X_j$ properly contains $U_i \cap X_{j-1}$ for all j, which means that $\dim U_i \geq \dim X$, and hence $\dim X = \sup \dim U_i$.

    (c) Let $X = \{a, b\}$ be a two point space whose open sets are $\{\varnothing, \{a\}, \{a, b\}\}$. Then $\dim X = 1$ by the chain $\{b\} \subset X$, but $U = \{a\}$ has dimension 0 and is dense in X.

    (d) Suppose that $r = \dim Y$ and $Y \neq X$. If $Y_0 \subset \cdots \subset Y_r$ is a chain of irreducible closed subsets of Y, then $Y_0 \subset \cdots \subset Y_r \subset X$ is a chain of irreducible closed subsets of X, so $\dim X \geq r + 1$. Hence if $\dim Y = \dim X$, then $Y = X$.

    (e) Let X_n be a Noetherian space of dimension n. Then the disjoint union $X = \coprod_{n \geq 0} X_n$ is Noetherian, but is infinite-dimensional. $\square$

2 Projective Varieties

1. **Exercise.** Prove the "homogeneous Nullstellensatz," which says that if $\mathfrak{a} \subseteq S$ is a homogeneous ideal, and if $f \in S$ is a homogeneous polynomial with $\deg f > 0$, such that $f(P) = 0$ for all $P \in Z(\mathfrak{a})$, then $f^q \in \mathfrak{a}$ for some $q > 0$.

   Solution. Let $\mathfrak{a} \subseteq S$ be a homogeneous ideal and $f \in S$ a homogeneous polynomial with $\deg f > 0$, such that $f(P) = 0$ for all $P \in Z(\mathfrak{a})$ in $\mathbf{P}^n$. Then f vanishes for all representatives for points in $Z(\mathfrak{a})$, so in particular, $f^r \in \mathfrak{a}$ when thinking of $Z(\mathfrak{a})$ as being in $\mathbf{A}^{n+1}$ (Theorem 1.3A). $\square$

2. **Exercise.** For a homogeneous ideal $\mathfrak{a} \subseteq S$, show that the following conditions are equivalent:

   (i) $Z(\mathfrak{a}) = \varnothing$;

   (ii) $\sqrt{\mathfrak{a}} =$ either S or the ideal $S_+ = \bigoplus_{d>0} S_d$;

   (iii) $\mathfrak{a} \supseteq S_d$ for some $d > 0$.

   Solution. First suppose (i) holds. By (Ex. 2.1), every polynomial $f \in S$ with $\deg f > 0$ satisfies $f^r \in \mathfrak{a}$ for some r, so $S_+ \subseteq \sqrt{\mathfrak{a}}$. If $\sqrt{\mathfrak{a}}$ contains any element of S_0, then it contains them all by virtue of being an ideal, so either $\sqrt{\mathfrak{a}} = S_+$ or $\sqrt{\mathfrak{a}} = S$.

   Now suppose that (ii) holds. Then for each x_i, there is an integer r_i such that $x_i^{r_i} \in \mathfrak{a}$. So for $N = \max r_i$, we have $x_i^N \in \mathfrak{a}$, and hence for d sufficiently large ($d \geq (n+1)(N-1)+1$ where $S = k[x_0, \ldots, x_n]$), every monomial of degree d contains a multiple of x_i^N for some i, so $S_d \subseteq \mathfrak{a}$.

   Finally, suppose that (iii) holds. Then $\mathfrak{a}$ contains x_i^d for $i = 0, \ldots, n$, which means that $Z(\mathfrak{a})$ consists of those points which are 0 for all x_i, and hence is empty. □

3. **Exercise.**

   (a) If $T_1 \subseteq T_2$ are subsets of S^h, then $Z(T_1) \supseteq Z(T_2)$.

   (b) If $Y_1 \subseteq Y_2$ are subsets of $\mathbf{P}^n$, then $I(Y_1) \supseteq I(Y_2)$.

   (c) For any two subsets Y_1, Y_2 of $\mathbf{P}^n$, $I(Y_1 \cup Y_2) = I(Y_1) \cap I(Y_2)$.

   (d) If $\mathfrak{a} \subseteq S$ is a homogeneous ideal with $Z(\mathfrak{a}) \neq \varnothing$, then $I(Z(\mathfrak{a})) = \sqrt{\mathfrak{a}}$.

   (e) For any subset $Y \subseteq \mathbf{P}^n$, $Z(I(Y)) = \overline{Y}$.

   Solution.

   (a) If $P \in Z(T_2)$, then $f(P) = 0$ for all $f \in T_2$, so in particular, this is true for all $f \in T_1$, so $P \in Z(T_1)$.

   (b) If $f \in I(Y_2)$, then $f(P) = 0$ for all $P \in Y_2$, so in particular, $f(P) = 0$ for all $P \in Y_1$, which means that $f \in I(Y_1)$.

   (c) We have $f \in I(Y_1 \cup Y_2)$ if and only if $f(P) = 0$ for all $P \in Y_1 \cup Y_2$ if and only if $f \in I(Y_1)$ and $f \in I(Y_2)$.

   (d) The inclusion $I(Z(\mathfrak{a})) \subseteq \sqrt{\mathfrak{a}}$ follows from (Ex. 2.1) because constant polynomials are not contained in $\mathfrak{a}$ if $Z(\mathfrak{a}) \neq \varnothing$. Conversely, if $f^r \in \mathfrak{a}$ for some r, then for $P \in Z(\mathfrak{a})$, $f^r(P) = 0$, so $f(P) = 0$.

   (e) For this we can pass to the affine case using Corollary 2.3. □

4. **Exercise.**

   (a) There is a 1-1 inclusion-reversing correspondence between algebraic sets in $\mathbf{P}^n$, and homogeneous radical ideals of S not equal to S_+, given by $Y \mapsto I(Y)$ and $\mathfrak{a} \mapsto Z(\mathfrak{a})$.

   (b) An algebraic set $Y \subseteq \mathbf{P}^n$ is irreducible if and only if $I(Y)$ is a prime ideal.

   (c) Show that $\mathbf{P}^n$ itself is irreducible.

   Solution.

   (a) We know that $Z(I(Y)) = \overline{Y} = Y$ (Ex. 2.3(e)) and that $I(Z(\mathfrak{a})) = \sqrt{\mathfrak{a}} = \mathfrak{a}$ (Ex. 2.3(d)) for $\mathfrak{a} \neq S_+$ (Ex. 2.2). The inclusion-reversing part is the content of (Ex. 2.3(a,b)).

(b) This is the same as the proof of Corollary 1.4 by using the fact that a homogeneous ideal $\mathfrak{a}$ is prime if for any two homogeneous elements f and g, $fg \in \mathfrak{a}$ implies either $f \in \mathfrak{a}$ or $g \in \mathfrak{a}$.

(c) The ideal $I(\mathbf{P}^n)$ is the zero ideal, which is prime in S, so $\mathbf{P}^n$ is irreducible (Ex. 2.4(b)). □

5. **Exercise.**

(a) $\mathbf{P}^n$ is a Noetherian topological space.

(b) Every algebraic set in $\mathbf{P}^n$ can be written uniquely as a finite union of irreducible algebraic sets, no one containing another. These are called its *irreducible components*.

Solution.

(a) Let $Y_1 \supseteq Y_2 \supseteq \cdots$ be a descending chain of closed subsets of $\mathbf{P}^n$. Then $I(Y_1) \subseteq I(Y_2) \subseteq \cdots$ is an ascending chain of ideals in S, which terminates because S is Noetherian, so using the fact $Y_i = Z(I(Y_i))$, we get that $\mathbf{P}^n$ is Noetherian.

(b) A closed subset of a Noetherian space is Noetherian, so an algebraic set in $\mathbf{P}^n$ is Noetherian, and hence can be expressed uniquely as a finite union of irreducible algebraic sets, no one containing another (Proposition 1.5). □

6. **Exercise.** If Y is a projective variety with homogeneous coordinate ring $S(Y)$, show that $\dim S(Y) = \dim Y + 1$.

Solution. Let $Y \subseteq \mathbf{P}^n$ be a projective variety with homogeneous coordinate ring $S(Y)$, and let U_i be the standard open affines of $\mathbf{P}^n$ with homeomorphisms $\varphi_i \colon U_i \to \mathbf{A}^n$. Let Y_i be the affine variety $\varphi_i(Y \cap U_i)$ with affine coordinate ring $A(Y_i) = k[x_1, \ldots, x_n]/I(Y_i)$. We claim that $A(Y_i)$ is isomorphic to the degree 0 part of the localization $S(Y)_{x_i}$. If we think of U_i as having coordinates $(a_0/a_i, \ldots, a_n/a_i)$ where a_i/a_i is omitted, then the polynomials of $A(Y_i)$ are equivalent to homogeneous polynomials of $S(Y)_{x_i}$ of degree 0 via homogenization (i.e., the map β used in the proof of Proposition 2.2). Then it follows that $S(Y)_{x_i} \cong A(Y_i)[x_i, x_i^{-1}]$. By Theorem 1.8A(a), the dimension of $A(Y_i)[x_i, x_i^{-1}]$ is $\dim A(Y_i) + 1$, which is equal to $\dim Y_i + 1$ by Proposition 1.7. Using (Ex. 1.10(b)), we see that $\dim Y = \sup \dim Y_i$. Finally, for $Y_i \neq \varnothing$, $\dim Y_i + 1 = \dim S(Y)_{x_i}$, and $\dim S(Y)_{x_i} = \dim S(Y)$ by Theorem 1.8A(a). In conclusion, $\dim Y + 1 = \dim S(Y)$. □

7. **Exercise.**

(a) $\dim \mathbf{P}^n = n$.

(b) If $Y \subseteq \mathbf{P}^n$ is a quasi-projective variety, then $\dim Y = \dim \overline{Y}$.

Solution.

(a) This is a direct consequence of (Ex. 2.6) because the coordinate ring of $\mathbf{P}^n$ is $k[x_0, \ldots, x_n]$.

(b) Following the proof of (Ex. 2.6), we see that it is enough to know that the dimension of a quasi-affine variety is the same as its closure, and this is the content of Proposition 1.10. □

8. **Exercise.** A projective variety $Y \subseteq \mathbf{P}^n$ has dimension $n-1$ if and only if it is the zero set of a single irreducible homogeneous polynomial f of positive degree. Y is called a *hypersurface* in $\mathbf{P}^n$.

Solution. Using (Ex. 2.6), this statement follows as it does in the proof of Proposition 1.13. □

9. **Exercise (Projective closure of an affine variety).** If $Y \subseteq \mathbf{A}^n$ is an affine variety, we identify $\mathbf{A}^n$ with an open set $U_0 \subseteq \mathbf{P}^n$ by the homeomorphism φ_0. Then we can speak of $\overline{Y}$, the closure of Y in $\mathbf{P}^n$, which is called the *projective closure* of Y.

   (a) Show that $I(\overline{Y})$ is the ideal generated by $\beta(I(Y))$, using the notation of the proof of Proposition 2.2.

   (b) Let $Y \subseteq \mathbf{A}^3$ be the twisted cubic of (Ex. 1.2). Its projective closure $\overline{Y} \subseteq \mathbf{P}^3$ is called the *twisted cubic curve* in $\mathbf{P}^3$. Find generators for $I(Y)$ and $I(\overline{Y})$, and use this example to show that if $f_1, \dots, f_r$ generate $I(Y)$, then $\beta(f_1), \dots, \beta(f_r)$ do *not* necessarily generate $I(\overline{Y})$.

   Solution.

   (a) Choose $f \in I(\overline{Y})$. Note that $I(Y)$ and $I(\overline{Y})$ have the same coordinate rings by (Ex. 2.3(e)) and (Ex. 2.4(a)). Then f vanishes on all $(a_1/a_0, \dots, a_n/a_0)$ where $(a_0, \dots, a_n) \in Y$. Hence f is a multiple of a polynomial of degree e of the form $x_0^e g(x_1/x_0, \dots, x_n/x_0)$, and so f is in the ideal generated by $\beta(I(Y))$.

   (b) The polynomials $f = z - x^3$ and $g = y - x^2$ are generators for $I(Y)$. Generators for $I(\overline{Y})$ are $\{wz^2 - x^3, wy - x^2, wz^2 - y^3\}$. Note that $wz^2 - y^3$ is not generated by the first two generators because there is no way to get the y^3. □

10. **Exercise (The cone over a projective variety).** Let $Y \subseteq \mathbf{P}^n$ be a nonempty algebraic set, and let $\theta\colon \mathbf{A}^{n+1} \setminus \{(0, \dots, 0)\} \to \mathbf{P}^n$ be the map which sends the point with affine coordinates $(a_0, \dots, a_m)$ to the point with homogeneous coordinates $(a_0, \dots, a_m)$. We define the *affine cone* over Y to be
$$C(Y) = \theta^{-1}(Y) \cup \{(0, \dots, 0)\}.$$

   (a) Show that $C(Y)$ is an algebraic set in $\mathbf{A}^{n+1}$, whose ideal is equal to $I(Y)$, considered as an ordinary ideal in $k[x_0, \dots, x_n]$.

   (b) $C(Y)$ is irreducible if and only if Y is.

   (c) $\dim C(Y) = \dim Y + 1$.

   Sometimes we consider the projective closure $\overline{C(Y)}$ of $C(Y)$ in $\mathbf{P}^{n+1}$. This is called the *projective cone* over Y.

   Solution.

   (a) Pick $f \in I(Y)$. Then f is homogeneous and hence vanishes at the origin. Also, f vanishes at every point in $\theta^{-1}(Y)$ because the vanishing of f at a point $P \in \mathbf{P}^n$ is independent of the choice of its representative. Finally, if $g \in I(C(Y))$, then g must be homogeneous since it vanishes at the origin, and it vanishes at each representative of $P \in Y$, and hence is a polynomial in $I(Y)$.

   (b) We know that $C(Y)$ is irreducible in $\mathbf{A}^{n+1}$ if and only if $I(Y)$ is a prime ideal (Corollary 1.4) and this is if and only if Y irreducible in $\mathbf{P}^n$ (Ex. 2.4(b)).

   (c) Let $A = k[x_0, \dots, x_n]$. We have $\dim C(Y) = \dim A/I(Y)$ (Proposition 1.7), and $\dim Y + 1 = \dim A/I(Y)$ (Ex. 2.6), so $\dim C(Y) = \dim Y + 1$. □

11. **Exercise (Linear varieties in $\mathbf{P}^n$).** A hypersurface defined by a linear polynomial is called a *hyperplane*.

(a) Show that the following two conditions are equivalent for a variety Y in $\mathbf{P}^n$:

(i) $I(Y)$ can be generated by linear polynomials.

(ii) Y can be written as an intersection of hyperplanes.

In this case we say that Y is a *linear variety* in $\mathbf{P}^n$.

(b) If Y is a linear variety of dimension r in $\mathbf{P}^n$, show that $I(Y)$ is minimally generated by $n-r$ linear polynomials.

(c) Let Y, Z be linear varieties in $\mathbf{P}^n$, with $\dim Y = r$, $\dim Z = s$. If $r+s-n \geq 0$, then $Y \cap Z \neq \varnothing$. Furthermore, if $Y \cap Z \neq \varnothing$, then $Y \cap Z$ is a linear variety of dimension $\geq r+s-n$.

Solution.

(a) First suppose that $I(Y)$ can be generated by linear polynomials, $I(Y) = (f_1, \dots, f_r)$. Then $Y = V(f_1) \cap \cdots V(f_r)$, so Y can be written as an intersection of hyperplanes. Conversely, if Y can be written as an intersection of hyperplanes $Y = H_1 \cap \cdots \cap H_s$, then write $I(H_i) = f_i$; we have $I(Y) = (f_1, \dots, f_s)$.

(b) Let Y be a linear variety of dimension r in $\mathbf{P}^n$, and write $I(Y) = (f_1, \dots, f_s)$ where the f_i are linear polynomials and $s \geq n-r$. We induct on $n-r$. If $r = n-1$, then we can take $s = 1$ by (Ex. 2.8). Otherwise, up to reordering, we can assume that $Y' = V(f_1) \cap \cdots \cap V(f_{s-1})$ is a linear variety properly containing Y such that $Y' \cap V(f_s) = Y$ because a linear variety of dimension $r+1$ is isomorphic to $\mathbf{P}^{r+1}$, which can be seen by the isomorphism $k[x_0, \dots, x_n]/(f_1, \dots, f_{n-r-1}) \cong k[y_0, \dots, x_{r+1}]$ (since by induction, $I(Y')$ can be generated by $n-r-1$ linear polynomials $f_1, \dots, f_{n-r-1}$). So Y is a codimension one linear variety in $\mathbf{P}^{r+1}$, so we are done.

(c) Now let Y, Z be linear varieties in $\mathbf{P}^n$ with $\dim Y = r$ and $\dim Z = s$. Consider the projection $\pi\colon \mathbf{A}^{n+1} \setminus 0 \to \mathbf{P}^n$ and let $Y' = \pi^{-1}(Y) \cup 0$ and $Z' = \pi^{-1}(Z) \cup 0$. Then $Y \cap Z = \pi(Y' \cap Z')$, and $Y' \cap Z'$ is a linear subspace of $\mathbf{A}^{n+1}$ (considering it as a k-vector space) which has dimension $\geq r+s-n$ by basic linear algebra, so $Y \cap Z \neq \varnothing$. Conversely, if $Y \cap Z \neq \varnothing$, then $\dim Y \cap Z = \dim Y' \cap Z' \geq r+s-n$. $\square$

12. **Exercise (The d-uple embedding).** For given $n, d > 0$, let $M_0, M_1, \dots, M_N$ be all the monomials of degree d in the $n+1$ variables $x_0, \dots, x_n$, where $N = \binom{n+d}{n} - 1$. We define a mapping $\rho_d\colon \mathbf{P}^n \to \mathbf{P}^N$ by sending the point $P = (a_0, \dots, a_n)$ to the point $\rho_d(P) = (M_0(a), \dots, M_N(a))$ obtained by substituting the a_i in the monomials M_j. This is called the d-uple *embedding* of $\mathbf{P}^n$ in $\mathbf{P}^N$. For example, if $n = 1$, $d = 2$, then $N = 2$, and the image Y of the 2-uple embedding of $\mathbf{P}^1$ in $\mathbf{P}^2$ is a conic.

(a) Let $\theta\colon k[y_0, \dots, y_N] \to k[x_0, \dots, x_n]$ be the homomorphism defined by sending y_i to M_i, and let $\mathfrak{a}$ be the kernel of θ. Then $\mathfrak{a}$ is a homogeneous prime ideal, and so $Z(\mathfrak{a})$ is a projective variety in $\mathbf{P}^N$.

(b) Show that the image of ρ_d is exactly $Z(\mathfrak{a})$.

(c) Now show that ρ_d is a homeomorphism of $\mathbf{P}^n$ onto the projective variety $Z(\mathfrak{a})$.

(d) Show that the twisted cubic curve in $\mathbf{P}^3$ is equal to the 3-uple embedding of $\mathbf{P}^1$ in $\mathbf{P}^3$, for suitable choice of coordinates.

Solution.

(a) Since the image of θ is a subring of $k[x_0, \dots, x_n]$, it must be a domain, so $\mathfrak{a}$ is a prime ideal. Also, if $p_1 M_1 + \cdots + p_N M_N$ is any polynomial relation among the monomials of degree N, then $\deg p_1 = \cdots = \deg p_N$, so $\mathfrak{a}$ is also homogeneous.

(b) It is immediate that the image of ρ_d is contained in $Z(\mathfrak{a})$. Note that $\mathfrak{a}$ is a binomial ideal generated by elements of the form $y_i y_j - y_k y_\ell$ where the subscripts are chosen so that $M_i M_j = M_k M_\ell$. This shows that any point that satisfies all of these equations must be in the image of ρ_d.

(c) Given $(M_0(a), \dots, M_N(a)) \in Z(\mathfrak{a})$, we can recover $(a_0, \dots, a_n)$ up to multiplication by an nth root of unity as follows: some M_{j_i} is the monomial x_i^n, so set $a_i = \sqrt[n]{M_{j_i}(a)}$. By examining other monomials such as $x_i x_j^{n-1}$, $x_i^2 x_j^{n-2}$, etc., we can determine which nth roots of unity to take, but we can still always scale all a_i by a common nth root of unity. Since $(a_0, \dots, a_n) \in \mathbf{P}^n$, this is sufficient, so θ is bijective. Also note that $\theta(Z(T)) = Z(T) \cap Z(\mathfrak{a})$, so θ is a closed map, and hence is a homeomorphism.

(d) The 3-uple embedding of $\mathbf{P}^1$ in $\mathbf{P}^3$ is parameterized by $(t^3, t^2 s, t s^2, s^3)$. Identifying $\mathbf{A}^3$ with the open subset of $\mathbf{P}^3$ defined by $x_0 \neq 0$, we see that this is a parameterization of the twisted cubic curve. □

14. **Exercise (The Segre embedding).** Let $\psi\colon \mathbf{P}^r \times \mathbf{P}^s \to \mathbf{P}^N$ be the map defined by sending the ordered pair $(a_0, \dots, a_r) \times (b_0, \dots, b_s)$ to $(\dots, a_i b_j, \dots)$ in lexicographic order, where $N = rs + r + s$. Note that ψ is well-defined and injective. It is called the *Segre embedding*. Show that the image of ψ is a *subvariety* of $\mathbf{P}^N$.

Solution. Note that $\psi(\lambda_1 a, \lambda_2 b) = \lambda_1 \lambda_2 \psi(a, b)$, so it is well-defined. If $\psi(a, b) = \psi(a', b')$, we can show $b = b'$ by finding some nonzero a_i and looking at the images $a_i b_j$ for $j = 0, \dots, s$ and also by finding some nonzero a'_k and looking at the images $a'_k b'_j$. Similarly, one can show that $a = a'$.

Let $\mathfrak{a}$ be the kernel of the map $k[\{z_{ij}\}] \to k[x_0, \dots, x_r, y_0, \dots, y_s]$ (here the z_{ij} are indexed by $i = 0, \dots, r$ and $j = 0, \dots, s$) defined by $z_{ij} \mapsto x_i y_j$. Then $\operatorname{image} \psi \subseteq Z(\mathfrak{a})$. It is not hard to see that a set of generators for $\mathfrak{a}$ is the set $\{z_{ij} z_{k\ell} - z_{kj} z_{i\ell} \mid 0 \leq i, k \leq r,\ 0 \leq j, \ell \leq s\}$. From this, the other inclusion $Z(\mathfrak{a}) \subseteq \operatorname{image} \psi$ follows. □

3 Morphisms

2. **Exercise.** A morphism whose underlying map on the topological spaces is a homeomorphism need not be an isomorphism.

(a) For example, let $\varphi\colon \mathbf{A}^1 \to \mathbf{A}^2$ be defined by $t \mapsto (t^2, t^3)$. Show that φ defines a bijective bicontinuous morphism of $\mathbf{A}^1$ onto the curve $y^2 = x^3$, but that φ is not an isomorphism.

(b) For another example, let the characteristic of the base field k be $p > 0$, and define a map $\varphi\colon \mathbf{A}^1 \to \mathbf{A}^1$ by $t \mapsto t^p$. Show that φ is bijective and bicontinuous but not an isomorphism. This is called the *Frobenius morphism*.

Solution.

(a) It is clear that φ is injective. Any point (x, y) that satisfies $y^2 = x^3$ can be written as (t^2, t^3) where t is a square root of x, and the sign is chosen so that $t^3 = y$, so φ is also surjective. The inverse image of a closed point of the curve $y^2 = x^3$ is a closed point of $\mathbf{A}^1$, so φ is a continuous map. Since φ maps closed points of $\mathbf{A}^1$ to closed points of the curve $y^2 = x^3$, φ^{-1} is a continuous map. Now let $f\colon V \to k$ be a regular function where V is an open set of the curve $y^2 - x^3$. Then $f \circ \varphi\colon \varphi^{-1}(V) \to k$ is obtained by substituting t^2 and t^3 into f, so $f \circ \varphi$ is a regular function; the denominator of this substitution does not vanish on $\varphi^{-1}(V)$ since the denominator of f does not vanish on V. However, using Corollary 3.7, φ cannot be an isomorphism because $A(\mathbf{A}^1) = k[t]$, whereas the coordinate ring of the curve $y^2 = x^3$ is $k[x,y]/(y^2 - x^3)$, which is not a UFD.

(b) If $t^p = s^p$, then $(t-s)^p = t^p - s^p = 0$, so $t = s$, so φ is injective. Since k is algebraically closed, φ is also surjective. Bicontinuity follows from the fact that closed points are mapped to closed points. Verification that φ is a morphism follows as in (a). However, the inverse map $t \mapsto t^{1/p}$ is not a morphism because substitution of $t^{1/p}$ into a rational function need not yield another rational function. □

3. **Exercise.**

(a) Let $\varphi\colon X \to Y$ be a morphism. Then for each $P \in X$, φ induces a homomorphism of local rings $\varphi_P^*\colon \mathcal{O}_{\varphi(P),Y} \to \mathcal{O}_{P,X}$.

(b) Show that a morphism φ is an isomorphism if and only if φ is a homeomorphism, and the induced map φ_P^* on local rings is an isomorphism, for all $P \in X$.

(c) Show that if $\varphi(X)$ is dense in Y, then the map φ_P^* is *injective* for all $P \in X$.

Solution.

(a) Given $f \in \mathcal{O}_{\varphi(P),Y}$, we define $\varphi_P^*(f) = f \circ \varphi$. This is clearly a map of rings, and if f vanishes on $\varphi(P)$, then $\varphi_P^*(f)$ vanishes on P, so φ_P^* maps the maximal ideal to the maximal ideal, and hence is a homomorphism of local rings.

(b) If φ is an isomorphism, then it is certainly a homeomorphism, and its inverse induces inverses for φ_P^* for all $P \in X$. Conversely, suppose that φ is a homeomorphism and that φ_P^* is an isomorphism for all $P \in X$. Then $\psi = \varphi^{-1}$ is a morphism because for any regular function f in a neighborhood of $Q \in Y$, $f \circ \psi \in \mathcal{O}_{\psi(Q),X} \to \mathcal{O}_{Q,Y}$, so is regular.

(c) Suppose that $\varphi_P^*(f) = 0$ for some regular function f on Y. Then $f \circ \varphi = 0$, and hence f is 0 on a dense subset of Y, which means $f = 0$, so φ_P^* is injective. □

6. **Exercise.** There are quasi-affine varieties which are not affine. For example, show that $X = \mathbf{A}^2 \setminus \{(0,0)\}$ is not affine.

Solution. The map $\alpha\colon k[x,y] \to \mathcal{O}(X)$ given by interpreting a polynomial as a rational function on X is injective because any vanishing polynomial on X would also vanish at the origin by continuity. Any rational function with a nonconstant denominator has a zero somewhere other than the origin, so a rational function on X must be polynomial. Hence α is also surjective, and $\mathcal{O}(X) \cong k[x,y]$. If X were affine, then Theorem 3.2(a) implies that $I(X) = 0$, but this is not the case. □

9. **Exercise.** The homogeneous coordinate ring of a projective variety is not invariant under isomorphism. For example, let $X = \mathbf{P}^1$, and let Y be the 2-uple embedding of $\mathbf{P}^1$ in $\mathbf{P}^2$. Then $X \cong Y$ (Ex. 3.4). But show that $S(X) \not\cong S(Y)$.

Solution. First, $S(X) \cong k[x,y]$. By (Ex. 2.12(b)), $S(Y)$ is isomorphic to the image of $\theta\colon k[x,y,z] \to k[x,y]$ given by $x \mapsto x^2$, $y \mapsto xy$, and $z \mapsto y^2$. However, the ring $k[x^2, xy, y^2]$ is not a UFD, so $S(X) \not\cong S(Y)$. □

11. **Exercise.** Let X be any variety and let $P \in X$. Show there is a 1-1 correspondence between the prime ideals of the local ring $\mathcal{O}_P$ and the closed subvarieties of X containing P.

    Solution. If X is affine, then by Theorem 3.2(c), $\mathcal{O}_P \cong A(X)_{\mathfrak{m}_P}$, so the prime ideals of $\mathcal{O}_P$ are in bijection with the prime ideals of $A(X)$ contained in $\mathfrak{m}_P$. These prime ideals are in bijection with the closed subvarieties of X that contain P. If X is quasi-affine, then embed Y as an open subset of an affine variety X'. Then the same reasoning holds, just noting that for a closed subvariety Y of X', $Y \cap X$ is a closed subvariety of X. The analogous argument for projective and quasi-projective varieties follows from Theorem 3.4(b). $\square$

12. **Exercise.** If P is a point on a variety X, then $\dim \mathcal{O}_P = \dim X$.

    Solution. If X is an affine variety, then this follows from Theorem 3.2(c). If X is a projective variety, then we use (Ex. 2.6) to conclude that $S(X) = \dim X + 1$. Then Theorem 3.4(b) shows that $\mathcal{O}_P = S(X)_{(\mathfrak{m}_P)}$, so $\dim \mathcal{O}_P = \dim S(X) - 1$ (Theorem 1.8A). In the case that X is quasi-affine, we use Proposition 1.10 to get $\dim X = \dim \overline{X}$. Since X is dense in $\overline{X}$, $\mathcal{O}_{P,X} = \mathcal{O}_{P,\overline{X}}$, so we can still conclude that $\dim \mathcal{O}_P = \dim X$. Finally, the case for quasi-projective varieties follows from the quasi-affine case. $\square$

13. **Exercise (The local ring of a subvariety).** Let $Y \subseteq X$ be a subvariety. Let $\mathcal{O}_{Y,X}$ be the set of equivalence classes $\langle U, f \rangle$ where $U \subseteq X$ is open, $U \cap Y \neq \varnothing$, and f is a regular function on U. We say that $\langle U, f \rangle$ is equivalent to $\langle V, g \rangle$, if $f = g$ on $U \cap V$. Show that $\mathcal{O}_{Y,X}$ is a local ring, with residue field $K(Y)$ and dimension $= \dim X - \dim Y$. It is the *local ring* of Y on X. Note if $Y = P$ is a point we get $\mathcal{O}_P$, and if $Y = X$ we get $K(X)$. Note also that if Y is not a point, then $K(Y)$ is not algebraically closed, so in this way, we get local rings whose residue fields are not algebraically closed.

    Solution. The set of all functions which vanish at Y forms the unique maximal ideal of $\mathcal{O}_{Y,X}$. To see this, note that if $f(P) \neq 0$ for some $P \in Y$, then there is a neighborhood U of P for which f is nowhere zero, so its inverse is given by $\langle U, 1/f \rangle$, so it is a unit. We define a function $\mathcal{O}_{Y,X} \to K(Y)$ by interpreting an element $f \in \mathcal{O}_{Y,X}$ as a rational function on Y via restriction. The kernel is the set of all f that vanish on Y, i.e., the maximal ideal, so $K(Y)$ is the residue field of $\mathcal{O}_{Y,X}$.

    Using an argument similar to (Ex. 2.6), we can assume that Y and X are affine if we just want to compute the dimension of $\mathcal{O}_{Y,X}$. Then as in the proof of Theorem 3.2, let $\mathfrak{p}_Y \subset A(X)$ be the prime ideal of functions vanishing on Y. Then $A(X)/\mathfrak{p}_Y \cong A(Y)$, so by Theorem 1.8A, $\dim A(X) = \operatorname{height} \mathfrak{p} + \dim A(Y)$. Finally, $\operatorname{height} \mathfrak{p} = \dim \mathcal{O}_{Y,X}$ since $\mathcal{O}_{Y,X} \cong A(X)_{\mathfrak{p}_Y}$ (this is similar to Theorem 3.2), so we conclude that $\dim \mathcal{O}_{Y,X} = \dim A(X) - \dim A(Y) = \dim X - \dim Y$. $\square$

15. **Exercise (Products of affine varieties).** Let $X \subseteq \mathbf{A}^n$ and $Y \subseteq \mathbf{A}^m$ be affine varieties.

    (a) Show that $X \times Y \subseteq \mathbf{A}^{n+m}$ with its induced topology is irreducible. The affine variety $X \times Y$ is called the *product* of X and Y. Note that its topology is in general not equal to the product topology.

    (b) Show that $A(X \times Y) \cong A(X) \otimes_k A(Y)$.

    (c) Show that $X \times Y$ is a product in the category of varieties.

    (d) Show that $\dim X \times Y = \dim X + \dim Y$.

    Solution.

(a) Suppose that $X \times Y = Z_1 \cup Z_2$ with Z_i closed and proper. Then define $X_i = \{x \in X \mid x \times Y \subset Z_i\}$. If $x \in X \setminus (X_1 \cup X_2)$, define $Y_i = (x \times Y) \cap Z_i$. Then $x \times Y = Y_1 \cup Y_2$, and both are closed in Y by virtue of being the intersections of closed subsets. So without loss of generality, $x \times Y = Y_1$, which implies $x \in X_1$, a contradiction. Hence $X = X_1 \cup X_2$, and both are closed in X (we can write down equations for X_i given equations for Z_i), so without loss of generality, $X = X_1$. This means that $X \times Y = Z_1$, so $X \times Y$ is irreducible.

(b) Let $I(X) \subset k[x_1, \ldots, x_n]$ and $I(Y) \subset k[y_1, \ldots, y_m]$ be the ideals of X and Y. Then identifying $k[x_1, \ldots, x_n]$ and $k[y_1, \ldots, y_m]$ as subrings of $k[x_1, \ldots, x_n, y_1, \ldots, y_m]$, we see that $I(X \times Y) = I(X)I(Y)$. The conclusion follows from

$$\begin{aligned} A(X \times Y) &\cong k[x_1, \ldots, x_n, y_1, \ldots, y_m]/I(X)I(Y) \\ &\cong k[x_1, \ldots, x_n]/I(X) \otimes_k k[y_1, \ldots, y_m]/I(Y) \\ &\cong A(X) \otimes_k A(Y). \end{aligned}$$

(c) The fact that the projections $X \times Y \to X$ and $X \times Y \to Y$ are morphisms follows from the fact that the maps $A(X) \to A(X) \otimes_k A(Y)$ and $A(Y) \to A(X) \otimes_k A(Y)$ given by $x \mapsto x \otimes 1$ and $y \mapsto 1 \otimes y$ are k-algebra homomorphisms. The fact that $X \times Y$ is a categorical product follows from Proposition 3.5, Corollary 3.8, and the fact that tensor product is the coproduct in the category of k-algebras.

(d) The formula $\dim X \times Y = \dim X + \dim Y$ follows from (b). □

21. **Exercise (Group varieties).** A group variety consists of a variety Y together with a morphism $\mu\colon Y \times Y \to Y$, such that the set of points of Y with the operation given by μ is a group, and such that the inverse map $y \mapsto y^{-1}$ is also a morphism of $Y \to Y$.

(a) The *additive group* $\mathbf{G}_a$ is given by the variety $\mathbf{A}^1$ and the morphism $\mu\colon \mathbf{A}^2 \to \mathbf{A}^1$ defined by $\mu(a, b) = a + b$. Show it is a group variety.

(b) The *multiplicative group* $\mathbf{G}_m$ is given by the variety $\mathbf{A}^1 \setminus \{(0)\}$ and the morphism $\mu(a, b) = ab$. Show it is a group variety.

(c) If G is a group variety, and X is any variety, show that the set $\operatorname{Hom}(X, G)$ has a natural group structure.

(d) For any variety X, show that $\operatorname{Hom}(X, \mathbf{G}_a)$ is isomorphic to $\mathcal{O}_X$ as a group under addition.

(e) For any variety X, show that $\operatorname{Hom}(X, \mathbf{G}_m)$ is isomorphic to the group of units in $\mathcal{O}(X)$, under multiplication.

Solution.

(a) Given a rational function $f(t)$, substituting $a + b$ and $-a$ for t results in another rational function, so $\mathbf{G}_a$ is a group variety.

(b) Given a rational function $f(t)$ whose denominator does not vanish, substituting ab and a^{-1} for t results in another rational function whose denominator does not vanish, so $\mathbf{G}_m$ is a group variety.

(c) Given $f, g \in \operatorname{Hom}(X, G)$, define fg to be the function $x \mapsto \mu(f(x), g(x))$. And define the inverse by $f^{-1}(x) = f(x)^{-1}$. Then $\operatorname{Hom}(X, G)$ has a natural group structure from G.

(d) By Proposition 3.5, we have a bijection of sets

$$\alpha\colon \operatorname{Hom}(X, \mathbf{G}_a) \to \operatorname{Hom}(k[t], \mathcal{O}(X)).$$

A k-algebra homomorphism $k[t] \to \mathcal{O}(X)$ is determined by the image of t, so the second set is in bijection with $\mathcal{O}(X)$. Looking through the proof of Proposition 3.5, we see that a map $h\colon k[t] \to \mathcal{O}(X)$ is mapped to the function $\alpha^{-1}(h)\colon X \to \mathbf{G}_a$ defined by $P \mapsto h(t)(P)$. Given another map $h'\colon k[t] \to \mathcal{O}(X)$, we have $h + h'\colon k[t] \to \mathcal{O}(X)$ defined by $t \mapsto h(t)+h'(t)$. Then $\alpha^{-1}(h+h')$ is the function $X \to \mathbf{G}_a$ defined by $P \mapsto (h(t)+h'(t))(P)$, which is the same as $h(t)(P) + h'(t)(P)$, so in fact, α is a group homomorphism. So $\mathrm{Hom}(X, \mathbf{G}_a) \cong \mathcal{O}(X)$ as groups.

(e) This follows as in the discussion of (d). □

4 Rational maps

7. **Exercise.** Let X and Y be two varieties. Suppose there are points $P \in X$ and $Q \in Y$ such that the local rings $\mathcal{O}_{P,X}$ and $\mathcal{O}_{Q,Y}$ are isomorphic as k-algebras. Then show that there are open sets $P \in U \subseteq X$ and $Q \in V \subseteq Y$ and an isomorphism of U to V which sends P to Q.

Solution. Since this is a local question, we may assume without loss of generality that X and Y are affine varieties. Let $A = \mathcal{O}(X)$ and $B = \mathcal{O}(Y)$ be the coordinate rings of X and Y, and $\mathfrak{p} \subset A$ and $\mathfrak{q} \subset Q$ the prime ideals corresponding to the points P and Q. The assumption that X and Y are varieties implies that A and B are domains. By assumption, we can find an isomorphism $\varphi\colon B_{\mathfrak{q}} \to A_{\mathfrak{p}}$. Since this is an isomorphisms of local rings, we necessarily have $\varphi^{-1}(\mathfrak{p}) = \mathfrak{q}$. Let $\{f_1, \dots, f_r\}$ be generators for A as a k-algebra. Since A is a domain, we can identify A as a subring of $A_{\mathfrak{p}}$, so let $\{g_1, \dots, g_r\} \subset B_{\mathfrak{q}}$ be such that $\varphi(g_i) = f_i$. By finding a common denominator, the subring of $B_{\mathfrak{q}}$ generated by B and $\{g_1, \dots, g_r\}$ is contained in some $B[g^{-1}]$ since B is a domain. Then $\varphi(B[g^{-1}]) = A[\varphi(g)^{-1}]$, and the restriction $\varphi\colon B[g^{-1}] \to A[\varphi(g)^{-1}]$ is an isomorphism, and hence the induced maps $D(\varphi(g)) \to D(g)$ is an isomorphism of open neighborhoods of P and Q, which sends P to Q by a previous remark. □

Solutions to *Algebraic Geometry*[*]

Steven V Sam

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Contents

1 Sheaves

1. **Exercise.** Let A be an Abelian group, and define the *constant presheaf* associated to A on the topological space X to be the presheaf $U \mapsto A$ for all $U \neq \varnothing$, with restriction maps the identity. Show that the constant presheaf $\mathscr{A}$ defined in the text is the sheaf associated to this presheaf.

   Solution. This will follow from the proof of (1.2). Let $\overline{A}$ denote the constant presheaf. The stalks $\overline{A}_P$ for all $P \in X$ are equal to A, so the sheafification of $\overline{A}$ on an open set U is defined to be the set of all maps to A that are continuous, and this is precisely the definition of the constant sheaf. □

2. **Exercise.**

   (a) For any morphism of sheaves $\varphi \colon \mathscr{F} \to \mathscr{G}$, show that for each point P, $(\ker \varphi)_P = \ker(\varphi_P)$ and $(\operatorname{image} \varphi)_P = \operatorname{image}(\varphi_P)$.

   (b) Show that φ is injective (respectively, surjective) if and only if the induced map on the stalks φ_P is injective (respectively, surjective) for all P.

   (c) Show that a sequence

   $$\cdots \mathscr{F}^{i-1} \xrightarrow{\varphi^{i-1}} \mathscr{F}^{i} \xrightarrow{\varphi^{i}} \mathscr{F}^{i+1} \longrightarrow \cdots$$

   of sheaves and morphisms is exact if and only if for each $P \in X$ the corresponding sequence of stalks is exact as a sequence of Abelian groups.

   Solution. Choose $x \in (\ker \varphi)_P$. Then there is an open set $U \ni P$ such that x is the class of $x' \in (\ker \varphi)(U)$. Then $\varphi_P(x')$ is represented by 0 in $\mathscr{G}_P$, so $x' \in \ker(\varphi_P)$. Conversely, pick $x \in \ker(\varphi_P)$. Then there is an open set $U \ni P$ such that x is represented by $x' \in \mathscr{F}(U)$,

[*] by Robin Hartshorne

and $\varphi(x')$ is represented by 0 in $\mathscr{G}_P$. This means that there is an open set $V \subseteq U$ such that $\varphi(x')|_V = 0$, so $x \in (\ker\varphi)_P$. Hence $(\ker\varphi)_P = \ker(\varphi_P)$.

Now pick $x \in (\operatorname{image}\varphi)_P$. Since the stalk of a presheaf at P is the same as the stalk of its sheafification at P, we see that there exists an open set $U \ni P$ and $y \in \mathscr{F}(U)$ such that $\varphi(y) = x$. Then $\varphi(y_P) = x_P$, so $x \in \operatorname{image}(\varphi_P)$. Now suppose $x \in \operatorname{image}(\varphi_P)$. Then there is an open set $U \ni P$ and $x' \in \mathscr{G}(U)$ representing x such that there exists $y \in \mathscr{F}(U)$ with $\varphi(y) = x'$, so $\varphi(y_P) = x$. Thus $(\operatorname{image}\varphi)_P = \operatorname{image}(\varphi_P)$, which finishes (a).

For (b), note that φ is injective if and only if $\ker\varphi = 0$, which is equivalent to $(\ker\varphi)_P = 0$ for all $P \in X$. By (a), this is equivalent to φ_P being injective for all $P \in X$. Surjectivity of φ is similar.

Now (c) is a direct consequence of (a) since kernels and images are preserved by taking stalks. □

3. **Exercise.**

(a) Let $\varphi\colon \mathscr{F} \to \mathscr{G}$ be a morphism of sheaves on X. Show that φ is surjective if and only if the following condition holds: for every open set $U \subseteq X$, and for every $s \in \mathscr{G}(U)$, there is a covering $\{U_i\}$ of U, and there are elements $t_i \in \mathscr{F}(U_i)$, such that $\varphi(t_i) = s|_{U_i}$ for all i.

(b) Give an example of a surjective morphism of sheaves $\varphi\colon \mathscr{F} \to \mathscr{G}$, and an open set U such that $\varphi(U)\colon \mathscr{F}(U) \to \mathscr{G}(U)$ is not surjective.

Solution. Let $\varphi\colon \mathscr{F} \to \mathscr{G}$ be a morphism of sheaves. By (Ex. 1.2(b)), φ is surjective if and only if φ_P is surjective for every point $P \in X$. Suppose φ_P is surjective for every $P \in X$. Let $U \subseteq X$ be an open set and $s \in \mathscr{G}(U)$. There exists an element $t_P \in \mathscr{F}_P$ such that $\varphi_P(t_P) = s_P$, which means there is some neighborhood U_P of P such that $\varphi(t|_{U^P}) = s|_{U^P}$. Then $\{U^P\}$ is a covering of U, and the condition holds. Conversely, suppose that the condition holds and let $s \in \mathscr{G}(X)$. For any point $P \in X$, there is an open covering $\{U_i\}$ of X and elements $t^i \in \mathscr{F}(U_i)$ such that $\varphi(t^i) = s|_{U_i}$ for all i. There is some i such that $P \in U_i$, which gives $\varphi_P(t^i_P) = s_P$. So each φ_P is surjective, and thus φ is surjective, which gives (a).

As for (b), let $\mathscr{F}$ be the sheaf on $\mathbf{C}$ with the usual topology that sends an open set U to the group of all analytic functions on U with the obvious restriction maps. Let $D\colon \mathscr{F} \to \mathscr{F}$ be differentiation; that is, for an open set U, $D(U)$ sends f to its derivative, which is also analytic. We use the fact that an analytic function f on an open set U has an antiderivative if and only if the integral of f over any closed contour in U is 0. In particular, the integral of f over a closed contour C is 0 if f is analytic in the region bounded by C. For any open set U and analytic function f on U, there exists an open covering $\{U_i\}$ of U such that each U_i is simply connected (take a small enough neighborhood around each point). Thus the integral of f along any contour in U_i is 0, so f has an antiderivative, which is of course analytic. Using part (a), this means that D is a surjective morphism of sheaves. However, let $U = \mathbf{C} \setminus \{0\}$. The function $f(z) = 1/z$ is analytic on U, but has no antiderivative since the integral of $1/z$ going counterclockwise along the unit circle is $2\pi i$, so $D(U)\colon \mathscr{F}(U) \to \mathscr{F}(U)$ is not surjective. □

4. **Exercise.**

(a) Let $\varphi\colon \mathscr{F} \to \mathscr{G}$ be a morphism of presheaves such that $\varphi(U)\colon \mathscr{F}(U) \to \mathscr{G}(U)$ is injective for each U. Show that the induced map $\varphi^+\colon \mathscr{F}^+ \to \mathscr{G}^+$ of associated sheaves is injective.

(b) Use part (a) to show that if $\varphi\colon \mathscr{F} \to \mathscr{G}$ is a morphism of sheaves, then $\operatorname{image}\varphi$ can be naturally identified with a subsheaf of $\mathscr{G}$, as mentioned in the text.

Solution. Part (a) follows from the fact that the sheafification process does not change stalks. Namely, if φ is an injective map of presheaves, then the stalks are all injective functions, so this implies that the map of sheafifications is also injective by (Ex. 1.2(b)).

Let $\mathscr{I}$ be the presheaf $U \mapsto \operatorname{image}(\varphi(U))$. It is enough to show that $\mathscr{I}$ is a sheaf since $\operatorname{image}(\varphi(U)) \subseteq \mathscr{G}(U)$ for all U. Let $\{U_i\}$ be a covering of U, and suppose there is an element $s \in \operatorname{image}(\varphi(U))$ such that $s|_{U_i} = 0$ for all i. There is an element $t \in \mathscr{F}(U)$ such that $\varphi(U)(t) = s$. Since $s|_{U_i} \in \operatorname{image}(\varphi(U_i))$, there exists $t_i \in \mathscr{F}(U_i)$ such that $\varphi(U_i)(t_i) = s|_{U_i}$, and $t_i = 0$ since $\varphi(U_i)$ is injective. Since $\varphi(U_i) \circ \rho^{\mathscr{F}}_{U,U_i} = \rho^{\mathscr{G}}_{U,U_i} \circ \varphi(U)$ for all i, $t|_{U_i} = t_i = 0$, so $t = 0$, and thus $s = 0$.

Now suppose there are elements $s_i \in \operatorname{image}(\varphi(U_i))$ such that for all i and j, $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$. Since φ is injective, there are unique t_i and t_j such that $\varphi(U_i \cap U_j)(t_i) = s_i|_{U_i \cap U_j}$ and $\varphi(U_i \cap U_j)(t_j) = s_j|_{U_i \cap U_j}$. Again by the commutative relation mentioned earlier, $t_i|_{U_i \cap U_j} = t_j|_{U_i \cap U_j}$, so there is a t such that $t|_{U_i} = t_i$ for all i. Setting $s = \varphi(U)(t)$, one gets $s|_{U_i} = s_i$, so $\mathscr{I}$ is a sheaf. The isomorphism is a consequence of the universal property of sheafification. This finishes (b). □

5. **Exercise.** Show that a morphism of sheaves is an isomorphism if and only if it is both injective and surjective.

   Solution. A morphism of sheaves is an isomorphism if and only if the induced maps on stalks are isomorphisms (1.1). This is equivalent to the induced maps on stalks being injective and surjective, which in turn is equivalent to the morphism of sheaves being both injective and surjective (Ex. 1.2(b)). □

6. **Exercise.**

   (a) Let $\mathscr{F}'$ be a subsheaf of a sheaf $\mathscr{F}$. Show that the natural map of $\mathscr{F}$ to the quotient sheaf $\mathscr{F}/\mathscr{F}'$ is surjective, and has kernel $\mathscr{F}'$. Thus there is an exact sequence

   $$0 \longrightarrow \mathscr{F}' \longrightarrow \mathscr{F} \longrightarrow \mathscr{F}/\mathscr{F}' \longrightarrow 0\,.$$

   (b) Conversely, if $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ is an exact sequene, show that $\mathscr{F}'$ is isomorphic to a subsheaf of $\mathscr{F}$, and that $\mathscr{F}''$ is isomorphic to the quotient of $\mathscr{F}$ by this subsheaf.

   Solution. Let $\mathscr{F}'$ be a subsheaf of the sheaf $\mathscr{F}$. There is an inclusion $\mathscr{F}'(U) \to \mathscr{F}(U)$ for all open sets U, which gives an injective morphism $\mathscr{F}' \to \mathscr{F}$. For all open sets U, there is a canonical projection $\mathscr{F}(U) \to \mathscr{F}(U)/\mathscr{F}'(U)$, which gives a morphism of presheaves. Composing this with the canonical morphism from the quotient presheaf to its sheafification gives a morphism of sheaves $\varphi\colon \mathscr{F} \to \mathscr{F}/\mathscr{F}'$. For any point $P \in X$, one has $\varphi_P\colon \mathscr{F}_P \to (\mathscr{F}/\mathscr{F}')_P$, and from the definition of direct limit, $(\mathscr{F}/\mathscr{F}')_P = \mathscr{F}_P/\mathscr{F}'_P$, so φ_P is canonical projection. By (Ex. 1.2(b)), φ is surjective. In each case, $\ker(\varphi_P) = \mathscr{F}'_P$, so by (Ex. 1.2(a)), $(\ker\varphi)_P = \mathscr{F}'_P$, which gives $\ker\varphi = \mathscr{F}'$. This gives (a).

   For (b), let φ denote the injective morphism $\mathscr{F}' \to \mathscr{F}$, and let $\mathscr{I}$ be the presheaf $U \mapsto \operatorname{image}(\varphi(U))$. By (Ex. 1.4(b)), $\mathscr{I}$ is a subsheaf of $\mathscr{F}$. The morphism $\mathscr{F}' \to \mathscr{I}$, where $\mathscr{F}'(U) \to \operatorname{image}(\varphi(U))$ is induced by $\varphi(U)$ for all U, is an isomorphism, so $\mathscr{F}'$ is isomorphic to a subsheaf of $\mathscr{F}$. Let ψ be the surjective morphism $\mathscr{F} \to \mathscr{F}''$. By exactness, $\operatorname{image}\varphi = \ker\psi$. By (Ex. 1.7(a)), $\operatorname{image}\psi \cong \mathscr{F}/\ker\psi$, which means $\mathscr{F}'' \cong \mathscr{F}/\ker\psi$. Since $\mathscr{F}'$ can be identified with $\ker\psi$ via isomorphism, $\mathscr{F}'' \cong \mathscr{F}/\mathscr{F}'$. □

7. **Exercise.** Let $\varphi\colon \mathscr{F} \to \mathscr{G}$ be a morphism of sheaves.

(a) Show that $\operatorname{image}\varphi \cong \mathscr{F}/\ker\varphi$.

(b) Show that $\operatorname{coker}\varphi \cong \mathscr{G}/\operatorname{image}\varphi$.

Solution. Let $\mathscr{I}$ be the presheaf $U \mapsto \operatorname{image}(\varphi(U))$. Letting $\varphi'(U)\colon \operatorname{image}(\varphi(U)) \to \mathscr{F}(U)/\ker(\varphi(U))$ be the canonical map for all open sets U defines a morphism of presheaves. This gives a morphism $\mathscr{I} \to \mathscr{F}/\ker\varphi$ by composing with the canonical morphism from the quotient presheaf to its sheafification, and this induces a morphism of sheaves $\psi\colon \operatorname{image}\varphi \to \mathscr{F}/\ker\varphi$. For any point P, the induced map $\psi_P\colon (\operatorname{image}\varphi)_P \to (\mathscr{F}/\ker\varphi)_P$ can be rewritten, using (Ex. 1.2(a)), as $\psi_P\colon \operatorname{image}(\varphi_P) \to \mathscr{F}_P/\ker(\varphi_P)$, and is the canonical isomorphism. Thus ψ is an isomorphism, so $\operatorname{image}\varphi \cong \mathscr{F}/\ker\varphi$.

Let $\mathscr{C}$ be the presheaf $U \mapsto \operatorname{coker}(\varphi(U))$. The canonical map $\mathscr{I} \to \operatorname{image}\varphi$ induces a homomorphism

$$\mathscr{G}(U)/\operatorname{image}(\varphi(U)) \to \mathscr{G}(U)/(\operatorname{image}\varphi)(U)$$

since there is a natural embedding $\operatorname{image}(\varphi(U)) \subseteq (\operatorname{image}\varphi)(U)$. Composing this with the identity map

$$\operatorname{coker}(\varphi(U)) \to \mathscr{G}(U)/\operatorname{image}(\varphi(U))$$

induces a morphism $\psi\colon \operatorname{coker}\varphi \to \mathscr{G}/\operatorname{image}\varphi$ as in (a). For any point P, the induced map

$$\psi_P\colon (\operatorname{coker}\varphi)_P \to (\mathscr{G}/\operatorname{image}\varphi)_P$$

can be rewritten, using (Ex. 1.2(a)), as

$$\psi_P\colon \mathscr{G}_P/\operatorname{image}(\varphi_P) \to \mathscr{G}_P/\operatorname{image}(\varphi_P),$$

and is the identity. Thus ψ is an isomorphism, so $\operatorname{coker}\varphi \cong \mathscr{G}/\operatorname{image}\varphi$. □

8. **Exercise.** For any open subset $U \subseteq X$, show that the functor $\Gamma(U,\cdot)$ from sheaves on X to Abelian groups is a left exact functor, i.e., if $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}''$ is an exact sequence of sheaves, then $0 \to \Gamma(U,\mathscr{F}') \to \Gamma(U,\mathscr{F}) \to \Gamma(U,\mathscr{F}'')$ is an exact sequence of groups.

   Solution. Let $\varphi\colon \mathscr{F}' \to \mathscr{F}$ and $\psi\colon \mathscr{F} \to \mathscr{F}''$ be the morphisms in the sequence. Since $\ker\varphi = 0$, $\ker(\varphi(U)) = 0$, so $\varphi(U)\colon \mathscr{F}'(U) \to \mathscr{F}(U)$ is injective. By (Ex. 1.4(b)), $U \mapsto \operatorname{image}(\varphi(U))$ is a sheaf isomorphic to $\operatorname{image}\varphi$. Since $\operatorname{image}\varphi = \ker\psi$, the isomorphism $\operatorname{image}(\varphi(U)) \cong \ker(\psi(U))$ follows, so $\Gamma(U,\cdot)$ is left exact. □

9. **Exercise (Direct Sum).** Let $\mathscr{F}$ and $\mathscr{G}$ be sheaves on X. Show that the presheaf $U \mapsto \mathscr{F}(U) \oplus \mathscr{G}(U)$ is a sheaf. It is called the *direct sum* of $\mathscr{F}$ and $\mathscr{G}$, and is denoted by $\mathscr{F} \oplus \mathscr{G}$. Show that it plays the role of direct sum and of direct product in the category of sheaves of Abelian groups on X.

   Solution. Let $\{U_i\}$ be an open cover of U. Given $(s_i,t_i) \in \mathscr{F}(U_i) \oplus \mathscr{G}(U_i)$ such that for all i,j, we have $(s_i,t_i) = (s_j,t_j)$ on $U_i \cap U_j$, then there exists a unique $(s,t) \in \mathscr{F}(U) \oplus \mathscr{G}(U)$ such that $(s,t) = (s_i,t_i)$ on U_i. Namely, we take s to be the gluing of the $\{s_i\}$ and t to be the gluing of the $\{t_i\}$. Hence $\mathscr{F} \oplus \mathscr{G}$ is a sheaf.

   That $\mathscr{F} \oplus \mathscr{G}$ plays the role of direct sum and direct product in the category of sheaves of Abelian groups on X follows immediately from its description and the fact that direct sum plays this role in the category of Abelian groups. □

14. **Exercise (Support).** Let $\mathscr{F}$ be a sheaf on X, and let $s \in \mathscr{F}(U)$ be a section over an open set U. The *support* of s, denoted $\operatorname{Supp} s$, is defined to be $\{P \in U \mid s_P \neq 0\}$, where s_P denotes the germ of s in the stalk $\mathscr{F}_P$. Show that $\operatorname{Supp} s$ is a closed subset of U. We define the *support* of $\mathscr{F}$, $\operatorname{Supp} \mathscr{F}$, to be $\{P \in X \mid \mathscr{F}_P \neq 0\}$. It need not be a closed subset.

    Solution. We show that the set $T = \{P \in U : s_P = 0\}$ is an open set of U to get the desired conclusion. Pick $P \in T$. Then there exists an open neighborhood V of P such that $s|_V = 0$. For any other point $Q \in V$, this means that $s_Q = 0$, so T is open.

    Now let $X = \mathbf{R}$ with the standard topology. For every open set U, define $\mathscr{F}(U)$ to be the group of all functions $f\colon U \to \mathbf{R}$ subject to $f(0) = 0$ if $0 \in U$ where the group operation is pointwise addition. For $V \subseteq U$, the restriction map $\mathscr{F}(U) \to \mathscr{F}(V)$ is just restriction of domain. If $\{U_i\}$ is a covering of U, and we have functions in $\mathscr{F}(U_i)$ for all i such that they agree on overlaps, then they uniquely determine a function in $\mathscr{F}(U)$, so $\mathscr{F}$ is a sheaf. Note that $\mathscr{F}_P \neq 0$ means that there is some $f \in \mathscr{F}(X)$ and some open neighborhood U_P of P such that $f|_{U_P}$ is nonzero on all of U_P. It is clear then that $\mathscr{F}_P = 0$ if $P = 0$ and $\mathscr{F}_P \neq 0$ otherwise, so the support of $\mathscr{F}$ is $\mathbf{R} \setminus \{0\}$, which is not a closed set. □

15. **Exercise (Sheaf $\mathscr{H}om$).** Let $\mathscr{F}$, $\mathscr{G}$ be sheaves of Abelian groups on X. For any open set $U \subseteq X$, show that the set $\operatorname{Hom}(\mathscr{F}|_U, \mathscr{G}|_U)$ of morphisms of the restricted sheaves has a natural structure of Abelian group. Show that the presheaf $U \mapsto \operatorname{Hom}(\mathscr{F}|_U, \mathscr{G}|_U)$ is a sheaf. It is called the *sheaf of local morphisms* of $\mathscr{F}$ into $\mathscr{G}$, "sheaf hom" for short, and is denoted $\mathscr{H}om(\mathscr{F}, \mathscr{G})$.

    Solution. For two morphisms $\varphi, \psi \in \operatorname{Hom}(\mathscr{F}|_U, \mathscr{G}|_U)$, we define $\varphi + \psi$ to be the map such that for every open set $V \subseteq U$,

    $$(\varphi + \psi)(V) = \varphi(V) + \varphi(V).$$

    For any inclusion of open sets $W \subseteq V$, the equality

    $$(\varphi(W) + \psi(W)) \circ \rho^{\mathscr{F}|_U}_{V,W} = \rho^{\mathscr{G}|_U}_{V,W} \circ (\varphi(V) + \psi(V)),$$

    where the ρ are the restriction maps, holds because composition of homomorphisms distributes with respect to addition for Abelian groups, so $\varphi + \psi$ is a morphism of sheaves. The identity element is the morphism 0 such that $0(V)$ is the zero map for all open sets V, and the inverse of φ is the morphism that sends an open set V to the map $-\varphi(V)$. Commutativity follows from commutativity of adding homomorphisms of Abelian groups. This gives $\operatorname{Hom}(\mathscr{F}|_U, \mathscr{G}|_U)$ a natural Abelian group structure induced by the Abelian group structure from $\operatorname{Hom}(\mathscr{F}(V), \mathscr{G}(V))$ for all $V \subseteq U$.

    Now let $\mathscr{H}om$ be the presheaf $U \mapsto \operatorname{Hom}(\mathscr{F}|_U, \mathscr{G}|_U)$. The restriction map $\rho_{U,V}\colon \mathscr{H}om(U) \to \mathscr{H}om(V)$ is defined as follows. For any open set $W \subseteq U$, $(\mathscr{F}|_V)(W \cap V) = \mathscr{F}(W \cap V)$, and $(\mathscr{F}|_U)(W) = \mathscr{F}(W)$, so $\rho_{U,V}$ is the family of restriction maps $\mathscr{F}(V) \to \mathscr{F}(V \cap W)$. Let $\{U_i\}$ be an open covering of U and choose $\psi \in \mathscr{H}om(U)$ such that $\psi|_{U_i} = 0$ for all i. Then for any open set $W \subseteq U$, the map $\psi(W \cap U_i)$ is 0. Since $\{W \cap U_i\}$ is a covering of W, for any $x \in \mathscr{F}(W)$, $\psi(x|_{W \cap U_i}) = 0$. Since $\mathscr{G}$ is a sheaf, $\psi(x) = 0$, so this means that $\psi(W) = 0$ for all $W \subseteq U$, so $\psi = 0$ in the first place.

    Now suppose there are elements $\psi_i \in \mathscr{H}om(U_i)$ such that for all i and j, $\psi_i|_{U_i \cap U_j} = \psi_j|_{U_i \cap U_j}$. For any open set $W \subseteq U$, the compatibility of the ψ_i gives rise to a map $\psi \in \mathscr{H}om(U)$ such that $\psi|_{U_i} = \psi_i$ for all i because $\mathscr{G}$ is a sheaf. Therefore, $\mathscr{H}om$ satisfies the additional sheaf axioms, so is a sheaf. □

16. **Exercise (Flasque Sheaves).** A sheaf $\mathscr{F}$ on a topological space X is *flasque* if for every inclusion $V \subseteq U$ of open sets, the restriction map $\mathscr{F}(U) \to \mathscr{F}(V)$ is surjective.

(a) Show that a constant sheaf on an irreducible topological space is flasque.

(b) If $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ is an exact sequence of sheaves, and if $\mathscr{F}'$ is flasque, then for any open set U, the sequence $0 \to \mathscr{F}'(U) \to \mathscr{F}(U) \to \mathscr{F}''(U) \to 0$ of Abelian groups is also exact.

(c) If $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ is an exact sequence of sheaves, and if $\mathscr{F}'$ and $\mathscr{F}$ are flasque, then $\mathscr{F}''$ is flasque.

(d) If $f\colon X \to Y$ is a continuous map, and if $\mathscr{F}$ is a flasque sheaf on X, then $f_*\mathscr{F}$ is a flasque sheaf on Y.

(e) Let $\mathscr{F}$ be any sheaf on X. We define a new sheaf $\mathscr{G}$, called the sheaf of *discontinuous sections* of $\mathscr{F}$ as follows. For each open set $U \subseteq X$, $\mathscr{G}(U)$ is the set of maps $s\colon U \to \bigcup_{P\in U} \mathscr{F}_P$ such that for each $P \in U$, $s(P) \in \mathscr{F}_P$. Show that $\mathscr{G}$ is a flasque sheaf, and that there is a natural injective morphism of $\mathscr{F}$ to $\mathscr{G}$.

Solution. Let X be an irreducible topological space, let A be an Abelian group, and let $\mathscr{A}$ be the constant sheaf on X determined by A. We claim that every open set $U \subseteq X$ is connected. If $U = X$ or $U = \varnothing$, this is clear. Otherwise, if $U \neq X$ is nonempty and not connected, then there is a nonempty proper subset $U_1 \subseteq U$ such that both U_1 and $U \setminus U_1$ are closed relative to U. This means that there are closed subsets $X_1, X_2 \subseteq X$ such that $U_1 = U \cap X_1$ and $U \setminus U_1 = U \cap X_2$. Since $U \neq X$, X_1 and X_2 are both proper subsets of X. It must be that $X \neq X_1 \cup X_2$, or else X is reducible. But $X \setminus U$ is closed, so we can write $X = (X \setminus U) \cup (X_1 \cup X_2)$ as the union of two closed proper subsets, which is a contradiction. Now let $V \subseteq U$ be an inclusion of open sets. Then any $f \in \mathscr{A}(V)$ is a constant map since V is connected. Thus f can be extended to U, so $\rho_{U,V}\colon \mathscr{A}(U) \to \mathscr{A}(V)$ is surjective, and $\mathscr{A}$ is flasque. This gives (a).

Now we show (b). Let φ be the map $\mathscr{F} \to \mathscr{F}''$ and ψ be the map $\mathscr{F}' \to \mathscr{F}$. By (Ex. 1.8), the functor $\Gamma(U, \cdot)$ is left exact, so it remains to show that $\varphi(U)\colon \mathscr{F}(U) \to \mathscr{F}''(U)$ is surjective. Pick $s \in \mathscr{F}''(U)$. Consider the set S of pairs (V, t) with $V \subseteq U$ open and $t \in \mathscr{F}(V)$ such that $\varphi(t) = s|_V$. This is nonempty because for each point $P \in X$, φ_P is surjective by (Ex. 1.2(b)). We partially order such pairs by $(V, t) \leq (V', t')$ if $V \subseteq V'$ and $t = t'|_V$. It is clear that for any chain $\{(V_i, t_i)\}$, the element $(\bigcup_i V_i, t)$ is a maximal element, where t is the element such that $t|_{V_i} = t_i$ for all i (which exists because $\mathscr{F}$ is a sheaf). By Zorn's lemma, there is a maximal element of S, which we denote (W, x). Suppose $W \neq U$. Then pick $P \in U \setminus W$. By the surjectivity of φ_P, there is a neighborhood W' containing P and an element $y \in \mathscr{F}(W')$ such that $\varphi(y) = s|_{W'}$. Then $\varphi(x|_{W\cap W'} - y|_{W\cap W'}) = 0$, so by exactness, there is an element $a \in \mathscr{F}'(W \cap W')$ such that $\psi(a) = x|_{W\cap W'} - y|_{W\cap W'}$. Since $\mathscr{F}$ is flasque, $\rho^{\mathscr{F}'}_{W,W\cap W'}\colon \mathscr{F}'(W) \to \mathscr{F}'(W \cap W')$ is surjective, so we can lift a to an element $b \in \mathscr{F}'(W)$. Then

$$(y + \psi(b)|_{W'})|_{W\cap W'} = y|_{W\cap W'} + \psi(b)|_{W\cap W'} = x|_{W\cap W'},$$

where the second equality follows because $\psi(W \cap W') \circ \rho^{\mathscr{F}'}_{W,W\cap W'} = \rho^{\mathscr{F}}_{W,W\cap W'} \circ \psi(W)$. Since $\{W, W'\}$ is a cover of $W \cup W'$, there is an element $c \in \mathscr{F}(W \cup W')$ such that $c|_{W'} = y + \psi(b)|_W$ and $c|_W = x$. This implies that $\varphi(c) = s|_{W\cup W'}$, which contradicts the maximality of (W, x), so in fact $W = U$, and thus $\varphi(U)$ is surjective.

Let $V \subseteq U$ be an inclusion of open sets. Then by (b), since $\mathscr{F}'$ is flasque, $\mathscr{F}(V) \to \mathscr{F}''(V)$ is surjective. Since $\mathscr{F}$ is flasque, $\mathscr{F}(U) \to \mathscr{F}(V)$ is also surjective. Their composition is surjective, and is the same map as $\mathscr{F}(U) \to \mathscr{F}''(U) \to \mathscr{F}''(V)$. This implies $\mathscr{F}''(U) \to \mathscr{F}''(V)$ is surjective, so $\mathscr{F}''$ is flasque, so we have (c).

If $f\colon X \to Y$ is continuous and $\mathscr{F}$ is a flasque sheaf on X, then for any inclusion of open sets $V \subseteq U$, $\rho_{U,V}\colon (f_*\mathscr{F})(U) \to (f_*\mathscr{F})(V)$ is exactly the restriction map $\mathscr{F}(f^{-1}(U)) \to \mathscr{F}(f^{-1}(V))$, which is surjective since $\mathscr{F}$ is flasque, so $f_*\mathscr{F}$ is flasque. This gives (d).

Finally, we prove (e). The restriction maps of $\mathscr{G}$ are restriction in the usual sense. Let $\{U_i\}$ be a covering of U, and choose $s \in \mathscr{G}(U)$ such that $s|_{U_i} = 0$ for all i. This means that $s|_{U_i}\colon U_i \to \bigcup_{P \in U_i} \mathscr{F}_P$ is just the zero map. Since $\{U_i\}$ is a covering, each $P \in U$ is mapped to 0, so $s = 0$. Now suppose we have elements $s_i \in \mathscr{G}(U_i)$ such that for any two i and j, $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$. Define $s \in \mathscr{G}(U)$ in the obvious way. That is, for a point P, there is an i such that $P \in U_i$, so let $s(P) = s_i(P)$. Since the s_i agree on their overlaps, s is well-defined, and $s|_{U_i} = s_i$ for all i, so $\mathscr{G}$ is a sheaf. Now let $V \subseteq U$ be an inclusion of open sets. For any $s \in \mathscr{G}(V)$, we can extend s to an element $t \in \mathscr{G}(U)$ by setting $t(P) = 0$ if $P \in U \setminus V$, and $t(P) = s(P)$ if $P \in U$, so $\mathscr{G}$ is flasque.

For an open set $U \subseteq X$, define $\mathscr{F}(U) \to \mathscr{G}(U)$ by $x \mapsto (P \mapsto x_P)$. It is immediate that this defines a morphism $\mathscr{F} \to \mathscr{G}$. Suppose $P \mapsto x_P$ is the zero map for $x \in \mathscr{F}(U)$. Then for every point $P \in U$, there is an open neighborhood U_P such that $x|_{U_P} = 0$. Since $\{U_P\}$ is a cover of U, and $\mathscr{F}$ is a sheaf, $x = 0$. Thus $\mathscr{F}(U) \to \mathscr{G}(U)$ is injective for all U, so $\mathscr{F} \to \mathscr{G}$ is injective. □

17. **Exercise (Skyscraper Sheaves).** Let X be a topological space, let P be a point, and let A be an Abelian group. Define a sheaf $i_P(A)$ on X as follows: $i_P(A)(U) = A$ if $P \in U$, 0 otherwise. Verify that the stalk of $i_P(A)$ is A at every point $Q \in \{P\}^-$, and 0 elsewhere, where $\{P\}^-$ denotes the closure of the set consisting of the point P. Hence the name "skyscraper sheaf." Show that this sheaf could also be described as $i_*(A)$, where A denotes the constant sheaf A on the closed subspace $\{P\}^-$, and $i\colon \{P\}^- \to X$ is the inclusion.

    Solution. If Q is in the closure of P, then every open set containing Q contains P, so the stalk at Q of $i_P(A)$ is A. If Q is not in the closure of P, then there is some open set of Q not containing P, and its value under $i_P(A)$ is 0, so the stalk of $i_P(A)$ at Q is also zero.

    Since every open set of $\{P\}^-$ contains P, the constant presheaf A on $\{P\}^-$ is a sheaf. By definition, $i_*(A)$ is A on every open set of X containing P, so is exactly the sheaf described above. □

19. **Exercise (Extending a Sheaf by Zero).** Let X be a topological space, let Z be a closed subset, let $i\colon Z \to X$ be the inclusion, let $U = X \setminus Z$ be the complementary open subset, and let $j\colon U \to X$ be its inclusion.

    (a) Let $\mathscr{F}$ be a sheaf on Z. Show that the stalk $(i_*\mathscr{F})_P$ of the direct image sheaf on X is $\mathscr{F}_P$ if $P \in Z$, 0 if $P \notin Z$. Hence we call $i_*\mathscr{F}$ the sheaf obtained by extending $\mathscr{F}$ by zero outside Z. By abuse of notation we will sometimes write F instead of $i_*\mathscr{F}$, and say "consider $\mathscr{F}$ as a sheaf on X," when we mean "consider $i_*\mathscr{F}$."

    (b) Now let $\mathscr{F}$ be a sheaf on U. Let $j_!(\mathscr{F})$ be the sheaf on X associated to the presheaf $V \mapsto \mathscr{F}(V)$ if $V \subseteq U$, $V \mapsto 0$ otherwise. Show that the stalk $(j_!(\mathscr{F}))_P$ is equal to $\mathscr{F}_P$ if $P \in U$, 0 if $P \notin U$, and show that $j_!\mathscr{F}$ is the only sheaf on X which has this property, and whose restriction to U is $\mathscr{F}$. We call $j_!\mathscr{F}$ the sheaf obtained by *extending $\mathscr{F}$ by zero* outside U.

    (c) Now let $\mathscr{F}$ be a sheaf on X. Show that there is an exact sequence of sheaves on X,

$$0 \longrightarrow j_!(\mathscr{F}|_U) \longrightarrow \mathscr{F} \longrightarrow i_*(\mathscr{F}|_Z) \longrightarrow 0 .$$

Solution. If $\mathscr{F}$ is a sheaf on Z, then for any open set $V \subseteq X$, $(i_*\mathscr{F})(V) = \mathscr{F}(i^{-1}(V)) = \mathscr{F}(V \cap Z)$. For any point $P \in Z$, $(i_*\mathscr{F})_P$ is the direct limit of the groups $(i_*\mathscr{F})(V)$ for all open sets $V \subseteq X$ containing P. Equivalently, this is the direct limit of the groups $\mathscr{F}(V \cap Z)$ for all open sets $V \subseteq X$ containing P. On the other hand, $\mathscr{F}_P$ is the direct limit of $\mathscr{F}(W)$ for all open sets W of Z containing P. The open sets of Z are exactly those of the form $V \cap Z$ for some open set V of X, so these direct limits are the same, and hence $(i_*\mathscr{F})_P = \mathscr{F}_P$. For any point $P \notin Z$, there is a neighborhood W containing P such that $W \cap Z = \varnothing$ since Z is closed. Then $(i_*\mathscr{F})(W) = 0$, so every germ of $(i_*\mathscr{F})_P$ is 0, and $(i_*\mathscr{F})_P = 0$. This gives (a).

Let $\mathscr{F}$ be a sheaf on $U = X \setminus Z$, and let $\mathscr{I}$ be the presheaf $V \mapsto \mathscr{F}(V)$ if $V \subseteq U$ and $V \mapsto 0$ otherwise. For any $P \in X$, $(j_!(\mathscr{F}))_P = \mathscr{I}_P$. If $P \in U$, then $\mathscr{I}_P = \mathscr{F}_P$ since $\mathscr{F}$ is the restriction of $\mathscr{I}$ on U. If $P \notin U$, then $\mathscr{I}_P = 0$ because any neighborhood W of P is not contained in U, which means $\mathscr{I}(W) = 0$. Suppose $\mathscr{G}$ is another sheaf on X whose restriction to U is $\mathscr{F}$ and such that $\mathscr{G}_P = 0$ for all $P \notin U$. Then we can define a morphism $\mathscr{I} \to \mathscr{G}$ by letting $\mathscr{I}(V) \to \mathscr{G}(V)$ be the identity for $V \subseteq U$ and letting $0 \to \mathscr{G}(V)$ be the zero map otherwise. This induces a morphism $\psi\colon j_!(\mathscr{F}) \to \mathscr{G}$. If $P \in U$, ψ_P is a map $\mathscr{F}_P \to \mathscr{G}_P = \mathscr{F}_P$, which is an isomorphism, and similarly, if $P \notin U$, then $\psi_P\colon 0 \to 0$ is also an isomorphism, so ψ is an isomorphism. This gives that $j_!(\mathscr{F})$ is the unique sheaf up to isomorphism subject to the properties described. So (b) is proven.

Let $\mathscr{I}$ be the presheaf $V \mapsto (\mathscr{F}|_U)(V) = \mathscr{F}(V)$ if $V \subseteq U$ and $V \mapsto 0$ otherwise. Then there is a natural map $\mathscr{I} \to \mathscr{F}$ where for $V \subseteq U$, $\mathscr{I}(V) = \mathscr{F}(V) \to \mathscr{F}(V)$ is the identity, and $\mathscr{I}(V) = 0 \to \mathscr{F}(V)$ is the zero map otherwise. This induces a unique morphism $j_!(\mathscr{F}|_U) \to \mathscr{F}$. For any open set $V \subseteq X$, $(i_*(\mathscr{F}|_Z))(V) = (\mathscr{F}|_Z)(V \cap Z)$, which is equal to the direct limit of $\mathscr{F}(W)$ over all open sets $W \subseteq X$ containing $V \cap Z$, but we lose nothing by only taking the limit over those W that are also contained in V. Using the restriction maps of $\mathscr{F}$ gives a map of $\mathscr{F}(V)$ to this direct limit, and also gives a natural morphism $\mathscr{F} \to i_*(\mathscr{F}|_Z)$ by composing with the canonical morphism from the direct limit presheaf to its sheafification. By (Ex. 1.2(c)), it is enough to show that for every $P \in X$,

$$0 \to (j_!(\mathscr{F}|_U))_P \to \mathscr{F}_P \to (i_*(\mathscr{F}|_Z))_P \to 0$$

is exact. If $P \in U$, then $(j_!(\mathscr{F}|_U))_P = (\mathscr{F}|_U)_P = \mathscr{F}_P$ and $(i_*(\mathscr{F}|_Z))_P = 0$, and in this case, $(j_!(\mathscr{F}_P|_U))_P \to \mathscr{F}_P$ is an isomorphism since this is the identity map. If $P \notin U$, then $(j_!(\mathscr{F}|_U))_P = 0$ and $(i_*(\mathscr{F}|_Z))_P = (\mathscr{F}|_Z)_P = \mathscr{F}_P$, and $\mathscr{F}_P \to (i_*(\mathscr{F}|_Z))_P$ is an isomorphism because these two sheaves behave the same on open sets of X contained in Z, so again it is the identity map. In both cases the corresponding sequence is exact. □

20. **Exercise (Subsheaf with Supports).** Let Z be a closed subset of X, and let $\mathscr{F}$ be a sheaf on X. We define $\Gamma_Z(X, \mathscr{F})$ to be the subgroup of $\Gamma(X, \mathscr{F})$ consisting of all sections whose support is contained in Z.

    (a) Show that the presheaf $V \mapsto \Gamma_{Z \cap V}(V, \mathscr{F}|_V)$ is a sheaf. It is called the subsheaf of $\mathscr{F}$ with supports in Z, and is denoted by $\mathscr{H}_Z^0(\mathscr{F})$.

    (b) Let $U = X \setminus Z$, and let $j\colon U \to X$ be the inclusion. Show there is an exact sequence of sheaves on X

    $$0 \longrightarrow \mathscr{H}_Z^0(\mathscr{F}) \longrightarrow \mathscr{F} \longrightarrow j_*(\mathscr{F}|_U)\ .$$

    Furthermore, if $\mathscr{F}$ is flasque, the map $\mathscr{F} \to j_*(\mathscr{F}|_U)$ is surjective.

Solution. Let $V \subseteq X$ be an open set with a covering $\{V_i\}$. Choose $s \in \Gamma_{Z\cap V}(V, \mathscr{F}|_V)$ such that $s|_{V_i} = 0$ for all i. This means that the support of $s|_{V_i}$ in V_i is empty. Since $\{V_i\}$ covers V, the support of s in V is also empty since $s_P = (s|_{V_i})_P$ for all i. Then $s_P = 0$ for all $P \in V$, which means that $s = 0$ since $\mathscr{F}$ is a sheaf. Now suppose we have elements $s_i \in \Gamma_{Z\cap V_i}(V_i, \mathscr{F}|_{V_i})$ such that for all i and j, $s_i|_{V_i \cap V_j} = s_j|_{V_i\cap V_j}$. Since $\mathscr{F}$ is a sheaf, there is a unique element $s \in \mathscr{F}(V)$ such that $s|_{V_i} = s_i$. We wish to show that $s \in \Gamma_{Z\cap V}(V, \mathscr{F}|_V)$. For any $P \in V \setminus Z$, $P \in V_i$ for some i, so $s_P = (s|_{V_i})_P = (s_i)_P$. Since the support of s_i in V_i is contained in $V_i \cap Z$, $(s_i)_P = 0$, so the support of s is contained in $Z \cap V$, and thus $\mathscr{H}^0_Z(\mathscr{F})$ is a sheaf.

For every open set $V \subseteq X$, $(\mathscr{H}^0_Z(\mathscr{F}))(V)$ is a subgroup of $\mathscr{F}(V)$, so define $\varphi\colon \mathscr{H}^0_Z(\mathscr{F}) \to \mathscr{F}$ by inclusion, which is injective. Also, $(j_*(\mathscr{F}|_U))(V) = (\mathscr{F}|_U)(j^{-1}(V)) = (\mathscr{F}|_U)(U \cap V) = \mathscr{F}(U \cap V)$, so let $\psi\colon \mathscr{F} \to j_*(\mathscr{F}|_U)$ be given by the restriction maps of $\mathscr{F}$. If $\mathscr{F}$ is flasque, then by definition, ψ is surjective on each open set and thus surjective. Since φ is injective, the presheaf $V \mapsto \operatorname{image}(\varphi(V))$ is a sheaf by (Ex. 1.4(b)), and it is enough to show that $\operatorname{image}(\varphi(V)) = \ker(\psi(V))$ for all V to show that $\operatorname{image}\varphi = \ker\psi$. If $x \in \ker(\psi(V))$, then $x|_{U\cap V} = 0$, which means its support in V must be contained in $Z \cap V$, so $x \in \operatorname{image}(\varphi(V))$. On the other hand, if $x \in \operatorname{image}(\varphi(V))$, then for every $Q \in V \setminus Z = U \cap V$, $x_Q = 0$, so there is some neighborhood $V_Q \subseteq U \cap V$ containing Q such that $x|_{V_Q} = 0$. Since $\{V_Q\}$ is a cover of $U \cap V$ and $j_*(\mathscr{F}|_U)$ is a sheaf, $\psi(V)(x) = x|_{U\cap V} = 0$, so $x \in \ker(\psi(V))$. □

22. **Exercise (Gluing Sheaves).** Let X be a topological space, let $\mathfrak{U} = \{U_i\}$ be an open cover of X, and suppose we are given for each i a sheaf $\mathscr{F}_i$ on U_i, and for each i, j an isomorphism $\varphi_{ij}\colon \mathscr{F}_i|_{U_i\cap U_j} \to \mathscr{F}_j|_{U_i\cap U_j}$ such that (1) for each i, $\varphi_{ii} = \mathrm{id}$, and (2) for each i, j, k, $\varphi_{ik} = \varphi_{jk} \circ \varphi_{ij}$ on $U_i \cap U_j \cap U_k$. Then there exists a unique sheaf $\mathscr{F}$ on X, together with isomorphisms $\psi_i\colon \mathscr{F}|_{U_i} \to \mathscr{F}_i$ such that for each i, j, $\psi_j = \varphi_{ij} \circ \psi_i$ on $U_i \cap U_j$. We say loosely that $\mathscr{F}$ is obtained by *gluing* the sheaves $\mathscr{F}_i$ via the isomorphisms φ_{ij}.

Solution. For every open set $V \subseteq X$, $\{V \cap U_i\}$ is a covering. Consider the group $\prod_i \mathscr{F}_i(V \cap U_i)$, and for an element s, let s_i be the component of s in $\mathscr{F}_i(V \cap U_i)$. Define $\mathscr{F}(V)$ to be the subgroup of components s such that for all i and j, one has $\varphi_{i,j}(s_i|_{V\cap U_i\cap U_j}) = s_j|_{V\cap U_i\cap U_j}$. For $W \subseteq V$, there is a map $\mathscr{F}(V) \to \mathscr{F}(W)$ induced by each $\mathscr{F}_i(V \cap U_i) \to \mathscr{F}_i(W \cap U_i)$, which is well-defined because of the compatibility of the φ on each triple intersection. We let these be the restriction maps of $\mathscr{F}$, so it is clear that $\mathscr{F}$ is a presheaf. Now let $\{V_j\}$ be a covering of V, and suppose that $s \in \mathscr{F}(V)$ is such that $s|_{V_j} = 0$ for all j. More precisely, for each component $s_i \in \mathscr{F}_i(V \cap U_i)$ of s, $s_i|_{V_j} = 0$ for all j. For any given i, $\{U_i \cap V_j\}$ is a covering of $U_i \cap V$, and $\mathscr{F}_i$ is a sheaf, so this implies $s_i = 0$ for all i, and hence $s = 0$. Now suppose there are $s^j \in \mathscr{F}(V_j)$ such that for all j and k, $s^j|_{V_j\cap V_k} = s^k|_{V_j\cap V_k}$. For fixed i, $\{U_i \cap V_j\}$ is a covering of $U_i \cap V$, and $s^j_i|_{V_j\cap V_k} = s^k_i|_{V_j\cap V_k}$. Since $\mathscr{F}_i$ is a sheaf, there is an element s_i such that $s_i|_{V_j} = s^j_i$ for all j. Furthermore, these elements satisfy the condition $\varphi_{i,j}(s_i|_{V\cap U_i\cap U_j}) = s_j|_{V\cap U_i\cap U_j}$, so they are the components of some $s \in \mathscr{F}(V)$, and therefore $\mathscr{F}$ is a sheaf.

For every inclusion of open sets $V \subseteq U_i$, $(\mathscr{F}|_{U_i})(V) = \mathscr{F}(V)$, so there is a morphism $\psi_i(V)\colon (\mathscr{F}|_{U_i})(V) \to \mathscr{F}_i(V)$ by $s \mapsto s_i$. To see this is injective, suppose there is t such that the component of t in $\mathscr{F}_i(V)$ is s_i. Then for any j, $\varphi_{j,i}(t_j|_{V\cap U_j}) = t_i|_{V\cap U_j} = s_i|_{V\cap U_j}$. Since $\varphi_{j,i}$ is an isomorphism, $t_j|_{V\cap U_j} = s_j|_{V\cap U_j}$, so $t = s$. For surjectivity, we can define $s_j = \varphi_{i,j}(s_i|_{V\cap U_i})$, which is an element of $V \cap U_j$, and by definition this gives an element of $\mathscr{F}(V)$. The map $s \mapsto s_i$ gives rise to an isomorphism $\psi_i\colon \mathscr{F}|_{U_i} \to \mathscr{F}_i$. That $\psi_j = \varphi_{i,j} \circ \psi_i$ on $U_i \cap U_j$ for all i and j is a consequence of the definition of the elements in $\mathscr{F}(X)$. Finally, suppose there is another sheaf $\mathscr{G}$ on X such that there are isomorphisms $\psi'_i\colon \mathscr{G}|_{U_i} \to \mathscr{F}_i$ satisfying $\psi'_j = \varphi_{i,j} \circ \psi'_i$ on $U_i \cap U_j$ for all i and j. This gives isomorphisms $\mathscr{F}_{U_i} \to \mathscr{G}_{U_i}$ via $(\psi'_i)^{-1} \circ \psi_i$. By (Ex. 1.15), $\mathscr{H}om(\mathscr{F}, \mathscr{G})$ is a sheaf, and

$\{U_i\}$ is a covering, so there is a morphism $\theta\colon \mathscr{F} \to \mathscr{G}$ such that $\theta|_{U_i} = (\psi_i')^{-1} \circ \psi_i$. By the same reasoning we get a morphism $\theta'\colon \mathscr{G} \to \mathscr{F}$ such that $\theta'|_{U_i} = \psi_i^{-1} \circ \psi_i'$. Since $(\theta \circ \theta')|_{U_i}$ restricts to the identity map, we conclude $\theta \circ \theta'$ is the identity. Thus $\mathscr{F}$ and $\mathscr{G}$ are isomorphic, so $\mathscr{F}$ is unique up to isomorphism. □

2 Schemes

1. By definition, $D(f)$ is the set of prime ideals of A not containing f. Let $\varphi\colon A \to A_f$ be the natural map $a \mapsto a/1$. The map $I \mapsto \varphi^{-1}(I)$ is an inclusion preserving injection from the set of ideals of A_f and the set of ideals of A, and is also a bijection between the set of prime ideals A_f and the primes of A that don't contain f, so the map $\psi\colon D(f) \to \operatorname{Spec} A_f$ given by this bijection preserves closed sets and hence is a homeomorphism. We wish to define a map $\psi^{\#}\colon \mathcal{O}_{\operatorname{Spec} A_f} \to \psi_* \mathcal{O}_X|_{D(f)}$. For each open set $U \subseteq \operatorname{Spec} A_f$, we need

$$\psi^{\#}(U)\colon \mathcal{O}_{\operatorname{Spec} A_f}(U) \to \mathcal{O}_X|_{D(f)}(\psi^{-1}(U)).$$

But we can identify $\psi^{-1}(U)$ with U via the homeomorphism, and since $D(f)$ is an open set, $\mathcal{O}_X|_{D(f)}(U) = \mathcal{O}_X(U)$. We also remark that for any prime $\mathfrak{p} \in D(f)$, $(A_f)_{\mathfrak{p}} \cong A_{\mathfrak{p}}$ in a natural way via φ; call this isomorphism φ'. By the construction of $\mathcal{O}$, we can give an isomorphism $\mathcal{O}_{\operatorname{Spec} A_f}(U) \to \mathcal{O}_X|_{D(f)}(\psi^{-1}(U))$. That is, for a function $s\colon U \to \coprod_{\mathfrak{p} \in U} (A_f)_{\mathfrak{p}}$ in $\mathcal{O}_X|_{D(f)}(U)$, map it to

$$\psi^{-1}(U) \to \coprod_{\varphi^{-1}(\mathfrak{p}) \in \psi^{-1}(U)} \varphi'((A_f)_{\mathfrak{p}}),$$

which is defined by composing s with the appropriate maps. This is an element in $\mathcal{O}_X|_{D(f)}(U)$; the condition of being locally a quotient of elements follows because it is true of s, and $\psi^{\#}(U)(s)$ is nothing more than a renaming of variables of s. Thus we have given the desired isomorphism $\psi^{\#}$. The stalk at any point is then also an isomorphism, so is automatically a local homomorphism. Thus, $(\psi, \psi^{\#})$ is an isomorphism of locally ringed spaces $(D(f), \mathcal{O}_X|_{D(f)}) \to (\operatorname{Spec} A_f, \mathcal{O}_{\operatorname{Spec} A_f})$.

3. (a) We will show that $\mathcal{O}_{X,P}$ has no nilpotent elements for all $P \in X$ if and only if $\mathcal{O}_X(U)$ has no nilpotent elements for all open sets $U \subseteq X$. Suppose that there exists $P \in X$ such that $\mathcal{O}_{X,P}$ has a nilpotent element $f_P \neq 0$. Let $f \in \mathcal{O}_X(X)$ be a representative of f_P. There exists n and an open neighborhood $V \subseteq X$ containing P such that $f^n|_V = 0$, which means $(f|_V)^n = 0$. If $f|_V = 0$, then $f_P = 0$, contrary to hypothesis, so $f|_V$ is a nilpotent element in $\mathcal{O}_X(V)$.

   Conversely, suppose that there is some open set $U \subseteq X$ such that $\mathcal{O}_X(U)$ has a nilpotent element $f \neq 0$. If $f_P \neq 0$, then it is a nilpotent element of $\mathcal{O}_{X,P}$. Suppose that $f_P = 0$ for all $P \in U$. Then for all $P \in U$, there exists a neighborhood $V_P \subseteq U$ of P such that $f|_{V_P} = 0$. Since $\{V_P\}$ is a covering of U, this implies $f = 0$, contrary to hypothesis, so $\mathcal{O}_{X,P}$ has a nilpotent element.

   (b) For every point $P \in X$, there is a neighborhood U^P such that $(U^P, \mathcal{O}_X|_{U^P})$ is isomorphic to $(\operatorname{Spec} A^P, \mathcal{O}_{\operatorname{Spec} A^P})$ for some ring A^P. Also, $((\mathcal{O}_X)_{\mathrm{red}})_P$ is the direct limit of $\mathcal{O}_X(U)_{\mathrm{red}}$ over all open sets U containing P. Each such ring is $\mathcal{O}_X(U)/I(U)$ where $I(U)$ is the ideal of nilpotent elements of $\mathcal{O}_X(U)$, so $((\mathcal{O}_X)_{\mathrm{red}})_P = (\mathcal{O}_X)_P/I$ where I is the ideal of nilpotent germs of $(\mathcal{O}_X)_P$ (this follows because by (a) we know that a representative of a germ is nilpotent if and only if that germ is nilpotent). Dividing by an ideal I preserves inclusions of ideals containing I, so $((\mathcal{O}_X)_{\mathrm{red}})_P$ is local since $(\mathcal{O}_X)_P$ is, and $(X, (\mathcal{O}_X)_{\mathrm{red}})$ is a locally ringed space.

Let $Y^P = \operatorname{Spec} A^P_{\mathrm{red}}$. We claim that $(U^P, (\mathcal{O}_X)_{\mathrm{red}}|_{U^P})$ is isomorphic as a locally ringed space to $(Y^P, \mathcal{O}_{Y^P})$. Any prime ideal of A^P must contain the ideal of its nilpotent elements I, so there is a bijection of ideals of A^P containing I and ideals of A^P_{red} given by projection which preserves inclusions and primes, and this bijection induces a homeomorphism $f\colon Y^P \to \operatorname{Spec} A^P$. For an open set $U \subseteq \operatorname{Spec} A^P = U^P$, define

$$\mathcal{O}_X|_{U^P}(U) \to \mathcal{O}_{Y^P}(f^{-1}(U))$$

in the following way. An element of $\mathcal{O}_X|_{U^P}(U)$ is a function $s \to \coprod_{\mathfrak{p}\in U} A^P_{\mathfrak{p}}$ with $s(\mathfrak{p}) \in A^P_{\mathfrak{p}}$ and that locally is a quotient of elements of A^P. We can describe $\mathcal{O}_{Y^P}(f^{-1}(U))$ in a similar way. Localization preserves quotients; that is, $(A/I)_{\mathfrak{p}}$ is canonically isomorphic to $A_{\mathfrak{p}}/(IA_{\mathfrak{p}})$. Thus, an element of $\mathcal{O}_X|_{U^P}(U)$ can be mapped naturally to an element of $\mathcal{O}_{Y^P}$ by composing with

$$A^P_{\mathfrak{p}} \to (A^P_{\mathfrak{p}})_{\mathrm{red}} \to (A^P_{\mathrm{red}})_{\mathfrak{p}}.$$

If a function $s \in \mathcal{O}_X|_{U^P}(U)$ is nilpotent, this means that $s(\mathfrak{p})$ is nilpotent for all $\mathfrak{p} \in U$, which means that its image in $\mathcal{O}_{Y^P}(U)$ is 0, so we have a map $\mathcal{O}_X(U)_{\mathrm{red}} \to \mathcal{O}_{Y^P}(f^{-1}(U))$. This defines a map of presheaves, which induces a morphism of sheaves $f^{\#}\colon (\mathcal{O}_X)_{\mathrm{red}}|_{U^P} \to f_*\mathcal{O}_{Y^P}$. Taking the stalk at a prime $\mathfrak{p} \in Y^P$, we get from our description above that

$$((\mathcal{O}_X)_{\mathrm{red}}|_{U^P})_{f(\mathfrak{p})} = ((\mathcal{O}_X)_{\mathrm{red}})_{f(\mathfrak{p})} = ((\mathcal{O}_X)_{f(\mathfrak{p})})_{\mathrm{red}} = (A^P_{f(\mathfrak{p})})_{\mathrm{red}},$$

and we get the map of stalks $f^{\#}_{\mathfrak{p}}\colon (A^P_{f(\mathfrak{p})})_{\mathrm{red}} \to (A^P_{\mathrm{red}})_{\mathfrak{p}}$, which is the canonical isomorphism described above, and therefore a local homomorphism. This also implies that $f^{\#}$ is an isomorphism of sheaves, so $(f, f^{\#})$ is an isomorphism of locally ringed spaces, and thus $(X, (\mathcal{O}_X)_{\mathrm{red}})$ is a scheme.

Since X and X_{red} have the same underlying topological space, the identity $f\colon X_{\mathrm{red}} \to X$ is a homeomorphism. For every open set $U \subseteq X$, there is a projection $\mathcal{O}_X(U) \to \mathcal{O}_X(U)_{\mathrm{red}}$, which defines a morphism of presheaves $\mathcal{O}_X \to (U \mapsto \mathcal{O}_X(U)_{\mathrm{red}})$. Composing this with the sheafification morphism $(U \mapsto \mathcal{O}_X(U)_{\mathrm{red}}) \to (\mathcal{O}_X)_{\mathrm{red}}$ gives a morphism of sheaves $f^{\#}\colon \mathcal{O}_X \to f_*(\mathcal{O}_X)_{\mathrm{red}}$. For any point $P \in X$, the stalk at P gives $f^{\#}_P\colon \mathcal{O}_{X,f(P)} \to ((\mathcal{O}_X)_{\mathrm{red}})_P$. Note that $((\mathcal{O}_X)_{\mathrm{red}})_P = (\mathcal{O}_{X,P})_{\mathrm{red}}$, and that $f^{\#}_P$ is projection. Projection of a local ring is a local homomorphism by the ideal inclusion preserving property of division, so $(f, f^{\#})$ gives a morphism of schemes $X_{\mathrm{red}} \to X$ that is a homeomorphism on the underlying spaces.

(c) Let $f\colon X \to Y$ be a morphism of schemes where X is a reduced scheme, and let $\varphi\colon Y_{\mathrm{red}} \to Y$ be the natural map described in (b). We wish to define a morphism $g\colon X \to Y_{\mathrm{red}}$ such that $f = \varphi \circ g$. Since φ is a homeomorphism on the underlying topological spaces, there is no choice but to define the map of topological spaces $g\colon X \to Y_{\mathrm{red}}$ to be $\varphi^{-1} \circ f$. For any open set $U \subseteq Y$, we have a ring homomorphism $f^{\#}\colon \mathcal{O}_Y(U) \to \mathcal{O}_X(f^{-1}(U))$. Since $\mathcal{O}_X(f^{-1}(U))$ is reduced, any nilpotent element of $\mathcal{O}_Y(U)$ must be in the kernel of $f^{\#}$. By the universal property of the kernel, this induces a unique homomorphism $\mathcal{O}_Y(U)_{\mathrm{red}} \to \mathcal{O}_X(f^{-1}(U))$ which commutes with the projection $\mathcal{O}_Y(U) \to \mathcal{O}_Y(U)_{\mathrm{red}}$. This gives a morphism of presheaves $(U \mapsto \mathcal{O}_Y(U)_{\mathrm{red}}) \to f_*\mathcal{O}_X$. Since the preimage of $\varphi^{-1}(U)$ under g is the same as the preimage of U under f, and by the universal property of sheafification, this induces a unique morphism of sheaves $g^{\#}\colon (\mathcal{O}_Y)_{\mathrm{red}} \to g_*\mathcal{O}_X$ such that $g^{\#} \circ \varphi^{\#} = f^{\#}$. Finally, for any point $P \in X$, the stalk at P induces maps $f^{\#}_P\colon \mathcal{O}_{Y,f(P)} \to \mathcal{O}_{X,P}$ and

$$\mathcal{O}_{Y,\varphi(g(P))} \xrightarrow{\varphi^{\#}_{g(P)}} ((\mathcal{O}_Y)_{\mathrm{red}})_{g(P)} \xrightarrow{g^{\#}_P} \mathcal{O}_{X,P}$$

which are equal. Since $f_P^\#$ and $\varphi_{\varphi(g(P))}^\#$ are local homomorphisms, $g_P^\#$ is also local. If not, then the preimage of the maximal ideal in $\mathcal{O}_{X,P}$ under $g_P^\#$ would not be maximal, and in turn gives that the preimage under the composition $g_P^\# \circ \varphi_{\varphi(g(P))}^\#$ isn't maximal, which contradicts that $f_P^\#$ is local since it is equal to the composition of those two maps. Thus, $(g, g^\#)$ is a morphism of schemes such that $\varphi \circ g = f$. To see that it is unique with respect to this property, let h be another such morphism. Then $h^\#\colon (\mathcal{O}_Y)_{\text{red}} \to h_*\mathcal{O}_X$ gives a morphism of presheaves $(U \mapsto \mathcal{O}_Y(U)_{\text{red}}) \to f_*\mathcal{O}_X$ by composing with the sheafification morphism. But this presheaf map is unique, so h and g are the same.

4. We construct an inverse map α^{-1} to show that α is a bijection. Let $\varphi\colon A \to \Gamma(X, \mathcal{O}_X)$ be a ring homomorphism. For each $P \in X$, there is an open neighborhood $U^P \subseteq X$ such that $(U^P, \mathcal{O}_X|_{U^P})$ is isomorphic as a locally ringed space to $(\operatorname{Spec} A^P, \mathcal{O}_{\operatorname{Spec} A^P})$ for some ring A^P. There is a restriction map $\Gamma(X, \mathcal{O}_X) \to \Gamma(U^P, \mathcal{O}_X)$. Since U^P is open, $\Gamma(U^P, \mathcal{O}_X|_{U^P}) \cong \Gamma(U^P, \mathcal{O}_X)$, which is in turn isomorphic to A^P, so using these isomorphisms, we get maps $\varphi^P\colon A \to A^P$. These induce morphisms of schemes $\operatorname{Spec} A^P \to \operatorname{Spec} A$ whose global sections $A \to A^P$ are precisely φ^P. The map of topological spaces $f\colon X \to \operatorname{Spec} A$ induced by the φ^P is well-defined because on any intersection $\operatorname{Spec} A^P \cap \operatorname{Spec} A^Q$, the values are induced by the restriction maps of $\mathcal{O}_X$, which force compatibility. That this map is continuous follows because each map $\operatorname{Spec} A^P \to \operatorname{Spec} A$ is continuous. Now we need to define $\mathcal{O}_{\operatorname{Spec} A}(V) \to \mathcal{O}_X(f^{-1}(V))$ for an arbitrary open set $V \subseteq \operatorname{Spec} A$. We can cover V by $\{V \cap U^P\}$, so that we have maps $\mathcal{O}_{\operatorname{Spec} A}(V \cap U^P) \to \mathcal{O}_{\operatorname{Spec} A^P}(f^{-1}(V \cap U^P))$. We claim that the images of a fixed element $x \in \mathcal{O}_{\operatorname{Spec} A}(V \cap U^P)$ agree on their overlaps and hence glue to give an image of x in $\mathcal{O}_X(f^{-1}(V))$. This follows because the restriction maps of each $\mathcal{O}_{\operatorname{Spec} A^P}$ are the restriction maps of $\mathcal{O}_X$. This also gives compatibility of our defined map with the restriction maps, so we have given a morphism of sheaves $f^\#\colon \mathcal{O}_{\operatorname{Spec} A} \to f_*\mathcal{O}_X$. Also, for any point $P \in X$, $f_P^\#$ is the same as taking the stalk of P of $\mathcal{O}_{\operatorname{Spec} A} \to \mathcal{O}_{\operatorname{Spec} A^P}$, so is a local homomorphism. Hence $(f, f^\#)$ is a morphism of schemes.

   We want that $\mathcal{O}_{\operatorname{Spec} A}(\operatorname{Spec} A) \to \Gamma(X, \mathcal{O}_X)$ is φ after identifying $\mathcal{O}_{\operatorname{Spec} A}(\operatorname{Spec} A)$ with A. This map is defined for each A^P via the restriction maps of $\mathcal{O}_X$ and φ^P, so the images of any $s \in \mathcal{O}_{\operatorname{Spec} A}(\operatorname{Spec} A)$ under φ^P is the restriction of $\Gamma(X, \mathcal{O}_X)$ to $\Gamma(U^P, \mathcal{O}_X|_{U^P})$, so they glue together to $\varphi(s)$.

   Now we need to show that $\alpha^{-1} \circ \alpha$ is also the identity. Suppose $(f, f^\#)$ is a morphism of schemes from X to $\operatorname{Spec} A$. Taking global sections gives a ring homomorphism $\varphi\colon A \to \Gamma(X, \mathcal{O}_X)$. The morphism of schemes $\alpha^{-1}(\varphi)$ at agrees with f on all open sets U^P by construction. Since the rest of the ring homomorphisms are determined by these values, $\alpha^{-1}(\varphi) = (f, f^\#)$, so we have the desired bijection.

5. Since $\mathbf{Z}$ is a principal ideal domain, the topological space $\operatorname{Spec} \mathbf{Z}$ consists of one point for every prime number p, and one point for the zero ideal. Every closed set is of the form $V((n))$ for some integer n, and $(n) \subseteq (p)$ if and only if $p|n$, so the closed sets of $\operatorname{Spec} \mathbf{Z}$ are all finite sets consisting of the ideals generated by prime numbers, the whole set and the empty set. This implies that every open set is of the form $D((n))$. In particular, $\mathcal{O}(D(n))$ is isomorphic to the localized ring $\mathbf{Z}_n$ where n is any integer. For any scheme X, there is a bijection of sets

$$\alpha\colon \operatorname{Hom}_{\text{Sch}}(X, \operatorname{Spec} \mathbf{Z}) \to \operatorname{Hom}_{\text{Ring}}(\mathbf{Z}, \Gamma(X, \mathcal{O}_X))$$

   by (Ex. 2.4). There is a unique morphism $\mathbf{Z} \to \Gamma(X, \mathcal{O}_X)$ because 1 is sent to 1 and this determines the image of all elements in $\mathbf{Z}$, so there is a unique morphism $X \to \operatorname{Spec} \mathbf{Z}$.

7. Since K is a field, $\operatorname{Spec} K$ is a one point space, so for a morphism $f\colon \operatorname{Spec} K \to X$ call the image of this point x. There is also a morphism of sheaves $f^{\#}\colon \mathcal{O} \to f_*\mathcal{O}_{\operatorname{Spec} K}$. Considering the stalk at x, we get $f_x^{\#}\colon \mathcal{O}_x \to (f_*\mathcal{O}_{\operatorname{Spec} K})_x$, which is a local homomorphism by definition. Also, $(f_*\mathcal{O}_{\operatorname{Spec} K})_x$ is the direct limit of $\mathcal{O}_{\operatorname{Spec} K}(f^{-1}(U))$ where we range over open sets U containing x. In each case, $f^{-1}(U) = \operatorname{Spec} K$, and $\mathcal{O}_{\operatorname{Spec} K}(\operatorname{Spec} K) = K$, so $(f_*\mathcal{O}_{\operatorname{Spec} K})_x = K$. Since $f_x^{\#}$ is a local homomorphism, $(f_x^{\#})^{-1}(0) = \mathfrak{m}_x$. This means the kernel of $f_x^{\#}$ is $\mathfrak{m}_x$, so there is an inclusion $\mathcal{O}_x/\mathfrak{m}_x \to K$.

   Conversely, suppose we are given a point $x \in X$ and an inclusion $k(x) \to K$. We define a continuous map $f\colon \operatorname{Spec} K \to X$ by sending the one point of $\operatorname{Spec} K$ to x. There is a projection $\mathcal{O}_x \to \mathcal{O}_x/\mathfrak{m}_x$, which we compose with the given inclusion to get a map $\mathcal{O}_x \to K$. We need to define a morphism of sheaves $f^{\#}\colon \mathcal{O} \to f_*\mathcal{O}_{\operatorname{Spec} K}$. However, if U does not contain x, then $f_*\mathcal{O}_{\operatorname{Spec} K}(U) = 0$, so we only need to specify $f^{\#}$ on open sets U containing x. On such open sets, $f_*\mathcal{O}_{\operatorname{Spec} K}(U) = K$. Then $f^{\#}$ is induced by the map $\mathcal{O}_x \to K$ since $\mathcal{O}_x$ is the direct limit of $\mathcal{O}(U)$ for all U containing x, and since the direct limit uses the restriction maps of $\mathcal{O}$, these induced maps on $\mathcal{O}(U)$ define a morphism of ringed spaces $f^{\#}$. Finally, we need to check that $f_P^{\#}$ is a local homomorphism for all $P \in X$. If every open set of P contains x, then this property is given by the fact that $\mathcal{O}_P = \mathcal{O}_x$ and that $\mathcal{O}_x \to K$ has kernel $\mathfrak{m}_x$. Otherwise, $\mathcal{O}_P = 0$, and there is nothing to show. Thus $(f, f^{\#})$ is a morphism of schemes. These two processes described are inverse to one another, so giving a morphism $\operatorname{Spec} K \to X$ is equivalent to giving a point $x \in X$ and an inclusion $k(x) \to K$.

8. The ring $k[\varepsilon]/\varepsilon^2$ has one prime ideal. To see this, note that any prime ideal contains $0 = \varepsilon^2$, so must also contain ε. Since (ε) is maximal and the smallest ideal containing ε, we get the claim. If we have a k-morphism $f\colon \operatorname{Spec} k[\varepsilon]/\varepsilon^2 \to X$, let x be the image of (ε). There is also a morphism of sheaves
$$f^{\#}\colon \mathcal{O}_X \to f_*\mathcal{O}_{\operatorname{Spec} k[\varepsilon]/\varepsilon^2}.$$
   Taking the stalk at (ε), $\mathcal{O}_{\operatorname{Spec} k[\varepsilon]/\varepsilon^2,(\varepsilon)} = k[\varepsilon]/\varepsilon^2$, so we get a local homomorphism
$$f_{(\varepsilon)}^{\#}\colon \mathcal{O}_{X,x} \to k[\varepsilon]/\varepsilon^2.$$
   Then the preimage of (ε) is $\mathfrak{m}_x$, and $k[\varepsilon]/\varepsilon = k$, so composing $f_{(\varepsilon)}^{\#}$ with this projection gives a map $\mathcal{O}_{X,x} \to k$ whose kernel contains $\mathfrak{m}_x$ and hence induces an injection $i\colon \mathcal{O}_{X,x}/\mathfrak{m}_x \to k$. Since f is a k-morphism, the following diagram
$$\begin{tikzcd} \operatorname{Spec} k[\varepsilon]/\varepsilon^2 \arrow[r, "f"] \arrow[d, "\varphi"] & X \arrow[dl, "\psi"] \\ \operatorname{Spec} k & \end{tikzcd}$$
   Note that $\mathcal{O}_{\operatorname{Spec} k,\varphi(x)} = k$, so we get local homomorphisms $\psi_x^{\#}\colon k \to \mathcal{O}_{X,x}$ and $\varphi_{(\varepsilon)}^{\#}\colon k \to k[\varepsilon]/\varepsilon^2$. Since $(\psi_x^{\#})^{-1}(\mathfrak{m}_x) = 0$, by composing with the projection $\mathcal{O}_{X,x} \to \mathcal{O}_{X,x}/\mathfrak{m}_x$, we get an injection $i'\colon k \to \mathcal{O}_{X,x}/\mathfrak{m}_x$ such that $i \circ i' = \psi_{(\varepsilon)}^{\#}$. But $\psi_{(\varepsilon)}^{\#}$ is the inclusion $k \to k[\varepsilon]/\varepsilon^2$, so $\mathcal{O}_{X,x}/\mathfrak{m}_x = k$, and hence x is rational over k. The image of $\mathfrak{m}_x^2$ under $f_{(\varepsilon)}^{\#}$ is 0 because $\varepsilon^2 = 0$, so this gives a k-vector space homomorphism $\mathfrak{m}_x/\mathfrak{m}_x^2 \to (\varepsilon)$. Also, there is an isomorphism $(\varepsilon) \to k$ via $\varepsilon \mapsto 1$, so this map gives an element of $T_x = \operatorname{Hom}_k(\mathfrak{m}_x/\mathfrak{m}_x^2, k)$.

Conversely, suppose we are given a point $x \in X$ such that $\mathcal{O}_{X,x}/\mathfrak{m}_x = k$, and a k-vector space homomorphism $\mathfrak{m}_x/\mathfrak{m}_x^2 \to k$. This extends to a map $\mathfrak{m}_x \to (\varepsilon)$, which we can further extend to $\mathcal{O}_{X,x} \to k[\varepsilon]/\varepsilon^2$ by sending an element not in $\mathfrak{m}_x$ to its image under the projection $\mathcal{O}_{X,x} \to k$. Then define a continuous map $f\colon \operatorname{Spec} k[\varepsilon]/\varepsilon^2 \to X$ by sending (ε) to x. This is a local homomorphism because the preimage of (ε) is $\mathfrak{m}_x$.

To define $f^\#\colon \mathcal{O}_X \to f_* \operatorname{Spec} k[\varepsilon]/\varepsilon^2$, we need only specify its values on open sets containing x since otherwise $f_*\mathcal{O}_{\operatorname{Spec} k[\varepsilon]/\varepsilon^2}(U) = 0$. We already have homomorphisms compatible with the restriction maps for all such open sets given by $\mathcal{O}_{X,x} \to k[\varepsilon]/\varepsilon^2$, so this gives $f^\#$. The map $f_P^\#$ is the map $\mathcal{O}_{X,x} \to k[\varepsilon]/\varepsilon^2$ as defined above if every open set of P contains x; otherwise, $\mathcal{O}_{X,P} = 0$. In either case, the map on stalks is a local homomorphism, so $(f, f^\#)$ is a morphism of schemes. We now need to check that this is a morphism over $\operatorname{Spec} k$. This follows because the maps of rings on open sets are k-algebra homomorphisms by construction. The process described here of getting a k-morphism from a point $x \in X$ rational over k and an element of T_x is inverse to the process described in the previous paragraph, so we have the desired equivalence.

13. (a) Let X be a Noetherian topological space. Let $U \subseteq X$ with a covering $\{U_i\}$. We will build a sequence of open sets V_i as follows. Let $V_0 = \varnothing$. Assuming that V_n has been constructed, and that $V_n \neq U$, choose U_j not entirely contained in V_n, and let $V_{n+1} = V_n \cup U_j$. If $V_n = U$, then we stop. In this case, $\{U_i\}$ has a finite subcover. If not, then there is an infinite descending chain of closed sets of X given by $U \setminus V_1 \supsetneq U \setminus V_2 \supsetneq \cdots$, which contradicts that X is Noetherian. Thus, every open set U is quasi-compact.

    Conversely, suppose that we have a descending chain of closed sets $V_1 \supseteq V_2 \supseteq \cdots$ in X. Let $U = \bigcup_{i \geq 0} X \setminus V_i$. Since U is open, there is a finite subcover $\{X \setminus V_{i_1}, \ldots, X \setminus V_{i_n}\}$. Then for all j, $X \setminus V_j \subseteq \bigcup_{k=1}^n X \setminus V_{i_k} = X \setminus \bigcap_{k=1}^n V_{i_k}$. This implies that $\bigcap_{i=1}^n V_{i_k} \subseteq V_j$ for all j. If $N = \max_{k=1}^n i_k$, then $\bigcap_{i=1}^n V_{i_k} = V_N$, so $V_N = V_j$ for all $j \geq N$, which implies that X is Noetherian.

    (b) Since X is an affine scheme, let A be a ring such that $X = \operatorname{Spec} A$. Suppose that $\{U_i\}$ is a covering of X. Since the open sets of the form $D(f)$, where $f \in A$, are a basis for the topology of X, each U_i can be written as the union of $D(f_j)$. This gives a finer covering. If we can show that this finer covering has a finite subcover, it will imply that $\{U_i\}$ has a finite subcover, so without loss of generality, assume $U_i = D(f_i)$ for some $f_i \in A$. Since $\bigcup D(f_i)$ is the set of prime ideals not containing any of the f_i and X is the set of all prime ideals of A, we get that $\bigcup D(f_i) = X$ if and only if the f_i generate the unit ideal. Thus, there is some finite sum $\sum_{i=1}^n a_i f_i = 1$ for some $a_i \in A$. This gives that $\bigcup_{i=1}^n D(f_i) = X$, so X is quasi-compact.

    Let k be a field and let $R = k[x_1, x_2, \ldots]$ be the polynomial ring over k in infinitely many variables. The infinite ascending chain of prime ideals $(x_1) \subsetneq (x_1, x_2) \subsetneq \cdots$ in R gives an infinite descending chain of closed sets $V(x_1) \supsetneq V(x_1, x_2) \supsetneq \cdots$ in $\operatorname{Spec} R$, so this is an example of a ring whose spectrum is not a Noetherian space.

    (c) Let A be a Noetherian ring and $V_1 \supseteq V_2 \supseteq \cdots$ be a descending chain of closed sets in $\operatorname{Spec} A$. Each closed set is of the form $V(\mathfrak{a}_i)$ for some ideal $\mathfrak{a}_i \subseteq A$, and $V(\mathfrak{a}_i) \supseteq V(\mathfrak{a}_j)$ if and only if $\sqrt{\mathfrak{a}_i} \subseteq \sqrt{\mathfrak{a}_j}$, so this chain gives an ascending chain of ideals $\sqrt{\mathfrak{a}_1} \subseteq \sqrt{\mathfrak{a}_2} \subseteq \cdots$ in A. Since A is Noetherian, there exists a number N such that $j \geq N$ implies that $\sqrt{\mathfrak{a}_j} = \sqrt{\mathfrak{a}_N}$, and this implies that $V_j = V_N$, so $\operatorname{Spec} A$ is a Noetherian space.

    (d) Let k be a field, and let $A = k[x_1, x_2, \ldots]/(x_1^2, x_2^2, \ldots)$. Each x_i is nilpotent, so every prime ideal must contain x_i for all i, and the smallest such ideal $(x_1, x_2, \ldots)$ is maximal since $A/(x_1, x_2, \ldots) = k$. So there is one prime ideal, and hence $\operatorname{Spec} A$ is Noetherian. However, since the x_i are independent of one another, we get an infinite ascending chain of ideals $(x_1) \underset{\neq}{\subsetneq} (x_1, x_2) \underset{\neq}{\subsetneq} \cdots$, so A is not Noetherian.

14. (a) Suppose that some element $x \in S_+$ is not nilpotent. Then form the subset $A = \{x, x^2, x^3, \dots\}$, and let P be the set of homogeneous ideals not meeting A. This is nonempty because $0 \in P$, and if $\{P_i\}$ is a chain in P, then their union is an ideal, and it is homogeneous because it is generated by the generators of each P_i, so by Zorn's lemma, P has a maximal element $\mathfrak{p}$. We claim that $\mathfrak{p}$ is prime. It is enough to show that for any two homogeneous elements a and b, $ab \in \mathfrak{p}$ implies either $a \in \mathfrak{p}$ or $b \in \mathfrak{p}$. Suppose a and b are homogeneous elements such that $ab \in \mathfrak{p}$ but $a \notin \mathfrak{p}$ and $b \notin \mathfrak{p}$. Then $(\mathfrak{p}, a)$ and $(\mathfrak{p}, b)$ are homogeneous ideals properly containing $\mathfrak{p}$, so must meet A. Then for some elements $p_1, p_2 \in \mathfrak{p}$ and $c_1, c_2 \in S$, and numbers n and m, we have $p_1 + c_1 a = x^n$ and $p_2 + c_2 b = x^m$. Multiplying them together, we get
$$p_1 p_2 + c_1 a p_2 + p_1 c_2 b + c_1 a c_2 b = x^{n+m},$$
but the sum on the left hand side is an element of $\mathfrak{p}$, which is a contradiction. Then $\mathfrak{p}$ is a homogeneous prime ideal which does not contain all of S_+, so $\operatorname{Proj} S \neq \varnothing$.

Suppose every element of S_+ is nilpotent. If $\mathfrak{p}$ is a prime ideal of S and $f \in S$ is a nilpotent element, then $f^n = 0$ for some n. Then $f^n \in \mathfrak{p}$, which implies $f \in \mathfrak{p}$, so every prime ideal must contain the set of nilpotent elements, and hence contain S_+, so in this case $\operatorname{Proj} S = \varnothing$.

(b) The set $\operatorname{Proj} T \setminus U = \{\mathfrak{p} \in \operatorname{Proj} T : \mathfrak{p} \supseteq \varphi(S_+)\}$ is the same set if we replace $\varphi(S_+)$ by the ideal it generates. Any element $f \in S$ can be expressed as a sum $f_1 + \cdots + f_n$ where the f_i are homogeneous, so $\varphi(f) = \varphi(f_1) + \cdots + \varphi(f_n)$ where each $\varphi(f_i)$ is homogeneous, so the ideal generated by $\varphi(S_+)$ is generated by homogeneous elements. Thus $\operatorname{Proj} T \setminus U$ is a closed set, so U is an open set.

As a map of topological spaces, define $f\colon U \to \operatorname{Proj} S$ by $\mathfrak{p} \mapsto \varphi^{-1}(\mathfrak{p})$. Since $\mathfrak{p} \not\supseteq \varphi(S_+)$, $\varphi^{-1}(\mathfrak{p}) \not\supseteq S_+$, so this is well-defined. Consider the localized map $\varphi_{(\mathfrak{p})}\colon S_{(\varphi^{-1}(\mathfrak{p}))} \to T_{(\mathfrak{p})}$ where $T_{(\mathfrak{p})}$ is the ring of elements of degree zero in the localized ring $A^{-1}T$ where A is the multiplicative system consisting of all homogeneous elements of T not in $\mathfrak{p}$, as defined in Hartshorne. Note that $S_{(\varphi^{-1}(\mathfrak{p}))}$ is a local ring whose maximal ideal is the image of $\varphi^{-1}(\mathfrak{p})$, and similarly with $T_{(\mathfrak{p})}$. From this, we see that $\varphi_{(\mathfrak{p})}$ is a local homomorphism. Now we need to define a morphism of sheaves $f^\#\colon \mathcal{O}_{\operatorname{Proj} S} \to f_* U$. If $V \subseteq \operatorname{Proj} S$ is an open set, then $\mathcal{O}_{\operatorname{Proj} S}(V)$ consists of functions $s\colon V \to \coprod_{\mathfrak{q} \in V} S_{(\mathfrak{q})}$ such that $s(\mathfrak{q}) \in S_{(\mathfrak{q})}$ and s is locally a quotient of elements of S. By composing with the localized maps, we can turn each such function into a function $t\colon f^{-1}(V) \to \coprod_{\mathfrak{q} \in f^{-1}(V)} T_{(\mathfrak{q})}$ such that $t(\mathfrak{q}) \in T_{(\mathfrak{q})}$ and t is locally a quotient of elements of T. The restriction maps of $\mathcal{O}_{\operatorname{Proj} S}$ and $\mathcal{O}_U$ are restriction of domain, so the map just defined gives a morphism of sheaves. The stalk at any point $\mathfrak{p} \in U$ is the map $\varphi_{(\mathfrak{p})}$, which is local, so f is a morphism of schemes.

(c) Let $\mathfrak{p}$ be any homogeneous prime ideal of T, and suppose that $\mathfrak{p}$ contains $\varphi(S_+)$. Let $x \in T$ be a homogeneous element of degree $\alpha > 0$. For some n, $n\alpha \geq d_0$, so $x^n \in T_{n\alpha} = \varphi(S_{n\alpha}) \subseteq \mathfrak{p}$, so $x \in \mathfrak{p}$. This implies that $T_+ \subseteq \mathfrak{p}$, so $U = \operatorname{Proj} T$. The induced map of topological spaces $f\colon \operatorname{Proj} T \to \operatorname{Proj} S$ is given by $\mathfrak{p} \mapsto \varphi^{-1}(\mathfrak{p})$. Suppose that $\varphi^{-1}(\mathfrak{p}) = \varphi^{-1}(\mathfrak{q})$ for two ideals $\mathfrak{p}, \mathfrak{q} \in \operatorname{Proj} T$. Then $\mathfrak{p}_d = \mathfrak{q}_d$ for all $d \geq d_0$. For any homogeneous element $x \in \mathfrak{q}$ of positive degree, some large power x^n has degree $\geq d_0$, so $x^n \in \mathfrak{p}$, which implies $x \in \mathfrak{p}$. By symmetry, $\mathfrak{p}_d = \mathfrak{q}_d$ for all $d > 0$. If $x \in \mathfrak{p}_0$, then since $\mathfrak{p} \in \operatorname{Proj} T$, there exists some homogeneous element $y \notin \mathfrak{p}$ with $\deg y > 0$. This means that $\deg xy > 0$, so $xy \in \mathfrak{q}$. Since $y \notin \mathfrak{q}$, we have $x \in \mathfrak{q}$, so by symmetry, $\mathfrak{p}_0 = \mathfrak{q}_0$, and f is injective.

Now we show that f is surjective. Let $\{\alpha_i\}$ be a set of homogeneous generators for T_+. Then $\{D_+(\alpha_i)\}$ is a cover of $\operatorname{Proj} T$. For any prime $\mathfrak{p} \in \operatorname{Proj} T$, $\alpha_i^{d_0} \in \mathfrak{p}$ if and only if $\alpha_i \in \mathfrak{p}$, so we may assume that $\deg \alpha_i \geq d_0$ for all i. We claim that $\{D_+(\varphi^{-1}(\alpha_i))\}$ forms a cover of $\operatorname{Proj} S$. If not, then there exists $\mathfrak{p}$ such that $\varphi^{-1}(\alpha_i) \in \mathfrak{p}$ for all i. However, the α_i generate $\bigoplus_{d \geq d_0} T_d$, so its preimage is contained in $\mathfrak{p}$. However, this preimage at least contains S_+ because φ_d is an isomorphism for $d \geq d_0$, so $S_+ \subseteq \mathfrak{p}$, which contradicts that $\mathfrak{p} \in \operatorname{Proj} S$. Then we have maps $f_{\alpha_i} \colon D_+(\alpha_i) \to D_+(\varphi^{-1}(\alpha_i))$ for all i. This map can be rewritten as $\operatorname{Spec} T_{(\alpha_i)} \to \operatorname{Spec} S_{(\varphi^{-1}(\alpha_i))}$, which is induced by the localized map $\psi \colon S_{(\varphi^{-1}(\alpha_i))} \to T_{(\alpha_i)}$. We claim that this localized map ψ is an isomorphism. If $\psi(a/u) = 0$, then

$$\psi(\varphi^{-1}(\alpha_i)a/\varphi^{-1}(\alpha_i)u) = 0,$$

which gives

$$\varphi(\varphi^{-1}(\alpha_i)a)/\varphi(\varphi^{-1}(\alpha_i)u) = 0$$

in $T_{(\alpha_i)}$. This means that

$$\alpha_i^n \varphi(\varphi^{-1}(\alpha_i)a) = \varphi(\varphi^{-1}(\alpha_i)a\alpha_i^n) = 0$$

for some n, and we may take n large enough so that the degree is higher than d_0, so $\varphi^{-1}(\alpha_i)a\alpha_i^n = 0$, which means $a = 0$ in $S_{(\varphi^{-1}(\alpha_i))}$. For surjectivity, choose $b/\alpha_i^n \in T_{(\alpha_i)}$. Then for some m, $\deg b\alpha_i^m \geq d_0$, so there is an element $\varphi^{-1}(b\alpha_i^m)/\varphi^{-1}(\alpha_i^{n+m})$ in $S_{(\varphi^{-1}(\alpha_i))}$, which maps to b/α_i^n by ψ. Thus, ψ is an isomorphism, so $f^\#$ is an isomorphism of sheaves because the $D_+(\alpha_i)$ form a cover. Then f is a homeomorphism because inverse image preserves ideal inclusion and hence closed sets, so $(f, f^\#)$ is an isomorphism of schemes.

An example of such a $\varphi \colon S \to T$ is given by letting T be a polynomial ring in n variables where each variable has degree 1, letting S be T except the degree 1 part is replaced by 0, and letting φ be the inclusion. Then φ is degree preserving and an isomorphism for $d \geq 2$, but is not surjective, so is not an isomorphism.

16. (a) We write $U \cap X_f$ as the set of $\mathfrak{p} \in \operatorname{Spec} B$ for which $f_\mathfrak{p} \notin \mathfrak{m}_\mathfrak{p}$ in the local ring $\mathcal{O}_\mathfrak{p}$. Also, $D(\bar{f})$ is the set of prime ideals of B not containing f, which is the set of prime ideals $\mathfrak{p}$ of B for which f is invertible in $B_\mathfrak{p} = \mathcal{O}_\mathfrak{p}$. Notice that $\mathcal{O}_\mathfrak{p} \setminus \mathfrak{m}_\mathfrak{p}$ is the set of units of $\mathcal{O}_\mathfrak{p}$. It is clear that every unit of $\mathcal{O}_\mathfrak{p}$ is not in $\mathfrak{m}_\mathfrak{p}$. For the converse, an element $x \notin \mathfrak{m}_\mathfrak{p}$ generates an ideal not contained in $\mathfrak{m}_\mathfrak{p}$, so must be the unit ideal since $\mathcal{O}_\mathfrak{p}$ is local, so is a unit. Thus, $U \cap X_f$ is the set of elements $\mathfrak{p} \in \operatorname{Spec} B$ for which $f_\mathfrak{p}$ is invertible in $\mathcal{O}_\mathfrak{p}$, so $U \cap X_f = D(\bar{f})$. Now cover X with open affine subschemes $\operatorname{Spec} A_i$. Then $\operatorname{Spec} A_i \cap X_f$ is an open set since it equals $D(f_i)$ where f_i is the restriction of f to $\Gamma(\operatorname{Spec} A_i, \mathcal{O}_{\operatorname{Spec} A_i})$, and $\bigcup(\operatorname{Spec} A_i \cap X_f) = X_f$, so X_f is an open subset of X.

(b) Since X is quasi-compact, let $U_1, \ldots, U_k$ be a covering of X such that $U_i \cong \operatorname{Spec} A_i$ for some ring A_i. Let ρ_i be the restriction map $\Gamma(X, \mathcal{O}_X) \to \Gamma(U_i, \mathcal{O}_X|_{U_i})$. Since the restriction of a to X_f is 0, the restriction of a in $U_i \cap X_f$ is 0 for all i. If f_i is the restriction of f to $A_i = \Gamma(U_i, \mathcal{O}_X|_{U_i})$, then $U_i \cap X_f = D(f_i)$ by (a). By (Ex. 2.1), $D(f_i) \cong (A_i)_{f_i}$, so there exists $n_i > 0$ such that $f_i^{n_i}\rho_i(a) = 0$. If $n = \max_{i=1}^k n_i$, then $f_i^n \rho_i(a) = 0$ for all i, and each such element is the restriction of $f^n a$. Since the U_i form a cover, this implies $f^n a = 0$.

(c) Write $U_i = \operatorname{Spec} A_i$ for the finite cover of X, and let f_i be the restriction of f in U_i. For each i, let b_i be the restriction of b in $U_i \cap X_f$. By (a), $U_i \cap X_f = D(f_i)$, so we can write $b_i = c_i/f_i^{n_i}$ for some $c_i \in A_i$ and integer n_i. On each intersection $U_i \cap U_j$, let $N_{i,j} = \max(n_i, n_j)$. Then $f_i^N b_i - f_j^N b_j$ restricts to 0 in $U_i \cap U_j \cap X_f$, so by (b), there is some

$n_{i,j}$ so that $f_i^{N_{i,j}+n_{i,j}} b_i = f_j^{N_{i,j}+n_{i,j}} b_j$. Letting $N = \max\{N_{i,j} + n_{i,j}\}$ (which is over a finite set), the elements $f_i^N b_i \in U_i$ all agree on overlaps, so lift to an element x of A. Since the restriction of x is $f^N b_i$ on each $U_i \cap X_f$, its restriction is $f^N b$ on X_f because $f^N b_i - x|_{X_f}$ restricts to 0 on each $U_i \cap X_f$.

(d) With the hypothesis of (c), cover X with open affines $U_i = \operatorname{Spec} A_i$. Let f_i be the restriction of f in $\Gamma(U_i, \mathcal{O}_X)$. By (a), $U_i \cap X_f = D(f_i)$. Since $D(f_i) = \operatorname{Spec}(A_i)_{f_i}$, f_i^{-1} exists in $\Gamma(U_i \cap X_f, \mathcal{O}_X)$. They are all restrictions of the same element, so they agree on overlaps. The $U_i \cap X_f$ cover X_f, so they lift to an element g. Then $fg|_{U_i \cap X_f} = 1$ for all i, so $fg|_{X_f} = 1$, and $f|_{X_f}$ is invertible. Thus, the map $A \to \Gamma(X_f, \mathcal{O}_{X_f})$ induces a ring homomorphism $A_f \to \Gamma(X_f, \mathcal{O}_{X_f})$. The injectivity follows from part (b); in that proof we only needed that X has a finite cover by open affines. The surjectivity follows from part (c), so we conclude that $A_f \cong \Gamma(X_f, \mathcal{O}_{X_f})$.

17. (a) For $x, x' \in X$, if $f(x) = f(x')$, then $f(x) \in U_i$ for some U_i, so $x, x' \in f^{-1}(U_i)$. Since f induces a homeomorphism $f^{-1}(U_i) \to U_i$, we have $x = x'$. Also, for any $y \in Y$, $y \in U_i$ for some i, and there is a homeomorphism $f^{-1}(U_i) \to U_i$ which means y has a preimage, so f is bijective. To see that f is a homeomorphism, consider an open set $V \subseteq X$. Then V is covered by $V \cap f^{-1}(U_i)$, which are open sets in $f^{-1}(U_i)$. By the homeomorphism $f^{-1}(U_i) \to U_i$, each $V \cap f^{-1}(U_i)$ is mapped to an open set by f, so V is mapped to their union, so f is a homeomorphism.

Now let $V \subseteq Y$ be an open set. We get a map $f^{\#} \colon \mathcal{O}_Y(V) \to \mathcal{O}_X(f^{-1}(V))$, which we claim is an isomorphism. Since the induced map $f^{-1}(U_i) \to U_i$ is an isomorphism, we have $\mathcal{O}_{U_i}(V \cap U_i) \to \mathcal{O}_{f^{-1}(U_i)}(f^{-1}(V \cap U_i))$ is an isomorphism, which we can rewrite as $\mathcal{O}_Y(V \cap U_i) \to \mathcal{O}_X(f^{-1}(V \cap U_i))$. Note that the $f^{-1}(V \cap U_i)$ form a cover for $f^{-1}(V)$. If $a \in \mathcal{O}_Y(V)$ maps to 0, then $f^{\#}(a)$ restricts to 0 in $\mathcal{O}_X(f^{-1}(V \cap U_i))$, so must come from 0 in $\mathcal{O}_Y(V \cap U_i)$. But then this implies that a restricts to 0 in each $\mathcal{O}_Y(V \cap U_i)$ so $a = 0$, and $f^{\#}$ is injective. For any $b \in \mathcal{O}_X(f^{-1}(V))$, let b_i be the restriction of b in $\mathcal{O}_X(f^{-1}(V \cap U_i))$. For each b_i, there is a corresponding $a_i \in \mathcal{O}_Y(V \cap U_i)$ that maps to it. The a_i agree on overlaps because their images in $\mathcal{O}_X(f^{-1}(V \cap U_i))$ do (and their overlaps are isomorphic), so they lift to an element $a \in \mathcal{O}_Y(V)$, and $f^{\#}(a) = b$, so $f^{\#}$ is surjective. Thus $f^{\#}$ is an isomorphism on all open sets, so is an isomorphism of sheaves, and f is an isomorphism of schemes.

(b) If $X = \operatorname{Spec} A$ is an affine scheme, then the identity 1 generates the unit ideal, and $X_1 = X$ since 1_x is the multiplicative identity for any point $x \in X$, so cannot be in the maximal ideal $\mathfrak{m}_x$ of $\mathcal{O}_x$.

Conversely, suppose there are elements $f_1, \ldots, f_r \in A = \Gamma(X, \mathcal{O}_X)$ such that the open subsets X_{f_i} are affine, and $f_1, \ldots, f_r$ generate the unit ideal in A. This means that for any $x \in X$, $(f_1)_x, \ldots, (f_r)_x$ generate the unit ideal of $\mathcal{O}_x$, so there is some i such that $(f_i)_x \notin \mathfrak{m}_x$. Then $x \in X_{f_i}$, so the X_{f_i} cover X. By (Ex. 2.16(d)), $\Gamma(X_{f_i}, \mathcal{O}_{X_{f_i}}) \cong A_{f_i}$, so since X_{f_i} is affine, $X_{f_i} \cong \operatorname{Spec} A_{f_i}$. By (Ex. 2.4), the identity $A \to \Gamma(X, \mathcal{O}_X)$ induces a morphism of schemes $\varphi \colon X \to \operatorname{Spec} A$. The map φ of topological spaces is given by taking an affine covering of X, say X_{f_i}, and mapping $\operatorname{Spec} A_{f_i} \to \operatorname{Spec} A$ by the induced map of

$$A \xrightarrow{=} \Gamma(X, \mathcal{O}_X) \xrightarrow{\rho_i} \Gamma(X_{f_i}, \mathcal{O}_{X_{f_i}}) \xrightarrow{\sim} A_{f_i}.$$

That is, if we call the above map ρ_i, then $\operatorname{Spec} A_{f_i} \to \operatorname{Spec} A$ is defined by $\mathfrak{p} \mapsto \rho_i^{-1}(\mathfrak{p})$. Now consider $D(f_i) \subseteq \operatorname{Spec} A$. Its preimage under φ is the set $\bigcup_{j=1}^{r} \{\mathfrak{p} \in \operatorname{Spec} A_{f_j} :$

$f_i \notin \rho_j^{-1}(\mathfrak{p})\}$. Certainly, this set contains $\operatorname{Spec} A_{f_i}$ because f_i is invertible in A_{f_i}. If this set contains $\mathfrak{p} \in \operatorname{Spec} A_{f_j}$, then $\mathfrak{p} \in X_{f_j}$, which means $f_\mathfrak{p} \notin \mathfrak{m}_\mathfrak{p}$ in $\mathcal{O}_\mathfrak{p} = (A_{f_j})_\mathfrak{p}$. Since $f_i \notin \rho_j^{-1}(\mathfrak{p})$, we also have $f_\mathfrak{p}$ is not contained in the maximal ideal of $(A_{f_i})_\mathfrak{p}$, so $\mathfrak{p} \in X_{f_i}$. Then $\varphi^{-1}(D(f_i)) = \operatorname{Spec} A_{f_i}$, and the induced map $\operatorname{Spec} A_{f_i} \to D(f_i)$ is an isomorphism. Then φ is an isomorphism by (a) because the $D(f_i)$ cover $\operatorname{Spec} A$ since the f_i generate the unit ideal. Therefore, X is an affine scheme.

18. (a) The intersection of all prime ideals of A is equal to the nilradical of A. Thus f is nilpotent if and only if f is contained in every prime ideal, which is equivalent to saying that $D(f) = \varnothing$.

(b) Suppose that φ is injective, and let $f\colon Y \to X$ be the induced morphism of schemes. Taking the stalk at a point $\mathfrak{p} \in X$, we get $f_\mathfrak{p}^\#\colon \mathcal{O}_{X,\mathfrak{p}} \to (f_*\mathcal{O}_Y)_\mathfrak{p}$. We know that $\mathcal{O}_{X,\mathfrak{p}} = A_\mathfrak{p}$ and $(f_*\mathcal{O}_Y)_\mathfrak{p}$ is the colimit of $\mathcal{O}_Y(f^{-1}(U))$ over all open sets U containing $\mathfrak{p}$, which is the same as considering just basic open sets $D(f)$ containing $\mathfrak{p}$. Since $\mathcal{O}_Y(f^{-1}(D(f))) = \mathcal{O}_Y(D(\varphi(f)))$, this colimit is equal to B localized at $\mathfrak{p}$ (thinking of B as an A-module). To see that $f_\mathfrak{p}^\#$ is injective, suppose that a/u maps 0. Then there exists $s \notin \varphi^{-1}(\mathfrak{p})$ such that $s\varphi(a) = 0$, which means that $\varphi(sa) = 0$. Since φ is injective, $sa = 0$, so $a/u = 0$.

If $f^\#\colon \mathcal{O}_X \to f_*\mathcal{O}_Y$ is injective, then taking global sections, $\Gamma(X, \mathcal{O}_X) \to \Gamma(Y, \mathcal{O}_Y)$ is injective, but this is φ by the correspondence of induced maps.

To see that $f(Y)$ is dense in X, we show that the intersection of all closed sets containing $f(Y)$ is X, which is acheived by showing that any closed set containing $f(Y)$ is X. This is further reduced to showing that if $\mathfrak{a}$ is an ideal contained in $\bigcap_{\mathfrak{p}\in Y} \varphi^{-1}(\mathfrak{p})$, then $V(\mathfrak{a}) = X$. For any $x \in \mathfrak{a}$, $\varphi(x)$ is contained in all prime ideals of B, so $\varphi(x)$ is nilpotent and there exists n such that $\varphi(x)^n = 0$. Since φ is injective, this means that $x^n = 0$, so x is contained in every prime ideal of A, which gives that $V(\mathfrak{a}) = X$.

(c) The map $f\colon Y \to X$ is defined by $\mathfrak{p} \mapsto \varphi^{-1}(\mathfrak{p})$, which we claim is injective. Suppose $\varphi^{-1}(\mathfrak{p}) = \varphi^{-1}(\mathfrak{q})$ for two prime ideals $\mathfrak{p}, \mathfrak{q} \in Y$. If $\mathfrak{p} \neq \mathfrak{q}$, then choose $x \in \mathfrak{q} \setminus \mathfrak{p}$. Since φ is surjective, $\varphi^{-1}(x)$ is nonempty. If $\varphi^{-1}(x) \subseteq \mathfrak{p}$, then $\varphi(\varphi^{-1}(\mathfrak{p}))$ is strictly bigger than $\mathfrak{p}$, which is a contradiction, so f is injective. We now claim that $f(Y) = V(\mathfrak{a})$ where $\mathfrak{a} = \bigcap_{\mathfrak{p}\in Y} \varphi^{-1}(\mathfrak{p})$. Suppose $\mathfrak{q}$ contains $\mathfrak{a}$, and let $\mathfrak{q}'$ be the inverse image of $\varphi(\mathfrak{q})$. Note that $\varphi(\mathfrak{q})$ is a prime ideal of B because φ is surjective. That is, if $ab \in \varphi(\mathfrak{q})$, then both a and b have preimages whose product is contained in $\mathfrak{q}$, which means that at least one of a and b is contained in $\varphi(\mathfrak{q})$. By definition, $\mathfrak{q}' \supseteq \mathfrak{q}$. If the inclusion is proper, then pick $x \in \mathfrak{q}' \setminus \mathfrak{q}$. There is some $y \in \mathfrak{q}$ such that $\varphi(x) = \varphi(y)$. But then $x - y \in \mathfrak{q}' \setminus \mathfrak{q}$ and $\varphi(x-y) = 0$. However, 0 is contained in every prime ideal of B, and hence $x - y$ is contained in $\mathfrak{a}$, which is a contradiction, so $\mathfrak{q}' = \mathfrak{q}$, which proves the claim and shows that $f(Y)$ is a closed set. So f is a bijection, and φ preserves inclusion of ideals, so f is a homeomorphism.

The proof that $f^\#$ is surjective is similar to the one in (a) showing that $f^\#$ is injective. The map on stalks is the same as the localization map of A-modules $\varphi_\mathfrak{p}\colon A_\mathfrak{p} \to B_\mathfrak{p}$, which is surjective because φ is surjective.

(d) There is a canonical injective ring homomorphism $\psi\colon A/\ker\varphi \to B$. Letting $X' = \operatorname{Spec} A/\ker\varphi$, this induces a map $f\colon Y \to X'$ and a map $f^\#\colon \mathcal{O}_{X'} \to f_*\mathcal{O}_Y$. The prime ideals of X' are in bijection with the prime ideals of A which contain $\ker\varphi$, so $X' \cong V(\ker\varphi)$. By assumption, $f(Y)$ is homeomorphic to a closed subset of X'. By (b), $f(Y)$ is dense in X', so $f(Y)$ is homeomorphic to $V(\ker\varphi)$. For any $\mathfrak{p} \in Y$, the map on stalks $f_\mathfrak{p}^\#\colon \mathcal{O}_{X',\psi^{-1}(\mathfrak{p})} \to \mathcal{O}_{Y,\mathfrak{p}}$ is the same as the map on stalks $\mathcal{O}_{X,\varphi^{-1}(\mathfrak{p})} \to \mathcal{O}_{Y,\mathfrak{p}}$ induced by φ. These maps are surjective by assumption, so $f_\mathfrak{p}^\#$ is also surjective. By (b), $f_\mathfrak{p}^\#$

is also injective, so each is an isomorphism, which means $(f, f^\#)$ is an isomorphism of schemes. Then there is an inverse morphism of schemes, which corresponds to an inverse ring homomorphism $B \to A/\ker\varphi$, so $A/\ker\varphi \cong B$, which means that φ is surjective.

19. If $A \cong A_1 \times A_2$, let $f = (1, 0)$ and $g = (0, 1)$. The localization $\operatorname{Spec} A_f = \operatorname{Spec} A_1$ is an open set in $\operatorname{Spec} A$ whose complement is $\operatorname{Spec} A_g = \operatorname{Spec} A_2$, so $\operatorname{Spec} A$ is disconnected. Conversely, suppose U is an open set in $\operatorname{Spec} A$ whose complement V is also open. Then we have maps $A \to \mathcal{O}(U)$ and $A \to \mathcal{O}(V)$ with no relations because $U \cap V = \varnothing$. In this case, $A \cong \mathcal{O}(U) \cong \mathcal{O}(V)$. This establishes the equivalence of (i) and (iii).

    It is obvious that (iii) implies (ii), jus ttake e_1 to be f as above and e_2 to be g. Given elements $e_1, e_2 \in A$ with the described properties, we claim that $A \cong Ae_1 \oplus Ae_2$. If $ae_1 = a'e_2$ for some $a, a' \in A$, then multiplying both sides by e_1 gives $ae_1 = 0$ and similarly, multiplying by e_2 gives $0 = a'e_2$, so $Ae_1 \cap Ae_2 = 0$. Also, Ae_1 and Ae_2 generate A because $e_1 + e_2 = 1$, so (ii) implies (iii).

3 First Properties of Schemes

If $f\colon \operatorname{Spec} A \to \operatorname{Spec} B$ is a morphism of schemes, then for any $g \in B$, one has $f^{-1}(D(g)) = D(f^\# g)$. This follows because there is a map $B \to A$ given by the structure sheaf, and also a localization map $A \to A \otimes_B B_g = A_{f^\# g}$. The composition of these maps is the same as first localizing $B \to B_g$ and then using the map given by the structure sheaf.

Lemma 1. Let X be a scheme and P a property of open affines of X such that

(1) For any $\operatorname{Spec} A \subseteq X$, if $\operatorname{Spec} A$ has property P, then so does $D(f) = \operatorname{Spec} A_f$ for all $f \in A$.

(2) If $f_1, \dots, f_n$ generate A and each $D(f_i) \subseteq \operatorname{Spec} A$ has property P, then so does $\operatorname{Spec} A$.

Then if there is an open affine covering of X such that each affine has property P, then every open affine of X has property P.

Proof. Let $\operatorname{Spec} A_i$ be an open affine covering of X such that each affine has property P, and let $\operatorname{Spec} B$ be any open affine. Then $\operatorname{Spec} B \cap \operatorname{Spec} A_i$ is an open covering of $\operatorname{Spec} B$. Each intersection can be covered with open affines that are localizations of both B and A_i by an element. By (1), these localizations also have property P. Now we have a covering of $\operatorname{Spec} B$ as in (2), which finishes the proof. ☐

1. Let $f\colon X \to Y$ be a morphism that is locally of finite type. We invoke Lemma 1 where P is the property that the preimage of an open affine $\operatorname{Spec} B$ has a covering by open affines $\operatorname{Spec} A_i$ such that each A_i is a finitely generated B-algebra. Let $\operatorname{Spec} B \subseteq Y$ be an open affine so that $f^{-1}(\operatorname{Spec} B)$ has a covering $\operatorname{Spec} A_i$ such that each A_i is a finitely generated B-algebra. Choose $g \in B$. If $f_i\colon \operatorname{Spec} A_i \to \operatorname{Spec} B$ is the restriction of f, then $f_i^{-1}(D(g)) = \operatorname{Spec}(A_i)_g$, so $f^{-1}(D(g)) = \bigcup \operatorname{Spec}(A_i)_g$. Each $(A_i)_g$ is a finitely generated B_g-algebra whose generators are the images of the generating set of A_i as a B-algebra, so this verifies (1) of the lemma.

    To verify (2), suppose $g_1, \dots, g_n$ generate the unit ideal of B, and that B_{g_i} has property P. Then the $D(g_i)$ cover $\operatorname{Spec} B$, so the $f^{-1}(\operatorname{Spec} B_{g_i})$ cover $f^{-1}(\operatorname{Spec} B)$. Notice that B_{g_i} is a finitely generated B-algebra with the generating set $1/g_i$, so any finitely generated B_{g_i}-algebra is also a finitely generated B-algebra. Since each $f^{-1}(\operatorname{Spec} B_{g_i})$ can be covered by finitely generated B_{g_i}-algebras, we take the union over all i to get a covering of $f^{-1}(\operatorname{Spec} B)$ by finitely generated B-algebras. We conclude that every open affine $\operatorname{Spec} R \subseteq Y$ has property P.

The converse follows by definition.

2. Let $f\colon X \to Y$ be a quasi-compact morphism, and let V_i be an open covering by affines of Y such that $f^{-1}(V_i)$ is quasi-compact for all i. Given an open affine $U \subseteq Y$, we can cover $U \cap V_i$ by open sets that are distinguished open sets in both U and V_i. Since U is affine and hence quasi-compact, we can take a finite number of such distinguished open sets. Then $f^{-1}(U)$ is the finite union of the preimages of these distinguished open sets, so it is enough to show that each distinguished open set has a quasi-compact preimage. We reduce to the case $f\colon X \to Y$ where X is quasi-compact and Y is affine, and showing that the preimages of distinguished opens are quasi-compact. Cover X with finitely many affines $\operatorname{Spec} A_i$ and let $Y = \operatorname{Spec} B$. Let $f_i\colon \operatorname{Spec} A_i \to Y$ be the restriction of f, and choose $D(g) \subseteq Y$. Then $f_i^{-1}(D(g)) = D(f_i^{\#}g)$. Finally, $f^{-1}(D(g)) = \bigcup f_i^{-1}(D(g))$, and each $D(f_i^{\#}g)$ is quasi-compact because it is isomorphic to $\operatorname{Spec}(A_i)_{f_i^{\#}g}$, so $f^{-1}(D(g))$ is the finite union of quasi-compact spaces and hence quasi-compact.

   The converse follows by definition.

3. (a) Suppose $f\colon X \to Y$ is of finite type. By definition, it is locally of finite type. Since f is of finite type, there is a covering by open affines of $Y = \bigcup V_i$ such that $f^{-1}(V_i)$ can be covered by a finite number of open affine subsets U_{ij}. Let W_k be an open covering of $f^{-1}(V_i)$. Then $W_k \cap U_{ij}$ is an open covering for each U_{ij}, and since affine schemes are quasi-compact, we can select finitely many of the $W_k \cap U_{ij}$ to be a cover. Taking the finite union of all such W_k for each U_{ij} gives a finite cover for $f^{-1}(V_i)$. Thus $f^{-1}(V_i)$ is quasi-compact, so f is quasi-compact.

   Conversely, suppose that f is locally of finite type and quasi-compact. By (Ex. 3.1), for any covering by open affines $Y = \bigcup \operatorname{Spec} B_i$, $f^{-1}(\operatorname{Spec} B_i)$ can be covered by open affines $\operatorname{Spec} A_{ij}$ such that A_{ij} is a finitely generated B_i-algebra. By (Ex. 3.2), the $f^{-1}(\operatorname{Spec} B_i)$ are quasi-compact, so we can choose finitely many $\operatorname{Spec} A_{ij}$, which means that f is of finite type.

   (b) Let $f\colon X \to Y$ be of finite type, and choose an open affine subset $V = \operatorname{Spec} B$ of Y. Using (a), we know that f is locally of finite type and quasi-compact. By (Ex. 3.1), $f^{-1}(V)$ can be covered by open affines $U_j = \operatorname{Spec} A_j$ such that A_j is a finitely generated B-algebra. By (Ex. 3.2), $f^{-1}(V)$ is quasi-compact, so we only need finitely many U_j.

   The converse follows from the definition of finite type.

   (c) Let $V = \operatorname{Spec} B$ be an open affine of Y. We use Lemma 1 on $f^{-1}(V)$ with P being the property that if $\operatorname{Spec} A \subseteq f^{-1}(V)$ is an open affine, then A is a finitely generated B-algebra. Let A be a finitely generated B-algebra with generators $\{a_1, \ldots, a_r\}$. For any $g \in A$, $\{1/g, a_1/1, \ldots, a_r/1\}$ generate A_g as a B-algebra, so $D(g)$ has property P.

   Now suppose that $(g_1, \ldots, g_m) = A$, and that each A_{g_i} is a finitely generated B-algebra. Let $\{a_{i1}/g_i^{n_{i1}}, \ldots, a_{ik}/g_i^{n_{ik}}\}$ be a set of generators for A_i (each one is finitely generated, so we assume for convenience that each is generated by k elements since there are only finitely many algebras to consider). Choose any $r \in A$. For each g_i, there is a polynomial in the generators that makes it equal to $r/1$. Combining these fractions and getting a common denominator, we can express $g_i^{n_i} r$ as a polynomial in the a_{ij} with coefficients in B for some n_i, call this polynomial p_i. Let $N = \max_{i=1}^m n_i$. Then $(g_1^N, \ldots, g_m^N) = A$, so we can write $c_1 g_1^N + \cdots + c_m g_m^N = 1$ for some $c_i \in B$. Then $c_1 g_1^{N-n_i} p_1 + \cdots + c_m g_m^{N-n_m} p_m = c_1 g_1^N r + \cdots + c_m g_m^N r = r$. Thus, $\{a_{ij}\} \cup \{g_i\}$ gives a finite generating set for A as a B-algebra. We conclude that for every open affine $\operatorname{Spec} A \subseteq f^{-1}(\operatorname{Spec} B)$, A is a finitely generated B-algebra.

4. Suppose $f\colon X \to Y$ is a finite morphism. We use Lemma 1 where $\operatorname{Spec} B$ has property P if $f^{-1}(\operatorname{Spec} B)$ is affine, equal to some $\operatorname{Spec} A$, and A is a finitely generated B-module. Let $\operatorname{Spec} B \subseteq Y$ be an open affine with property P, so that $f^{-1}(\operatorname{Spec} B) = \operatorname{Spec} A$ where A is a finitely generated B-module, and choose $g \in B$. The B-module structure on A is given by the map $\varphi\colon B \to A$, which is induced by the restriction $f\colon \operatorname{Spec} A \to \operatorname{Spec} B$. Localizing at g, we get $\varphi_g\colon B_g \to A_g$ where $A_g = A \otimes_B B_g$. If $\{g_1, \ldots, g_r\}$ is a generating set for A as a B-module, then $\{g_1/1, \ldots, g_r/1\}$ is a generating set for A_g as a B_g-module, so A_g is a finitely generated B_g-module. This gives that $\operatorname{Spec} B_g$ has property P because $f^{-1}(\operatorname{Spec} B_g) = \operatorname{Spec} A_{f^\# g}$.

   Now suppose that there are elements $g_1, \ldots, g_r$ that generate B such that each B_{g_i} has property P. That is, $f^{-1}(\operatorname{Spec} B_{g_i}) = \operatorname{Spec} A_i$, where each A_i is a finitely generated B_{g_i}-module. By abuse of notation, we shall use g_i to also mean the image of g_i under the map $B \to \Gamma(X, \mathcal{O}_X)$. Note that $\Gamma(X, \mathcal{O}_X)$ is generated by the g_i. Let $X' = f^{-1}(\operatorname{Spec} B)$. Using the notation of (Ex. 2.16), we claim that $X'_{g_i} = A_i$ for all i. By (Ex. 2.16a), $\operatorname{Spec} A_i \cap X'_{g_i}$ is the set of primes of A_i that do not contain g_i. Since we have a map $B_{g_i} \to A_i$, it must be that g_i is invertible in A_i, so no primes can contain it, and $\operatorname{Spec} A_i \subseteq X'_{g_i}$. For any other j, $\operatorname{Spec} B_{g_i} \cap \operatorname{Spec} B_{g_j} = \operatorname{Spec} B_{g_i g_j}$. This gives that $\operatorname{Spec} A_i \cap \operatorname{Spec} A_j$ is

$$\{\mathfrak{p} \in \operatorname{Spec} A_j : g_i \notin \mathfrak{p}\} = \{\mathfrak{p} \in \operatorname{Spec} A_i : g_j \notin \mathfrak{p}\},$$

   which means that $X_{g_i} \cap \operatorname{Spec} A_j = \operatorname{Spec} A_i \cap \operatorname{Spec} A_j$, which proves the claim. By (Ex. 2.17), this means that $f^{-1}(\operatorname{Spec} B)$ is affine, say equal to $\operatorname{Spec} A$.

   Then we get a map $\operatorname{Spec} A \to \operatorname{Spec} B$, which means that each A_i is a localization of A as B-modules. In particular, $A_i = A_{g_i}$. Let $\{a_{ij}\}$ be a finite generating set for A_i as a B_{g_i}-module. Then the set $\{a_{ij}/1, a_{ij}/g_i\}$ is a generating set for A_i as a B-module. Then for any $c \in A$, we can write $c/1 = p_i/g_i^{n_i}$ in A_{g_i} where p_i is some linear combination of the a_{ij} using coefficients from B. Then $g_i^{n_i} c = p_i$ for all i. Let $N = \max_{k=1}^r n_k$; then there are coefficients $b_i \in B$ such that $b_1 g_1^N + \cdots + b_r g_r^N = 1$ in A. This gives $b_1 g_1^{N-n_1} p_1 + \cdots + b_r g_r^{N-n_r} p_r = c$, so $\{a_{ij}\} \cup \{g_i^k\}_{k=1}^N$ gives a finite generating set for A as a B-module, which verifies (2) of Lemma 1, so we're done.

   The converse follows by definition.

5. (a) Let $f\colon X \to Y$ be a finite morphism, and $y \in Y$ some point. There is an open affine $U = \operatorname{Spec} B$ containing y, and by (Ex. 3.4), $f^{-1}(U)$ is an open affine $\operatorname{Spec} A$ such that the map $\varphi\colon B \to A$ induced by $f\colon \operatorname{Spec} A \to \operatorname{Spec} B$ makes A a finitely generated B-module. Then y corresponds to a prime ideal of B, and $f^{-1}(y)$ is the set of prime ideals in A whose preimage under φ is y. There is a bijection between the prime ideals of A whose preimage is y and the prime ideals of $A \otimes_B B_y = A_y$. Also, A_y is a finitely-generated B_y-module via the localized map $\varphi_y\colon B_y \to A_y$. Now B_y is a local ring and we are concerned with the number of primes of A_y whose preimage is the maximal ideal yB_y. Thus, we divide by this ideal to get $B_y/yB_y \to A_y/\varphi_y(y)A_y$. Then $A_y/\varphi_y(y)A_y$ is a finitely generated B_y/yB_y-module. In particular, B_y/yB_y is a field, so $A_y/\varphi_y(y)A_y$ is an Artinian ring, and hence has finitely many prime ideals. So $f^{-1}(y)$ is a finite set, and thus f is quasi-finite.

   (b) Let $f\colon X \to Y$ be a finite morphism. We claim that it is enough to show that $f(X)$ is closed to show that $f(V)$ is closed for any closed $V \subseteq X$. To see this, note that there is a closed immersion $V \hookrightarrow X$ which is finite because $V = \operatorname{Spec} A/I$ for some ideal I, and the composition of finite maps is finite. Thus if we know the above fact, then the image of V is closed under the map $V \hookrightarrow X \to Y$.

   To show that $f(X)$ is closed, it is enough to show that for any open affine $U \subseteq Y$, $f(X) \cap U$ is closed in U. To see why, take a covering U_i of Y. Then if $f(X) \cap U_i$ is closed relative to U_i, then $U_i \setminus f(X)$ is open in Y, and $\bigcup U_i \setminus f(X) = Y \setminus f(X)$, so $f(X)$ is closed. Furthermore, $f^{-1}(U)$ is an open affine of X, $V \cap f^{-1}(U)$ is closed in $f^{-1}(U)$ and has image $U_i \cap f(X)$. Let $U_i = \operatorname{Spec} B$ and $f^{-1}(U) = \operatorname{Spec} A$. We reduce to the case of showing that if $f\colon \operatorname{Spec} A \to \operatorname{Spec} B$ is a finite morphism, then $f(\operatorname{Spec} A)$ is closed in $\operatorname{Spec} B$ and $f(\operatorname{Spec} A) = \operatorname{Spec} B \cap f(X)$.

The morphism of schemes $f\colon \operatorname{Spec} A \to \operatorname{Spec} B$ induces a ring homomorphism $\varphi\colon B \to A$. If $I = \ker \varphi$, then there is a factorization $B \to B/I \hookrightarrow A$ that induces a morphism of schemes

$$\operatorname{Spec} A \to \operatorname{Spec} B/I \to \operatorname{Spec} B.$$

Then $\operatorname{Spec} B/I = V(I) \subseteq B$, which is closed in B. Also, $V(I) = f(X) \cap \operatorname{Spec} B$ because $f^{-1}(\operatorname{Spec} B) = \operatorname{Spec} A$. The morphism $\operatorname{Spec} A \to \operatorname{Spec} B/I$ is finite because if A is a finitely generated B-module given by the action $B \to A$, and the kernel of the action is I, then A is a finitely generated B/I-module because everything in I acts trivially on A. So if $\{a_1, \dots, a_n\}$ is a generating set as a B-module, and $a \in A$ can be written $b_1a_1 + \cdots + b_na_n$, then $a = \bar{b}_1a_1 + \cdots + \bar{b}_na_n$ where $\bar{b}_i$ is b_i modulo I. If $\operatorname{Spec} A$ is closed in $\operatorname{Spec} B/I$, we're done, so we reduce to the case that $B \to A$ is an injection and $f\colon \operatorname{Spec} A \to \operatorname{Spec} B$ is finite.

Then $B \hookrightarrow A$ is an integral extension. Any closed set of $\operatorname{Spec} A$ is of the form $V(\mathfrak{a})$, and the image of this set under f is $\{\mathfrak{p} \cap B : \mathfrak{p} \supseteq \mathfrak{a}\}$. We claim that $f(V(\mathfrak{a})) = V(\mathfrak{a} \cap B)$. The inclusion $f(V(\mathfrak{a})) \subseteq V(\mathfrak{a} \cap B)$ is clear; if $\mathfrak{p} \supseteq \mathfrak{a}$, then $B \cap \mathfrak{p} \supseteq B \cap \mathfrak{a}$. The reverse direction is a consequence of the going up theorem. That is, for a prime $\mathfrak{q}$ in B such that $\mathfrak{q} \supseteq B \cap \mathfrak{a}$, there is a prime $\mathfrak{p}$ such that $\mathfrak{q} = \mathfrak{p} \cap B$. Thus $f(V(\mathfrak{a}))$ is closed, which finishes the proof.

(c) Let k be a field, and let X be the scheme obtained by gluing two copies of $\mathbf{A}^1_k$ at the complement of a point P, and let $Y = \operatorname{Spec} k[x] = \mathbf{A}^1_k$. We get a morphism of schemes $f\colon X \to Y$ by gluing the identity morphisms $\mathbf{A}^1_k \to \mathbf{A}^1_k$ along the complement of P. Then f is surjective and quasi-finite because $f^{-1}(P)$ is two points and $f^{-1}(x)$ is one point for every $x \neq P$. To see that f is of finite type, note that Y is affine and that $f^{-1}(Y) = X$ is covered by the two copies of $\mathbf{A}^1_k$, each of which is equal to $\operatorname{Spec} k[x]$, and $k[x]$ is a finitely generated $k[x]$-algebra. Since $f^{-1}(Y)$ is not affine, f is not finite by (Ex. 3.4).

6. We first show that any integral scheme has a unique generic point. For some $\operatorname{Spec} A \subseteq X$, we claim that the point ξ corresponding to the 0 ideal of A is the desired generic point. By $\bar{\xi}$, we mean the closure of the set $\{\xi\}$. Note first that $\bar{\xi} \supseteq \operatorname{Spec} A$ because every prime ideal of A contains 0. Then we can write $X = \bar{\xi} \cup (X \setminus \operatorname{Spec} A)$ as a union of closed sets. Since X is an irreducible space, it must be that $\bar{\xi} = X$, so ξ is a generic point. Now we show that ξ is independent of A. Choose two open affines $\operatorname{Spec} A, \operatorname{Spec} B \subseteq X$. Since X is irreducible, $\operatorname{Spec} A \cap \operatorname{Spec} B$ is nonempty. Then there is an open set $\operatorname{Spec} C$ in $\operatorname{Spec} A \cap \operatorname{Spec} B$ that is a distinguished open in both $\operatorname{Spec} A$ and $\operatorname{Spec} B$. This means that the point corresponding to 0 in A is the same as the point corresponding to 0 in C since C is a localization of A. Likewise for B, so the points corresponding to 0 in A and B are the same, so we have shown uniqueness. Then for any $\operatorname{Spec} A$, $\mathcal{O}_\xi = (\mathcal{O}|_{\operatorname{Spec} A})_\xi$, which is isomorphic to $A_{(0)}$, the quotient field of A, so $\mathcal{O}_\xi$ is a field.

7. Let $f\colon X \to Y$ be a dominant, generically finite morphism of finite type of integral schemes. Let ξ be the generic point of X and η be the generic point of Y. We claim that $f(\xi) = \eta$. The closure of $f(\xi)$ is the intersection of all closed sets containing $f(\xi)$. The preimage of each such closed set is a closed set of X containing ξ, and hence is all of X. Then any closed set containing

$f(\xi)$ contains $f(X)$. Since $f(X)$ is dense, any closed set containing $f(\xi)$ must be Y. This means that $f(\xi)$ is a generic point for Y; by uniqueness, $f(\xi) = \eta$.

Choose $\operatorname{Spec} B \subseteq Y$ whose preimage is nonempty, and choose $\operatorname{Spec} A \subseteq f^{-1}(\operatorname{Spec} B)$. Then A is a finitely-generated B-algebra, so A is also a finitely-generated $B_{(0)}$-algebra, and $B_{(0)} = K(Y)$, the function field of Y by (Ex. 3.6). By Noether normalization, there exists n such that A is an integral extension of $K(Y)[x_1, \dots, x_n]$. If $n > 0$, then there are infinitely many primes in $K(Y)[x_1, \dots, x_n]$, each of which lies over 0 in $K(Y)$. By the going up theorem, each of these primes corresponds to a prime in A that lies over 0 in $K(Y)$, but this contradicts that the fiber of η is finite. Thus $n = 0$, so A is a finite $K(Y)$-module. Since $K(X) = \mathcal{O}_{X,\xi}$ is the colimit over these A (because $f(\xi) = \eta$ implies $\xi \in f^{-1}(\operatorname{Spec} A)$), we conclude that $K(X)$ is a finite $K(Y)$-module, which means $K(X)/K(Y)$ is a finite field extension.

Now let $\{g_1, \dots, g_r\}$ be a generating set for A as a B-algebra. Since $K(X)/K(Y)$ is a finite field extension, there exist polynomials $p_i(x)$ with coefficients in $K(Y)$ such that $p_i(g_i) = 0$. Multiplying denominators, we can assume these coefficients are in B. Let b_i be the leading coefficient of each p_i. Let $b = b_1 \cdots b_r$ and localize to get A_b as a finitely generated B_b-algebra. Then the generating set is $\{g_1/1, \dots, g_r/1\}$, and they satisfy the localized versions of p_i. However, each $b_i/1$ is now invertible, so we have that each $g_i/1$ satisfies a monic polynomial with coefficients in B_b. Thus, A_b is a finitely generated B_b-module.

It remains to be shown that there is some open affine in $\operatorname{Spec} B$ whose preimage is an open affine. We can cover $f^{-1}(\operatorname{Spec} B)$ by open affines $\operatorname{Spec} A_1, \dots, \operatorname{Spec} A_n$ such that each A_i is a finitely generated B-algebra. By localizing, and the comments above, we can assume without loss of generality that each A_i is a finitely generated B-module. If $n = 1$, then we're done, so assume otherwise. Let V be the intersection of the $\operatorname{Spec} A_i$. There exists $a_i \in A_i$ for each i such that $D(a_i) \subseteq V$. Since each A_i is a finitely generated B-module, a_i satisfies a monic polynomial with coefficients in B. The maps $B \to A_i$ are injective because $f(\xi) = \eta$, and B is an integral domain, so we may assume that the constant term of each polynomial is nonzero; call these constant terms c_i. Let $c = c_1 \cdots c_{n-1}$. Each c_i is a multiple of a_i, so any prime ideal of A_i that contains a_i also contains c. Thus $\operatorname{Spec}(A_i)_{a_i} \supseteq \operatorname{Spec}(A_i)_c$ for $i < n$. This means that for $i < n$,

$$\operatorname{Spec}(A_i)_c \subseteq V \cap f^{-1}(\operatorname{Spec} B_c) \subseteq \operatorname{Spec} A_n \cap f^{-1}(\operatorname{Spec} B_c) = \operatorname{Spec}(A_n)_c.$$

We also have $f^{-1}(\operatorname{Spec} B_c) = \bigcup_{i=1}^n \operatorname{Spec}(A_i)_c$, but by the previous computation, this union is just $\operatorname{Spec}(A_n)_c$. Finally, by previous remarks, $(A_n)_c$ is a finitely generated B_c-module, so $f^{-1}(\operatorname{Spec} B_c) \to \operatorname{Spec} B_c$ is a finite morphism. Also, $\operatorname{Spec} B_c$ is nonempty because $c_i \neq 0$, and B is an integral domain, which means $c \neq 0$. Since Y is an irreducible space, this gives that $\operatorname{Spec} B_c$ must be dense.

8. For any two open affines $\operatorname{Spec} A$ and $\operatorname{Spec} B$ of X, we show how to glue together $\operatorname{Spec} \widetilde{A}$ and $\operatorname{Spec} \widetilde{B}$. The inclusions $A \hookrightarrow \widetilde{A}$ and $B \hookrightarrow \widetilde{B}$ induce morphisms of schemes $f\colon \operatorname{Spec} \widetilde{A} \to \operatorname{Spec} A$ and $g\colon \operatorname{Spec} \widetilde{B} \to \operatorname{Spec} B$. If $\operatorname{Spec} A \cap \operatorname{Spec} B = \varnothing$, there is no gluing to do, so assume otherwise. We can cover $\operatorname{Spec} A \cap \operatorname{Spec} B$ by open sets that are distinguished open sets in both. Consider one such open set $\operatorname{Spec} C \subseteq \operatorname{Spec} A \cap \operatorname{Spec} B$. Then both $f^{-1}(\operatorname{Spec} C)$ and $g^{-1}(\operatorname{Spec} C)$ are distinguished opens in $\operatorname{Spec} \widetilde{A}$ and $\operatorname{Spec} \widetilde{B}$, respectively, and we claim they are isomorphic. Both of these rings can be thought of as localizations of normalizations. However, normalization commutes with localization [2, Proposition 4.13], so $f^{-1}(\operatorname{Spec} C)$ and $g^{-1}(\operatorname{Spec} C)$ can be obtained by first localizing A and B, and then taking the normalization. By our choice of $\operatorname{Spec} C$, the localizations are equal, so we get the isomorphism. These isomorphisms glue to

give an isomorphism of schemes

$$f^{-1}(\operatorname{Spec} A \cap \operatorname{Spec} B) \to g^{-1}(\operatorname{Spec} A \cap \operatorname{Spec} B),$$

and we use this isomorphism to glue $\operatorname{Spec} \widetilde{A}$ and $\operatorname{Spec} \widetilde{B}$ together. The last thing to verify is that if we have a third scheme $\operatorname{Spec} \widetilde{D}$, then it is irrelevant in which order we glue it to $\operatorname{Spec} \widetilde{A}$ and $\operatorname{Spec} \widetilde{B}$. But this follows because we are gluing along intersections using localization to identify isomorphic pieces (i.e., if we localize three times, it's irrelevant in which order it is done). Thus, we have specified a gluing along intersections that is compatible along triple intersections, so we glue all the normalizations to get $\widetilde{X}$. Any local ring of $\widetilde{X}$ is a localization of some open affine in $\widetilde{X}$. All such affines are integrally closed domains, and hence remain integrally closed under localization, so $\widetilde{X}$ is a normal scheme.

To get a morphism $\varphi\colon \widetilde{X} \to X$, we glue the morphisms $\operatorname{Spec} \widetilde{A} \to \operatorname{Spec} A$ induced by the inclusion $A \hookrightarrow \widetilde{A}$ for all open affines $\operatorname{Spec} A \subseteq X$. Let Z be a normal integral scheme with a dominant morphism $f\colon Z \to X$. Then for $\operatorname{Spec} A \subseteq X$, and $\operatorname{Spec} B \subseteq f^{-1}(\operatorname{Spec} A)$, we have an injective ring homomorphism $A \hookrightarrow B$ where B is integrally closed. The injectivity follows because f being dominant means it maps the generic point of Z to the generic point of X, so the preimage of 0 is 0. Then B is the integral closure of A in some field extension L of $K(A)$, the quotient field of A. Since we have specified an embedding of A in L, there is a unique way to extend this embedding to $K(A)$. Restricting this map to $\widetilde{A}$, we get a factorization $A \to \widetilde{A} \to B$, which gives

$$\operatorname{Spec} B \to \operatorname{Spec} \widetilde{A} \to \operatorname{Spec} A.$$

The maps $\operatorname{Spec} B \to \operatorname{Spec} \widetilde{A}$ are compatible on overlaps (since we can cover them with distinguished opens) so glue together to give a morphism $Z \to \widetilde{X}$, and f factors uniquely through $\widetilde{X}$.

Now suppose X is of finite type over a field k. For any $\operatorname{Spec} A \subseteq X$, we have $\varphi^{-1}(\operatorname{Spec} A) = \operatorname{Spec} \widetilde{A}$ by construction. Then A is an integral domain that is a finitely generated k-algebra and $\widetilde{A}$ is the integral closure of A in $K(A)$. A theorem of Noether [2, Theorem 4.14] then says that $\widetilde{A}$ is a finitely generated A-module, so φ is a finite morphism. This result is also the content of Theorem I.3.9A in Hartshorne.

12. (a) If $\varphi\colon S \to T$ is surjective and degree preserving, then $\varphi(S_+) = T_+$. By definition, $U = \{\mathfrak{p} \in \operatorname{Proj} T : \mathfrak{p} \not\supseteq \varphi(S_+)\}$, so we see that $U = \operatorname{Proj} T$. The map $f\colon \operatorname{Proj} T \to \operatorname{Proj} S$ is defined by $\mathfrak{p} \mapsto \varphi^{-1}(\mathfrak{p})$, which we claim is injective. Suppose $\varphi^{-1}(\mathfrak{p}) = \varphi^{-1}(\mathfrak{q})$ for $\mathfrak{p}, \mathfrak{q} \in \operatorname{Proj} T$. If $\mathfrak{p} \neq \mathfrak{q}$, then choose $x \in \mathfrak{q} \setminus \mathfrak{p}$. Since φ is surjective, $\varphi^{-1}(x)$ is nonempty. If $\varphi^{-1}(x) \subseteq \mathfrak{p}$, then $\varphi(\varphi^{-1}(\mathfrak{p}))$ is strictly bigger than $\mathfrak{p}$, which is a contradiction, so f is injective. We now claim that $f(\operatorname{Proj} T) = V(\mathfrak{a})$ where $\mathfrak{a} = \bigcap_{\mathfrak{p} \in \operatorname{Proj} T} \varphi^{-1}(\mathfrak{p})$. Suppose $\mathfrak{q} \supseteq \mathfrak{a}$, and let $\mathfrak{q}'$ be the inverse image of $\varphi(\mathfrak{q})$. Note that $\varphi(\mathfrak{q})$ is a homogeneous prime ideal of B because φ is surjective. That is, if $ab \in \varphi(\mathfrak{q})$, where both a and b are homogeneous, then a and b have homogeneous preimages whose product is contained in $\mathfrak{q}$, which means that at least one of a and b is contained in $\varphi(\mathfrak{q})$. By definition, $\mathfrak{q}' \supseteq \mathfrak{q}$. If the inclusion is proper, then pick $x \in \mathfrak{q}' \setminus \mathfrak{q}$. There is some $y \in \mathfrak{q}$ such that $\varphi(x) = \varphi(y)$. But then $x - y \in \mathfrak{q}' \setminus \mathfrak{q}$ and $\varphi(x-y) = 0$. However, 0 is contained in every prime ideal of B, and hence $x - y$ is contained in $\mathfrak{a}$, which is a contradiction, so $\mathfrak{q}' = \mathfrak{q}$, which proves the claim and shows that $f(\operatorname{Proj} T)$ is a closed set. So f is a bijection, and φ preserves inclusion of ideals, so f is a homeomorphism. Finally, the map on stalks is the same as the localization map $\varphi_{(\mathfrak{p})}\colon S_{(\mathfrak{p})} \to T \otimes_S S_{(\mathfrak{p})}$, which is surjective because φ is surjective. Thus, f is a closed immersion.

(b) There is a commutative diagram of graded rings where the maps are projection.

$$\begin{tikzcd} S \arrow[r] \arrow[d] & S/I' \arrow[dl] \\ S/I & \end{tikzcd}$$

This induces a commutative diagram of schemes.

$$\begin{tikzcd} \operatorname{Proj} S & \operatorname{Proj} S/I' \arrow[l] \\ \operatorname{Proj} S/I \arrow[u] \arrow[ur] & \end{tikzcd}$$

The map $S/I' \to S/I$ is an isomorphism on the degree d part for $d \geq d_0$, so by 2.14c, the map $\operatorname{Proj} S/I \to \operatorname{Proj} S/I'$ is an isomorphism. The commutative diagram above shows that I and I' determine the same closed subscheme.

13. (a) Let $f\colon X \to Y$ be a closed immersion. Then we identify X with a closed subset $V \subseteq Y$. Cover Y by open affines $U_i = \operatorname{Spec} A_i$. Locally on each U_i, we have a closed immersion $f^{-1}(V \cap U_i) \to U_i$ which looks like $A_i \to A_i/I_i$ for some ideal $I_i \subseteq A_i$. Then A_i/I_i is a finitely generated A_i-algebra, so f is a morphism of finite type.

(b) Let $f\colon X \to Y$ be a quasi-compact open immersion. Then we identify X with an open affine $U \subseteq Y$. For any open affine $V \subseteq Y$, $f^{-1}(V) = U \cap V$. We can cover this intersection with open sets that are distinguished in both U and V, and since f is quasi-compact, we can choose finitely many to cover. If $V = \operatorname{Spec} A$, then each such distinguished open in $U \cap V$ is $\operatorname{Spec} A_f$ for some $f \in A$, and A_f is a finitely generated A-algebra with generating set $\{1/f\}$, so f is of finite type.

(c) Let $f\colon X \to Y$ and $g\colon Y \to Z$ be two morphisms of finite type, and let $h = g \circ f$. Let $U = \operatorname{Spec} C$ be an open affine of Z. By (Ex. 3.3(b)), $g^{-1}(U)$ can be covered by finitely many $\operatorname{Spec} B_i$ such that B_i is a finitely generated C-algebra. Then $f^{-1}(\operatorname{Spec} B_i)$ can be covered by finitely many $\operatorname{Spec} A_{ij}$ such that A_{ij} is a finitely generated B_i-algebra. Then we have $C \to B_i \to A_{ij}$, so A_{ij} is a finitely generated C-algebra. To see this, it is enough to note that for some n and m, there are surjective homomorphisms $B_i[x_1, \ldots, x_n] \to A_{ij}$ and $C[y_1, \ldots, y_m] \to B_i$, so this gives a surjective homomorphism $C[x_1, \ldots, x_n, y_1, \ldots, y_m] \to A_{ij}$. Since $h^{-1}(U)$ is the union of the $\operatorname{Spec} A_{ij}$, we see that h is a morphism of finite type.

(d) Let $f\colon X \to S$ and $g\colon S' \to S$ be morphisms such that f is of finite type. Let f' be the morphism $X' \to S'$ where $X' = X \times_S S'$. Choose an open affine $U = \operatorname{Spec} A \subseteq S$ with $g^{-1}(U)$ nonempty, and $U' = \operatorname{Spec} A' \subseteq g^{-1}(U)$ such that $f'^{-1}(U')$ is nonempty. We can cover $f^{-1}(U)$ (which is nonempty) by finitely many open affines $V_i = \operatorname{Spec} B_i$ such that B_i is a finitely generated A-algebra. By the comments in Hartshorne's construction of the fiber product, $f'^{-1}(U')$ is covered by $V_i \times_U U' = \operatorname{Spec}(B_i \otimes_A A')$. If $\{b_1, \ldots, b_r\}$ is a finite generating set for B_i as an A-algebra, then $\{b_1 \otimes_A 1, \ldots, b_r \otimes_A 1\}$ is a finite generating set for $B_i \otimes_A A'$ as an A'-algebra. We can cover S with open affines U_i, and $g^{-1}(U_i)$ is a cover for S'. We have just showed that we can cover each $g^{-1}(U_i)$ with open affines $V_{ij} = \operatorname{Spec} A'_{ij}$ whose preimage under f' can be covered by finitely many $W_{ijk} = \operatorname{Spec} B'_{ijk}$ such that each B'_{ijk} is a finitely generated A'_{ij}-algebra, so f' is a morphism of finite type. Therefore, morphisms of finite type are stable under base extension.

(e) The morphism $X \times_S Y \to S$ can be factored $X \times_S Y \to Y \to S$. The first map is of finite type since $X \to S$ is of finite type and by (d), and the second map is of finite type by assumption. Part (c) gives that their composition $X \times_S Y \to S$ is of finite type.

(f) Let $f\colon X \to Y$ be a quasi-compact morphism, and let $g\colon Y \to Z$ be a morphism such that $h = g \circ f$ is of finite type. Pick $\operatorname{Spec} C \subseteq Z$, $\operatorname{Spec} B \subseteq g^{-1}(\operatorname{Spec} C)$, and $\operatorname{Spec} A \subseteq f^{-1}(\operatorname{Spec} B)$ (assuming these are nonempty; there's nothing to do in the case of empty preimage). Then $\operatorname{Spec} A \subseteq h^{-1}(\operatorname{Spec} C)$, so by (Ex. 3.3(c)), A is a finitely generated C-algebra, and we have ring homomorphisms $C \to B \to A$. If $\{a_1, \dots, a_n\}$ are the generators for A as a C-algebra, then there is a surjective homomorphism $C[x_1, \dots, x_n]$ given by the map $C \to A$ and mapping each x_i to a_i. Then this factors through a map $B[x_1, \dots, x_n] \to A$ where each x_i maps to a_i and $B \to A$ is given. Since

$$C[x_1, \dots, x_n] \to B[x_1, \dots, x_n] \to A$$

is equal to $C[x_1, \dots, x_n] \to A$ and hence surjective, this implies that $B[x_1, \dots, x_n] \to A$ is surjective, so A is a finitely generated B-algebra. Finally, if $\operatorname{Spec} C_i$ is a cover of Z, then we can find a cover $\operatorname{Spec} B_j$ of Y such that $\operatorname{Spec} B_j \subseteq g^{-1}(\operatorname{Spec} C_i)$ for some i, and such that by the above remarks, $f^{-1}(\operatorname{Spec} B_j)$ can be covered by finitely many $\operatorname{Spec} A_{jk}$ such that A_{jk} is a finitely generated B_j-algebra, so f is locally of finite type. By assumption, f is also quasi-compact, so f is of finite type by (Ex. 3.3(a)).

(g) Since Y is Noetherian, it is quasi-compact, so we can cover it with finitely many open affines $\operatorname{Spec} B_i$. Then each $f^{-1}(\operatorname{Spec} B_i)$ can be covered by finitely many open affines $\operatorname{Spec} A_{ij}$ each of which is quasi-compact, and the $f^{-1}(\operatorname{Spec} B_i)$ cover X. So X is a finite union of quasi-compact open sets, which means X is quasi-compact. Also, each A_{ij} is a finitely generated B_i-algebra. Then $A_{ij} \cong B_i[x_1, \dots, x_n]/I$ for some n and some ideal I. Since Y is Noetherian, B_i is a Noetherian ring, so by the Hilbert basis theorem, $B_i[x_1, \dots, x_n]$ is Noetherian, and homomorphic images of Noetherian rings are Noetherian. Thus we have covered X by Noetherian rings and shown it is quasi-compact, which means X is a Noetherian scheme.

14. We show that every open subset of X contains a closed point of X. Let $U = \operatorname{Spec} A \subseteq X$ be an open affine, and $p \in U$ a point corresponding to a maximal ideal in A. Then p is closed relative to U. We claim that p is closed in X. Let $V = \operatorname{Spec} B$ be any other open affine that contains x. Then we can cover $U \cap V$ by an open set containing x that is distinguished in both U and V. Call it $\operatorname{Spec} A_f = \operatorname{Spec} B_g$. If $\mathfrak{p}$ is the ideal corresponding to p, then $\mathfrak{p}A_f$ is maximal in A_f, and corresponds to a maximal ideal in B_g of the form $\mathfrak{q}B_g$. We wish to show that $\mathfrak{q} \subseteq B$ is maximal. Note that $B_g/\mathfrak{q}B_g = (B/\mathfrak{q})_g$. We know that B is a finitely generated k-algebra by (Ex. 3.3(c)), so $B/\mathfrak{q}B$ is also a finitely generated k-algebra. It is also an integral domain, so the Krull dimension of $B/\mathfrak{q}B$ is the same as the transcendence degree of its quotient field over k. However, its quotient field is $(B/\mathfrak{q})_g = B_g/\mathfrak{q}B_g$, which is a finitely generated k-algebra also, so the Krull dimension of $B_g/\mathfrak{q}B_g$ is the same as its transcendence degree over k. However, the Krull dimension of $B_g/\mathfrak{q}B_g$ is 0 because it is a field, so the Krull dimension of $B/\mathfrak{q}$ is 0, which means 0 is a maximal ideal and hence $B/\mathfrak{q}$ is a field, and implies that $\mathfrak{q} \subseteq B$ is a maximal ideal as desired.

We have thus shown that p is closed relative to any open affine U_i that contains p. This means that $(X \setminus U_i) \cup \{p\}$ is a closed set of X for all U, and their intersection is $(X \setminus U) \cup \{p\}$ where U is the union of the U_i, so no point of U_i that is not p can be a limit point of p. Also, if $x \in X \setminus U$, then no open affine containing x contains p, so x is also not a limit point of p. We conclude that the closure of p in X is just p, so it is a closed point. Finally, any open set can be covered by open affines, so contains a closed point of X. Thus, the set of closed points of X is dense.

To see that this is not true for an arbitrary scheme, let k be a field, and consider $\operatorname{Spec} k[x]_{(x)}$, which contains two elements corresponding to the 0 ideal and (x). Then (x) is a closed point and 0 is not, so the set of closed points is not dense in this case.

16. Let S be the set of nonempty closed subsets $V \subseteq X$ such that V does not have property $\mathscr{P}$. If S is nonempty, it has a minimal element Y with respect to inclusion; if not, then we would have an infinite descending chain of closed sets that never stabilized. Then Y must have proper nonempty closed subsets or else Y would vacuously have property $\mathscr{P}$. But each proper nonempty closed subset must then have property $\mathscr{P}$ by minimality, so Y has property $\mathscr{P}$, which is a contradiction. Thus S is empty and X has property $\mathscr{P}$.

17. (a) If X is a Noetherian scheme, then X can be covered by finitely many open affines $\operatorname{Spec} A_i$ such that each A_i is a Noetherian ring. By (Ex. 2.13(c)), each $\operatorname{Spec} A_i$ is a Noetherian topological space, so by (Ex. 2.13(a)), each open subset of $\operatorname{Spec} A_i$ is quasi-compact. We can intersect any open subset $U \subseteq X$ with each $\operatorname{Spec} A_i$ to write it as the finite union of quasi-compact open sets, which means U is quasi-compact. Using (Ex. 2.13(a)) again, X is a Noetherian topological space.

    Let $V \subseteq X$ be a closed irreducible subset. Giving V the structure of a closed subscheme, it is enough to show that an irreducible scheme has a unique generic point. First note that an irreducible affine scheme $\operatorname{Spec} A$ has a unique generic point corresponding to the nilradical $\operatorname{nil} A$. That $\operatorname{nil} A$ is prime follows because $\operatorname{Spec} A$ is homeomorphic to $\operatorname{Spec}(A/\operatorname{nil} A)$ since every prime of A contains $\operatorname{nil} A$, so $\operatorname{Spec}(A/\operatorname{nil} A)$ is irreducible. It is also reduced, and hence integral, so $\operatorname{nil} A$ is prime. That this is a generic point follows because every other prime ideal contains $\operatorname{nil} A$, and this also gives that it is unique. If X is an irreducible scheme, then any open affine is also irreducible (if $U = U_1 \cup U_2$ with U_1 and U_2 proper closed subsets relative to U, then $U_1 \cup (X \setminus U)$ and $U_2 \cup (X \setminus U)$ are proper closed subsets of X). For any two open affines $\operatorname{Spec} A$ and $\operatorname{Spec} B$ of X, they have nonempty intersection since X is irreducible. We can find an open set in their intersection that is distinguished in both $\operatorname{Spec} A$ and $\operatorname{Spec} B$, call it $\operatorname{Spec} A_f = \operatorname{Spec} B_g$. Then note that $(\operatorname{nil} A)A_f = \operatorname{nil} A_f$ and also $(\operatorname{nil} B)B_g = \operatorname{nil} B_g$, so the nilradical in A and B correspond to the same point in X. Also, the closure of this point contains both A and B. Since A and B were arbitrary, this means this same point corresponds to the nilradical of any open affine, and hence is a generic point for X. Finally, it is unique because any other generic point must be a generic point relative to any open affine that contains it, where we know that it is unique. So X is a Zariski space.

    (b) Let $V \subseteq X$ be a minimal nonempty closed subset. Then V is irreducible, and the closure of any point of V is V, which means each point is a generic point of V. Since X is a Zariski space, V has a unique generic point, so V contains only one point.

    (c) Choose $x, y \in X$, and assume that an open set contains x if and only if it contains y. Since X is Noetherian, we can find a closed set Y minimal among those that contain x and y. Then Y is irreducible; if not, then write $Y = Y_1 \cup Y_2$ where Y_1 and Y_2 are proper closed subsets of Y. Then neither can contain both x and y by the minimality of Y, so say $x \in Y_1$ and $y \in Y_2$. Then $X \setminus Y_2$ is an open set that contains x but not y, which is a contradiction, so Y is irreducible. Then $\bar{x} = \bar{y} = Y$. However, this means that both x and y are generic points of Y, which contradicts that X is a Zariski space. So either there is an open set containing x but not y or vice versa, which means that X satisfies the T_0 separation axiom.

(d) Since the closure of the generic point is X, every other point of X is a limit point of it. Thus if U is a nonempty open set of X, then U either contains the generic point or another point that is a limit point of the generic point; either way U contains the generic point.

(e) Let $\bar{x}$ denote the closure of $\{x\}$. For a point $x \in X$, the set of points $y \in \bar{x}$ is the set of y such that $x \rightsquigarrow y$, so x is a closed point if and only if x is minimal with respect to $\rightsquigarrow$.

Now suppose x is a generic point of an irreducible component $X' \subseteq X$. If $y \rightsquigarrow x$, then $x \in \bar{y}$. This means that $\bar{y} \supseteq X'$. If $y \in X'$, then $\bar{y} \subseteq X'$ since X' is a closed set containing y. Then $y = x$ since X is a Zariski space, so closed irreducible subsets have unique generic points. If $y \notin X'$, then there is some closed irreducible subset X'' containing y. If z is the generic point of X'', then $y \in \bar{z}$, so $z \rightsquigarrow y$. This implies $z \rightsquigarrow x$, which means $x \in \bar{z}$. Then $X' \subseteq X''$, which contradicts that X' is an irreducible component of X. We conclude that x is maximal with respect to $\rightsquigarrow$.

Conversely, suppose x is maximal with respect to $\rightsquigarrow$. Then $Y = \bar{x}$ is irreducible because no proper closed subset of Y can contain x. So x is a generic point of Y. Now we claim that Y is an irreducible component of X. Write $X = X_1 \cup \cdots \cup X_s$ where each X_i is a closed irreducible set such that $X_i \nsubseteq X_j$. If $Y \subseteq X_i$ for some i, and ξ_i is the generic point of X_i, then $x \in \bar{\xi_i}$, which means $\xi_i \rightsquigarrow x$ and thus $\xi_i = x$, which implies $Y = X_i$. So if $Y \nsubseteq X_i$ for all i, then $Y = (Y \cap X_1) \cup \cdots \cup (Y \cap X_s)$ gives Y as a union of proper closed subsets, contradicting that Y is irreducible. We conclude that x is the generic point of an irreducible component of X.

Finally, let $Y \subseteq X$ be a closed set and choose $x \in Y$. If y is a specialization of x, then $y \in \bar{x} \subseteq Y$, so every closed set contains the specializations of all of its points. Dually, if $U \subseteq X$ is an open set, $x \in U$, and y is a generization of x, then x is a limit point of y, so $y \in U$.

(f) Suppose X is a Noetherian topological space. Any descending chain of closed sets in $t(X)$ is of the form $t(V_1) \supseteq t(V_2) \supseteq \cdots$ where the V_i are closed sets in X. Then this gives a descending chain of closed sets in X: $V_1 \supseteq V_2 \supseteq \cdots$. Since X is Noetherian, this chain stabilizes, so the chain in $t(X)$ also stabilizes, which means $t(X)$ is Noetherian.

If V is reducible, say $V = V_1 \cup V_2$ where V_1 and V_2 are proper closed subsets, then $t(V) = t(V_1) \cup t(V_2)$ and $t(V_1)$ and $t(V_2)$ are proper closed subsets, so $t(V)$ is reducible. If $t(V)$ is reducible, then $t(V) = t(V_1) \cup t(V_2)$ where $t(V_1)$ and $t(V_2)$ are proper closed subsets since every closed set in $t(X)$ is of the form $t(U)$ for some closed $U \subseteq X$. Then $V = V_1 \cup V_2$ and $V_i \neq V$ because otherwise $t(V_i) = t(V)$. So a closed set $V \subseteq X$ is irreducible if and only if $t(V) \subseteq t(X)$ is irreducible. Let $V \subseteq X$ be a closed irreducible subset. The set $t(V)$ is the set of closed irreducible subsets of $V \subseteq X$. We claim that the point p_V in $t(V)$ corresponding to V is the unique generic point of $t(V)$. That it is a generic point follows because any closed set containing p_V is a closed irreducible subset of V containing V, and the only such one is V. So the closure of p_V is $t(V)$. Any other point of $t(V)$ corresponds to a proper closed subset of $V \subseteq X$, so cannot have $t(V)$ as its closure. This gives the claim, so $t(X)$ is a Zariski space.

If X is not a Zariski space, then there is some closed irreducible subset $V \subseteq X$ that either has no generic point, or more than one. In the first case, the point in $t(X)$ associated to V has an empty preimage under α, and in the second case its preimage has more than one element under α. Either way, α is not bijective, so is not a homeomorphism. Conversely, suppose X is a Zariski space. If $\alpha(x) = \alpha(y)$, this means that the closure of x and y are the same. However, the closure of a point is an irreducible set, so by uniqueness of generic points, $x = y$, and α is injective. Also, α is surjective because every closed irreducible set has a generic point. Every closed set of $t(X)$ is of the form $t(V)$ for some closed subset $V \subseteq X$. The closure of a point $p \in X$ is contained in V if and only if $p \in V$, so $\alpha^{-1}(t(V)) = V$. Also, for any closed subset $U \subseteq X$, we can write $U = U_1 \cup \cdots \cup U_r$ as a union of irreducible closed subsets. Then $\alpha(U) = t(U_1) \cup \cdots \cup t(U_r)$ because the closure of any point in U is an irreducible closed subset of some U_i. Each $t(U_i)$ is a closed set by definition, so α takes closed sets to closed sets. Thus, α is a homeomorphism.

18. (a) Let $\mathfrak{F}_0$ be the set of open sets of X. Inductively, define $\mathfrak{F}_n$ to be the union of $\mathfrak{F}_{n-1}$ and the set of subsets that are finite intersections or complements of subsets in $\mathfrak{F}_{n-1}$. Then $\mathfrak{F} = \bigcup_{n=0}^{\infty} \mathfrak{F}_n$ because this is the smallest such family of subsets satisfying (1), (2), and (3). Thus, any constructible set can be obtained by a finite number of operations involving finite intersections and complements.

Suppose U is a constructible set. Then U is obtained by using a finite number of operations of finite intersection and taking complements. We induct on the number n of such steps to show that U is a disjoint union of locally closed subsets. If $n = 0$, then U is an open set of X, and thus locally closed. Otherwise, let n be the minimal number of operations needed. Then either U was obtained as the intersection of finitely many subsets of $\mathfrak{F}_{n-1}$ or $X \setminus U \in \mathfrak{F}_{n-1}$. In the first case, we have $U = U_1 \cap \cdots \cap U_r$ where $U_i \in \mathfrak{F}_{n-1}$ and each U_i is a finite disjoint union of locally closed subsets. To show U is a disjoint union of locally closed subsets, we can assume $r = 2$ and use induction. So write $U_1 = V_1 \cup \cdots \cup V_s$ and $U_2 = W_1 \cup \cdots \cup W_t$ where the V_i and W_i are locally closed subsets. Then $U = U_1 \cap U_2 = \bigcup V_i \cap W_j$, so U is a finite union of locally closed subsets. In fact this union is disjoint since the V_i are disjoint as are the W_i.

In the second case, $X \setminus U$ is a finite disjoint union of locally closed subsets $V_1, \ldots, V_r$. Then $U = (X \setminus V_1) \cap \cdots \cap (X \setminus V_r)$. We have $V_i = O_i \cap C_i$ for some open set O_i and closed set C_i, so

$$X \setminus V_i = X \setminus (O_i \cap C_i) = (O_i \setminus C_i) \cup (C_i \setminus O_i) \cup X \setminus (O_i \cup C_i).$$

Since $X \setminus (O_i \cup C_i) = (X \setminus O_i) \cap (X \setminus C_i)$, we have written $X \setminus V_i$ as a finite disjoint union of locally closed subsets, which reduces to the first case, so U is a finite disjoint union of locally closed subsets. This finishes our inductive step, so we conclude all constructible sets can be written as a finite disjoint union of locally closed subsets.

Now suppose U is a finite disjoint union of locally closed subsets $U_1, \ldots, U_n$. Then each U_i is constructible, as is $X \setminus U_i$. So $\bigcap_{i=1}^n X \setminus U_i = X \setminus U$ is constructible, which means U is constructible.

(b) Suppose U is a dense constructible set. By (a), U is a finite disjoint union of locally closed subsets U_i. Then the closure of U, which is X, is equal to the union of the closures of the U_i. Since X is irreducible, there is some U_i whose closure is X. Then $U_i = O_i \cap C_i$ where O_i is open and C_i is closed. Since the closure of U_i is X, we get $C_i = X$, so U_i is an open set of X. Then U_i contains the generic point, so U does too. This also shows that U contains a nonempty open subset of X. If a constructible set contains the generic point of X, then it is dense because every open set contains that generic point.

(c) If S is closed then it is the complement of an open set and hence constructible, and by (Ex. 3.17(e)), it is stable under specialization. Now suppose S is constructible and stable under specialization. By (a), S is the finite disjoint union of locally closed subsets $S_1, \ldots, S_r$. Then S_i is the intersection of an open set O_i and a closed set C_i. For any irreducible component Z of C_i, $S_i \cap Z$ is an open set of Z, so contains the generic point of Z. Since S is stable under specialization, it contains the closure of this generic point, so contains

Z. Thus, each S_i is closed because X is Zariski and hence C_i can be written as a union of finitely many irreducible components. So S is a finite union of closed sets and therefore closed.

If T is open, then it is constructible, and by (Ex. 3.17(e)), it is also stable under generization. Now suppose T is constructible and stable under generization. Then $X \setminus T$ is constructible. We claim that $X \setminus T$ is stable under specialization. If not, then there is some $x_0 \notin T$ such that some $x_1 \in T$ is a specialization of x_0. Since T is stable under generization, this cannot happen. By what we have shown above, $X \setminus T$ is closed, so T is open.

(d) Let $f \colon X \to Y$ be a continuous map of Zariski spaces, and let U be a constructible subset of Y. By (a), we can write $U = U_1 \cup \cdots \cup U_r$ where each U_i is a locally closed subset, so write $U_i = O_i \cap C_i$ where O_i is an open subset of Y and C_i is a closed subset of Y. Since preimage preserves unions and intersections, we get

$$f^{-1}(U) = (f^{-1}(O_i) \cap f^{-1}(C_i)) \cup \cdots \cup (f^{-1}(O_r) \cap f^{-1}(C_r)).$$

Then each $f^{-1}(O_i)$ is open and each $f^{-1}(C_i)$ is closed. Finally,

$$(f^{-1}(O_i) \cap f^{-1}(C_i)) \cap (f^{-1}(O_j) \cap f^{-1}(C_j)) = f^{-1}(U_i \cap U_j) = \varnothing$$

for $i \neq j$ since $U_i \cap U_j = \varnothing$, so $f^{-1}(U)$ is a finite disjoint union of locally closed subsets, and hence constructible by (a).

19. (a) Suppose we can show that $f(X)$ is constructible under the given hypotheses. If U is a constructible subset of X, then using (Ex. 3.18(a)), write $U = U_1 \cup \cdots \cup U_r$ where U_i is a locally closed subset. Write $U_i = O_i \cap C_i$ where O_i is open and C_i is closed. Then U_i has a closed subscheme structure in O_i. We have a restriction $U_i \to Y$, and this morphism is locally of finite type since O_i is Noetherian, so for any $\operatorname{Spec} B \subseteq Y$, $f^{-1}(\operatorname{Spec} B) \cap C_i$ locally looks like quotients of finitely-generated B-algebras by (Ex. 3.11(b)), and are hence finitely-generated. Since X is Noetherian, so is U, and closed subsets of quasi-compact sets are themselves quasi-compact. Then the restricted morphism is of finite type between Noetherian schemes, so $f(U_i)$ is constructible by our assumption. Then $f(U) = f(U_1) \cup \cdots \cup f(U_r)$ and $Y \setminus f(U) = (Y \setminus f(U_1)) \cap \cdots \cap (Y \setminus f(U_r))$, so $f(U)$ is constructible. Thus we reduce to showing that $f(X)$ is constructible.

We may also assume that X is affine because we can cover X by finitely many open affines V_i, so $f(U \cap V_i)$ is constructible implies $f(U)$ is constructible. Similarly, we may assume that Y is affine because if we cover Y by finitely many open affines W_i, then we have maps $U \cap f^{-1}(W_i) \to f(U) \cap W_i$. Then the union of the images of $U \cap f^{-1}(W_i)$ in $f(U) \cap W_i$ is $f(U)$ and constructibility preserves finite unions. In addition, an open subscheme of a Noetherian scheme is locally Noetherian by Proposition 3.2, and is quasi-compact by (Ex. 3.17(a)) and (Ex. 2.13(a)), so is Noetherian. Also, a closed set of an affine scheme $\operatorname{Spec} A$ looks like $\operatorname{Spec} A/I$ for some ideal I, and A being Noetherian implies A/I is Noetherian, so we may assume that both X and Y are also Noetherian.

By (Ex. 2.3(b)), there is a morphism $X_{\text{red}} \to X$ that is a homeomorphism on the underlying spaces. Since constructibility is a topological property, we can replace X with X_{red} and get a map $X_{\text{red}} \to X \to Y$. By (Ex. 2.3(c)), we can also replace Y with Y_{red} and get a map $X_{\text{red}} \to Y_{\text{red}}$. So we may assume that both X and Y are reduced schemes. Also, since X and Y are Noetherian, we can focus on the irreducible components of X and Y by similar reasoning as above, so we may further assume that X and Y are irreducible schemes.

By Proposition 3.1, this means we may assume that X and Y are integral schemes. If $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$, then we have a morphism (which is of finite type because each of the reductions thus far have preserved this) $f\colon X \to Y$ which is equivalent to a ring homomorphism $\varphi\colon B \to A$. Then the map $B/\ker\varphi \to A$ is an inclusion which gives a dominant morphism $f'\colon X \to \operatorname{Spec}(B/\ker\varphi) \subseteq \operatorname{Spec} B$. Thus proving that $f'(X) \subseteq \operatorname{Spec}(B/\ker\varphi)$ is constructible gives that $f(X)$ is constructible in Y because $\operatorname{Spec}(B/\ker\varphi)$ is a closed subset of Y.

Therefore, we have reduced to showing that $f(X)$ is constructible in the case that $f\colon X \to Y$ is a dominant morphism of finite type of integral Noetherian affine schemes.

(b) Let n be the number of generators of B as an A-algebra. We split the proof of the algebraic result into two cases when $n = 1$ and when $n > 1$. This proof is a rewording of the one found in [1].

If $n = 1$, write $B = A[t]$ where $t \in B$ generates B as an A-algebra. Pick nonzero $b \in B$, and write it as $b = c_d t^d + c_{d-1}t^{d-1} + \cdots + c_0$ where $c_d \neq 0$ and $c_i \in B$. We consider two subcases (1) when t has no relations (i.e., B is the polynomial ring in one variable over A) and (2) when t satisfies some relation, so that $t \in K(B)$ is algebraic over $K(A)$ where $K(A)$ means the quotient field of A.

In the first case, let $a = a_d$. Let K be an algebraically closed field and $\varphi\colon A \to K$ such that $\varphi(a) \neq 0$. The polynomial $\varphi(a_d)x^d + \cdots + \varphi(a_0)$ has d roots and K is infinite since it is algebraically closed (if K were finite and equal to $\{k_1, \ldots, k_r\}$, then the polynomial $(x - k_1)\cdots(x - k_r) + 1$ has no roots in K), so there is some $r \in K$ such that $\varphi(a_d)r^d + \cdots + \varphi(a_0) \neq 0$. Extend φ to $\varphi'\colon A[t] \to K$ by mapping t to r.

Now suppose that t is algebraic over $K(A)$. Then there are equations

$$a_d t^d + a_{d-1}t^{d-1} + \cdots + a_0 = 0$$

and

$$a'_e(b^{-1})^e + a'_{e-1}(b^{-1})^{e-1} + \cdots + a'_0 = 0$$

where $a_i, a'_i \in K(A)$ and $a_d \neq 0$ and $a'_e \neq 0$. Let $a = a_d a'_e$. Let K be an algebraically closed field and $\varphi\colon A \to K$ such that $\varphi(a) \neq 0$. We first extend φ to $A_a \to K$ in the obvious way by sending $1/a$ to $1/\varphi(a)$. We can next extend φ to some valuation ring R containing A_a [1, Theorem 5.21]. From the equations we know that t and b^{-1} are both integral over A_a. Since the integral closure of A_a is the intersection of all valuation rings of $K(A_a)$ containing it, this means t and b^{-1} are elements of R. Since $t \in R$, so is b, so b is a unit of R, which means our extension $R \to K$ maps b to something nonzero. Since R contains t and A, we can restrict it to B to get a map $\varphi'\colon B \to K$ that maps b to something nonzero.

If $n > 1$, we use induction on n. Suppose B is a finitely-generated A-algebra with generators $b_1, \ldots, b_n$. Pick any nonzero $b \in B$. Note that B is a finitely-generated $A[b_1]$-algebra with generators $b_2, \ldots, b_n$ and that $A[b_1]$ is a Noetherian domain. By induction, there is a nonzero $c \in A[b_1]$ such that for all homomorphisms $\varphi\colon A[b_1] \to K$ where K is an algebraically closed field such that $\varphi(c) \neq 0$, φ extends to a homomorphism $\varphi'\colon B \to K$ with $\varphi'(b) \neq 0$. Now note that $A[b_1]$ is a finitely-generated A-algebra so we apply induction again to get an element $a \in A$ such that for any homomorphism $\psi\colon A \to K$ where K is an algebraically closed field such that $\psi(a) \neq 0$, ψ extends to a homomorphism $\psi'\colon A[b_1] \to K$ with $\psi'(c) \neq 0$. Putting this together, we get the following: if K is algebraically closed and $\varphi\colon A \to K$ is any homomorphism such that $\varphi(a) \neq 0$, then φ extends to a homomorphism $\varphi'\colon B \to K$ such that $\varphi'(b) \neq 0$. This finishes the proof of the algebraic result.

Let $X = \operatorname{Spec} B$ and $Y = \operatorname{Spec} A$. In our case, $f\colon \operatorname{Spec} B \to \operatorname{Spec} A$ is a dominant morphism of finite type of integral Noetherian affine schemes, so the induced map $A \to B$ is injective and makes B a finitely-generated A-algebra, and both A and B are Noetherian domains. So we can use the algebraic result above on $1 \in B$, to get an element $a \in A$ with the appropriate properties. We claim that $D(a) \subseteq f(\operatorname{Spec} B)$. Choose $\mathfrak{p} \in D(a)$, i.e., $a \notin \mathfrak{p}$. Then $A/\mathfrak{p}$ is an integral domain, so we can take the algebraic closure of its quotient field, call it k. Then we have a map $A \to A/\mathfrak{p} \hookrightarrow K(A/\mathfrak{p}) \hookrightarrow k$ which maps a to something nonzero and has kernel $\mathfrak{p}$, call it φ. Then φ extends to a map $\varphi'\colon B \to k$ such that $\varphi'(1) \neq 0$, so we have the following commutative diagram of schemes

$$\begin{tikzcd}
\operatorname{Spec} B \arrow[r, "f"] & \operatorname{Spec} A \\
\operatorname{Spec} k \arrow[u] \arrow[ur] &
\end{tikzcd}$$

where $\operatorname{Spec} k \to \operatorname{Spec} A$ maps the one point of $\operatorname{Spec} k$ to $\mathfrak{p}$ and $\operatorname{Spec} k \to \operatorname{Spec} B$ maps the one point to $\ker \varphi'$. This means that $f(\ker \varphi') = \mathfrak{p}$, so $\mathfrak{p} \in f(\operatorname{Spec} B)$ implies $D(a) \subseteq f(\operatorname{Spec} B)$ as claimed. Thus $f(\operatorname{Spec} B)$ contains a nonempty open subset of Y.

(c) By (b), there exists $a \in A$ such that $D(a) \subseteq f(X)$. We will show that $f(X) \cap V(a)$ is constructible in Y. If this intersection is empty, there is nothing to do, so assume otherwise. Note that $V(a) = \operatorname{Spec} A/(a)$, so consider the map $f'\colon \operatorname{Spec} B/aB \to \operatorname{Spec} A/(a)$ induced by f, whose image is $f(X) \cap V(a)$. Since $A \to B$ is injective, we have $A/(a) \to B/aB$ injective also, so f' is dominant. Also, both are Noetherian rings. We know that (a) has a primary decomposition because A is a Noetherian ring, so we can write (a) as the intersection of some primary ideals. Furthermore, the radicals of these primary ideals are prime, call these primes $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$. Then $\sqrt{(a)} = \bigcap \mathfrak{p}_i$, so $V(a) = \bigcup_i V(\mathfrak{p}_i)$ as topological spaces since $V(a) = V(\sqrt{(a)})$ as topological spaces. For each $\mathfrak{p}_i B$, we can do the same thing since B is Noetherian, so we have maps $\operatorname{Spec} B/\mathfrak{q}_j \to \operatorname{Spec} A/\mathfrak{p}_i$ for primes $\mathfrak{q}_j \in \operatorname{Spec} B$, and the union of their images is $f(X) \cap V(a)$. While the scheme structure may be different, constructibility is a topological property and we are preserving the underlying topological spaces. These maps now involve integral domains, so each image contains a nonempty open subset by (b), and hence is constructible in $V(\mathfrak{p}_i)$ by Noetherian induction. A locally closed subset of $V(\mathfrak{p}_i)$ is also a locally closed subset of $\operatorname{Spec} B$, so in fact the images of $\operatorname{Spec} B/\mathfrak{q}_j \to \operatorname{Spec} A/\mathfrak{p}_i$ are constructible in $\operatorname{Spec} B$. Since constructibility is closed under finite unions, we conclude that $f(X) \cap V(a)$, and therefore $f(X)$, are constructible.

(d) Let $f\colon \mathbf{A}^1_k \to \mathbf{P}^2_k$ be the morphism given by $x \mapsto (x, 1, 0)$. Then $f(\mathbf{A}^1_k)$ is neither open nor closed because $(x, 1, 0)$ is not the zero set of any ideal of homogeneous polynomials, and neither is its complement.

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Solutions to *Algebraic Geometry*[*]

Steven V Sam
ssam@mit.edu

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Contents

4 Separated and Proper Morphisms

1. Let $f\colon X \to Y$ be a finite morphism of schemes. Then we can cover Y with open affines U_i such that $f^{-1}(U_i) \to U_i$ is a map of affine schemes. By Proposition 4.1, each morphism $f^{-1}(U_i) \to U_i$ is separated. Then
$$\Delta\colon f^{-1}(U_i) \to f^{-1}(U_i) \times_{U_i} f^{-1}(U_i)$$
is a closed immersion for all i, so $\Delta\colon X \to X \times_Y X$ is also, which means f is separated. Now let $g\colon X' \to Y$ be a morphism. Choose an open affine $\operatorname{Spec} A \subseteq Y$, let $f^{-1}(\operatorname{Spec} A) = \operatorname{Spec} B$, and choose an open affine $\operatorname{Spec} C \subseteq g^{-1}(\operatorname{Spec} A)$. Then the preimage of $\operatorname{Spec} C$ in $X \times_Y X'$ is $\operatorname{Spec}(B \otimes_A C)$, which is a finitely generated C-module since B is a finitely generated A-module. We can cover X' with such open affines, so finite morphisms are stable under base change. By (Ex. 3.5(b)), finite morphisms are closed, so f is universally closed. Finally, finite implies finite type, so f is proper.

2. Let U be the dense open subset such that $f|_U = g|_U$. The maps $f\colon X \to Y$ and $g\colon X \to Y$ are S-morphisms, so they induce a morphism $h\colon X \to Y \times_S Y$. Let p_1 and p_2 be the projection maps $Y \times_S Y \to Y$ given by the fiber product. The product of the morphisms $f|_U$ and $g|_U$ is $h|_U$ because $h|_U = h \circ i$ where i is the inclusion $U \hookrightarrow X$, and this is the unique morphism such that $p_1 \circ h|_U = f|_U$ and $p_2 \circ h|_U = g|_U$. By the same reasoning, $h|_U = f|_U \circ \Delta$, or we could

[*]by Robin Hartshorne

appeal to the following diagram

$$\begin{tikzcd}
U \arrow[r, "f|_U"] \arrow[drrr, bend left, "f|_U"] \arrow[drr, "h|_U"] \arrow[ddrr, bend right, "f|_U"'] & Y \arrow[dr, "\Delta"] \arrow[drr, bend left, "="] \arrow[ddr, bend right, "="'] & & \\
& & Y \times_S Y \arrow[r, "p_2"] \arrow[d, "p_1"] & Y \arrow[d] \\
& & Y \arrow[r] & S
\end{tikzcd}$$

and note that everything clearly commutes except maybe $h|_U = f|_U \circ \Delta$, but this follows from the definition of fiber product. This means that $h(U) \subseteq \Delta(Y) \subseteq Y \times_S Y$. Since $Y \to S$ is separated, $\Delta(Y)$ is closed, so $h^{-1}(\Delta(Y))$ is a closed subset containing U. We are given that U is dense, so we conclude that $h(X) \subseteq \Delta(Y)$. Now consider the image subscheme of h, which we call image(h). By (Ex. 3.11(d)), since X is reduced, image(h) is $\overline{h(X)}$ with the reduced induced subscheme structure. By (Ex. 3.11(c)), there is a factorization $X \to \text{image}(h) \to \text{image}(h) \cap \Delta(Y) \to Y \times_S Y$. In particular, this gives a map $X \to Y$ since Δ is a closed immersion, so we can factor h as $X \to Y \xrightarrow{\Delta} Y \times_S Y$; call this first map h'. By definition, $p_1 \circ \Delta$ and $p_2 \circ \Delta$ are the identity morphism on Y. Since $f = p_1 \circ h$ and $g = p_2 \circ h$, we get $f = p_1 \circ \Delta \circ h' = h$ and $g = p_2 \circ \Delta \circ h' = h$, so $f = g$.

(a) Let $R = \mathbf{C}[x,y]/(x^2, xy)$, let $S = \operatorname{Spec} \mathbf{C}$ and let $X = Y = \operatorname{Spec} R$. Since Y is affine, $Y \to S$ is separated. The set $U = D(y)$ is dense because it contains the nilradical (x), which is the generic point. Let $f \colon X \to Y$ be the map corresponding to the identity $R \to R$, and let $g \colon X \to Y$ be the map corresponding to the map $R \to R$ defined by $x \mapsto 0, y \mapsto y$. Since f and g do not give the same map on global sections, $f \neq g$. However, we claim that f and g agree on U. Note that $D(y) = \operatorname{Spec} R_y = \operatorname{Spec} \mathbf{C}[y, y^{-1}]$, where the last equality follows because $x = 0$ since $xy = 0$ and y becomes invertible. Since our ring homomorphisms only differed in where they sent x, and now x is gone, it is clear that f and g agree on U.

(b) Let $X = \operatorname{Spec} \mathbf{C}[x]$ and let Y be two copies of X glued along the complement of the point $P = (x-1)$. Write $Y = U_1 \cup U_2$ where both U_1 and U_2 are isomorphic copies of X. Let $f \colon X \to U_1$ and $g \colon X \to U_2$ be the respective isomorphisms. Then X is reduced over $\operatorname{Spec} k$ and f and g agree on $\operatorname{Spec} k[x] \setminus \{P\}$, which is open because $(x-1)$ is maximal by the Nullstellensatz. This set is dense because it contains the generic point, but $f \neq g$.

3. Let X be a separated scheme over $S = \operatorname{Spec} A$, and let $U = \operatorname{Spec} B$ and $V = \operatorname{Spec} C$ be open affines of X. The fiber product $U \times_S V$ is equal to $\Delta(U \cap V) \subseteq X \times_S X$. Since X is separated, Δ is a closed immersion, and in particular, this implies that $U \cap V$ and $U \times_S V$ are isomorphic as schemes. Finally, $U \times_S V = \operatorname{Spec}(B \otimes_A C)$, so $U \cap V$ is affine.

   Let k be a field, and let X_1 and X_2 be copies of $\mathbf{A}^2_k$. Let $U_i \subseteq X_i$ be the open set $\mathbf{A}^2_k \setminus \{(x,y)\} = D(x) \cup D(y)$, and let X be the result of gluing U_1 to U_2 via the identity morphism. Then X is a nonseparated scheme over $\operatorname{Spec} k$, and X_1 and X_2 are open affines of X. However, their intersection is isomorphic to $D(x) \cup D(y)$, which is not an affine scheme.

4. Let Z be a closed subscheme of X which is proper over S. The diagram

$$\begin{tikzcd}
Z \arrow[r] \arrow[dr] & X \arrow[r] \arrow[d] & Y \arrow[dl] \\
& S &
\end{tikzcd}$$

commutes. Since $Z \to S$ is proper and $Y \to S$ is separated, we get that $Z \to X \to Y$ is proper by Corollary 4.8e. Proper implies closed, so $f(Z)$ is closed in Y. Now put the image subscheme structure on $f(Z)$. Then $f(Z) \hookrightarrow Y$ is a closed immersion and hence a finite type morphism by (Ex. 3.13(a)). Since Y is Noetherian, $f(Z)$ is Noetherian by (Ex. 3.13(g)). Closed immersions are separated by Corollary 4.6a, so by Corollary 4.6b, $f(Z) \to S$ is separated.

Now we show that $f(Z) \to S$ is universally closed. Let $V \to S$ be any morphism. We get the following commutative diagram

$$\begin{tikzcd} V \times_S Z \arrow[r] \arrow[d] & Z \arrow[d] \\ V \times_S f(Z) \arrow[r] \arrow[d] & f(Z) \arrow[d] \\ V \arrow[r] & S \end{tikzcd} .$$

We first show that $V \times_S Z \to V \times_S f(Z)$ is surjective. Pick any $x \in V \times_S f(Z)$, and let y be its image in $f(Z)$. Since $Z \to f(Z)$ is surjective, let $y' \in Z$ be an element in its preimage. Then we have maps of residue fields $k(y) \to k(x)$ and $k(y) \to k(y')$. Then let L be some field containing both $k(x)$ and $k(y')$. The inclusions $k(x) \hookrightarrow L$ and $k(y') \hookrightarrow L$ give the following diagram

$$\begin{tikzcd} \operatorname{Spec} L \arrow[dr, dashed] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\ & V \times_S Z \arrow[r] \arrow[d] & Z \arrow[d] \\ & V \times_S f(Z) \arrow[r] & f(Z) \end{tikzcd} .$$

The image of $\operatorname{Spec} L$ in $V \times_S Z$ then maps to x, which shows the surjectivity. To see that $V \times_S f(Z) \to V$ is closed, let U be a closed subset of $V \times_S f(Z)$. Its preimage U' in $V \times_S Z$ is closed, and we know that the composite map $V \times_S Z \to V$ is closed because $Z \to S$ is proper, so the image of U' in V is closed. Since $V \times_S Z \to V \times_S f(Z)$ is surjective, the image of U' in the composite map $V \times_S Z \to V$ is the same as the image of U in the map $V \times_S f(Z) \to V$. Thus, $f(Z) \to S$ is universally closed, so is proper.

5. (a) Let R be some valuation ring of K/k. Since X is an integral scheme over a field, it is Noetherian and irreducible, so has a generic point ξ. By Lemma 4.4, a point x such that R dominates $\mathcal{O}_{X,x}$ is equivalent to a morphism $\operatorname{Spec} R \to X$ that sends the maximal ideal of R to x. Such a morphism is equivalent to a dotted arrow in the following commutative diagram

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d] & X \arrow[d] \\ \operatorname{Spec} R \arrow[r] \arrow[ur, dashed] & \operatorname{Spec} k \end{tikzcd}$$

where the map $\operatorname{Spec} K \to X$ sends the point of $\operatorname{Spec} K$ to ξ. By the valuative criterion, at most one such arrow exists. So if a center of a given valuation of K/k exists, then it is unique.

(b) This is the same as above, except that the valuative criterion now tells us that exactly one such dotted arrow exists, so every valuation of K/k has a unique center.

(d) The map $X \to \operatorname{Spec} k$ induces an injective map of rings $k \to \Gamma(X, \mathcal{O}_X)$, so identify k as a subring of $\Gamma(X, \mathcal{O}_X)$. If $k \neq \Gamma(X, \mathcal{O}_X)$, then choose $a \in \Gamma(X, \mathcal{O}_X) \setminus k$. Since k is algebraically closed, $k[a^{-1}] \subseteq K$ is a transcendental extension of k and thus (a^{-1}) is a maximal ideal. Localizing at this ideal, we get a local ring A such that $a^{-1} \in \mathfrak{m}_A$. By Theorem I.6.1A, there exists a valuation ring R of K/k such that R dominates A. In particular, $a^{-1} \in \mathfrak{m}_R$ because $\mathfrak{m}_R \cap A = \mathfrak{m}_A$.

We claim that the image of a is nonzero in every local ring of X. First suppose that X is affine, say $X = \operatorname{Spec} A$. To compute the local ring at a point $x \in X$, it is enough to take limits over distinguished opens containing x. Distinguished opens are the spectra of localizations of A, and X is integral, so the image of a is nonzero in each. Now for X not necessarily affine, cover it with affine schemes. Suppose that the image of a under the restriction maps is zero in some open affine U. Then the intersection with U and any other open affine V is nonempty since X is irreducible, so cover their intersection with open sets distinguished in both U and V. Since V is an integral affine, the image of a in $\Gamma(V, \mathcal{O}_X)$ must be zero, and hence the image of a is zero in all open affines of X. By the sheaf property, $a = 0$ in the global section. This proves the claim.

Now pick any $x \in X$. If $a^{-1} \in \mathcal{O}_{X,x}$, then a^{-1} is a unit, since $a \in \mathcal{O}_{X,x}$, so $a^{-1} \notin \mathfrak{m}_x$. But $a^{-1} \in R \cap \mathcal{O}_{X,x}$, which means x is not a center for R. If $a^{-1} \notin \mathcal{O}_{X,x}$, then a is not a unit, so $a \in \mathfrak{m}_x$. However, $a \notin R$, so $R \cap \mathcal{O}_{X,x} \neq \mathfrak{m}_x$, so x is not a center for R in this case either. This means R has no center, which contradicts (b). We conclude that $k = \Gamma(X, \mathcal{O}_X)$.

6. If X and Y are affine varieties over k, we can write $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$ where A and B are finitely generated k-domains. Let $f\colon X \to Y$ be a proper morphism. Let $\varphi\colon B \to A$ be the induced map of rings, which we can factor as $B \to B/\ker\varphi \to A$. This gives a factorization $X \to \operatorname{Spec}(B/\ker\varphi) \to Y$. The second map is finite and a composition of finite maps is finite, so to show that f is finite, it is enough to show that $X \to \operatorname{Spec}(B/\ker\varphi)$ is finite. Also, by Corollary 4.8(e), $X \to \operatorname{Spec}(B/\ker\varphi)$ is proper. Thus, we may assume that φ is injective.

In this case, to show that f is finite means to show that A is a finitely generated B-module. Let K be the fraction field of A. Then it is enough to show that A is contained in the integral closure of B in K. By Lemma 4.4, there is a morphism $\operatorname{Spec} K \to X$ that sends the one point of $\operatorname{Spec} K$ to the generic point of X. Now let R be any valuation ring of K that contains B, and let $\operatorname{Spec} R \to Y$ be any morphism such that the following diagram

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d, "i"] & X \arrow[d, "f"] \\ \operatorname{Spec} R \arrow[r] & Y \end{tikzcd}$$

commutes, where i is the map induced by the inclusion $R \hookrightarrow K$. By the valuative criterion, there is a unique morphism $\operatorname{Spec} R \to X$ filling in the diagram, and this corresponds to a map of rings $A \to R$. Then the generic point of $\operatorname{Spec} R$ is mapped to the generic point of X, which means $A \to R$ is injective. Thus A is contained in the intersection of all valuation rings containing B. By Theorem 4.11A, this intersection is the integral closure of B in K, so A is an integral extension, as desired.

8. Let $f\colon X \to Y$ and $f'\colon X' \to Y'$ be two S-morphisms having $\mathscr{P}$. Define $Z := Y \times_S Y'$. The projection map $Z \to Y$ gives a base extension $X \times_Y Z \to Z$ having $\mathscr{P}$ and similarly, the base

extension $X' \times_{Y'} Z \to Z$ has $\mathscr{P}$. In the fiber square

$$\begin{tikzcd} (X \times_Y Z) \times_Z (X' \times_{Y'} Z) \arrow[r, "p_2"] \arrow[d, "p_1"] & X' \times_{Y'} Z \arrow[d] \\ X \times_Y Z \arrow[r] & Z \end{tikzcd}$$

both p_1 and p_2 have $\mathscr{P}$. By transitivity of fiber products,

$$(X \times_Y Z) \times_Z (X' \times_{Y'} Z) \cong X \times_S X'.$$

Thus the composition $X \times_S X' \to Z$ has property $\mathscr{P}$, which is the product morphism $f \times f'$.

Now let $f\colon X \to Y$ and $g\colon Y \to Z$ be two morphisms such that $g \circ f$ has $\mathscr{P}$ and g is separated. The following diagram

$$\begin{tikzcd} X \arrow[dr, dashed, "\Gamma_f"] \arrow[drr, bend left, "f"] \arrow[ddr, bend right, "\mathrm{id}_X"'] & & \\ & X \times_Z Y \arrow[r, "p_2"] \arrow[d, "p_1"] & Y \arrow[d, "g"] \\ & X \arrow[r, "g \circ f"] & Z \end{tikzcd}$$

commutes by definition. The morphism $\Gamma_f\colon X \to X \times_Z Y$ is a base extension of the diagonal morphism $\Delta\colon Y \to Y \times_Z Y$. Since g is separated, Δ is a closed immersion and hence has $\mathscr{P}$, and so Γ_f also has $\mathscr{P}$. Also, p_2 is a base extension of $g \circ f$ so has $\mathscr{P}$. Thus the composition $f = p_2 \circ \Gamma_f$ has $\mathscr{P}$.

Note that the map $X_{\mathrm{red}} \to X$ is a closed immersion so has $\mathscr{P}$. First we know from (Ex. 2.3(b)) that it is a homeomorphism. To see that the map of sheaves is surjective, it is enough to check on stalks. But $\mathcal{O}_{X_{\mathrm{red}}}$ is defined as the sheafification of $U \mapsto \mathcal{O}_X(U)_{\mathrm{red}}$. The presheaf that it comes from has the same stalks as $\mathcal{O}_{X_{\mathrm{red}}}$, and now it is obvious that the map on stalks is surjective because they are colimits of maps of rings $\mathcal{O}_X(U) \to \mathcal{O}_X(U)_{\mathrm{red}}$, which are surjective. So the composition $X_{\mathrm{red}} \to X \to Y$ has $\mathscr{P}$ and is equal to $X_{\mathrm{red}} \to Y_{\mathrm{red}} \to Y$. But $Y_{\mathrm{red}} \to Y$ is a closed immersion and hence separated, so $X_{\mathrm{red}} \to Y_{\mathrm{red}}$ has $\mathscr{P}$.

9. Let $f\colon X \to Y$ and $g\colon Y \to Z$ be projective morphisms. Let $\mathbf{P}^n_Y$ denote $\mathbf{P}^n_{\mathbf{Z}} \times Y$. Then we have

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "f'"] & Y \arrow[r, "g"] \arrow[d, "g'"] & Z \\ \mathbf{P}^n_Y \arrow[ur, "p_f"'] & \mathbf{P}^m_Z \arrow[ur, "p_g"'] & \end{tikzcd}$$

for some n and m where f' and g' are closed immersions and p_f and p_g are projections. The projection $\mathbf{P}^n_Z \to Z$ and $g\colon Y \to Z$ give a projection $\alpha\colon \mathbf{P}^n_Z \times_Z Y \to \mathbf{P}^n_Z$, and note that $\mathbf{P}^n_Z \times_Z Y \cong \mathbf{P}^n_Y$. So the following diagram

$$\begin{tikzcd} \mathbf{P}^n_Y \arrow[r, "p_f"] \arrow[dr, dashed, "\beta"] \arrow[ddr, bend right, "\alpha"'] & Y \arrow[dr, "g'"] & \\ & \mathbf{P}^n_Z \times_Z \mathbf{P}^m_Z \arrow[r, "p_Z"] \arrow[d] & \mathbf{P}^m_Z \arrow[d] \\ & \mathbf{P}^n_Z \arrow[r] & Z \end{tikzcd}$$

commutes. Now note that $(\mathbf{P}^n_Z \times_Z \mathbf{P}^m_Z) \times_{\mathbf{P}^m_Z} Y \cong \mathbf{P}^n_Z \times_Z Y \cong \mathbf{P}^n_Y$, so in fact, β is a base extension of g' and hence a closed immersion.

Now we show that there is a closed immersion $\gamma\colon \mathbf{P}^n_Z \times_Z \mathbf{P}^m_Z \to \mathbf{P}^N_Z$ where $N = nm + n + m$. First suppose that Z is affine, say $Z = \operatorname{Spec} A$. Write $R = A[x_0, \ldots, x_n]$ and $S = A[y_0, \ldots, y_n]$. Then $\mathbf{P}^n_Z = \operatorname{Proj} R$, $\mathbf{P}^m_Z = \operatorname{Proj} S$, and we can write $\mathbf{P}^N_Z = \operatorname{Proj} A[x_0y_0, \ldots, x_iy_j, \ldots, x_ny_m]$. We claim that $\mathbf{P}^n_Z \times_Z \mathbf{P}^m_Z = \operatorname{Proj} B$ where $B = \bigoplus_{i=0}^\infty R_i \otimes_A S_i$, and R_i denotes the degree i part of a graded ring R. This follows because for any homogeneous element $r \otimes s$ in B_+, one has an isomorphism $B_{(r\otimes s)} \cong R_{(r)} \otimes_A S_{(s)}$. Then we have a surjection $A[x_0y_0, \ldots, x_iy_j, \ldots, x_ny_m] \to B$, which gives the desired closed immersion. In the case that Z is not affine, we can cover it with open affines and give a closed immersion on the preimage of each open affine in $\mathbf{P}^n_Z \times_Z \mathbf{P}^m_Z$, and our construction of the map is compatible on overlaps.

Thus, the composition $X \to \mathbf{P}^n_Y \to \mathbf{P}^n_Z \times_Z \mathbf{P}^m_Z \to \mathbf{P}^N_Z$ is a closed immersion. Combining all of the above information, the following diagram

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "f'"] & Y \arrow[r, "g"] \arrow[d, "g'"] & Z \\
\mathbf{P}^n_Y \arrow[ur, "p_f"] \arrow[d, "\beta"] & \mathbf{P}^m_Z \arrow[ur, "p_g"] & \\
\mathbf{P}^n_Z \times_Z \mathbf{P}^m_Z \arrow[ur, "p_Z"] \arrow[r, "\gamma"] & \mathbf{P}^N_Z \arrow[u, "p_N"] &
\end{tikzcd}$$

commutes where p_N is a projection map. To see that the triangle involving p_N commutes, it is enough to consider open affines and to think of the map of rings (which commutes because p_Z and p_N become inclusions), so gf is projective.

Now say that a morphism has property $\mathscr{P}$ if that morphism is projective. To see that $\mathscr{P}$ satisfies (a)-(f) of (Ex. 4.8), it is enough to check (a)-(c). We have done (b) above. For (a), let $f\colon X \to Y$ be a closed immersion. Since $\mathbf{P}^0_Y \cong Y$, we can factor f as $X \to \mathbf{P}^0_Y \to Y$, and hence closed immersions are projective. For (c), let $f\colon X \to Y$ be a projective morphism, and $g\colon Y' \to Y$ be any other morphism. For some n, we can factor f as $X \to \mathbf{P}^n_Y \to Y$ where the first map is a closed immersion and the second is a projection. Then the following diagram

$$\begin{tikzcd}
X \times_Y Y' \arrow[d] \arrow[dr, dashed, "h"] \arrow[drr, bend left, "f'"] & & \\
X \arrow[dr] \arrow[drr, "f"] & \mathbf{P}^n_Y \times_Y Y' \arrow[r] \arrow[d] & Y' \arrow[d, "g"] \\
& \mathbf{P}^n_Y \arrow[r] & Y
\end{tikzcd}$$

commutes. First note $X \times_{\mathbf{P}^n_Y} (\mathbf{P}^n_Y \times_Y Y') \cong X \times_Y Y'$, so h is a base extension of $X \to \mathbf{P}^n_Y$, and hence is a closed immersion. Finally, $\mathbf{P}^n_Y \times_Y Y' \cong \mathbf{P}^n_{Y'}$, so f' is a projective morphism.

5 Sheaves of Modules

1. Let $\mathscr{E}^*$ denote the dual sheaf $\mathscr{H}om_{\mathcal{O}_X}(\mathscr{E}, \mathcal{O}_X)$. Since $\mathscr{E}$ is locally free, let U be an open affine such that $\mathscr{E}|_U$ is free. We will define canonical isomorphisms in the following problems. Hence

they will agree on overlaps (since they are independent of a choice of basis), so it is enough to define canonical isomorphisms and assume that $\mathscr{E}$ is free to begin with.

(a) Let U be an open set. We will exhibit an isomorphism $\mathscr{E}(U) \cong \mathscr{E}^{**}(U)$. Pick $x \in \mathscr{E}(U)$. We define $\hat{x} \in \mathscr{E}^{**}(U)$ by mapping $f \in \mathscr{E}^*(U)$ to $f(x) \in \mathcal{O}_X(U)$. If $x \neq 0$, then since $\mathscr{E}(U)$ is free, there exists a basis of $\mathscr{E}(U)$ that contains x, so there is a homomorphism $f \in \mathscr{E}^*$ that sends x to something nonzero. Hence $x \mapsto \hat{x}$ is injective. Furthermore, $x \mapsto \hat{x}$ is an isomorphism because the double dual of a module has the same rank as the module. Thus the map $\mathscr{E} \to \mathscr{E}^{**}$ given by $\mathscr{E}(U) \to \mathscr{E}^{**}(U)$ for all open sets U as above is an isomorphism.

(b) We have $\mathrm{Hom}_{\mathcal{O}_X(U)}(\mathscr{E}(U), \mathscr{F}(U)) \cong \mathrm{Hom}_{\mathcal{O}_X(U)}(\mathscr{E}(U), \mathcal{O}_X(U)) \otimes_{\mathcal{O}_X(U)} \mathscr{F}(U)$ by the universal property of the tensor product of modules and this isomorphism is canonical. This gives an isomorphism $\mathrm{Hom}_{\mathcal{O}_X}(\mathscr{E}, \mathscr{F}) \cong (\mathrm{Hom}_{\mathcal{O}_X}(\mathscr{E}, \mathcal{O}_X) \otimes_{\mathcal{O}_X} \mathscr{F})$ where by the right hand side we mean tensor presheaf. Now we use the universal property of sheafification to get a (canonical) isomorphism of sheaves.

(c) For modules, $\mathscr{E}(U) \otimes_{\mathcal{O}_X(U)} -$ is the left adjoint of $\mathrm{Hom}_{\mathcal{O}_X(U)}(\mathscr{E}(U), -)$, so for $\mathcal{O}_X$-modules $\mathscr{F}$ and $\mathscr{G}$, we get the canonical isomorphism

$$\mathrm{Hom}_{\mathcal{O}_X(U)}(\mathscr{E}(U) \otimes_{\mathcal{O}_X(U)} \mathscr{F}(U), \mathscr{G}(U)) \cong \mathrm{Hom}_{\mathcal{O}_X(U)}(\mathscr{F}(U), \mathrm{Hom}_{\mathcal{O}_X(U)}(\mathscr{E}(U), \mathscr{G}(U))),$$

which by the above comments gives the desired isomorphism with the tensor presheaf in place of the tensor sheaf. We finish by using the fact that sheafification is left adjoint to the forgetful functor from the category of sheaves to the category of presheaves.

(d) Write $\mathscr{E} \cong \mathcal{O}_Y^n$ (from above comments we may assume $\mathscr{E}$ is free). We remark that direct sum acts as the product and coproduct in the category of $\mathcal{O}_X$-modules, and thus commutes with functors that are either left or right adjoints. Then we have

$$f_*(\mathscr{F}) \otimes_{\mathcal{O}_Y} \mathcal{O}_Y^n \cong (f_*(\mathscr{F}) \otimes_{\mathcal{O}_Y} \mathcal{O}_Y)^n \cong (f_*(\mathscr{F}))^n \cong f_*(\mathscr{F}^n).$$

Now we compute the other expression

$$\begin{aligned} f_*(\mathscr{F} \otimes_{\mathcal{O}_X} f^*\mathcal{O}_Y^n) &\cong f_*(\mathscr{F} \otimes_{\mathcal{O}_X} (f^{-1}\mathcal{O}_Y^n \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X)) \\ &\cong f_*(\mathscr{F} \otimes_{\mathcal{O}_X} (f^{-1}\mathcal{O}_Y \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X)^n) \\ &\cong f_*(\mathscr{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X^n) \\ &\cong f_*(\mathscr{F}^n), \end{aligned}$$

where the isomorphism $\mathcal{O}_X \cong f^*\mathcal{O}_Y$ follows from the fact that f^* is a left adjoint of f_* and using Yoneda's lemma. Thus, we have shown the desired isomorphism.

2. (a) Let $\mathscr{F}$ be an $\mathcal{O}_X$-module. Since R is a DVR, X has two points, t_0 and t_1, where t_0 corresponds to the zero ideal and t_1 corresponds to the maximal ideal. Let $M = \mathscr{F}(X)$, which is a module over $\Gamma(X, \mathcal{O}_X) = R$, and $L = \mathscr{F}(\{t_0\})$, which is a module over $\mathcal{O}_X(\{t_0\}) = K$, and hence a vector space. Finally, the restriction map $\mathscr{F}(X) \to \mathscr{F}(\{t_0\})$ gives the homomorphism $\rho\colon M \otimes_R K \to L$.

Conversely, suppose we are given an R-module M, a K-vector space L, and a homomorphism $\rho\colon M \otimes_R K \to L$. Let $\mathscr{F}(X) = M$ and $\mathscr{F}(\{t_0\}) = L$. Then the map $\rho\colon \mathscr{F}(X) \to \mathscr{F}(\{t_0\})$ makes $\mathscr{F}$ an $\mathcal{O}_X$-module, and it is clear that we have just defined a bijection of data.

(b) Use the notation from (a). If $\mathscr{F}$ is quasi-coherent, then by Proposition 5.4, $\mathscr{F} \cong \widetilde{M}$. Since R is a DVR, the maximal ideal is generated by an element m, and we can write $\{t_0\} = D(m)$. By Proposition 5.1(c), $\mathscr{F}(\{t_0\}) \cong M_m \cong M \otimes_R K$, so ρ is an isomorphism. Conversely, if ρ is an isomorphism, then $\mathscr{F}$ is given by $\mathscr{F}(X) = M$ and $\mathscr{F}(\{t_0\}) = M \otimes_R K$, so $\mathscr{F} \cong \widetilde{M}$ and is hence quasi-coherent.

4. Let $\mathscr{F}$ be an $\mathcal{O}_X$-module where X is a scheme. Suppose that $\mathscr{F}$ is quasi-coherent and pick a point $x \in X$. Then there exists an open set $U \ni x$ such that $U = \operatorname{Spec} A$ is affine and $\mathscr{F}|_U \cong \widetilde{M}$ where M is an A-module. Let $E \to G \to M \to 0$ be a presentation of M. Write $E = A^I$ and $G = A^J$ for indexing sets I and J, and let $\mathscr{E} = (\mathcal{O}_X|_U)^I$ and $\mathscr{G} = (\mathcal{O}_X|_U)^J$ where we mean a direct sum of $\mathcal{O}_X|_U$ with one copy for each element of I and J, respectively. The map $E \to G$ induces a morphism of sheaves $\mathscr{E} \to \mathscr{G}$ whose cokernel is $\mathscr{F}|_U$, which can be seen by looking at stalks. Conversely, suppose every point $x \in X$ has a neighborhood such that $\mathscr{F}|_U$ is isomorphic to a cokernel of a morphism $\mathscr{E} \to \mathscr{G}$ of free sheaves on U. If $M = \mathscr{G}(U)/\mathscr{E}(U)$, then $\mathscr{F}|_U \cong \widetilde{M}$ because we can define a map of presheaves induced by the identity $M \to \mathscr{G}(U)/\mathscr{E}(U)$ and appeal to the fact that stalks are the same after sheafification. So $\mathscr{F}$ is quasi-coherent.

If instead we assume that X is Noetherian and $\mathscr{F}$ is coherent, then we can find a finite presentation $E \to G \to M \to 0$, which shows that $\mathscr{F}$ is locally a cokernel of a morphism of free sheaves of finite rank. Conversely, if we know that X is Noetherian and $\mathscr{F}$ is locally a cokernel of a morphism of free sheaves of finite rank, then $M = \mathscr{G}(U)/\mathscr{E}(U)$ is a finitely generated A-module, so $\mathscr{F}$ is coherent.

5. (a) Let k be a field, let $X = \operatorname{Spec} k[x]$, and let $Y = \operatorname{Spec} k$. The inclusion $k \hookrightarrow k[x]$ induces a morphism $f\colon X \to Y$ of varieties over k. Let $\mathscr{F} = \mathcal{O}_X$, which is a coherent sheaf on X. However, $f_*\mathscr{F}$ is not a coherent sheaf on Y because $f_*\mathscr{F}(Y) = \mathscr{F}(X) = \widetilde{k[x]}$, and $k[x]$ is not a finitely generated k-module.

(b) Let $f\colon X \to Y$ be a closed immersion. Then we identify X with a closed subset $V \subseteq Y$. Cover Y by open affines $U_i = \operatorname{Spec} A_i$. Locally on each U_i, we have a closed immersion $f^{-1}(V \cap U_i) \to U_i$ which looks like $A_i \to A_i/I_i$ for some ideal $I_i \subseteq A_i$. Then A_i/I_i is a finitely generated A_i-module, so f is a finite morphism.

(c) Let $f\colon X \to Y$ be a finite morphism of Noetherian schemes, and let $\mathscr{F}$ be a coherent sheaf on X. Let $U_i = \operatorname{Spec} A_i$ be a covering by open affines of Y. For any i, $f_*\mathscr{F}(U_i) = \mathscr{F}(f^{-1}(U_i))$, and $f^{-1}(U_i) = \operatorname{Spec} B_i$ where B_i is a finitely generated A_i-module by (Ex. 3.4). By Proposition 5.4, $\mathscr{F}|_{f^{-1}(U_i)} \cong \widetilde{M_i}$ where M_i is a finitely generated B_i-module, hence is also a finitely generated A_i-module. Thus $f_*\mathscr{F}$ is a coherent sheaf on Y.

6. (a) By definition, $\operatorname{Supp} m = \{\mathfrak{p} \in X : m_\mathfrak{p} \neq 0\}$ where $m_\mathfrak{p}$ denotes the germ of m in $\mathscr{F}_\mathfrak{p} = M_\mathfrak{p}$ (the equality follows by Proposition 5.1(b)). We have $\mathfrak{p} \in \operatorname{Supp} m$ if and only if the image of m is nonzero in the localized module $M_\mathfrak{p}$. This is equivalent to $\operatorname{Ann} m \subseteq \mathfrak{p}$, which is equivalent to $\mathfrak{p} \in V(\operatorname{Ann} m)$. We conclude that $\operatorname{Supp} m = V(\operatorname{Ann} m)$.

(b) By definition, $\operatorname{Supp} \mathscr{F} = \{\mathfrak{p} \in X : \mathscr{F}_\mathfrak{p} \neq 0\}$. It is clear that if $\mathfrak{p} \in \operatorname{Supp} \mathscr{F}$, then $\operatorname{Ann} M \subseteq \mathfrak{p}$ because $\mathscr{F}_\mathfrak{p} = M_\mathfrak{p}$. Conversely, suppose $\mathfrak{p} \notin \operatorname{Supp} \mathscr{F}$, so that $M_\mathfrak{p} = 0$. Let $\{m_1, \ldots, m_r\}$ be a generating set for M. Then each m_i is annihilated by some $a_i \in A \setminus \mathfrak{p}$, so in particular, M is annihilated by $a_1 \cdots a_r$, so $\operatorname{Ann} M \not\subseteq \mathfrak{p}$. Thus $\operatorname{Supp} \mathscr{F} = V(\operatorname{Ann} M)$ if M is finitely generated.

(c) Let X be a Noetherian scheme and let $\mathscr{F}$ be a coherent sheaf on X. Cover X with finitely many open affines $U_i = \operatorname{Spec} A_i$. By Proposition 5.4, we know that $\mathscr{F}|_{U_i} \cong \widetilde{M_i}$ where M_i

is a finitely generated A_i-module. Since the stalks of $\mathscr{F}$ and $\mathscr{F}|_{U_i}$ agree at any $\mathfrak{p} \in U_i$, we see that $\operatorname{Supp} \mathscr{F} \cap U_i = \operatorname{Supp} \mathscr{F}|_{U_i}$. By (b), each $\operatorname{Supp} \mathscr{F}|_{U_i}$ is closed relative to U_i. Since the intersection of $\operatorname{Supp} \mathscr{F}$ with every U_i is closed, $\operatorname{Supp} \mathscr{F}$ is a closed subset of X.

(d) By (Ex. 1.20(b)), we have an exact sequence

$$0 \longrightarrow \mathscr{H}_Z^0(\mathscr{F}) \longrightarrow \mathscr{F} \longrightarrow j_*(\mathscr{F}|_U)$$

where $U = X \setminus Z$ and $j \colon U \hookrightarrow X$ is inclusion. Since $\mathscr{F}$ is quasi-coherent, so is $\mathscr{F}|_U$. Also, X is Noetherian, so by Proposition 5.8, $j_*(\mathscr{F}|_U)$ is quasi-coherent. Then $\mathscr{H}_Z^0(\mathscr{F})$ is the kernel of a morphism of quasi-coherent sheaves, so is itself quasi-coherent by Proposition 5.7. The global section of $\mathscr{H}_Z^0(\mathscr{F})$ is $\Gamma_Z(\mathscr{F})$, which is the submodule of $\Gamma(X, \mathscr{F}) = M$ consisting of all sections whose support is contained in $Z = V(\mathfrak{a})$. By (a), this is the set of $m \in M$ such that $V(\operatorname{Ann} m) \subseteq V(\mathfrak{a})$, or equivalently $\sqrt{\operatorname{Ann} m} \supseteq \sqrt{\mathfrak{a}}$. This also describes $\Gamma_{\mathfrak{a}}(M)$, so we have $\Gamma_{\mathfrak{a}}(M) \cong \Gamma_Z(\mathscr{F})$, which means that $\Gamma_{\mathfrak{a}}(M)^\sim \cong \mathscr{H}_Z^0(\mathscr{F})$.

(e) If $\mathscr{F}$ is a quasi-coherent $\mathcal{O}_X$-module, cover X with open affines $U_i = \operatorname{Spec} A_i$ such that $\mathscr{F}|_{U_i} \cong \widetilde{M_i}$ where M_i is an A_i-module. Then $\mathscr{F}|_{U_i}$ is a quasi-coherent $\mathcal{O}_{U_i}$-module. From (d), we know that $\mathscr{H}^0_{Z \cap U_i}(\mathscr{F}|_{U_i}) \cong \Gamma_{\mathfrak{a}_i}(M_i)^\sim$ where $Z \cap U_i = V(\mathfrak{a}_i)$ (the V taken in $\operatorname{Spec} A_i$). Furthermore, we have $\mathscr{H}_Z^0(\mathscr{F}) \cap U_i = \mathscr{H}^0_{Z \cap U_i}(\mathscr{F}|_{U_i})$, so $\mathscr{H}_Z^0(\mathscr{F})$ is quasi-coherent. If in addition we know that $\mathscr{F}$ is coherent, then we can take each M_i to be a finitely generated A_i-module. Since X is Noetherian, A_i is Noetherian, so $\Gamma_{\mathfrak{a}_i}(M_i)$ is a submodule of a Noetherian module and hence Noetherian. This gives that $\mathscr{H}_Z^0(\mathscr{F})$ is coherent.

7. (a) Let $V \ni x$ be an open affine $\operatorname{Spec} A$ such that $\mathscr{F}|_V \cong \widetilde{M}$ where M is a finitely generated A-module. Since $(\mathscr{F}|_V)_x = \mathscr{F}_x$ and $(\mathcal{O}|_V)_x = \mathcal{O}_x$, x corresponds to some prime ideal $\mathfrak{p} \subseteq A$, and we know that $M_{\mathfrak{p}}$ is a free $A_{\mathfrak{p}}$-module.

Let $a_1, \dots, a_n$ be a basis for $M_{\mathfrak{p}}$ as a $A_{\mathfrak{p}}$-module. Clearing denominators, we may assume that $a_i \in M$. Now let $b_1, \dots, b_m$ be a generating set for M as an A-module. Then we can write $b_i = \sum c_{i,j} a_j$ for all i, and after clearing denominators, we see that some multiple of b_i is generated by the a_j with coefficients in A. Denote this multiple $d_i b_i$, then $d_i \notin \mathfrak{p}$ because none of the denominators of the $c_{i,j}$ are in $\mathfrak{p}$. Now let $e = d_1 \cdots d_m$ be their product. Then M is contained in the A_e-module generated by $a_1, \dots, a_n$, which implies that M_e is generated by $a_1, \dots, a_n$ as an A_e-module. Since the a_i have no relations as an $A_{\mathfrak{p}}$-module, they have no relations as an A_e-module since $e \notin \mathfrak{p}$. Thus M_e is a free A_e-module. By Proposition 5.1(c), $\mathscr{F}|_{D(e)} \cong (M_e)^\sim$, which gives the desired result since $D(e)$ is open in X and contains x.

(b) If $\mathscr{F}$ is locally free, then cover X with open affines U such that $\mathscr{F}|_U$ is a free $\mathcal{O}|_U$-module. Then for any $x \in U$, $(\mathscr{F}|_U)_x$ is a free $(\mathcal{O}|_U)_x$-module because localization commutes with direct sums (because tensor product does). Using the fact that $(\mathscr{F}|_U)_x = \mathscr{F}_x$ and $(\mathcal{O}|_U)_x = \mathcal{O}_x$, we see that $\mathscr{F}_x$ is a free $\mathcal{O}_x$-module for all $x \in X$.

Conversely, suppose that $\mathscr{F}_x$ is a free $\mathcal{O}_x$-module for all $x \in X$. By (a), for every $x \in X$, there is an open neighborhood $U^x \ni x$ such that $\mathscr{F}|_{U^x}$ is free. Then by definition, $\mathscr{F}$ is locally free.

(c) First suppose that $\mathscr{F}$ is invertible. Then let $\mathscr{G} = \mathscr{H}om_{\mathcal{O}_X}(\mathscr{F}, \mathcal{O}_X)$, the dual sheaf of $\mathscr{F}$. For every open set V, we can define $\Gamma(V, \mathscr{F} \otimes \mathscr{G}) \to \mathcal{O}_X(V)$ by defining it on the presheaf $\mathscr{F}(V) \otimes \operatorname{Hom}(\mathscr{F}(V), \mathcal{O}_X(V))$ and using the universal property of sheafification. Define the presheaf morphism by $x \otimes f \mapsto f(x)$. This is a morphism because it is compatible with the restriction maps.

By (b), $\mathscr{F}_x$ is a free $\mathcal{O}_x$-module for every $x \in X$, and by (a), there is a neighborhood $U^x \ni x$ such that $\mathscr{F}|_{U^x}$ is free, so $\mathscr{F}$ is free when restricted to a distinguished open set of an open affine of X. To show that $\mathscr{F} \otimes \mathscr{G} \to \mathcal{O}_X$ is an isomorphism, it is enough to do so on these distinguished opens. Then we need to show for a ring R that the map $R \otimes \operatorname{Hom}(R, R) \to R$ given by $r \otimes f \mapsto f(r)$ is an isomorphism. Letting f be the identity homomorphism shows that this is surjective. If $f(r) = 0$, then $f(1) = 0$, which means f is the zero map, and hence $r \otimes f = 0$, which shows injectivity.

Conversely, suppose that there exists $\mathscr{G}$ and φ such that $\varphi\colon \mathscr{F} \otimes \mathscr{G} \to \mathcal{O}_X$ is an isomorphism. Write $\varphi(f_1 \otimes g_1 + \cdots + f_n \otimes g_n) = 1$. Then $\varphi_x\colon \mathscr{F}_x \otimes_{\mathcal{O}_{X,x}} \mathscr{G}_x \to \mathcal{O}_{X,x}$ is an isomorphism for all $x \in X$. Let $\mathfrak{m}_x$ denote the maximal ideal of $\mathcal{O}_{X,x}$. Then for some i, we have $\varphi_x(f_i \otimes g_i) \notin \mathfrak{m}_x$. Since $\mathcal{O}_{X,x}$ is a local ring, $\varphi_x(f_i \otimes g_i)$ is a unit, so define $f = (\varphi_x(f_i \otimes g_i))^{-1} f_i$. Then $\mathscr{F}_x \cong \mathcal{O}_{X,x} f$, so $\mathscr{F}_x$ is a free $\mathcal{O}_{X,x}$-module of rank 1. Using (b), $\mathscr{F}$ is locally free of rank 1.

8. (a) Cover X with finitely many open affines $U_i = \operatorname{Spec} A_i$ such that $\mathscr{F}|_{U_i} \cong \widetilde{M_i}$ where M_i is a finitely generated A_i-module. For any $x \in U_i$, we have $(\mathscr{F}|_{U_i})_x \cong \mathscr{F}_x$ and $(\mathcal{O}|_{U_i})_x \cong \mathcal{O}_x$, so $\varphi(x)$ can be computed in either X or U_i. If we can show that for all $n \in \mathbf{Z}$, $\{x \in U_i : \varphi(x) \geq n\}$ is a closed subset of U_i, then $\{x \in X : \varphi(x) \geq n\}$ is a closed subset of X, so we can reduce to the case that X is affine. For notation, let $X = \operatorname{Spec} A$ and $\mathscr{F} \cong \widetilde{M}$ with M a finitely generated A-module.

   Now pick $x \in X$ and let $\mathfrak{p} \subseteq A$ be the prime corresponding to x. We have

$$\mathscr{F}_x \otimes_{\mathcal{O}_X} k(x) = M_\mathfrak{p} \otimes_{A_\mathfrak{p}} A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p} = M_\mathfrak{p}/\mathfrak{p}M_\mathfrak{p},$$

   which is a $k(x)$-vector space with dimension $\varphi(x) =: n$. Let $a_1, \ldots, a_n$ be a basis for $M_\mathfrak{p}/\mathfrak{p}M_\mathfrak{p}$ over $A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}$. Clearing denominators, we may assume that $a_i \in A_\mathfrak{p}$. Since $A_\mathfrak{p}$ is a local ring, by Nakayama's lemma, $a_1, \ldots, a_n$ is a generating set for $M_\mathfrak{p}$ as an $A_\mathfrak{p}$-module. Now let $m_1, \ldots, m_r$ be a generating set of M as an A-module. We can write $m_i = \sum c_{i,j} a_j$ where $c_{i,j} \in A_\mathfrak{p}$. Clearing denominators, some multiple of m_i, say $d_i m_i$, is a linear combination of the a_i with coefficients in A. Let $e = d_1 \cdots d_r$ be their product, then $e \notin \mathfrak{p}$, so $x \in D(e)$. Now choose any other $x' \in D(e)$ and let $\mathfrak{q}$ be the prime ideal corresponding to x'. Since $e \notin \mathfrak{q}$, we see that $M_\mathfrak{q}/\mathfrak{q}M_\mathfrak{q}$ is generated by the images of the a_i as a $k(x')$-vector space, so $\varphi(x') \leq n$. Therefore, for all n, the set $\{x \in X : \varphi(x) < n\}$ is open, which means its complement is closed.

   (b) By (Ex. 5.7(b)), $\mathscr{F}_x$ is a free $\mathcal{O}_x$-module for all $x \in X$. The rank of $\mathscr{F}_x$ is the rank of $\mathscr{F}|_U$ where U is some open set of X containing x such that $\mathscr{F}|_U$ is free. Since X is connected, these ranks are the same for all $x \in X$. Write $\mathscr{F}_x = \mathcal{O}_x^n$ for some n. Then $\mathscr{F}_x \otimes_{\mathcal{O}_x} k(x) = (\mathcal{O}_x/\mathfrak{m}_x)^n$, so the rank of $\mathscr{F}_x$ as a free $\mathcal{O}_x$-module is the same as the dimension of $\mathscr{F}_x \otimes_{\mathcal{O}_x} k(x)$ as a $k(x)$-vector space, which means that $\varphi(x)$ is a constant function.

   (c) Since $\mathscr{F}$ being locally free is a local criterion, we can reduce to the case that $X = \operatorname{Spec} A$ is affine, $\mathscr{F} \cong \widetilde{M}$ with M a finitely generated A-module, and $A_f = \mathcal{O}_X(D(f))$ is reduced for all $f \in A$. Choose $\mathfrak{p} \in X$. As in part (a), we can choose $m_1, \ldots, m_n \in M_\mathfrak{p}$ such that they form a basis for $M_\mathfrak{p}/\mathfrak{p}M_\mathfrak{p}$ as a $k(\mathfrak{p})$-vector space and generate $M_\mathfrak{p}$ as an $A_\mathfrak{p}$-module.

   We claim that $M_\mathfrak{p}$ is a free $A_\mathfrak{p}$-module. To show this, it is enough to show that the m_i are linearly independent. Suppose we have a relation $\sum \frac{a_i}{b_i} m_i = 0$ where $a_i \in A$ and $b_i \notin \mathfrak{p}$ in $M_\mathfrak{p}$. Then there is an element $a \notin \mathfrak{p}$ such that $a(\sum_i (\prod_{j \neq i} b_j) m_i) = 0$. Since the m_i are linearly independent over $A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}$, we see that $a_i \in \mathfrak{p}$ for all i. We can choose $e \in A$

as in (a) such that if $\mathfrak{q} \in D(e)$, then $M_\mathfrak{q}/\mathfrak{q}M_\mathfrak{q}$ is generated by the images of the m_i. Now let $f = aeb_1 \cdots b_n$. From our choice of e, if $\mathfrak{q} \in D(f)$, then the images of $m_1, \ldots, m_n$ in $M_\mathfrak{q}/\mathfrak{q}M_\mathfrak{q}$ are generators. Since φ is a constant function, this implies that the images are in fact a basis. This implies that the relation $\sum \frac{a_i}{b_i} m_i = 0$ holds in $M_\mathfrak{q}$, which means that $a_i \in \mathfrak{q}$. Therefore, $a_i \in \bigcap_{\mathfrak{q} \ni f} \mathfrak{q}$, and this intersection is the nilradical of f, which is 0. Thus the $a_i = 0$, which gives the desired claim.

Then $\mathscr{F}_x$ is a free $\mathcal{O}_x$-module for all $x \in X$, so by (Ex. 5.7(b)), $\mathscr{F}$ is locally free.

15. (a) Let $X = \operatorname{Spec} A$ be a Noetherian affine scheme, and let $\mathscr{F} \cong \widetilde{M}$ be a quasi-coherent sheaf on X. We can write $M = \bigcup_{S \subseteq M} AS$ where AS is the A-submodule of M generated by S and where the union is taken over all finite subsets S of M, so M is the union of its finitely generated submodules. We claim that $\mathscr{F}$ is the union of $(AS)^\sim$. Let U be any open subset of X, and pick any $m \in \mathscr{F}(U)$. If $U = D(f)$ for some $f \in M$, then $\mathscr{F}(U) = M_f$, and it is clear that M_f is the union of $(AS)_f$. Otherwise, cover U with finitely many distinguished opens $D(f_i)$. If m_i is the image of m in M_{f_i}, then there is some subset $S_i \subseteq M$ such that $m_i \in \Gamma(D(f_i), (AS_i)^\sim)$. Let $S = \bigcup S_i$. Then $m_i \in \Gamma(D(f_i), (AS)^\sim)$, so by the sheaf property, $m \in \Gamma(U, (AS)^\sim)$. Note that S is a finite set, so $(AS)^\sim$ is a coherent sheaf. We conclude that $\mathscr{F}$ is the union of its coherent subsheaves.

(b) Let $i \colon U \to X$ be the inclusion morphism. An open subset of a Noetherian scheme is Noetherian, so by Proposition 5.8(c), $i_*\mathscr{F}$ is a quasi-coherent sheaf on X. By (a), $i_*\mathscr{F}$ is the union of its coherent subsheaves $\mathscr{F}_\alpha$. Then $\mathscr{F}(U)$ is the union of the $\mathscr{F}_\alpha(U)$. Since $\mathscr{F}$ is coherent, $\mathscr{F}(U)$ is a Noetherian module, there exists finitely many α_i such that $\mathscr{F}(U)$ is the union of $\mathscr{F}_{\alpha_i}(U)$, or else we could build an infinite ascending chain of submodules. Then the union of their finite generating sets is finite, so there exists α such that $\mathscr{F}_\alpha(U) = \mathscr{F}(U)$. Let $\mathscr{F}' = \mathscr{F}_\alpha$. Since $\mathscr{F}'|_U$ and $\mathscr{F}$ are both coherent and have the same global section, $\mathscr{F}'|_U \cong \mathscr{F}$.

(c) Let $i \colon U \to X$ be inclusion, and let $\rho \colon \mathscr{G} \to i_*(\mathscr{G}|_U)$ be the morphism such that for every open set $V \subseteq X$, the map $\mathscr{G}(V) \to \Gamma(V, i_*(\mathscr{G}|_U))$ is given by the restriction map $\mathscr{G}(V) \to \mathscr{G}(U \cap V)$. Let $\rho^{-1}(i_*\mathscr{F}) \subseteq \mathscr{G}$ be the set-theoretic inverse image of $i_*\mathscr{F}$ in $i_*(\mathscr{G}|_U)$. Then $\rho^{-1}(i_*\mathscr{F})$ is a subsheaf of a quasi-coherent sheaf and hence is a quasi-coherent sheaf. Now we proceed as in (b) to find a coherent subsheaf of $\rho^{-1}(i_*\mathscr{F})$, which we call $\mathscr{F}'$, such that $\mathscr{F}'|_U \cong \mathscr{F}$. We see that $\mathscr{F}'$ is a subsheaf of $\mathscr{G}$ necessarily.

(d) Cover X with finitely many open affines $V_1, \ldots, V_r$. Then $U \cap V_1$ is an open subset of V_1, and $\mathscr{F}|_{U \cap V_1}$ is a coherent sheaf and $\mathscr{G}|_{V_1}$ is a quasi-coherent sheaf. By (c), we can find a subsheaf $\mathscr{F}_1 \subseteq \mathscr{G}|_{V_1}$ on V_1 such that $\mathscr{F}_1|_{U \cap V_1} \cong \mathscr{F}|_{U \cap V_1}$. Now we consider the open subset $(U \cup V_1) \cap V_2 \subseteq V_2$. By construction, $\mathscr{F}$ and $\mathscr{F}_1$ agree on $(U \cup V_1) \cap V_2$, so glue to give a sheaf. Using (c) again, we may find a subsheaf $\mathscr{F}_2 \subseteq \mathscr{G}|_{V_2}$ on V_2 such that its restriction to $(U \cup V_1) \cap V_2$ is the same as our glued together sheaf. Again, we can glue $\mathscr{F}_2$ and $\mathscr{F}_1$ together because they agree on overlaps by our choice. Now it is clear how to repeat this process, and since there are only finitely many V_i, at the end we are left with a coherent subsheaf $\mathscr{F}' \subseteq \mathscr{G}$ such that $\mathscr{F}'|_U \cong \mathscr{F}$.

(e) Let U be an open set of X, and choose $s \in \mathscr{F}(U)$. Let $\mathscr{G}$ be the coherent subsheaf of $\mathscr{F}|_U$ generated by s. By (d), there exists a coherent subsheaf of $\mathscr{F}$, denote it $\mathscr{G}_s$ such that $\mathscr{G}_s|_U \cong \mathscr{G}$. Hence we can write $\mathscr{F}$ as the union of the $\mathscr{G}_s$ where s ranges over all sections over all open sets of $\mathscr{F}$.

16. (a) For any $x \in X$, we have $T^r(\mathscr{F})_x = (\mathscr{F} \otimes_{\mathcal{O}_X} \cdots \otimes_{\mathcal{O}_X} \mathscr{F})_x = \mathscr{F}_x^{\otimes_{\mathcal{O}_x} r}$ and similarly for $S^r(\mathscr{F})$ and $\bigwedge^r(\mathscr{F})$ because stalks commute with quotients (and the stalk of a presheaf and of its

sheafification are the same). So we have reduced to the case of showing that for a free module M of rank n, $T^r(M)$, $S^r(M)$, and $\bigwedge^r(M)$ are free of rank n^r, $\binom{n+r-1}{n-1}$, and $\binom{n}{r}$, respectively.

Write $M = R^n$. Then $T^r(M) = R^n \otimes \cdots \otimes R^n$. Since tensor product commutes with direct sum, we can expand this to get $T^r(M) = R^{n^r}$. Since M is a free module, $S(M)$ is isomorphic to the polynomial ring over R in n variables, and the degree r part is freely generated by the monomials of degree r. The number of such monomials is the number of ways to choose r things from a collection, without order, of n things allowing repetition, and this is given by $\binom{n+r-1}{n-1}$. Now choose a basis $e_1, \ldots, e_n$ of M. Then $\bigwedge^r(M)$ is freely generated by alternating forms of length r of the form $e_{i_1} \wedge \cdots \wedge e_{i_r}$ where $i_1 < \cdots < i_r$. This follows by considering the basis for $T^r(M)$, and then noting that the relations of the exterior algebra show that if $i_j = i_k$ for some j and k, then $e_{i_1} \wedge \cdots \wedge e_{i_r} = 0$, and also that if the i_j are not in increasing order, then one can anticommute them and get the same element, with the i_j in increasing order, up to a sign. So the number of basis elements is $\binom{n}{r}$.

(b) From (Ex. 5.1(b)), $\mathscr{H}om_{\mathcal{O}_X}(\bigwedge^{n-r}\mathscr{F}, \bigwedge^n \mathscr{F}) \cong (\bigwedge^{n-r}\mathscr{F})^* \otimes_{\mathcal{O}_X} \bigwedge^n \mathscr{F}$. The multiplication map $\bigwedge^r \mathscr{F} \otimes \bigwedge^{n-r}\mathscr{F} \to \bigwedge^n \mathscr{F}$ induces a map $\bigwedge^r \mathscr{F} \to \mathscr{H}om_{\mathcal{O}_X}(\bigwedge^{n-r}\mathscr{F}, \bigwedge^n \mathscr{F})$ in the obvious way. We claim that this is an isomorphism. From (a) and the isomorphism above, both are locally free and have the same rank, so it is enough to show that this induced map is injective. To check this we pass to stalks. Choose $x \in X$. If $f \in (\bigwedge^r \mathscr{F})_x$ induces the 0 map on stalks, then $f = 0$ because it kills any basis of $(\bigwedge^{n-r}\mathscr{F})_x$, so we get the desired claim.

(e) We proceed by induction on n, the case $n = 0$ being clear. For $n > 0$,

$$\begin{aligned}
T^n(f^*(\mathscr{F})) &= f^*(\mathscr{F}) \otimes_{\mathcal{O}_X} T^{n-1}(f^*(\mathscr{F})) \\
&= (f^{-1}(\mathscr{F}) \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X) \otimes_{\mathcal{O}_X} f^*(T^{n-1}(\mathscr{F})) \\
&\cong f^{-1}(\mathscr{F}) \otimes_{f^{-1}\mathcal{O}_Y} f^*(T^{n-1}(\mathscr{F})) \\
&= f^{-1}(\mathscr{F}) \otimes_{f^{-1}\mathcal{O}_Y} (f^{-1}(\mathscr{F}^{\otimes n-1}) \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X) \\
&\cong f^{-1}(\mathscr{F}^{\otimes n}) \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X \\
&= f^*(T^n(\mathscr{F})),
\end{aligned}$$

where the last isomorphism follows because f^{-1} is defined as a colimit, which commutes with left adjoints (in this case $\otimes$).

Let $\mathscr{I}$ be the degree n part of the sheaf ideal such that $T(\mathscr{F})/\mathscr{I} = S(\mathscr{F})$. Since f^* is a left adjoint, it is right exact, so

$$f^*\mathscr{I} \longrightarrow f^*(T^n(\mathscr{F})) \longrightarrow f^*(S^n(\mathscr{F})) \longrightarrow 0$$

is exact. In fact, for sections x, y of $\mathscr{I}$, one has $f^*(x \otimes y) = f^*x \otimes f^*y$ since tensor commutes with f^*, so we can write an exact sequence

$$0 \longrightarrow f^*\mathscr{I} \longrightarrow T^n(f^*(\mathscr{F})) \longrightarrow S^n(f^*(\mathscr{F})) \longrightarrow 0\,.$$

We have already shown that $T^n(f^*(\mathscr{F})) = f^*(T^n(\mathscr{F}))$, so we deduce that $S^n(f^*(\mathscr{F})) = f^*(S^n(\mathscr{F}))$. Showing that $\bigwedge$ commutes with f^* proceeds in the same way.

17. (a) Let $U = \operatorname{Spec} A \subseteq Y$ be an open affine. We claim that $f^{-1}(U) \to U$ is an affine morphism. Since $f\colon X \to Y$ is affine, there exists a covering of open affines $U_i = \operatorname{Spec} A_i \subseteq Y$ such that

$f^{-1}(U_i) = \operatorname{Spec} B_i$ is affine. We can cover $U \cap U_i$ with open sets V_{ij} that are distinguished in both U and U_i. Write $V_{ij} = D(g_{ij})$ where $g_{ij} \in A_i$, and let h_{ij} be the image of g_{ij} under the map $A_i \to B_i$. Then $f^{-1}(D(g_{ij})) = \operatorname{Spec}(B_i)_{h_{ij}}$. Since the $D(g_{ij})$ cover U, $f^{-1}(U) \to U$ is an affine morphism as desired.

So it is enough to show that if $f\colon X \to Y$ is an affine morphism and $Y = \operatorname{Spec} A$ is affine, then X is also affine. So there is a covering of Y by open affines such that their preimages are affines. By the above comments, we may assume that these open sets are in fact distinguished opens, and we can take finitely many, say $D(g_1), \ldots, D(g_n)$. Let h_i be the image of g_i under the map $A \to \Gamma(X, \mathcal{O}_X)$. Write $f^{-1}(D(g_i)) = \operatorname{Spec} B_i$. Then we claim that $X_{h_i} = \operatorname{Spec} B_i$. By (Ex. 2.16(a)), $\operatorname{Spec} B_i \cap X_{h_i}$ is the set of primes of B_i that do not contain h_i. Since we have a map $A_{g_i} \to B_i$, it must be that g_i is invertible in B_i, so no primes can contain it, and $\operatorname{Spec} B_i \subseteq X_{h_i}$. For any other j, $\operatorname{Spec} A_{g_i} \cap \operatorname{Spec} A_{g_j} = \operatorname{Spec} A_{g_i g_j}$. This gives that $\operatorname{Spec} B_i \cap \operatorname{Spec} B_j$ is $\{\mathfrak{p} \in \operatorname{Spec} B_j : h_i \notin \mathfrak{p}\} = \{\mathfrak{p} \in \operatorname{Spec} B_i : h_j \notin \mathfrak{p}\}$. This means that $X_{h_i} \cap \operatorname{Spec} B_j = \operatorname{Spec} B_i \cap \operatorname{Spec} B_j$, which proves the claim. Finally, $h_1, \ldots, h_n$ generate $\Gamma(X, \mathcal{O}_X)$, and each X_{h_i} is affine, so by (Ex. 2.17(b)), X is affine.

(b) Let $f\colon X \to Y$ be an affine morphism. There is a covering U_i of Y by open affines such that $f^{-1}(U_i)$ is affine. Since an affine scheme is quasi-compact, this means that f is quasi-compact. Since the maps $f^{-1}(U_i) \to U_i$ are between affine schemes, they are separated by Proposition 4.1. Then by Corollary 4.6f (note that the Noetherian condition is not necessary), f is separated. That a finite morphism is affine follows from the definition of finite.

(c) We wish to glue together the schemes $\operatorname{Spec} \mathscr{A}(U)$ as U ranges over all open affines of Y. Let $U = \operatorname{Spec} A$ and $V = \operatorname{Spec} B$ be two open affines. If $U \cap V$ is empty, there is nothing to do. Otherwise, cover $U \cap V$ with open sets that are distinguished in both U and V. Let $W = \operatorname{Spec} C$ be a distinguished open in $U \cap V$. Also, let $A' = \mathscr{A}(U)$, $B' = \mathscr{A}(V)$, and $C' = \mathscr{A}(W)$. Since $\mathscr{A}$ is an $\mathcal{O}_Y$-module,

$$\begin{tikzcd} \mathscr{A}(U) \arrow[r, "\rho_{UW}"] & \mathscr{A}(W) \\ \mathcal{O}_Y(U) \arrow[r] \arrow[u] & \mathcal{O}_Y(W) \arrow[u] \end{tikzcd}$$

is an $\mathcal{O}_Y(U)$-module homomorphism where ρ_{UW} is the restriction map given by $\mathscr{A}$. As C is a localization of both A and B, we also have that C' is a localization of both A' and B' since $\mathscr{A}$ is quasi-coherent, and hence we can identify A' and B' along C'. There are maps $A \to A'$ and $B \to B'$ given by the $\mathcal{O}_Y$-algebra structure of $\mathscr{A}$, and they induce morphisms $g\colon \operatorname{Spec} \mathscr{A}(U) \to U$ and $h\colon \operatorname{Spec} \mathscr{A}(V) \to V$.

In fact, the isomorphisms given by the distinguished covering of $U \cap V$ patch together to give an isomorphism $g^{-1}(U \cap V) \to h^{-1}(U \cap V)$. Since these isomorphisms come from restriction maps of a sheaf, it is clear that they agree on triple overlaps, so this gives a gluing, call the scheme X. The maps $\mathscr{A}(U) \to U$ for all open affines are compatible on overlaps, so glue together to give a morphism $f\colon X \to Y$. For an inclusion $U \subseteq V$ of open affines of Y, the morphism $f^{-1}(U) \to f^{-1}(V)$ is given by the restriction homomorphism $\mathscr{A}(V) \to \mathscr{A}(U)$ by construction above.

If there is an X' and $f'\colon X' \to Y$ with the same properties of X, then we can define a morphism $X \to X'$ by gluing together morphisms on open affines $\operatorname{Spec} \mathscr{A}(U)$ where U is an open affine of Y. Then this morphism will be an isomorphism, so we see that X is unique.

(d) By construction, for every open affine $U \subseteq Y$, $f^{-1}(U) \cong \operatorname{Spec} \mathscr{A}(U)$, so f is affine. Also, for every open set $U \subseteq Y$, we have $f_*\mathcal{O}_X(U) = \mathcal{O}_X(f^{-1}(U)) \cong \mathscr{A}(U)$. The isomorphism is clear if U is affine, or if U is contained in some open affine. In the general case, cover Y with open affines U_i, and for each $U \cap U_i$, we have $\mathcal{O}_X(f^{-1}(U \cap U_i)) \cong \mathscr{A}(U \cap U_i)$, which follows from the construction. Since these isomorphisms are canonical, they patch together to give the isomorphism for U.

Conversely, suppose that $f\colon X \to Y$ is an affine morphism, and set $\mathscr{A} = f_*\mathcal{O}_X$. For every open set $U \subseteq Y$, $\mathscr{A}(U) = \mathcal{O}_X(f^{-1}(U))$, so there is a morphism $\mathcal{O}_Y(U) \to \mathcal{O}_X(f^{-1}(U))$, which gives $\mathscr{A}(U)$ the structure of $\mathcal{O}_Y(U)$-module. For an inclusion $V \subseteq U$, it is clear that the restriction map $\mathcal{O}_X(f^{-1}(U)) \to \mathcal{O}_X(f^{-1}(V))$ is an $\mathcal{O}_X(U)$-module homomorphism. So $\mathscr{A}$ is an $\mathcal{O}_Y$-module.

In particular, for every open affine $U = \operatorname{Spec} A \subseteq Y$, $f^{-1}(U) = \operatorname{Spec} B$ is affine by (a). Considering B as an A-module, $\mathscr{A}|_U \cong \widetilde{B}$, so $\mathscr{A}$ is a quasi-coherent sheaf of $\mathcal{O}_Y$-algebras. Now if $V \subseteq U$ is an open affine, the morphism on spectra $f^{-1}(V) \to f^{-1}(U)$ is induced by the map of rings $\mathscr{A}(U) = \mathcal{O}_X(f^{-1}(U)) \to \mathcal{O}_X(f^{-1}(V)) = \mathscr{A}(V)$. From the uniqueness of $\mathbf{Spec}\,\mathscr{A}$ in (c), we conclude that $X \cong \mathbf{Spec}\,\mathscr{A}$.

(e) Let $\mathscr{M}$ be a quasi-coherent $\mathscr{A}$-module. We glue together the $\mathcal{O}_X(f^{-1}(U))$-modules $(\mathscr{M}(U))^\sim$ as U ranges over all open affines of Y. Given two open affines U and V of Y, we can cover their intersection with open sets that are distinguished in both. The sections of these distinguished open sets are given by localizing modules, and since they are the same in both $\mathscr{M}(U)$ and $\mathscr{M}(V)$, there is an isomorphism on their intersection. These isomorphisms are compatible with triple overlaps because they are given by localization. So we can glue these sheaves by (Ex. 1.22) to get an $\mathcal{O}_X$-module which we call $\widetilde{\mathscr{M}}$.

We claim that $\sim$ and f_* give an equivalence of categories between the category of quasi-coherent $\mathcal{O}_X$-modules and the category of quasi-coherent $\mathscr{A}$-modules. Let $\mathscr{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Then $(f_*\mathscr{F})^\sim$ is naturally isomorphic to $\mathscr{F}$ because they are isomorphic on open affines and using Corollary 5.5. Similarly, if $\mathscr{M}$ is a quasi-coherent $\mathscr{A}$-module, then $f_*\widetilde{\mathscr{M}}$ is naturally isomorphic to $\mathscr{M}$.

18. (a) Pick two open affines $U_1 = \operatorname{Spec} A_1$ and $U_2 = \operatorname{Spec} A_2$ such that $\mathscr{E}|_{U_1}$ and $\mathscr{E}|_{U_2}$ are free of rank n, and pick an open affine subset $V = \operatorname{Spec} B \subseteq U_1 \cap U_2$. Let $\psi = \psi_2 \circ \psi_1^{-1}$, which is an automorphism of $\mathbf{A}^n_V = \operatorname{Spec} B[x_1, \dots, x_n]$, and let θ_1, θ_2, and θ be the induced automorphisms from ψ_1, ψ_2, and ψ, respectively. The diagram

$$\begin{tikzcd}
\mathbf{A}^n_{U_1} \arrow[r, "\psi_1^{-1}"] & f^{-1}(U_1) & \\
\mathbf{A}^n_V \arrow[u] \arrow[r, "\psi_1^{-1}"] & f^{-1}(V) \arrow[u] \arrow[r, "\psi_2"] \arrow[d] & \mathbf{A}^n_V \arrow[d] \\
& f^{-1}(U_2) \arrow[r, "\psi_2"] & \mathbf{A}^n_{U_2}
\end{tikzcd}$$

commutes, where the unlabeled arrows are the ones induced by the ring homomorphisms $A_i \to B$, and their linear extensions $A_i[x_1, \dots, x_n] \to B[x_1, \dots, x_n]$. The identification of $S(\mathscr{E}(U_i))$ with $\mathcal{O}(U_i)[x_1, \dots, x_n]$ fixes the coefficients of $\mathcal{O}(U_i)$, and hence the automorphism of $B[x_1, \dots, x_n]$ induced by ψ_1^{-1} fixes B by the commutativity of the above diagram, and similarly with ψ_2. Let $e_1, \dots, e_n$ be the chosen basis for $\mathscr{E}(U_2)$. By the diagram, we see that

$\theta_1^{-1}(x_i)$ is the image of e_i under the map $S(\mathscr{E}(U_2)) \to S(\mathscr{E}(V))$, so $\theta_2(\theta_1^{-1}(x_i)) = \theta(x_i) = x_i$. This shows that $(X, f, \{U\}, \{\psi\})$ is a vector bundle of rank n over Y. A different choice of bases for the $\mathscr{E}(U)$ would result in different maps ψ', but $(X, f, \{U\}, \{\psi\})$ is isomorphic to $(X, f, \{U\}, \{\psi'\})$ via the identity morphism $X \to X$.

(b) First assume that $Y = \operatorname{Spec} A$ is affine. Since $f\colon X \to Y$ is a vector bundle of rank n, $X = \mathbf{A}^n_Y$. The sections $s \in \Gamma(Y, \mathscr{S}(X/Y))$ are in natural bijection with A-algebra homomorphisms $\theta_s\colon A[x_1, \ldots, x_n] \to A$, which are themselves in natural bijection with n-tuples $(\theta_s(x_1), \ldots, \theta_s(x_n))$, namely the images of x_i. The set of all n-tuples $(a_1, \ldots, a_n) \in A^n$ has a natural A-module structure, so we use this to determine an $\mathcal{O}_Y$-module structure on $\mathscr{S}(X/Y)$. Namely, for two sections s and t, let $s + t$ be the section determined by the A-algebra homomorphism $\theta_{s+t}\colon A[x_1, \ldots, x_n] \to A$ defined by $x_i \mapsto \theta_s(x_i) + \theta_t(x_i)$. For $a \in A$, define as to be the section determined by the A-algebra homomorphism $\theta_{as}\colon A[x_1, \ldots, x_n] \to A$ defined by $x_i \mapsto a\theta_s(x_i)$. From these constructions, it is clear that $\mathscr{S}(X/Y) \cong \mathcal{O}_Y^n$, and hence is free of rank n. It is also clear that if Y is not necessarily affine, then this local construction glues together to give a global $\mathcal{O}_Y$-module structure on $\mathscr{S}(X/Y)$ that makes it locally free of rank n.

(c) Let $V \subseteq Y$ be an open set. Define a map $\mathscr{E}^*(V) \to \mathscr{S}(V)$ in the following way. Any $s \in E^*(V)$ is an element of $\operatorname{Hom}(\mathscr{E}|_V, \mathcal{O}_V)$. This determines an $\mathcal{O}_V$-algebra homomorphism $S(\mathscr{E}|_V) \to \mathcal{O}_V$, which in turn determines a morphism $\mathbf{Spec}\, \mathcal{O}_V \to \mathbf{Spec}\, S(\mathscr{E}|_V)$. Since $\mathbf{Spec}\, \mathcal{O}_V = V$ and $\mathbf{Spec}\, S(\mathscr{E}|_V) = f^{-1}(V)$, this is an element of $\mathscr{S}(V)$. To show that this is an isomorphism, we can show that it is an isomorphism on stalks. Since $\mathscr{S}$ and $\mathscr{E}^*$ are both locally free of rank n, $\mathscr{S}_y$ and $\mathscr{E}^*_y$ are both free $\mathcal{O}_y$-modules of rank n for all $y \in Y$, so it is enough to check that the map given is injective on stalks.

The only part in question is getting an $\mathcal{O}_V$-algebra homomorphism $S(\mathscr{E}|_V) \to \mathcal{O}_V$ from an element of $\operatorname{Hom}(\mathscr{E}|_V, \mathcal{O}_V)$ since the rest of the operations are clearly invertible. On the level of stalks, $(\mathscr{E}|_V)_y$ is a free $\mathcal{O}_y$-module of rank n, and $S(\mathscr{E}|_V) \cong \mathcal{O}_y[x_1, \ldots, x_n]$. Some basis of $(\mathscr{E}|_V)_y$ has been chosen for the given vector bundle structure on Y. The $\mathcal{O}_y$-algebra homomorphism $\mathcal{O}_y[x_1, \ldots, x_n] \to \mathcal{O}_y$ is determined by the images of x_i, which are in turn determined by the images of the basis elements of $(\mathscr{E}|_V)_y$. Now it is clear that a nonzero map $\operatorname{Hom}(\mathscr{E}|_V, \mathcal{O}_V)$ gives a nonzero map $S(\mathscr{E}|_V) \to \mathcal{O}_V$, so we get the desired isomorphism.

(d) By (b), for every vector bundle $f\colon X \to Y$ of rank n, we have an associated sheaf of sections $\mathscr{S}(X/Y)$. It is clear from construction that two isomorphic vector bundles give isomorphic sheaves of sections. Also by (b), the sheaf of sections arising from a vector bundle of n is a locally free sheaf of rank n. Every locally free sheaf has an associated geometric vector bundle, and part (c) implies that two locally free sheaves are isomorphic if and only if they admit isomorphic vector bundles. Finally, part (a) says that the geometric vector bundle associated to a locally free sheaf is in fact a vector bundle. These comments give the following injections

$$\begin{pmatrix} \text{isomorphism classes} \\ \text{of vector bundles} \\ \text{of rank } n \text{ over } Y \end{pmatrix} \hookrightarrow \begin{pmatrix} \text{isomorphism classes of} \\ \text{sheaves of sections on } Y \end{pmatrix}$$
$$\hookrightarrow \begin{pmatrix} \text{isomorphism classes} \\ \text{of locally free sheaves} \\ \text{of rank } n \text{ on } Y \end{pmatrix}$$
$$\hookrightarrow \begin{pmatrix} \text{isomorphism classes} \\ \text{of associated vector} \\ \text{bundles of rank } n \text{ over } Y \end{pmatrix}$$
$$\hookrightarrow \begin{pmatrix} \text{isomorphism classes} \\ \text{of vector bundles} \\ \text{of rank } n \text{ over } Y \end{pmatrix},$$

which shows that there is a bijection between isomorphism classes of locally free sheaves of rank n on Y, and isomorphism classes of vector bundles of rank n over Y.

6 Divisors

1. Let X be a scheme satisfying $(*)$, that is, X is a Noetherian integral separated scheme which is regular in codimension one. The projection $X \times \mathbf{P}^n \to \mathbf{P}^n$ is a base extension of $X \to \operatorname{Spec} \mathbf{Z}$ and hence separated. The map $\mathbf{P}^n \to \operatorname{Spec} \mathbf{Z}$ is projective and thus also separated, so the composition $X \times \mathbf{P}^n \to \operatorname{Spec} \mathbf{Z}$ is separated. Also there is a finite covering of X by Noetherian open affines, and the same for $\mathbf{P}^n$, so their pairwise fiber products give a finite covering of $X \times \mathbf{P}^n$ of Noetherian open affines, which means $X \times \mathbf{P}^n$ is Noetherian.

   Let $\pi\colon X \times \mathbf{P}^n \to X$ be canonical projection and suppose that there are two generic points η_1 and η_2 of $X \times \mathbf{P}^n$. Then both map to η_X, the generic point of X. Choose some open affine $U = \operatorname{Spec} A$ of X containing η_X. Then $\pi^{-1}(U) = \mathbf{P}^n_A$ which contains η_1 and η_2. We know that $\mathbf{A}^n_A$ is a dense open subset of $\mathbf{P}^n_A$, and that it is irreducible, so this implies that $\mathbf{P}^n_A$ is also irreducible, and is a contradiction. Hence $X \times \mathbf{P}^n$ is integral.

   Now let $x \in X \times \mathbf{P}^n$ be a point of codimension one, and put $y = \pi(x)$ Then either y is a point of codimension one, or y is the generic point of X. In the first case, $\bar{x} = \pi^{-1}(y) = \operatorname{Spec} k(y) \times \mathbf{P}^n$, which we can write as the union of open affines isomorphic to $\mathbf{A}^n_{k(y)}$. This implies that $\mathcal{O}_x$ is isomorphic to a localization of $\mathcal{O}_y[t_1, \ldots, t_n]$. By assumption, $\mathcal{O}_x$ is dimension one. Since $\mathcal{O}_y$ is a DVR, it is integrally closed, and hence so is $\mathcal{O}_y[t_1, \ldots, t_n]$. So $\mathcal{O}_x$ is integrally closed and also a DVR. In the second case that y is the generic point of X, we see that $\mathcal{O}_x$ is a localization of $K[t_1, \ldots, t_n]$ where K is the function field of X. By the same reasons as above, this means $\mathcal{O}_x$ is a DVR. So we see that $X \times \mathbf{P}^n$ is regular in codimension one and therefore $X \times \mathbf{P}^n$ satisfies $(*)$.

   Now we can define maps $p_1^*\colon \operatorname{Cl} X \to \operatorname{Cl}(X \times \mathbf{P}^n)$ and $p_2^*\colon \operatorname{Cl} \mathbf{P}^n \to \operatorname{Cl}(X \times \mathbf{P}^n)$ coming from the projection maps. Both are injective and p_1^* comes from case 1 mentioned above, and p_2^* comes from case 2 mentioned above. If H is a hyperplane that generates $\operatorname{Cl} \mathbf{P}^n \cong \mathbf{Z}$, then p_2^* is defined by $H \mapsto X \times H$. Finally, these two maps induce a map $\varphi\colon \operatorname{Cl} X \oplus \operatorname{Cl} \mathbf{P}^n \to \operatorname{Cl}(X \times \mathbf{P}^n)$, which is surjective because the prime divisors of $X \times \mathbf{P}^n$ must be of either case 1 or case 2. The injectivity of φ follows because $\mathbf{P}^n_{k(y)}$ and $X \times H$ have different images when projected under the appropriate maps, and these are what come from p_1^* and p_2^*. Thus, φ is an isomorphism, so $\operatorname{Cl}(X \times \mathbf{P}^n) \cong \operatorname{Cl} X \times \mathbf{Z}$.

4. From the choice of f, $z^2 - f$ is an irreducible polynomial. Since $k[x_1, \dots, x_n, z]$ is a UFD, this means that $(z^2 - f)$ is prime, and hence A is a domain. Let K denote the quotient field of A. Then $K = k(x_1, \dots, x_n)[z]/(z^2 - f)$ because $z^{-1} = zf^{-1}$. Let $A' = k(x_1, \dots, x_n)$.

   Every element $\alpha \in K$ can be written $\alpha = g + hz$ with $g, h \in A'$. If $g, h \in k[x_1, \dots, x_n]$, then the equation $\alpha^2 - 2g\alpha + (g^2 - h^2 f) = 0$ shows that α is integral over $k[x_1, \dots, x_n]$. On the other hand, if $\alpha = g + hz$ is integral over $k[x_1, \dots, x_n]$, then there is a monic polynomial $F(X)$ such that $F(\alpha) = 0$. Since $X^2 - 2gX + (g^2 - h^2 f)$ is the minimal polynomial of α, it divides $F(X)$. In particular, the coefficients $-2g$ and $g^2 - h^2 f$ are integral over $k[x_1, \dots, x_n]$. Since $-2g, g^2 - h^2 f \in A'$, and $k[x_1, \dots, x_n]$ is a UFD, it must be that $-2g, g^2 - h^2 f \in k[x_1, \dots, x_n]$. In particular, this implies that $g \in k[x_1, \dots, x_n]$, and hence $h^2 f \in k[x_1, \dots, x_n]$. If $h \in A' \setminus k[x_1, \dots, x_n]$, then write $h = h_1/h_2$ in reduced terms. Then $f/h_2^2 \in k[x_1, \dots, x_n]$, but f is square-free, so this is a contradiction. Thus, $h \in k[x_1, \dots, x_n]$ also.

   We have thus shown that $g + hz$ is integral over $k[x_1, \dots, x_n]$ if and only if $g, h \in k[x_1, \dots, x_n]$. This implies that the integral closure of $k[x_1, \dots, x_n]$ in K is A, and hence A is integrally closed.

5. For the change of coordinates in part (b), we must at least assume that k has a square root of -1. For example, there can be no such change of coordinates if $k = \mathbf{R}$ because $x_0^2 + \cdots + x_r^2 = 0$ has one solution while $x_0 x_1 = x_2^2 + \cdots + x_r^2$ has infinitely many.

   (a) If $r \geq 2$, $f = -(x_1^2 + \cdots + x_r^2)$ is a square-free polynomial, so by (Ex. 6.4), the ring $k[x_0, \dots, x_n]/(x_0^2 + x_1^2 + \cdots + x_r^2)$ is integrally closed. The localization of an integrally closed domain is again an integrally closed domain, so X is normal.

   (b) Let i be a square root of -1. We do the change of coordinates $x_0 \mapsto (z_0 + z_1)/2$, $x_1 \mapsto i(z_0 - z_1)/2$, and $x_j \mapsto iz_j$ for $j \geq 2$. Then

$$\begin{aligned} x_0^2 + \cdots + x_r^2 &= (z_0^2/4 + z_0 z_1/2 + z_1^2/4) + (-z_0^2/4 + z_0 z_1/2 - z_1^2/4) - z_2^2 - \cdots - z_r^2 \\ &= z_0 z_1 - z_2^2 - \cdots - z_r^2, \end{aligned}$$

   so we have the desired change of coordinates.

   (1) Let $A = \operatorname{Spec} k[x_0, \dots, x_n]/(x_0 x_1 - x_2^2)$, and let Y be defined by $x_1 = x_2 = \cdots = x_n = 0$. Then Y is a prime divisor, so by Proposition 6.5(c), there is an exact sequence

$$\mathbf{Z} \longrightarrow \operatorname{Cl} X \longrightarrow \operatorname{Cl}(X \setminus Y) \longrightarrow 0$$

   where the first map is $1 \mapsto 1 \cdot Y$. Then Y is cut out set-theoretically by the function x_1, and the divisor of x_1 is $2 \cdot Y$ because $x_1 = 0$ implies $x_2^2 = 0$ and x_2 generates the maximal ideal of the local ring at the generic point at Y. So $X \setminus Y = A_{x_1}$ and $A_{x_1} = k[x_0, x_1, x_1^{-1}, \dots, x_n]/(x_0 x_1 - x_2^2)$. Then $x_0 = x^{-1} x_2^2$, so $A_{x_1} = k[x_1, x_1^{-1}, \dots, x_n]$. Since this is a UFD, $\operatorname{Cl}(X \setminus Y) = 0$ by Proposition 6.2. This implies that Y generates $\operatorname{Cl} X$.

   Finally, we show that Y is not principal. By (Ex. 6.4), A is integrally closed, it is enough to show that $\mathfrak{p} = (x_1, \dots, x_n)$ is not a principal ideal. The sufficiency can be found in the proof of Proposition 6.2. Let $\mathfrak{m} = (x_0, \dots, x_n)$, then $\mathfrak{m}/\mathfrak{m}^2$ is an n-dimensional vector space over k generated by the images of the x_i. Since $\mathfrak{p} \subseteq \mathfrak{m}$ and the image of $\mathfrak{p}$ in $\mathfrak{m}/\mathfrak{m}^2$ contains the images of the x_i, we conclude that $\mathfrak{p}$ is not principal. Hence Y generates $\operatorname{Cl} X$ and has order 2, so we conclude that $\operatorname{Cl} X = \mathbf{Z}/2$.

(2) Let V be the projective variety defined by the equation $x_0^2 + \cdots + x_3^2 = 0$. By the comments above, there is a change of coordinates so that V is given by $x_0x_1 = x_2x_3$. From Example 6.6.1, $\operatorname{Cl} V \cong \mathbf{Z} \oplus \mathbf{Z}$ for $n = 3$, but the ambient space is irrelevant, i.e., for $n > 3$, we can take the fiber product of $\mathbf{P}_k^1 \times_k \mathbf{P}_k^1$ with $\mathbf{A}_k^1$ $n-3$ times to get Q and use Proposition 6.6. By (I, Ex. 2.10(a)), A is the affine cone of V, so by (Ex. 6.3(b)), there is an exact sequence

$$0 \longrightarrow \mathbf{Z} \longrightarrow \operatorname{Cl} V \longrightarrow \operatorname{Cl} X \longrightarrow 0$$

where $\mathbf{Z} \to \operatorname{Cl} V$ is given by $1 \mapsto V.H$. The cokernel of this map is $\mathbf{Z}$, so $\operatorname{Cl} X \cong \mathbf{Z}$.

(3) By Proposition 6.2, it is equivalent to show that $A = k[x_0, \dots, x_n]/(x_0x_1 - x_2^2 - \cdots - x_r^2)$ is a UFD if $r \geq 4$. First note that $A/(x_0) = k[x_2, \dots, x_n]/(x_2^2 + \cdots + x_r^2)$ is normal by (a), so x_0 is prime in A. We now appeal to a theorem by Nagata that says that if A is a domain and S is a multiplicative set generated by prime elements in A, then A is a UFD if and only if $S^{-1}A$ is a UFD and A has the ascending chain condition for principal ideals. Here the ascending chain condition is clear because A is Noetherian, and we take S to be generated by x_0. Then $A_{x_0} = k[x_0, x_2, \dots, x_n]_{x_0}$, which is a localization of a UFD and hence a UFD. Thus A is a UFD.

(c) (1) Let X be the affine quadric hypersurface given by the equation $x_0x_1 = x_2^2$. Then by (Ex. 6.3(b)), there is a short exact sequence

$$0 \longrightarrow \mathbf{Z} \longrightarrow \operatorname{Cl} Q \longrightarrow \operatorname{Cl} X \longrightarrow 0$$

where $\mathbf{Z} \to \operatorname{Cl} Q$ is given by $1 \mapsto Q.H$. From (b) part 1, the cokernel of this map is $\mathbf{Z}/2$, which means that $\operatorname{Cl} Q \cong \mathbf{Z}$ or $\operatorname{Cl} Q \cong \mathbf{Z} \oplus \mathbf{Z}/2$. But $\operatorname{Cl} Q$ is cyclic and generated by the ruling of the projective quadric cone as in Example 6.5.2. So $\operatorname{Cl} Q \cong \mathbf{Z}$, and the class of a hyperplane section $Q.H$ is twice the generator.

(2) This was done in (b) part 2 above.

(3) Let X be the affine quadric hypersurface given by the equation $x_0x_1 = x_2^2 + \cdots + x_r^2$. Then by (Ex. 6.3(b)), there is a short exact sequence

$$0 \longrightarrow \mathbf{Z} \longrightarrow \operatorname{Cl} Q \longrightarrow \operatorname{Cl} X \longrightarrow 0$$

where $\mathbf{Z} \to \operatorname{Cl} Q$ is given by $1 \mapsto Q.H$. From (b) part 3, the cokernel of this map is 0, which means that it is an isomorphism, so $\operatorname{Cl} Q \cong \mathbf{Z}$ and it is generated by $Q.H$.

6. (a) If P, Q, R are three collinear points of X, then $P + Q + R \sim 3P_0$. Since P is associated to the divisor $P - P_0$ in $\operatorname{Cl}^\circ X$, we see that $P + Q + R = 0$ in the group law. Conversely, if $P + Q + R = 0$ in the group law, then $(P - P_0) + (Q - P_0) + (R - P_0)$ is principal, or equivalently, $P + Q + R \sim 3P_0$. Since $3P_0$ is linearly equivalent to $z = 0$, there is a line L meeting X in P, Q, R.

(b) If the tangent line at P passes through P_0, then $P + P + P_0 \sim 3P_0$, which we can rewrite as $P + P \sim 2P_0$. Then the image of $P + P$ in $\operatorname{Cl}^\circ X$ is $2(P_0 - P_0) = 0$, and thus P has order 2 in the group law. Conversely, suppose that P has order 2 in the group law. Since P_0 is the identity element in the group law, this means that $P + P + P_0 = 0$. By part (a), this means that P, P, P_0 are collinear counting multiplicity, so the tangent line at P passes through P_0.

(c) If P is an inflection point, then there the tangent line at P passes through P with multiplicity ≥ 3, so $P + P + P = 0$ in the group law by (a), so P has order 3. Conversely, suppose that P has order 3 in the group law. Then there is a line passing through P with multiplicity ≥ 3. In particular, the tangent line at P does this, so P is an inflection point.

(d) Because of the relation $P+Q+R=0$ if P,Q,R are collinear, we need to check that if P and Q correspond to rational points on X, then so $-R$ also corresponds to a rational point. Since $-R$ and R are reflections across the x-axis, it is enough to show that if P and Q are rational points, then the line PQ intersects X at a rational point. If the line PQ is tangent to either P or Q, then either $P+Q+Q=0$ or $P+P+Q=0$, and if PQ passes through P_0, then $P+Q=0$.

If $P=Q$, then PQ is the tangent line at P. If this tangent line PQ passes through P_0, then by (b), $P+P=0$. Otherwise, we consider the curve defined by $y^2=x^3-x$ in $\mathbf{C}^2$ and check that the tangent line of P intersects this curve in a rational point. Let P be of the form $(a,\sqrt{a^3-a})$ where $a\notin\{0,1,-1\}$. (If $a\in\{0,1,-1\}$, this corresponds to the tangent line passing through P_0. Also, the case $(a,-\sqrt{a^3-a})$ is similar, so we omit it.) The derivative of $y=\sqrt{x^3-x}$ is $\dfrac{3x^2-1}{2\sqrt{x^3-x}}$, so the equation of the tangent line through P is

$$y=\frac{3a^2-1}{2\sqrt{a^3-a}}(x-a)+\sqrt{a^3-a}. \tag{1}$$

Substituting this into $y^2=x^3-x$, we get (after simplification via Maxima):

$$\frac{(x-a)^2((4a^3-4a)x-(a^4+2a^2+1))}{4a^3-a}=0$$

and the solutions are rational, so the intersection is in fact a rational point since the equation (1) for PQ has rational coefficients.

The last case is when $P\neq Q$ and their third point of intersection is not P, Q, or P_0. Again, we can work in the complex plane $\mathbf{C}^2$ and consider the curve defined by the equation $y^2=x^3-x$. Then consider two points on this curve with rational coordinates $P=(a,\sqrt{a^3-a})$ and $Q=(b,\sqrt{b^3-b})$. There are two choices for the sign of the square root, but the other three cases are similar. Given these two points, with $a\neq b$ (since PQ does not intersect at P_0), we have the equation for the line PQ:

$$y-\sqrt{a^3-a}=\frac{\sqrt{b^3-b}-\sqrt{a^3-a}}{b-a}(x-a). \tag{2}$$

Solving for y and substituting this into $y^2=x^3-x$, we get (after simplification via Maxima):

$$\frac{(x-a)(x-b)((b-a)^2x+(2\sqrt{a^3-a}\sqrt{b^3-b}-ab^2-a^2b+b+a))}{(b-a)^2}=0$$

The three solutions for x are all rational, and the equation (2) for PQ has rational coefficients, we conclude that PQ intersects the curve defined by $y^2=x^3-x$ in a rational point. Thus, the points of X with coordinates in $\mathbf{Q}$ form a subgroup of the algebraic group structure on X.

There are an obvious four rational points on X, which are $(0,1,0)$, $(1,0,1)$, $(-1,0,1)$, and $(0,0,1)$, all of which have order 2 and form the group $\mathbf{Z}/2\oplus\mathbf{Z}/2$. We claim that these are the only rational points on X. Let (x,y,z) be a rational point with $z\neq 0$. Then we can work in the affine plane and consider a rational point (x,y).

We first show that $|x|$ is the square of a rational number. Write $|x|=a/b$. For a prime p, let p^n be the highest power of p dividing a and p^m be the highest power of p dividing b, and define $v_p(x) = n - m$. We claim that $v_p(x)$ is even for all primes p, which will show that x is a square. Choose a prime p and assume $v_p(x) \neq 0$. If $v_p(x) > 0$, then $v_p(x^2-1) = 0$, which implies that $v_p(y^2) = 2v_p(y) = v_p(x)$ because $y^2 = x(x^2-1)$. If $v_p(x) < 0$, then $v_p(x^2-1) = 2v_p(x)$, so $2v_p(y) = 3v_p(x)$. This gives the claim. Now write $|x| = u^2/v^2$. Since $y^2 = x(x^2-1)$, $|x^2-1|$ is a square. Then $|v^4(x^2-1)| = |u^4 - v^4|$ is a square, and is in fact an integer.

Lemma. The equation $u^4 - v^4 = \pm w^2$ has an integer solution only if at least one of u, v, w is zero.

Proof. Suppose u, v, w are positive and satisfy this equation, with w minimal with respect to these properties. By symmetry, we can assume the sign of w^2 is positive. Then (v^2, w, u^2) is a Pythagorean triple. If z is even, then we must have u and v odd by minimality. So $v^2 = p^2 - q^2$, $w = 2pq$, and $u^2 = p^2 + q^2$ for some odd integers p and q by the classification of Pythagorean triples. But now $p^4 - q^4 = (uv)^2$, which contradicts minimality.

So it must be that w is odd, in which case $v^2 = 2pq$, $w = p^2 - q^2$, and $u^2 = p^2 + q^2$ for odd p and q. Since $2pq$ is a square, we can write $p = r^2$ and $q = 2s^2$. We can also write $p = \alpha^2 - \beta^2$, $q = 2\alpha\beta$, and $u = \alpha^2 + \beta^2$ because (p, q, u) is a Pythagorean triple. Since $2s^2 = 2\alpha\beta$, we can write $\alpha = A^2$ and $\beta = B^2$. Substituting these into $p = \alpha^2 - \beta^2$ gives $r^2 = A^4 - B^4$. Since $w = (p+q)(p-q)$, $p + q < w$ implies $r < w$, which contradicts minimality. $\square$

We cannot have $v = 0$ since it is a denominator. If $w = 0$, then $u = \pm v$, so $x = \pm 1$. Finally, if $u = 0$, then $x = 0$. We conclude then that the four rational points mentioned above are the only ones.

7. We let X be the nodal cubic curve $y^2 z = x^3 + x^2 z$ in $\mathbf{P}^2_k$ for some field k with $\operatorname{char} k \neq 2, 3$. The proof that the Cartier divisors of X of degree 0, $\operatorname{CaCl}^\circ X$, are in bijection with the nonsingular closed points of X is similar to Example 6.11.4, so we omit it.

   The nontrivial part is providing an isomorphism of algebraic groups $\mathbf{G}_m \to X \setminus Z$ where Z is the singular point $(0,0,1)$. Let $P_0 = (0,1,0)$ be the identity of $X \setminus Z$. Imitating the proof of Example 6.11.4, we might try $t \mapsto \left(t, 1, \dfrac{t^3}{1-t^2}\right)$. However, we know that $(-1, 0, 1)$ is the unique point on X whose tangent line passes through P_0, so -1 must map to $(-1,0,1)$. Also, 1 must map to P_0, so this map does not work. We see that $0 \mapsto (0,1,0)$. Restricting to $z = 1$, we get the curve $t \mapsto \left(-1 + \dfrac{1}{t^2}, \dfrac{1}{t^3} - \dfrac{1}{t^2}, 1\right)$. Thinking of this as a map $\mathbf{P}^1_k \to \mathbf{P}^2_k$, we see that $\infty \mapsto (1,0,1)$ and $1 \mapsto (0,0,1)$.

   To fix the map, we should use a linear fractional transformation S such that $S(0) = 1$, $S(1) = 0$, $S(-1) = \infty$. Such an S is given by $t \mapsto \dfrac{1-t}{1+t}$. If we compose our original map with S, then we get

$$t \mapsto \left(\frac{1-t}{1+t}, 1, \frac{(1-t)^3}{4t(1+t)}\right).$$

   However, this isn't defined at $t = -1$, so we fix this by clearing denominators to get

$$\boxed{t \mapsto (4t(1-t), 4t(1+t), (1-t)^3).}$$

   Now this defines a morphism of varieties $f\colon \mathbf{G}_m \to X \setminus Z$, which is clearly an isomorphism. Finally, we need to check that f is a morphism of algebraic groups. Given two points $P =$

$(4t(1-t), 4t(1+t), (1-t)^3)$ and $Q = (4u(1-u), 4u(1+u), (1-u)^3)$, there are a few cases to check. Since P_0 is the identity in the group law, we can assume that t and u are different from 1. So in fact, we may assume that both P and Q live in $\mathbf{A}^2_k$, so treat them as points $P = \left(\frac{4t}{(1-t)^2}, \frac{4t(1+t)}{(1-t)^3}\right)$ and $Q = \left(\frac{4u}{(1-u)^2}, \frac{4u(1+u)}{(1-u)^3}\right)$.

If $t = u = -1$, then $P = Q = (-1, 0, 1)$, and the tangent line at PQ passes through P_0, so $P + Q = 0$, which agrees with the fact that -1 has order 2 in $\mathbf{G}_m$. If $t = u \neq -1$, then assume that P lies on the curve $y = \sqrt{x^3 + x^2}$ (By curve we mean the set of points (x, y) for which the square root exists. Also, the case $y = -\sqrt{x^3 + x^2}$ is similar). The derivative of this function is $\frac{3x^2 + 2x}{2\sqrt{x^3 + x^2}}$. The computation of the third point on PQ is too messy to include, but it is $R = \left(\frac{4t^2}{(1-t^2)^2}, -\frac{4t^2(1+t^2)}{(1-t^2)^3}\right)$, so $P + P = -R$, which means $f(t^2) = f(t) + f(t)$.

The last case is when $t \neq u$. In this case, the equation for the line PQ is

$$y - \frac{4t(1+t)}{(1-t)^3} = \frac{\frac{4t(1+t)}{(1-t)^3} - \frac{4u(1+u)}{(1-u)^3}}{\frac{4t}{(1-t)^2} - \frac{4u}{(1-u)^2}} \left(x - \frac{4t}{(1-t)^2}\right). \tag{3}$$

Substituting this into $y^2 = x^3 + x^2$ and simplifying with Maxima, we get

$$((t-1)^2 x - 4t)((u-1)^2 x - 4u)((tu-1)^2 x - 4tu) = 0$$

so the third point has x-coordinate $\frac{4tu}{(1-tu)^2}$. Plugging this into (3), we get that $P + Q = -R$ where $R = \left(\frac{4tu}{(1-tu)^2}, -\frac{4tu(1+tu)}{(1-tu)^3}\right)$. Therefore, $f(ut) = f(u) + f(t)$, so we have the desired isomorphism.

8. (a) If $\mathscr{L}$ is an invertible sheaf on Y, then we claim that $f^*\mathscr{L}$ is an invertible sheaf on X. The question is local, so we can assume $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$ are affine, and that $\mathscr{L} \cong \widetilde{B}$. Using Proposition 5.2(e) and Corollary 5.5, $f^*\mathscr{L} \cong (A \otimes_B B)^\sim \cong \widetilde{A}$, which gives the claim. It is clear that if $\mathscr{L} \cong \mathscr{M}$, then $f^*\mathscr{L} \cong f^*\mathscr{M}$ because functors preserve isomorphisms, so $f^* \colon \operatorname{Pic} Y \to \operatorname{Pic} X$ is a well-defined function. To see that it is a group homomorphism, we need to show that $f^*(\mathscr{L} \otimes_{\mathcal{O}_Y} \mathscr{M}) \cong f^*\mathscr{L} \otimes_{\mathcal{O}_X} f^*\mathscr{M}$ where $\mathscr{L}$ and $\mathscr{M}$ are invertible sheaves on Y. This is illustrated in the following steps:

$$\begin{aligned} f^*\mathscr{L} \otimes_{\mathcal{O}_X} f^*\mathscr{M} &= (f^{-1}\mathscr{L} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X) \otimes_{\mathcal{O}_X} (f^{-1}\mathscr{M} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X) \\ &\cong (f^{-1}\mathscr{L} \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathscr{M}) \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X \\ &\cong f^{-1}(\mathscr{L} \otimes_{\mathcal{O}_Y} \mathscr{M}) \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X \\ &= f^*(\mathscr{L} \otimes_{\mathcal{O}_Y} \mathscr{M}) \end{aligned}$$

where the second isomorphism follows because f^{-1} is defined as a sheafification of a colimit, and thus commutes with $\otimes$ as presheaves because it is a left adjoint. Using the universal property of sheafification gives the desired result.

(b) Since $f \colon X \to Y$ is a finite morphism of nonsingular curves, from Proposition 6.11 and Proposition 6.15, we have the following diagram

$$\begin{array}{ccccc} \operatorname{Cl} Y & \xrightarrow{\alpha_Y} & \operatorname{CaCl} Y & \xrightarrow{\beta_Y} & \operatorname{Pic} Y \\ \downarrow{\scriptstyle f^*} & & & & \downarrow{\scriptstyle f^*} \\ \operatorname{Cl} X & \xrightarrow{\alpha_X} & \operatorname{CaCl} X & \xrightarrow{\beta_X} & \operatorname{Pic} X \end{array}$$

where the horizontal arrows are isomorphisms. We claim that this diagram commutes. Denote the function fields of X and Y by $K(X)$ and $K(Y)$.

We first define a map $f^*\colon \operatorname{CaCl} Y \to \operatorname{CaCl} X$ induced by $f^*\colon \operatorname{Pic} Y \to \operatorname{Pic} X$. For any Cartier divisor $D \in \operatorname{CaCl} Y$ represented by $\{(U_i, a_i)\}$, let $f^*(D)$ be represented by $\{(f^{-1}(U_i), b_i)\}$ where b_i is the image of a_i in the inclusion of function fields $K(Y) \hookrightarrow K(X)$. It is clear that $\mathscr{L}(f^*(D)) = f^*(\mathscr{L}(D))$ because we can define canonical isomorphisms on $f^{-1}(U_i)$ and U_i since f^* is defined with a colimit over open sets containing $f(f^{-1}(U_i)) = U_i$ (which is open). Also, this map is a homomorphism because it is equal to $\beta_X^{-1} f^* \beta_Y$.

Using the isomorphisms α_Y and α_X, the homomorphism $f^*\colon \operatorname{CaCl} Y \to \operatorname{CaCl} X$ induces a homomorphism $\operatorname{Cl} Y \to \operatorname{Cl} X$, which we claim is the same defined in the text. Pick a Cartier divisor $D \in \operatorname{CaCl} Y$. By extending linearly, we can assume that under the isomorphism α_Y, D corresponds to a closed point $Q \in Y$. Under the isomorphism α_X, $f^*(D)$ is $\sum v_P(b_i) \cdot P$ where for each closed point P, the b_i corresponds to an index i for which $P \in f^{-1}(U_i)$. However, if $P \notin f^{-1}(Q)$, then $v_P(b_i) = 0$. This follows from the fact that $\varphi_P\colon \mathscr{O}_{Y,\varphi(P)} \to \mathscr{O}_{X,P}$ is a local homomorphism and the fact that valuation 0 corresponds to not being in the maximal ideal of a local ring. So this sum is exactly what is described in the text on p.137, i.e., $f^*(Q) = \sum_{P \in f^{-1}(Q)} v_P(t) \cdot P$ where $t \in K(Y)$ is such that $v_Q(t) = 1$. Thus the diagram above commutes, so the homomorphisms f^* correspond under the isomorphisms.

(c) Now let X be a locally factorial integral closed subscheme of $Y = \mathbf{P}^n_k$ and $f\colon X \hookrightarrow Y$ be the inclusion. As above, we have the following diagram

$$\begin{tikzcd} \operatorname{Cl} Y \arrow[r, "\varphi"] \arrow[d] & \operatorname{Pic} Y \arrow[d, "f_*"] \\ \operatorname{Cl} X \arrow[r, "\psi"] & \operatorname{Pic} X \end{tikzcd}$$

where φ and ψ are the isomorphisms of Proposition 6.16. We claim that this diagram commutes. First note that $\operatorname{Cl} Y$ is generated by a hyperplane H, which corresponds to $\mathscr{O}_Y(1)$ under φ. By Corollary 5.16(a), $X = \operatorname{Proj} k[x_0, \ldots, x_n]/I$ for some homogeneous ideal I. Then by Proposition 5.12(c), $f^*(\mathscr{O}_Y(1)) = \mathscr{O}_X(1)$ as elements of $\operatorname{Pic} X$. Locally, $\mathscr{O}_X(1)$ is generated by the the $\overline{x}_i$, which are the images of the x_i, on the open sets $D_+(\overline{x}_i)$. This describes the Cartier divisor D identified with $\mathscr{O}_X(1)$. The Weil divisor identified with D is $\sum v_Y(\overline{x}_i) \cdot Y \in \operatorname{Cl} X$ where Y ranges over all prime divisors of X and $Y \cap D_+(\overline{x}_i) \neq \varnothing$. This is precisely the image $H.X$ described in (Ex. 6.2(a)). Since the diagram commutes for H, which is a generator of $\operatorname{Cl} Y$, it commutes for all of $\operatorname{Cl} Y$.

10. For notation, we let $F(X)$ be the free Abelian group generated by all coherent sheaves on X, and we let $H(X)$ be the kernel of the canonical projection $F(X) \to K(X)$.

(a) Since X is affine, every coherent sheaf is isomorphic to $\widetilde{M}$ for some finitely generated $k[x]$-module M. By Proposition 5.2(a), Corollary 5.5, and Proposition 5.6, the sequence

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0 \tag{4}$$

is exact if and only if the sequence

$$0 \longrightarrow \widetilde{M}' \longrightarrow \widetilde{M} \longrightarrow \widetilde{M}'' \longrightarrow 0$$

is exact. So we can think of $K(X)$ as the quotient of the free Abelian group generated by all the finitely generated $k[x]$-modules, by the subgroup generated by all expressions

$M - M' - M''$ whenever an exact sequence as in (4) exists. So we claim that $k[x]$ generates $K(X)$. By induction, the direct sum of n copies of $k[x]$ is equivalent to $n \cdot k[x]$ in $K(X)$ by the exact sequence

$$0 \longrightarrow k[x]^{n-1} \longrightarrow k[x]^n \longrightarrow k[x] \longrightarrow 0\,.$$

For a general finitely generated module M, there exists a finite free resolution of length ≤ 1

$$0 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0$$

where F_0 and F_1 are finitely generated free $k[x]$-modules by the structure theorem of finitely generated modules over a PID. Namely, $M \cong k[x]^r \oplus k[x]/\mathfrak{a}_1 \oplus \cdots \oplus k[x]/\mathfrak{a}_s$ for some r and proper principal ideals $\mathfrak{a}_i$. So we can take F_1 to be $r+s$ copies of $k[x]$, and then s copies of $k[x]$ for F_0, where the map $F_0 \to F_1$ is given by multiplication by generators of the $\mathfrak{a}_i$. Thus M is equivalent to some multiple of $k[x]$, and we have a surjective map $f\colon \mathbf{Z} \to K(X)$. Suppose that $\ker f \neq 0$, and let $n > 0$ be in $\ker f$. If $n > 1$, then $k[x]^n = 0$ in $K(X)$. Since $k[x]^n = k[x]^{n-1} + k[x]$ and $k[x]^{n-1} = (n-1)\cdot k[x]$ by (4), then $k[x] = -(n-1)\cdot k[x]$, which implies that $k[x] = 0$. So either $K(X) = 0$ or $K(X) \cong \mathbf{Z}$. In part (b), we will show that there is a surjective homomorphism $K(X) \to \mathbf{Z}$, so in fact $K(X) \cong \mathbf{Z}$.

(b) Let X be an integral scheme with function field K and generic point ξ. If the sequence

$$0 \longrightarrow \mathscr{F}' \longrightarrow \mathscr{F} \longrightarrow \mathscr{F}'' \longrightarrow 0$$

is exact, then $\dim_K \mathscr{F}_\xi = \dim_K \mathscr{F}'_\xi + \dim_K \mathscr{F}''_\xi$ because the stalk functor is exact. It follows that for any expression $a_1\mathscr{F}_1 + \cdots + a_n\mathscr{F}_n \in H(X)$, we have $\sum a_i = 0$. Thus $\gamma(\mathscr{F}) = 0$ if and only if $\dim_K \mathscr{F}_\xi = 0$. So there is a well-defined map $\operatorname{rank}\colon K(X) \to \mathbf{Z}$ defined by $\gamma(\mathscr{F}) \mapsto \dim_K \mathscr{F}_\xi$ and extending linearly. This map is a homomorphism because for two coherent sheaves $\mathscr{F}$ and $\mathscr{G}$ on X, $\gamma(\mathscr{F}) + \gamma(\mathscr{G}) = \gamma(\mathscr{F} \oplus \mathscr{G})$, and $\dim_K \mathscr{F} + \dim_K \mathscr{G} = \dim_K(\mathscr{F} \oplus \mathscr{G})_\xi$ (we can extend linearly to see that rank preserves addition of arbitrary linear combinations of coherent sheaves). Finally, rank is surjective because $n \cdot \mathcal{O}_X \mapsto n$ for all n.

(c) Let $i\colon Y \to X$ be the given closed immersion, and let $Z = X \setminus Y$. Let $\alpha\colon K(Y) \to K(X)$ be defined by $\gamma(\mathscr{F}) \mapsto \gamma(i_*\mathscr{F})$ where $\mathscr{F}$ is a coherent sheaf on Y, and extending linearly, and let $\beta\colon K(X) \to K(Z)$ be defined by $\gamma(\mathscr{F}) \mapsto \gamma(\mathscr{F}|_Z)$ where $\mathscr{F}$ is a coherent sheaf on X, and extending linearly.

We first claim that α and β are well-defined homomorphisms. The map α defines a homomorphism $F(Y) \to F(X)$ by (Ex. 5.5(b)) and (Ex. 5.5(c)). Composing with the projection $F(X) \to K(X)$, we get a homomorphism $F(Y) \to F(X)$. To see that α is a well-defined homomorphism, it is enough to check that if $\gamma(\mathscr{F}) = 0$ where $\mathscr{F}$ is a coherent sheaf on Y, then $\gamma(i_*\mathscr{F}) = 0$. This follows because an exact sequence of coherent sheaves on Y remains exact after applying the functor i_*. We can check this by passing to stalks and using (Ex. 1.2(c)). By (Ex. 1.19(a)), if $P \in Y$, then $\mathscr{F}_P = (i_*\mathscr{F})_P$, and if $P \notin Y$, then $(i_*\mathscr{F})_P = 0$.

Similarly, to see that β is a well-defined homomorphism, it is enough to check that if $\mathscr{F}$ is a coherent sheaf on X and $\gamma(\mathscr{F}) = 0$, then $\gamma(\mathscr{F}|_Z) = 0$. This follows from the fact that an exact sequence of coherent sheaves on X remains exact after restricting to Z, which we can check by passing to stalks.

Next we claim that $\ker\beta = \operatorname{image}\alpha$ and that β is surjective. By (Ex. 1.19(a)) and (Ex. 1.19(b)), the stalk of the coherent sheaf that $\beta(\alpha(\gamma(\mathscr{F})))$ represents, where $\mathscr{F}$ is a coherent sheaf on Y, at any point is 0. Extending linearly, we see that $\operatorname{image}\alpha \subseteq \ker\beta$.

Now suppose that $\mathscr{F}$ is a coherent sheaf on X and that $\beta(\gamma(\mathscr{F}))=0$, i.e., that the support of $\mathscr{F}$ is contained in Y. We wish to show that $\gamma(\mathscr{F}) \in \operatorname{image}\alpha$. We can reduce to the affine case, so that $X=\operatorname{Spec} A$, $\mathscr{F}=\widetilde{M}$, $Y=V(I)$ for some ideal $I\subseteq A$, and $\mathscr{I}_Y=\widetilde{I}$. We shall show that $I^nM=0$ for some n. Since M is finitely generated, we can show that there is some n so that I^n annihilates each of the generators, and then take the maximum of all such n, so reduce to the case that M is cyclic, i.e., so that $M=A/\operatorname{Ann} M$. From (Ex. 5.6(b)), $\operatorname{Supp}\mathscr{F}=V(\operatorname{Ann} M)$. This implies that $V(I)\supseteq V(\operatorname{Ann} M)$, so $\sqrt{I}\subseteq\sqrt{\operatorname{Ann} M}$. Since A is Noetherian, this implies that $I^n\subseteq\operatorname{Ann} M$ for some n, which gives the claim. So then we have a finite filtration

$$\mathscr{F}=\mathscr{F}_0\supseteq\mathscr{F}_1\supseteq\mathscr{F}_2\supseteq\cdots\supseteq\mathscr{F}_{n-1}\supseteq\mathscr{F}_n=0$$

where $\mathscr{F}_i=\mathscr{I}_Y^i\mathscr{F}$. Also,

$$\mathscr{I}_Y^i\mathscr{F}/\mathscr{I}_Y^{i+1}\mathscr{F}\cong\mathscr{I}_Y^i\mathscr{F}\otimes_{\mathcal{O}_X}\mathcal{O}_X/\mathscr{I}_Y=\mathscr{I}_Y^i\mathscr{F}\otimes_{\mathcal{O}_X}\mathcal{O}_Y,$$

so $\mathscr{F}_i/\mathscr{F}_{i+1}$ is an $\mathcal{O}_Y$-module for all i. Now suppose that $\gamma(\mathscr{F}_{i+1})\in\operatorname{image}\alpha$. First, the $\mathscr{F}_i$ are subsheaves of $\mathscr{F}$ and hence coherent $\mathcal{O}_X$-modules. Since $\mathscr{F}_i/\mathscr{F}_{i+1}$ is an $\mathcal{O}_Y$-module, which is coherent, we have $\gamma(\mathscr{F}_i/\mathscr{F}_{i+1})\in\operatorname{image}\alpha$, so $\gamma(\mathscr{F}_i)=\gamma(\mathscr{F}_{i+1})+\gamma(\mathscr{F}_i/\mathscr{F}_{i+1})\in\operatorname{image}\alpha$. By induction, we see that $\gamma(\mathscr{F})\in\operatorname{image}\alpha$.

A general element of $K(X)$ is of the form $\sum\gamma(\mathscr{F}_i)-\sum\gamma(\mathscr{G}_i)$. Taking $\mathscr{F}=\bigoplus\mathscr{F}_i$ and $\mathscr{G}=\bigoplus\mathscr{G}_i$, this element is equal to $\gamma(\mathscr{F})-\gamma(\mathscr{G})$. For notation, let $U=X\setminus Y$. If $\gamma(\mathscr{F})-\gamma(\mathscr{G})\in\ker\beta$, then $\gamma(\mathscr{F}|_U)-\gamma(\mathscr{G}|_U)=0$ in $K(U)$. We first prove the following lemma:

Lemma. If $\mathscr{F}$ is a coherent sheaf on U and $\mathscr{F}'$ and $\mathscr{F}''$ are two coherent sheaves on X which are extensions of $\mathscr{F}$, then $\gamma(\mathscr{F}')-\gamma(\mathscr{F}'')\in K(Y)$.

Proof. Let $j\colon U\hookrightarrow X$ be the open immersion. Consider the exact sequence

$$0\longrightarrow\mathscr{H}_Y^0(\mathscr{F}')\longrightarrow\mathscr{F}'\longrightarrow j_*j^*(\mathscr{F}')\longrightarrow\mathscr{H}_Y^1(\mathscr{F}')\longrightarrow 0\,,$$

where we are taking $\mathscr{H}_Y^1(\mathscr{F}')$ to be the cokernel of the previous map. By the exactness of j^*, we get the exact sequence

$$0\longrightarrow\mathscr{H}_Y^0(\mathscr{F}')|_U\longrightarrow\mathscr{F}\longrightarrow\mathscr{F}\longrightarrow\mathscr{H}_Y^1(\mathscr{F}')|_U\longrightarrow 0$$

where the map $\mathscr{F}\to\mathscr{F}$ is the identity since j_* and j^* form an adjoint pair. This means that $\operatorname{Supp}(\mathscr{H}_Y^0(\mathscr{F}'))$ and $\operatorname{Supp}(\mathscr{H}_Y^1(\mathscr{F}'))$ are contained in Y, so $(\mathscr{F}'/\mathscr{H}_Y^0(\mathscr{F}'))|_U\cong\mathscr{F}$, which means that $\mathscr{F}'/\mathscr{H}_Y^0(\mathscr{F}')$ is coherent and $\mathscr{H}_Y^0(\mathscr{F}'/\mathscr{H}_Y^0(\mathscr{F}'))=0$.

Replacing $\mathscr{F}'$ with $\mathscr{F}'/\mathscr{H}_Y^0(\mathscr{F}')$, we can assume $\mathscr{H}_Y^0(\mathscr{F}')=0$, and get an exact sequence

$$0\longrightarrow\mathscr{F}'\longrightarrow j_*\mathscr{F}\longrightarrow\mathscr{H}_Y^1(\mathscr{F}')\longrightarrow 0\,.$$

Now let $\mathscr{F}'+\mathscr{F}''$ be the image of $\mathscr{F}'\oplus\mathscr{F}''\to j_*\mathscr{F}$, which is coherent. Replacing $j_*(\mathscr{F})$ by $\mathscr{F}'+\mathscr{F}''$, we get a short exact sequence

$$0\longrightarrow\mathscr{F}'\longrightarrow\mathscr{F}'+\mathscr{F}''\longrightarrow(\mathscr{F}'+\mathscr{F}'')/\mathscr{F}'\longrightarrow 0 \tag{5}$$

Then $(\mathscr{F}'+\mathscr{F}'')/\mathscr{F}'\subseteq\mathscr{H}_Y^1(\mathscr{F}')$ and $\operatorname{Supp}(\mathscr{F}'+\mathscr{F}'')/\mathscr{F}'\subseteq Y$ implies that $\gamma((\mathscr{F}'+\mathscr{F}'')/\mathscr{F}')\in K(Y)$, and similarly we get $\gamma((\mathscr{F}'+\mathscr{F}'')/\mathscr{F}'')\in K(Y)$. By the exact sequence (5), we have

$$\gamma(\mathscr{F}')+\gamma((\mathscr{F}'+\mathscr{F}'')/\mathscr{F}')=\gamma(\mathscr{F}'+\mathscr{F}'')$$

and
$$\gamma(\mathscr{F}'') + \gamma((\mathscr{F}' + \mathscr{F}'')/\mathscr{F}'') = \gamma(\mathscr{F}' + \mathscr{F}''),$$
so $\gamma(\mathscr{F}') - \gamma(\mathscr{F}'') \in K(Y)$. □

Returning to the proof, we write
$$\mathscr{F}|_U + \sum_{i=1}^{n}(-S_i' - S_i'' + S_i) = \mathscr{G}|_U + \sum_{i=1}^{n}(-R_i' - R_i'' + R_i)$$
as formal symbols where we have short exact sequences of coherent sheaves on U
$$0 \longrightarrow S_i' \longrightarrow S_i \longrightarrow S_i'' \longrightarrow 0$$
and
$$0 \longrightarrow R_i' \longrightarrow R_i \longrightarrow R_i'' \longrightarrow 0\ .$$
Without loss of generality, we can assume that $\mathscr{F}|_U = R_1$ and $\mathscr{G}|_U = S_1$, and can take extensions to coherent sheaves on X to get exact sequences
$$0 \longrightarrow \bar{R}_i' \longrightarrow \mathscr{F} \longrightarrow \bar{R}_i'' \longrightarrow 0$$
and
$$0 \longrightarrow \bar{S}_i' \longrightarrow \mathscr{G} \longrightarrow \bar{S}_i'' \longrightarrow 0\ .$$
Then we can write
$$\mathscr{F} - \bar{S}_1' - \bar{S}_1'' + \bar{S}_1 = \mathscr{G} - \bar{R}_1' - \bar{R}_1'' + \bar{R}_1$$
as sheaves on X. The sheaves $\bar{R}_1'$ and $\bar{S}_1'$ restrict to the same sheaf on U, and so do $\bar{R}_1''$ and $\bar{S}_1''$, so we can write
$$\gamma(\mathscr{F}) - \gamma(\mathscr{G}) = \gamma(\bar{R}_1') - \gamma(\bar{S}_1') + \gamma(\bar{R}_1'') - \gamma(\bar{S}_1'').$$
By the lemma above, this implies that $\gamma(\mathscr{F}) - \gamma(\mathscr{G}) \in K(Y)$. So $\ker\beta = \operatorname{image}\alpha$ as claimed. The surjectivity of β follows from (Ex. 5.15) because Z is an open subset of X. Therefore, the sequence
$$K(Y) \xrightarrow{\alpha} K(X) \xrightarrow{\beta} K(X \setminus Y) \longrightarrow 0$$
is exact.

11. (d) Since $n\gamma(\mathcal{O}_X) = \gamma(\mathcal{O}_X^n)$, and the rank of $\mathcal{O}_X^n$ is n, $\operatorname{rank}\circ f$ is the identity on $\mathbf{Z}$. This implies that $f\colon \mathbf{Z} \to K(X)$ defined by $1 \mapsto \gamma(\mathcal{O}_X)$ is an injection. Now identify $\operatorname{Pic} X = \operatorname{Cl} X$. We claim that $\psi\colon \operatorname{Pic} X \to K(X)$ is an injection. Indeed, by (b), $\det\circ\psi$ is the identity on $\operatorname{Pic} X$ because any divisor D is identified with $\mathscr{L}(D)$ under the identification $\operatorname{Pic} X = \operatorname{Cl} X$.

Now we show that $\operatorname{image} f \cap \operatorname{image}\psi = 0$. Since any divisor D is linearly equivalent to an effective divisor, $\operatorname{image}\psi$ consists of elements of the form $\gamma(\mathcal{O}_D)$ where D is an effective divisor. Since $\mathcal{O}_D$ is the structure sheaf of a subscheme of X of codimension one, we cannot have $\gamma(\mathcal{O}_D) = \gamma(\mathcal{O}_X)$, which gives the claim. Finally, we show that their images generate $K(X)$. By (c), for any coherent sheaf $\mathscr{F}$ of rank r, we have $\gamma(\mathscr{F}) - r\gamma(\mathcal{O}_X) \in \operatorname{image}\psi$. By definition, $r\gamma(\mathcal{O}_x) \in \operatorname{image} f$, so we see that $\gamma(\mathscr{F})$ is generated by $\operatorname{image} f$ and $\operatorname{image}\psi$. The elements $\gamma(\mathscr{F})$ are generators for $K(X)$, so we have the claim.

From the above comments, we can identify $\operatorname{Pic} X$ and $\mathbf{Z}$ as subgroups of $K(X)$ such that $\operatorname{Pic} X + \mathbf{Z} = K(X)$ and $\operatorname{Pic} X \cap \mathbf{Z} = 0$, so we conclude that $K(X) \cong \operatorname{Pic} X \oplus \mathbf{Z}$.

12. Consider $K(X) \to \operatorname{Pic} X \to \mathbf{Z}$ where the first map is projection via the isomorphism $K(X) \cong \operatorname{Pic} X \oplus \mathbf{Z}$, and for the second map, we write an invertible sheaf as a Weil divisor $\sum n_i P_i$ and map it to $\sum n_i$. Let deg be the composition $K(X) \to \mathbf{Z}$ where $\deg \mathscr{F} = \deg \gamma(\mathscr{F})$. It is immediately clear from the definition of $K(X)$ that condition (3) is satisfied. From the definition of degree of a divisor, it is also clear that condition (1) is satisfied.

    If $\mathscr{F}$ is a torsion sheaf, then $\gamma(\mathscr{F}) = \gamma(\mathscr{O}_D)$ for some effective divisor $D = \sum n_i P_i$. The stalk of $\mathscr{O}_D$ at P_i is k^{n_i}, whose length as a k-module is n_i. We claim that this is also the length of k^{n_i} as an $\mathscr{O}_{P_i}$-module. Since k is algebraically closed, we have an embedding $k \hookrightarrow \mathscr{O}_{P_i}$ and the residue field of $\mathscr{O}_{P_i}$ is k. So a filtration of k^{n_i} as an $\mathscr{O}_{P_i}$-module can be extended to a k-filtration. On the other hand, a maximal k-filtration of k^{n_i} has simple quotients, and we claim that such a filtration remains simple over $\mathscr{O}_{P_i}$. To see this, let $M \cong \langle a \rangle$ be a simple nonzero module. Then it is isomorphic to $\mathscr{O}_{P_i} / \operatorname{Ann} a$. Since $\mathscr{O}_{P_i}$ is local, $\operatorname{Ann} a \subseteq \mathfrak{m}_P$, which means that $\mathfrak{m}_P / \operatorname{Ann} a$ must be 0 since it is a submodule of M. Hence, $M \cong \mathfrak{m}_P / \operatorname{Ann} a \cong k$, which gives the claim. Thus, $\deg(\mathscr{F}) = \sum n_i = \sum_{P \in X} \operatorname{length}(\mathscr{F}_P)$, so this function also satisfies condition (2).

    Finally, the degree function must be unique. To see why, we can check by induction on the rank of a sheaf. If a sheaf has rank 0, then it is a torsion sheaf, and condition (2) forces uniqueness of degree. For invertible sheaves of rank 1, condition (1) forces uniqueness. For all other sheaves, we can find an exact sequence as in (Ex. 6.11(c)) and then condition (3) forces uniqueness by induction.

7 Projective Morphisms

1. Let $P \in X$ be a point. The map on stalks $f_P \colon \mathscr{L}_P \to \mathscr{M}_P$ is surjective. By assumption, $\mathscr{L}$ and $\mathscr{M}$ are invertible sheaves, so $\mathscr{L}_P$ and $\mathscr{M}_P$ are isomorphic to $\mathscr{O}_{X,P}$ as $\mathscr{O}_{X,P}$-modules. Then f_P is given by multiplication by an element $x \in \mathscr{O}_{X,P}$. If x belongs to the maximal ideal of $\mathscr{O}_{X,P}$, then the image of f_P is the maximal ideal and hence cannot be surjective. Hence x is a unit, so f_P is an isomorphism. We conclude that f is an isomorphism.

4. (a) Suppose there is an invertible sheaf $\mathscr{L}$ on X such that $\mathscr{L}$ is ample. Then by Theorem 7.6, $\mathscr{L}^m$ is very ample over $\operatorname{Spec} A$ for some $m > 0$. In particular, this means that there is an immersion $i \colon X \to \mathbf{P}^r_A$ that factors through the map $X \to A$ such that $i^*(\mathscr{O}(1)) \cong \mathscr{L}^m$. By definition, i factors as $X \to Y \to \mathbf{P}^r_A$ where Y is a closed subscheme of $\mathbf{P}^r_A$ and $X \to Y$ is an open immersion. This means that the morphism $X \to \operatorname{Spec} A$ is quasi-projective, so by Theorem 4.9, X is separated over A.

   (b) Let U_1 and U_2 be the two copies of the affine line glued together to form X, and choose $\mathscr{L} \in \operatorname{Pic} X$. Then $\mathscr{L}$ restricted to each U_i is also an invertible sheaf, and hence must be trivial by Proposition 6.2. So every invertible sheaf on X is the result of taking the structure sheaf on U_1 and U_2 and gluing on their intersection. We know that $U_1 \cap U_2 \cong \operatorname{Spec} k[x, x^{-1}]$, so again by Proposition 6.2 the restriction of $\mathscr{L}$ to $U_1 \cap U_2$ is trivial. Thus, we are asking about automorphisms of the structure sheaf of $\operatorname{Spec} k[x, x^{-1}]$. By Corollary 5.5, these are equivalent to the $k[x, x^{-1}]$-module automorphisms of $k[x, x^{-1}]$, namely the automorphisms of the free module on one generator. These are all given by multiplication by a unit, and the units of $k[x, x^{-1}]$ are of the form ax^n where $a \in k^*$ and $x \in \mathbf{Z}$. It is immediate that two sheaves obtained by the automorphisms ax^n and bx^m are not isomorphic if $n \neq m$. If $n = m$, however, the sheaves differ by multiplication of a unit in $k[x]$, so there are natural isomorphisms on open sections of the sheaves. Thus, $\operatorname{Pic} X \cong \mathbf{Z}$. To fix notation, let $\mathscr{L}_n$ be the invertible sheaf given by the automorphism that is multiplication by x^n.

We claim that $\mathscr{L}_n$ is not generated by global sections if $n \neq 0$. In U_1 and U_2, we have two different points corresponding to the prime ideal (x). The restriction of $\mathscr{L}_n$ to one of U_1 and U_2 is identified with $\mathcal{O}_{\operatorname{Spec} k[x]}$ identically, say U_1, and the other is twisted by this automorphism. In fact, considering $\mathscr{L}_{-n}$ switches the roles of U_1 and U_2, so we can assume that $n > 0$. The global section of X is $k[x]$, and taking a set of generators for it will generate U_1 at the stalk of (x). However, for any set of global sections of $k[x]$ that we take, their images in the stalk of the prime ideal corresponding to (x) in U_2 will be multiplied by x^n, and thus cannot generate, because for example, we don't get x, and the stalk is the ring $k[x]_{(x)}$. So we have the claim.

Now let $\mathscr{F} = \mathscr{L}_1$ be a generator of $\operatorname{Pic} X$, and let $\mathscr{L}$ be an invertible sheaf. Then $\mathscr{F}$ is coherent, but $\mathscr{F} \otimes \mathscr{L}^{\otimes n}$ is not generated by its global sections from the discussion above. Thus, X admits no ample invertible sheaves.

8. **Exercise.** Let X be a Noetherian scheme, let $\mathscr{E}$ be a coherent locally free sheaf on X, and let $\pi\colon \mathbf{P}(\mathscr{E}) \to X$ be the corresponding projective space bundle. Show that there is a natural 1-1 correspondence between sections of π and quotient invertible sheaves $\mathscr{E} \to \mathscr{L} \to 0$ of $\mathscr{E}$.

   Solution. Let $f\colon Y \to X$ be the identity map with $Y = X$. Then to give a section of π, it is equivalent to give a morphism $Y \to \mathbf{P}(\mathscr{E})$ over X, which is equivalent to an invertible sheaf $\mathscr{L}$ on Y together with a surjection $f^*\mathscr{E} \to \mathscr{L}$ (Proposition 7.12). Since f is the identity, this is exactly a quotient invertible sheaf $\mathscr{E} \to \mathscr{L} \to 0$. □

8 Differentials

3. (a) From the following fiber diagram

$$\begin{tikzcd} X \times_S Y \arrow[r, "p_2"] \arrow[d, "p_1"] & Y \arrow[d] \\ X \arrow[r] & S \end{tikzcd}$$

and Proposition 8.10, we have $\Omega_{X\times_S Y/Y} \cong p_1^*\Omega_{X/S}$. By Proposition 8.11, there is an exact sequence of sheaves on $X \times_S Y$

$$p_2^*\Omega_{Y/S} \longrightarrow \Omega_{X\times_S Y/S} \longrightarrow \Omega_{X\times_S Y/Y} \longrightarrow 0\ .$$

We claim that the first map is injective and that this sequence splits naturally, which will give the desired isomorphism

$$\Omega_{X\times_S Y/S} \cong \Omega_{X\times_S Y/Y} \oplus p_2^*\Omega_{Y/S} \cong p_1^*\Omega_{X/S} \oplus p_2^*\Omega_{Y/S}.$$

To see this, it is enough to construct natural splittings locally, so choose $\operatorname{Spec} A \subseteq X$, $\operatorname{Spec} B \subseteq Y$, and $\operatorname{Spec} C \subseteq S$. Then we have ring homomorphisms $C \to B$ and $B \to A \otimes_C B$, so by Proposition 8.3A, there is an exact sequence of C-modules

$$\Omega_{B/C} \otimes_B (A \otimes_C B) \longrightarrow \Omega_{A\otimes_C B/C} \longrightarrow \Omega_{A\otimes_C B/B} \longrightarrow 0\ .$$

The first map has a left inverse if and only if any derivation d of B over C into any $(A \otimes_C B)$-module T can be extended to a derivation $A \otimes_C B \to T$ by the universal property of Ω. But this is clear because we can define $A \otimes_C B \to T$ by $a \otimes b \mapsto d(b)$ and extend linearly. Thus the exact sequence splits naturally, so we are done.

(b) From (a), we have the isomorphism

$$\Omega_{X\times_k Y/k} \cong p_1^*\Omega_{X/k} \oplus p_2^*\Omega_{Y/k}.$$

By (Ex. 5.16(d)) and (Ex. 5.16(e)), we have an isomorphism

$$\bigwedge^n \Omega_{X\times_k Y/k} \cong \bigwedge^{n'} p_1^*\Omega_{X/k} \otimes \bigwedge^{n''} p_2^*\Omega_{Y/k} \cong p_1^* \bigwedge^{n'} \Omega_{X/k} \otimes p_2^* \bigwedge^{n''} \Omega_{Y/k}$$

of their highest exterior powers. The canonical sheaf is defined to be the highest exterior power of the sheaf of differentials, so this translates to the desired isomorphism

$$\omega_{X\times_k Y} \cong p_1^*\omega_X \otimes p_2^*\omega_Y.$$

(c) By Example 8.20.3, $\omega_Y \cong \mathscr{O}_Y(d-n-1)$ where $d=3$ is the degree of Y and $n=2$ is the dimension of the ambient space. This means that $\omega_Y \cong \mathscr{O}_Y$ is trivial. Let p_1 and p_2 be the projections $X = Y \times Y \to Y$. By (Ex. 6.8(a)), p_1^* and p_2^* are homomorphisms $\operatorname{Pic} Y \to \operatorname{Pic} X$, so $p_1^*\omega_Y$ and $p_2^*\omega_Y$ are also trivial, which means that ω_X is trivial by (b). This means that $p_g(X) = \dim_k \Gamma(X, \omega_X) = 1$.

Since Y is a plane cubic curve, it has degree 3 by Proposition I.7.6(d). Using (I, Ex. 7.2(b)), we see that $p_a(Y) = \frac{1}{2}(3-1)(3-2) = 1$. Now by (I, Ex. 7.2(e)), $p_a(Y\times Y) = p_a(Y)^2 + 2(-1)^r p_a(Y)$ where r is the dimension of Y. Since Y is a curve, and hence has dimension 1, and thus $p_a(Y \times Y) = -1$.

8. Let X and X' be two birationally equivalent nonsingular projective varieties, and consider the rational map $X \to X'$. Let $V \subseteq X$ be the largest open set for which there is a morphism $f\colon V \to X'$ representing this rational map. Proposition 8.11 gives a map $f^*\Omega_{X'/k} \to \Omega_{V/k}$. Since they are both locally free sheaves of the same rank, there is an induced map on their highest exterior powers $f^*\omega_{X'} \to \omega_V$, and also for any arbitrary exterior power $f^*\Omega^q_{X'/k} \to \Omega^q_{V/k}$ by (Ex. 5.16(e)). Using that f^* commutes with $\otimes$ gives a map $f^*\omega_{X'}^{\otimes n} \to \omega_V^{\otimes n}$. Then we have induced maps on global sections

$$f^*\colon \Gamma(X', \omega_{X'}^{\otimes n}) \to \Gamma(V, \omega_V^{\otimes n})$$

and

$$f^*\colon \Gamma(X', \Omega^q_{X'/k}) \to \Gamma(V, \Omega^q_{V/k}).$$

By Corollary I.4.5, there is an open set $U \subseteq V$ such that $f(U)$ is open in X' and f induces an isomorphism from U to $f(U)$. A nonzero global section of a locally free sheaf $\mathscr{F}$ cannot vanish on a dense open set. To see this, let U be a dense open set and suppose a global section s vanishes on it. Then for any open affine V on which $\mathscr{F}$ is free, we can take a distinguished open of V in $U \cap V$, call it W. But the map on modules $\Gamma(V, \mathscr{F}) \to \Gamma(W, \mathscr{F})$ is injective because X is an integral scheme and this map is just localization, so $s = 0$ to start with. So the first maps of vector spaces above are injective. The proof that the natural restriction map $\Gamma(X, \omega_X^{\otimes n}) \to \Gamma(V, \omega_V^{\otimes n})$ is a bijection is similar to the one presented in the proof of Theorem 8.19 with $n = 1$, as is the proof that $\Gamma(X, \Omega^q_{X/k}) \to \Gamma(V, \Omega^q_{V/k})$ is a bijection. Thus, $P_n(X') \leq P_n(X)$ and $h^{q,0}(X') \leq h^{q,0}(X)$. By symmetry, we get the other inequalities, which means $P_n(X) = P_n(X')$ and $h^{q,0}(X') = h^{q,0}(X)$.

Solutions to *Algebraic Geometry*[*]

Steven V Sam

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Contents

2 Cohomology of Sheaves

3. (a) We first check that $\Gamma_Y(X, -)$ is a functor. Let $\varphi\colon \mathscr{F} \to \mathscr{G}$ be a morphism of sheaves on X. Pick $s \in \Gamma_Y(X, \mathscr{F})$ and $P \in \operatorname{Supp}(\varphi(s))$. Then $\varphi(s)_P \neq 0$ implies that $s_P \neq 0$, so $\operatorname{Supp}(\varphi(s)) \subseteq \operatorname{Supp}(s) \subseteq Y$, which means that we have a map $\Gamma_Y(X, \mathscr{F}) \to \Gamma_Y(X, \mathscr{G})$. It is immediate that $\Gamma_Y(X, -)$ is left exact because $\Gamma(X, -)$ is.

   (b) Let $U = X \setminus Y$. Pick $s \in \Gamma_Y(X, \mathscr{F}'')$. By (Ex. II.1.16(b)), there is $\widetilde{s} \in \Gamma(X, \mathscr{F})$ that maps to s. We know that $s|_U = 0$, so $\widetilde{s}|_U \in \Gamma(U, \mathscr{F}')$. Since $\mathscr{F}'$ is flasque, there exists $t \in \Gamma(X, \mathscr{F}')$ such that $t|_U = \widetilde{s}|_U$. Then $(\widetilde{s} - t)|_U = 0$, so $\widetilde{s} - t \in \Gamma_Y(U, \mathscr{F})$, and this element maps to s because $t \in \Gamma(X, \mathscr{F}')$.

   (c) Let $\mathscr{F}$ be flasque and embed it in an injective sheaf $\mathscr{I}$. Then $\mathscr{I}$ is flasque by Lemma 2.4, and $\mathscr{G} = \mathscr{F}/\mathscr{I}$ is flasque by (Ex. II.1.16(c)). Since $\mathscr{I}$ is injective and $H^i_Y(X, -)$ is a derived functor, $H^i_Y(X, \mathscr{I}) = 0$ for $i > 0$. Hence $H^1(X, \mathscr{F})$ is the cokernel of $\Gamma_Y(X, \mathscr{I}) \to \Gamma_Y(X, \mathscr{G})$, which is 0 by (b). The long exact sequence on cohomology then gives $H^i(X, \mathscr{F}) \cong H^{i-1}(X, \mathscr{G})$ for all $i \geq 2$, so by induction and the fact that $\mathscr{G}$ is flasque, we conclude that $H^i(X, \mathscr{F}) = 0$ for all $i > 0$.

[*] by Robin Hartshorne

(d) Denote $U = X \setminus Y$. Let φ and ψ denote the maps $\Gamma_Y(X, \mathscr{F}) \to \Gamma(X, \mathscr{F})$ and $\Gamma(X, \mathscr{F}) \to \Gamma(U, \mathscr{F})$, respectively, where φ is an inclusion and ψ is the restriction map. Since $\mathscr{F}$ is flasque, we get that ψ is surjective. Pick $s \in \Gamma_Y(X, \mathscr{F})$. Then $\operatorname{Supp}(\psi(s)) \subseteq Y$, which means that for any $P \in U$, there is a neighborhood $V \subseteq U$ containing P such that $\psi(s)|_V = 0$. This implies by the sheaf property that $\psi(s) = 0$. Hence $\psi\varphi = 0$. On the other hand, suppose $s \in \ker \psi$. Then $\psi(s)_P = 0$ for all $P \in U$, which means that $s_P = 0$ for all $P \in U$, and hence $s \in \Gamma_Y(X, \mathscr{F})$.

(e) Let $\mathscr{I}^\bullet$ be a flasque resolution for $\mathscr{F}$. Then $\mathscr{I}^\bullet|_U$ is a flasque resolution for $\mathscr{F}|_U$. That it is a resolution can be checked at the stalk level, and this is clear because $(\mathscr{F}|_U)_P = \mathscr{F}_P$ for all $P \in U$, and $(\mathscr{F}|_U)_P = 0$ for $P \notin U$ (for a general sheaf $\mathscr{F}$). Then by (d),

$$0 \longrightarrow \Gamma_Y(X, \mathscr{I}^\bullet) \longrightarrow \Gamma(X, \mathscr{I}^\bullet) \longrightarrow \Gamma(U, \mathscr{I}^\bullet|_U) \longrightarrow 0$$

is a short exact sequence of cochain complexes; the fact that the maps are given by inclusion and restriction give that the appropriate squares commute. The long exact sequence induced on cohomology is the desired long exact sequence of cohomology groups.

(f) The restriction map of $\mathscr{F}$ induces a map $\Gamma_Y(X, \mathscr{F}) \to \Gamma_Y(V, \mathscr{F}|_V)$ which is a natural isomorphism of functors. To see this, if $s \in \Gamma_Y(X, \mathscr{F})$, then its image in $\Gamma(V, \mathscr{F}|_V)$ will have support in $Y \cap V = Y$. Conversely, if $t \in \Gamma_Y(V, \mathscr{F}|_V)$, we know that $t|_W = 0$ where $W = V \cap (X \setminus Y)$. Hence we can lift t to an element $t' \in \Gamma(X, \mathscr{F})$ such that $t'|_V = t$ and $t|_{X \setminus Y} = 0$. The functor on the left is a universal δ-functor. The functor on the right is also a δ-functor by virtue of being a derived functor. Furthermore, the functor on the right is effaceable. Given a sheaf $\mathscr{F}$ on X, we can embed $\mathscr{F}$ into a flasque sheaf, and the restriction of a flasque sheaf to an open set remains flasque, which means its cohomology vanishes on V.

4. Let $\mathscr{I}^\bullet$ be a flasque resolution for $\mathscr{F}$. Then we define

$$0 \longrightarrow \Gamma_{Y_1 \cap Y_2}(X, \mathscr{I}^\bullet) \xrightarrow{\varphi} \Gamma_{Y_1}(X, \mathscr{I}^\bullet) \oplus \Gamma_{Y_2}(X, \mathscr{I}^\bullet) \xrightarrow{\psi} \Gamma_{Y_1 \cup Y_2}(X, \mathscr{I}^\bullet) \longrightarrow 0$$

where φ is the inclusion $s \mapsto (s, -s)$ and ψ is the map $(s, t) \mapsto s + t$. It is immediate that φ is injective and that we have exactness in the middle.

For surjectivity, pick $\alpha \in \Gamma_{Y_1 \cup Y_2}(X, \mathscr{I}^r)$. The sections $\alpha|_{X \setminus Y_2} \in \Gamma(X \setminus Y_2, \mathscr{I}^r)$ and $0 \in \Gamma(X \setminus Y_1, \mathscr{I}^r)$ agree on overlaps, so lift to a section $\alpha' \in \Gamma(X \setminus (Y_1 \cap Y_2), \mathscr{I}^r)$. Since $\mathscr{I}^r$ is flasque, we can lift this to an element $\alpha'' \in \Gamma(X, \mathscr{I}^r)$. Now the stalk of α'' outside of Y_1 is 0, so $\alpha'' \in \Gamma_{Y_1}(X, \mathscr{I}^r)$, and the stalk of α'' and α agree outside of Y_2, so $\alpha - \alpha'' \in \Gamma_{Y_2}(X, \mathscr{I}^r)$, so we see that ψ is surjective.

It is not hard to see that φ and ψ define maps of cochain complexes, and hence the long exact sequence on cohomology gives the desired Mayer–Vietoris sequence.

3 Cohomology of a Noetherian Affine Scheme

1. If X is affine, say $X = \operatorname{Spec} A$, then $X_{\text{red}} = \operatorname{Spec} A_{\text{red}}$ where A_{red} is A modulo its nilpotent elements. Suppose conversely that X_{red} is affine. Let $\mathscr{N}$ be the sheaf of nilpotent elements on X. Pick any coherent sheaf $\mathscr{F}$ on X. If we can show that $H^1(X, \mathscr{F}) = 0$, then X is affine by Theorem 3.7. Note that we have a filtration

$$\mathscr{F} \supseteq \mathscr{N} \cdot \mathscr{F} \supseteq \mathscr{N}^2 \cdot \mathscr{F} \supseteq \cdots,$$

and that for some r, $\mathscr{N}^r \cdot \mathscr{F} = 0$ because $\mathscr{F}$ is coherent. More precisely, we can look at a finite open covering and pick a high enough r such that each finite generating set of $\mathscr{F}$ on each open set is annihilated by $\mathscr{N}^r$. Also note that

$$\mathscr{N}^i \cdot \mathscr{F} / \mathscr{N}^{i+1} \cdot \mathscr{F} \cong (\mathscr{F}/\mathscr{N}^{i+1} \cdot \mathscr{F})/(\mathscr{F}/\mathscr{N}^i \cdot \mathscr{F}),$$

and that $\mathscr{F}/\mathscr{N}^i \cdot \mathscr{F} \cong \mathscr{F} \otimes_{\mathcal{O}_X} \mathcal{O}^i_{X_{\text{red}}}$, so each quotient in the filtration is a quasi-coherent sheaf on X_{red}. In particular, their first cohomology vanishes on X_{red}. But X and X_{red} are homeomorphic, and sheaf cohomology depends only on the topology of the scheme, so in fact these quotients have vanishing first cohomology group on X as well. Now working by induction, we can show that $H^1(X, \mathscr{F}) = 0$ by knowing that $\mathscr{N}^r \cdot \mathscr{F} = 0$ for sufficiently large r.

2. Every irreducible component of X is a scheme via the reduced closed subscheme structure. If $X = \operatorname{Spec} A$ is affine, each closed subscheme looks like $\operatorname{Spec} A/I$ for some ideal $I \subset A$, so each irreducible component is affine.

   Conversely, suppose each irreducible component is affine. By putting the reduced structure on each irreducible component and using induction, it is enough to prove that if $X = Y \cup Z$ where Y and Z are reduced closed subschemes such that Y and Z are affine, then X is affine. Let $\mathscr{I}$ and $\mathscr{J}$ be the sheaves of ideals of Y and Z, respectively, and let $\mathscr{F}$ be a coherent sheaf of ideals on X. Since $\mathscr{I} \cap \mathscr{J}$ is the sheaf of ideals associated to a closed subscheme whose topological space is X, and X is assumed to be reduced, $\mathscr{I} \cap \mathscr{J} = 0$ and hence $\mathscr{I} \cdot \mathscr{J} = 0$. Then $\mathscr{F} \cdot \mathscr{I}$ is annihilated by $\mathscr{J}$ so is a coherent sheaf on Z, which means that $H^1(Z, \mathscr{F} \cdot \mathscr{I}) = 0$. Similarly, $\mathscr{F}/\mathscr{F} \cdot \mathscr{I}$ is annihilated by $\mathscr{I}$, so is a coherent sheaf on Y and $H^1(Y, \mathscr{F}/\mathscr{F} \cdot \mathscr{I}) = 0$. The short exact sequence

$$0 \longrightarrow \mathscr{F} \cdot \mathscr{I} \longrightarrow \mathscr{F} \longrightarrow \mathscr{F}/\mathscr{F} \cdot \mathscr{I} \longrightarrow 0$$

   gives

$$H^1(X, \mathscr{F} \cdot \mathscr{I}) \longrightarrow H^1(X, \mathscr{F}) \longrightarrow H^1(X, \mathscr{F}/\mathscr{F} \cdot \mathscr{I}) .$$

   We can compute sheaf cohomology of $\mathscr{F} \cdot \mathscr{I}$ in Z and similarly, we can compute sheaf cohomology of $\mathscr{F}/\mathscr{F} \cdot \mathscr{I}$ in Y, so the vanishing of the two cohomology groups on the outside implies $H^1(X, \mathscr{F}) = 0$, which gives that X is affine.

3. (a) Let $f \colon M \to N$ be a map of A-modules. To see that $\Gamma_{\mathfrak{a}}(-)$ is a functor, it is enough to show that $f(\Gamma_{\mathfrak{a}}(M)) \subseteq \Gamma_{\mathfrak{a}}(N)$. But this is obvious: if there is an $n > 0$ such that $am = 0$ for all $a \in \mathfrak{a}^n$ and $m \in M$, then $af(m) = f(am) = 0$, so $f(m) \in \Gamma_{\mathfrak{a}}(N)$. The left exactness is also obvious.

   (b) We first find an isomorphism for $i = 0$. In this case, $H^0_{\mathfrak{a}}(M) = \Gamma_{\mathfrak{a}}(M)$ and $H^0_Y(X, \widetilde{M}) = \Gamma_Y(X, \widetilde{M})$. Since these can both be thought of as submodules of M, we just need to check that they contain the same elements. Given $m \in \Gamma_{\mathfrak{a}}(M)$, it is killed by some power of $\mathfrak{a}$. If a prime ideal $\mathfrak{p}$ does not contain $\mathfrak{a}$, then it does not contain any power of $\mathfrak{a}$, so in the localization $M_{\mathfrak{p}}$, m is killed. Hence $\mathfrak{p} \in V(\mathfrak{a})$ implies $\mathfrak{p} \notin \operatorname{Supp} m$, which means $m \in \Gamma_Y(X, \widetilde{M})$.

   Conversely, if $m \in \Gamma_Y(X, \widetilde{M})$, then m is in the kernel of the localization map $M \to M_{\mathfrak{p}}$ for all primes $\mathfrak{p}$ which do not contain $\mathfrak{a}$. Let $x_1, \ldots, x_n$ be generators for the ideal $\mathfrak{a}$. Then $\{D(x_i)\}$ is an open cover of $X \setminus Y$. The image of m in each $D(x_i)$ is 0 because of the sheaf property of $\widetilde{M}$, which means that m is annihilated by x_i^N for N sufficiently large and all i. Hence $m \in \Gamma_{\mathfrak{a}}(M)$.

Since $H^i_{\mathfrak{a}}(-)$ is a universal δ-functor on the category of A-modules, to show the isomorphism, it is enough to show that $H^i_Y(X, \widetilde{-})$ is an effaceable functor for all i. To do this, let N be any A-module and embed it into an injective A-module I. Then $\widetilde{I}$ is flasque (Proposition 3.4), and hence $H^i_Y(X, \widetilde{I}) = 0$ for all i (Ex. 2.3(c)).

(c) Let $M \to I^\bullet$ be an injective resolution of M. Then

$$H^i_{\mathfrak{a}}(M) = (\mathrm{image}(\Gamma_{\mathfrak{a}}(I^{i-1}) \to \Gamma_{\mathfrak{a}}(I^i)))/(\ker(\Gamma_{\mathfrak{a}}(I^i) \to \Gamma_{\mathfrak{a}}(I^{i+1}))),$$

and being a quotient of modules consisting of elements which are annihilated by some power of $\mathfrak{a}$, it itself also has that property.

4. (a) If $\operatorname{depth}_{\mathfrak{a}} M \geq 1$, then there exists an element $f \in \mathfrak{a}$ such that the map $M \to M$ given by multiplication by f is injective. Hence we cannot have $\mathfrak{a}^n m = 0$ for any nonzero $m \in M$, so $\operatorname{depth}_{\mathfrak{a}} M \geq 1$ implies that $\Gamma_{\mathfrak{a}}(M) = 0$.

Now assume that M is finitely generated and that $\Gamma_{\mathfrak{a}}(M) = 0$. Then $\mathfrak{a}$ is not contained in any associated prime of M. Indeed, if $\mathfrak{p} \supseteq \mathfrak{a}$ is an associated prime, then $\mathfrak{p}$ annihilates some nonzero element $m \in M$, and hence so does $\mathfrak{a}$. If $\operatorname{depth}_{\mathfrak{a}} M = 0$, then every element of $\mathfrak{a}$ annihilates some nonzero element of M. Hence $\mathfrak{a}$ is contained in the union of the associated primes of M, and so must be contained in at least one of them by prime avoidance. We conclude from this that if M is finitely generated and $\Gamma_{\mathfrak{a}}(M) = 0$, then $\operatorname{depth}_{\mathfrak{a}} M \geq 1$.

(b) Let T_n be the statement that $\operatorname{depth}_{\mathfrak{a}} M \geq n$ if and only if $H^i_{\mathfrak{a}}(M) = 0$ for all $i < n$. We prove by induction on n that T_n is true for all n. The case $n = 0$ is (a), so suppose it true for n and choose M with $\operatorname{depth}_{\mathfrak{a}} M \geq n + 1$. Let $x_1, \dots, x_{n+1} \in \mathfrak{a}$ be an M-regular sequence; we get a short exact sequence

$$0 \longrightarrow M \xrightarrow{\cdot x_1} M \longrightarrow M/x_1M \longrightarrow 0$$

which gives rise to a long exact sequence on cohomology

$$\cdots \longrightarrow H^{n-1}_{\mathfrak{a}}(M/x_1M) \longrightarrow H^n_{\mathfrak{a}}(M) \longrightarrow H^n_{\mathfrak{a}}(M) \longrightarrow \cdots.$$

The first term vanishes since $\operatorname{depth}_{\mathfrak{a}} M/x_1M \geq n$. Also, the map $H^n_{\mathfrak{a}}(M) \to H^n_{\mathfrak{a}}(M)$ is multiplication by x_1, which is not injective (Ex. 3.3(c)) if $H^n_{\mathfrak{a}}(M) \neq 0$, so we conclude that $H^n_{\mathfrak{a}}(M) = 0$. So $\operatorname{depth}_{\mathfrak{a}} M \geq n+1$ implies that $H^i_{\mathfrak{a}}(M) = 0$ for all $i < n+1$.

Conversely, suppose that $H^i_{\mathfrak{a}}(M) = 0$ for all $i < n + 1$. Then the long exact sequence on cohomology gives that $H^i_{\mathfrak{a}}(M/x_1M) = 0$ for all $i < n$. By induction, $\operatorname{depth}_{\mathfrak{a}} M/x_1M \geq n-1$, so $\operatorname{depth}_{\mathfrak{a}} M \geq n$. Hence T_{n+1} is also true.

5. First suppose that $\operatorname{depth} \mathcal{O}_P \geq 2$. By (Ex. 2.3(e)) and (Ex. 2.3(f)), there is a long exact sequence

$$H^0_P(U, \mathcal{O}|_U) \longrightarrow \Gamma(U, \mathcal{O}|_U) \longrightarrow \Gamma(U \setminus P, \mathcal{O}|_U) \longrightarrow H^1_P(U, \mathcal{O}|_U).$$

By (Ex. 2.5), we have natural isomorphisms

$$H^i_P(U, \mathcal{O}|_U) = H^i_P(U_P, j^*\mathcal{O}|_U)$$

where U_P is the local space of P in U and $j\colon U_P \to U$ is the inclusion. But $j^*\mathcal{O}|_U$ is a quasi-coherent sheaf which locally on a neighborhood of $Q \in U_P$ looks like $\widetilde{\mathcal{O}_Q}$. Replacing U_P with an affine neighborhood X of P relative to U_P, we get

$$H^i_P(U_P, j^*\mathcal{O}|_U) = H^i_P(X, \widetilde{\mathcal{O}_P})$$

(Ex. 2.3(f)). Now (Ex. 3.4(b)) shows that

$$H^0_P(U, \mathcal{O}|_U) = H^1_P(U, \mathcal{O}|_U) = 0,$$

which implies that the restriction map $\Gamma(U, \mathcal{O}|_U) \to \Gamma(U \setminus P, \mathcal{O}|_U)$ is an isomorphism.

Conversely, pick U to be an open affine neighborhood of P. Then the long exact sequence becomes

$$0 \longrightarrow H^0_P(U, \mathcal{O}|_U) \longrightarrow \Gamma(U, \mathcal{O}|_U) \longrightarrow \Gamma(U \setminus P, \mathcal{O}|_U) \longrightarrow H^1_P(U, \mathcal{O}|_U) \longrightarrow 0$$

because higher cohomology vanishes for affine schemes. So if $\Gamma(U, \mathcal{O}|_U) \to \Gamma(U \setminus P, \mathcal{O}|_U)$ is an isomorphism, then we know that

$$H^0_P(U, \mathcal{O}|_U) = H^1_P(U, \mathcal{O}|_U) = 0.$$

Using the above, we see that this is equivalent to $\operatorname{depth} \mathcal{O}_P \geq 2$.

6. (a) We appeal to the general fact that a functor that has an exact left adjoint preserves injective objects. Since the $\sim$ and $\Gamma(\operatorname{Spec} A_i, -)$ functors are an equivalence of categories between the category of A_i-modules and $\mathbf{Qco}(\operatorname{Spec} A_i)$, we get that $\widetilde{I_i}$ is an injective object in $\mathbf{Qco}(\operatorname{Spec} A_i)$. For an open immersion $f \colon U_i \to X$, the functor f_* is right adjoint to the exact functor f^{-1}, so also preserves injectives, which means $f_*(\widetilde{I_i})$ is injective in $\mathbf{Qco}(X)$. Finally, finite direct sums are the same as finite products, which is the categorical product, and a product of injectives is injective, so we conclude that $\mathscr{G}$ is an injective object in $\mathbf{Qco}(X)$.

   (b) Let $\mathscr{I}$ be an injective object of $\mathbf{Qco}(X)$. We can embed $\mathscr{I}$ in a sheaf $\mathscr{G}$ as in (a). Since $\mathscr{I}$ is injective, this inclusion splits, so $\mathscr{I}$ is a direct summand of $\mathscr{G}$. We know that $\mathscr{G}$ is flasque (Corollary 3.6), and a direct summand of a flasque sheaf is flasque, so $\mathscr{I}$ is flasque.

   (c) We have just shown that the forgetful functor from $\mathbf{Qco}(X)$ to $\mathbf{Mod}(X)$ is effaceable. It is obvious that the global sections functor is unaffected by an application of the forgetful functor, so we conclude that sheaf cohomology of quasi-coherent sheaves can be computed within $\mathbf{Qco}(X)$.

4 Čech Cohomology

1. Let $\mathfrak{U}$ be an open affine cover of Y. Since $f_*\mathscr{F}$ is quasi-coherent (Proposition II.5.8(c)), we have an isomorphism

$$\check{H}^p(\mathfrak{U}, f_*\mathscr{F}) \cong H^p(Y, f_*\mathscr{F})$$

for all $p \geq 0$ (Theorem 4.5). By definition, the Čech cohomology groups are the homology of the cochain complex with groups

$$C^p(\mathfrak{U}, f_*\mathscr{F}) = \prod_{i_0 < \cdots < i_p} \mathscr{F}(f^{-1}(U_{i_0,\ldots,i_p})),$$

which is equal to $C^p(f^{-1}(\mathfrak{U}), \mathscr{F})$ where $f^{-1}(\mathfrak{U})$ is the inverse images of all open affines in $\mathfrak{U}$. So we have isomorphisms

$$\check{H}^p(\mathfrak{U}, f_*\mathscr{F}) \cong \check{H}^p(f^{-1}(\mathfrak{U}), \mathscr{F})$$

for all $p \geq 0$, and using Theorem 4.5 on $f^{-1}(\mathfrak{U})$ and $\mathscr{F}$, we conclude that

$$H^i(X, \mathscr{F}) \cong H^i(Y, f_*\mathscr{F})$$

for all $i \geq 0$.

2. (a) The function field $K(X)$ is a finite field extension of $K(Y)$ (cf. (Ex. II.3.7)). Let $r = \dim_{K(Y)} K(X)$, and choose a basis $\{x_1, \ldots, x_r\}$ of $K(X)$ as a $K(Y)$-vector space. Now let $\mathscr{M}$ be the $\mathcal{O}_X$-submodule of $K(X)$ (as a constant sheaf) generated by $\{x_1, \ldots, x_r\}$, i.e., on each open affine $\operatorname{Spec} A$ of X, $\mathscr{M}|_{\operatorname{Spec} A} = \widetilde{M_A}$ where M_A is the submodule of $K(X)$ generated over A with generators $\{x_1, \ldots, x_r\}$. Then $\mathscr{M}$ is coherent. Since $\mathscr{M}$ is generated by the global sections $\{x_1, \ldots, x_r\}$, so is $f_*\mathscr{M}$, so we get a morphism $\alpha\colon \mathcal{O}_Y^r \to f_*\mathscr{M}$. Since f is surjective, the generic point of X maps to the generic point of Y. At the generic point of Y, $\mathcal{O}_Y^r$ is an r-dimensional vector space over $K(Y)$, as is $f_*\mathscr{M}$. From our description, the map at the stalk of the generic point is surjective, and hence an isomorphism.

   (b) Let $\alpha\colon \mathcal{O}_Y^r \to f_*\mathscr{M}$ be the morphism from (a). Apply the functor $\mathscr{H}om(-, \mathscr{F})$ to this map to get $\beta\colon \mathscr{H}om(f_*\mathscr{M}, \mathscr{F}) \to \mathscr{F}^r$. The sheaf $\mathscr{H}om(f_*\mathscr{M}, \mathscr{F})$ is a coherent $f_*\mathcal{O}_X$-module and f is an affine morphism, so $\mathscr{H}om(f_*\mathscr{M}, \mathscr{F}) \cong f_*\mathscr{G}$ for some coherent sheaf $\mathscr{G}$ on X (Ex. II.5.17(e)) (though the statement is for quasi-coherent sheaves, we can insert "finitely generated" within the proof to get the desired result).

   (c) Let $f\colon X \to Y$ be a finite surjective morphism of Noetherian separated schemes with X affine. There is an induced morphism $f_{\text{red}}\colon X_{\text{red}} \to Y_{\text{red}}$ (Ex. II.2.3), and X_{red} is affine (Ex. 3.1). If we can show that Y_{red} is affine, then Y is also affine (Ex. 3.1). By the fact the construction of f_{red}, it is surjective. It is not hard to see that if A is a finite B-module, then A_{red} is a finite B_{red}-module, so f_{red} is also a finite morphism. Finally, the reduced structure preserves the Noetherian and separated properties (Ex. II.4.8), so we may reduce to the case that both X and Y are reduced.

   Now let X' be an irreducible component of X. Then $f(X')$ is closed in Y (Ex. II.3.5(b)). Since f is surjective, $f(X')$ is also irreducible, and every irreducible component of Y is of the form $f(X')$ for some irreducible component X' of X. Hence, we get a map $f\colon X' \to f(X')$ which is also finite, surjective, and between Noetherian separated schemes. So we may reduce to the case that X and Y are also irreducible (Ex. 3.2). Combined with the above, we may assume that both X and Y are integral schemes.

   Now let $\mathscr{I}$ be a coherent sheaf of ideals on Y. By (b), there exists $\mathscr{G} \in \mathbf{Coh}(X)$ and a morphism $\beta\colon f_*\mathscr{G} \to \mathscr{I}^r$ for some $r > 0$ such that β is an isomorphism at the generic point of Y. Then $H^i(Y, f_*\mathscr{G}) = 0$ for all $i > 0$ by (Ex. 4.1) and (Theorem 3.7). We have short exact sequences

$$0 \longrightarrow \ker\beta \longrightarrow f_*\mathscr{G} \longrightarrow f_*\mathscr{G}/\ker\beta \longrightarrow 0$$

   and

$$0 \longrightarrow f_*\mathscr{G}/\ker\beta \longrightarrow \mathscr{I}^r \longrightarrow \operatorname{coker}\beta \longrightarrow 0\,.$$

   Note that $f_*\mathscr{G} \in \mathbf{Coh}(Y)$ (Ex. II.5.5(c)), so $\ker\beta$ and $\operatorname{coker}\beta$ are coherent sheaves on Y (Proposition II.5.7). Since β is an isomorphism at the generic point, both $\ker\beta$ and $\operatorname{coker}\beta$ vanish at the generic point, so there exists some nonempty open set containing the generic point for which $\ker\beta$ and $\operatorname{coker}\beta$ are not supported, let Z be the complement of this open set. Then we can compute sheaf cohomology of $\ker\beta$ and $\operatorname{coker}\beta$ as sheaves on Z and get the same result (Ex. 2.3(e)), so by Noetherian induction, $H^i(X, \ker\beta) = H^i(X, \operatorname{coker}\beta) = 0$ for all $i > 0$. Since $H^1(Y, f_*\mathscr{G}) = H^1(X, \mathscr{G}) = 0$ (Ex. 4.1) and (Theorem 3.7), the long exact sequence on cohomology gives $H^1(Y, f_*\mathscr{G}/\ker\beta) = 0$, and hence $H^1(Y, \mathscr{I}^r) = 0$. Finally, direct sum commutes with cohomology (Proposition 2.9), so $H^1(Y, \mathscr{I})^r = 0$, which implies that $H^1(Y, \mathscr{I}) = 0$. Therefore, Y is affine (Theorem 3.7).

3. Choose the open covering $\mathfrak{U}$ given by $U = D(x) \cup D(y)$. Then

$$\begin{aligned} \Gamma(D(x), \mathcal{O}) &\cong k[x^{\pm}, y], \\ \Gamma(D(y), \mathcal{O}) &\cong k[x, y^{\pm}], \\ \Gamma(D(x) \cap D(y), \mathcal{O}) &\cong k[x^{\pm}, y^{\pm}]. \end{aligned}$$

and the restriction maps are the natural inclusions. Hence the image of $C^0(\mathfrak{U}, U)$ in $C^1(\mathfrak{U}, U)$ is generated as a k-vector space by monomials $x^i y^j$ where at least one of i and j is nonnegative. Since $H^1(U, \mathcal{O}_U) \cong H^1(\mathfrak{U}, U)$ (Theorem 4.5), we conclude that it is isomorphic to the k-vector space spanned by $\{x^i y^j \mid i, j < 0\}$.

6. To check exactness, it is enough to check at the stalks, so pick $x \in X$. Then we get

$$0 \longrightarrow \mathscr{I}_x \xrightarrow{\varphi} \mathcal{O}^*_{X,x} \xrightarrow{\psi} \mathcal{O}^*_{X_0,x} \longrightarrow 0$$

and φ is given by $a \mapsto 1 + a$, which is a homomorphism because $\mathscr{I}^2 = 0$, and is clearly injective. This is also well-defined because the multiplicative inverse of $1 + a$ is $1 - a$. Now we check exactness in the middle. The image of φ is contained in the kernel of ψ because $\psi(1 + a) = 1$. Conversely, if $\psi(b) = 1$ for $b \in \mathcal{O}^*_{X,x}$, then we can write $b = 1 + (1 - b)$ where $1 - b \in \mathscr{I}_x$.

Finally, we need to check surjectivity of ψ. Pick $b \in \mathcal{O}^*_{X_0,x}$ with inverse c. Then we can find lifts $\widetilde{b}$ and $\widetilde{c}$ in $\mathcal{O}_X$, and $bc \mapsto 1$ under the projection $\mathcal{O}_{X,x} \to \mathcal{O}_{X_0,x}$. So $1 - \widetilde{b}\widetilde{c} = a$ for some $a \in \mathscr{I}_x$. Hence we have $\widetilde{b}\widetilde{c}(a - 1) = (1 + a)(1 - a) = 1$, which means $\widetilde{b} \in \mathcal{O}^*_X$.

The exact sequence of Abelian groups

$$\cdots \longrightarrow H^1(X, \mathscr{I}) \longrightarrow \operatorname{Pic} X \longrightarrow \operatorname{Pic} X_0 \longrightarrow H^2(X, \mathscr{I}) \longrightarrow \cdots$$

is an immediate consequence of the long exact sequence of cohomology groups and (Ex. 4.5).

8. (a) Let N be the least integer such that $H^i(X, \mathscr{G}) = 0$ for all coherent sheaves $\mathscr{G}$ and $i > N$. Now let $\mathscr{F}$ be an arbitrary quasi-coherent sheaf. We can write $\mathscr{F}$ as the union of its coherent subsheaves (Ex. II.5.15(e)), or in other words, as a direct limit of its coherent subsheaves. Since direct limits commute with cohomology (Proposition 2.9), we conclude that $H^i(X, \mathscr{F}) = 0$ for $i > N$, which means $N = \operatorname{cd}(X)$.

   (b) If X is quasi-projective over k, then we can find an open immersion $X \to Y$ with Y projective over k. Let N be the least integer such that $H^i(X, \mathscr{G}) = 0$ for all locally free coherent sheaves $\mathscr{G}$ and $i > N$. Now pick $\mathscr{F} \in \mathbf{Qco}(X)$. We can find $\mathscr{F}' \in \mathbf{Qco}(Y)$ such that $\mathscr{F}'|_X \cong \mathscr{F}$ (Ex. II.5.15). Then we can write $\mathscr{F}'$ as the quotient of a sheaf $\mathscr{E}$ which is the direct sum of twisted structure sheaves $\mathcal{O}(n_i)$ for some integers n_i (Corollary II.5.18); let $\mathscr{K}$ be the kernel. Since exactness of sheaves is a local property, we get a short exact sequence

$$0 \longrightarrow \mathscr{K}|_X \longrightarrow \mathscr{E}|_X \longrightarrow \mathscr{F} \longrightarrow 0\,.$$

   Also, a sheaf $\mathscr{G}$ is locally free if and only if its stalks $\mathscr{G}_x$ are free $\mathcal{O}_x$-modules for all $x \in X$ (Ex. II.5.7(b)), so $\mathscr{K}|_X$ is a coherent sheaf. We get a long exact sequence

$$\cdots \longrightarrow H^i(X, \mathscr{E}|_X) \longrightarrow H^i(X, \mathscr{F}) \longrightarrow H^{i+1}(X, \mathscr{K}|_X) \longrightarrow H^{i+1}(X, \mathscr{E}|_X) \longrightarrow \cdots,$$

   and if $i > N$, then the terms on the outside vanish, and hence $H^i(X, \mathscr{F}) \cong H^{i+1}(X, \mathscr{K}|_X)$. Applying Grothendieck vanishing and descending induction, we get that $H^i(X, \mathscr{F}) = 0$ for all $i > N$, so $\operatorname{cd}(X) = N$.

(c) Since Čech cohomology agrees with sheaf cohomology when X is a Noetherian separated scheme and the open covering $\mathfrak{U}$ consists of open affines, and $\mathscr{F}$ is a quasi-coherent sheaf (Theorem 4.5), we have $H^i(X, \mathscr{F}) \cong \check{H}^i(\mathfrak{U}, \mathscr{F})$. It is obvious that $\check{H}^i(\mathfrak{U}, \mathscr{F}) = 0$ for $i > r$ if $\mathfrak{U}$ consists of $r+1$ open affines, so $\operatorname{cd}(X) \le r$.

(d)

(e)

5 The Cohomology of Projective Space

1. The short exact sequence

$$0 \longrightarrow \mathscr{F}' \longrightarrow \mathscr{F} \longrightarrow \mathscr{F}'' \longrightarrow 0$$

induces a long exact sequence on cohomology

$$0 \longrightarrow H^0(X, \mathscr{F}') \longrightarrow H^0(X, \mathscr{F}) \longrightarrow H^0(X, \mathscr{F}'') \longrightarrow \cdots \longrightarrow H^d(X, \mathscr{F}'') \longrightarrow 0$$

where $d = \dim X$. Then the cohomology groups are finitely-generated k-vector spaces (Theorem 5.2), so the alternating sum of their dimensions is 0. This is equivalent to saying that

$$\chi(\mathscr{F}) = \chi(\mathscr{F}') + \chi(\mathscr{F}'').$$

7. (a) Choose $\mathscr{F} \in \mathbf{Coh}(Y)$. Then $i_*\mathscr{F} \in \mathbf{Coh}(X)$ (Ex. II.5.5(c)). We know that

$$H^q(Y, \mathscr{F} \otimes (i^*\mathscr{L})^n) = H^q(X, i_*(\mathscr{F} \otimes (i^*\mathscr{L})^n))$$

for all $q \ge 0$ (Ex. 4.1). Also, $(i^*\mathscr{L})^n \cong i^*(\mathscr{L}^n)$ (Ex. II.6.8(a)), and $i_*(\mathscr{F} \otimes i^*(\mathscr{L}^n)) \cong i_*\mathscr{F} \otimes \mathscr{L}^n$ (Ex. II.5.1(d)), so we get

$$H^q(Y, \mathscr{F} \otimes (i^*\mathscr{L})^n) \cong H^q(X, i_*\mathscr{F} \otimes \mathscr{L}^n).$$

By Proposition 5.3, $\mathscr{L}$ is ample.

(b) Let $f\colon X_{\text{red}} \to X$ be the canonical map. Then $f^*\mathscr{L} = f^{-1}\mathscr{L} \otimes_{f^{-1}\mathcal{O}_X} \mathcal{O}_{X_{\text{red}}} = \mathscr{L}_{\text{red}}$, so by (a), if $\mathscr{L}$ is ample, then so is $\mathscr{L}_{\text{red}}$.
Conversely, suppose that $\mathscr{L}_{\text{red}}$ is ample. Let $\mathscr{N}$ be the sheaf of nilpotent elements on X. Then there exists r such that $\mathscr{N}^r = 0$ since X is Noetherian. Choose $\mathscr{F} \in \mathbf{Coh}(X)$. Then $\mathscr{N}^i \cdot \mathscr{F} / \mathscr{N}^{i+1} \cdot \mathscr{F} \in \mathbf{Coh}(X_{\text{red}})$ since it is the quotient of two coherent sheaves. Then there is an integer N_i such that

$$H^q(X_{\text{red}}, \mathscr{N}^i \cdot \mathscr{F} / \mathscr{N}^{i+1} \cdot \mathscr{F} \otimes_{\mathcal{O}_{X_{\text{red}}}} \mathscr{L}_{\text{red}}^n) = 0$$

for all $n \ge N_i$ and $q > 0$ (Proposition 5.3). Let $n_0 = \max\{N_0, \ldots, N_{r-1}\}$. Since $\mathscr{L}_{\text{red}} = \mathscr{L} \otimes_{\mathcal{O}_X} \mathcal{O}_{X_{\text{red}}}$, it follows that

$$\mathscr{N}^i \cdot \mathscr{F} / \mathscr{N}^{i+1} \cdot \mathscr{F} \otimes_{\mathcal{O}_{X_{\text{red}}}} \mathscr{L}_{\text{red}}^n \cong \mathscr{N}^i \cdot \mathscr{F} / \mathscr{N}^{i+1} \cdot \mathscr{F} \otimes_{\mathcal{O}_X} \mathscr{L}^n.$$

The latter is a sheaf on X, and since sheaf cohomology depends only on the topological space $X \cong X_{\text{red}}$, we conclude that

$$H^q(X_{\text{red}}, \mathscr{N}^i \cdot \mathscr{F} / \mathscr{N}^{i+1} \cdot \mathscr{F} \otimes_{\mathcal{O}_X} \mathscr{L}^n) = 0$$

for $n \geq n_0$ and $q > 0$. We have a short exact sequence

$$0 \longrightarrow \mathscr{N}^{i+1} \cdot \mathscr{F} \longrightarrow \mathscr{N}^i \cdot \mathscr{F} \longrightarrow \mathscr{N}^i \cdot \mathscr{F} / \mathscr{N}^{i+1} \cdot \mathscr{F} \longrightarrow 0$$

which becomes

$$0 \longrightarrow \mathscr{N}^{i+1} \cdot \mathscr{F} \otimes \mathscr{L}^n \longrightarrow \mathscr{N}^i \cdot \mathscr{F} \otimes \mathscr{L}^n \longrightarrow \mathscr{N}^i \cdot \mathscr{F} / \mathscr{N}^{i+1} \cdot \mathscr{F} \otimes \mathscr{L}^n \longrightarrow 0$$

because locally free sheaves are flat. To finish, we take the long exact sequence on cohomology and use descending induction on i to get that

$$H^q(X, \mathscr{F} \otimes \mathscr{L}^n) = 0$$

for all $n \geq n_0$ and $q > 0$ (the base case is $i > r$), so $\mathscr{L}$ is ample (Proposition 5.3).

(c) One direction follows from (a): if $\mathscr{L}$ is an ample sheaf on X, then for the inclusion $j_i \colon X_i \to X$ where X_i is an irreducible component of X, $j_i^* \mathscr{L} = \mathscr{L} \otimes \mathcal{O}_{X_i}$ is ample on X_i.

Suppose conversely that $j_i^* \mathscr{L}$ is ample for each inclusion $j_i \colon X_i \to X$ where the X_i are the irreducible components of X. To show that $\mathscr{L}$ is ample on X, we can proceed by induction on the number of irreducible components of X by putting the reduced structure on the irreducible components. Hence we need only prove that if $X = Y \cup Z$ with $f \colon Y \to X$ and $g \colon Z \to X$ and $f^* \mathscr{L}$ and $g^* \mathscr{L}$ are ample, then $\mathscr{L}$ is ample. Let $\mathscr{I}$ and $\mathscr{J}$ be the sheaves of ideals of Y and Z, respectively. The sheaf $\mathscr{I} \cap \mathscr{J}$ is the sheaf of ideals of a closed subscheme of X with the same underlying topological space as X. Since we assume X is reduced, $\mathscr{I} \cap \mathscr{J} = 0$, and hence $\mathscr{I} \cdot \mathscr{J} = 0$. Choose $\mathscr{F} \in \mathbf{Coh}(X)$. Then $\mathscr{F} \cdot \mathscr{I}$ is killed by $\mathscr{J}$, so is a coherent sheaf on Z. Hence there is some N_1 such that

$$H^q(Z, (\mathscr{F} \cdot \mathscr{I}) \otimes (g^* \mathscr{L})^n) = 0$$

for all $q > 0$ and $n \geq N_1$. Now consider the sheaf $\mathscr{G} = \mathscr{F} / \mathscr{F} \cdot \mathscr{I}$. It is killed by $\mathscr{I}$, so is a coherent sheaf on Y, so there is a N_2 such that

$$H^q(Y, \mathscr{G} \otimes (f^* \mathscr{L})^n) = 0$$

for all $q > 0$ and $n \geq N_2$. Now take $n_0 = \max(N_1, N_2)$. The short exact sequence

$$0 \longrightarrow \mathscr{F} \cdot \mathscr{I} \longrightarrow \mathscr{F} \longrightarrow \mathscr{G} \longrightarrow 0$$

gives rise to a long exact sequence on cohomology

$$\cdots \longrightarrow H^q(X, \mathscr{F} \cdot \mathscr{I} \otimes \mathscr{L}^n) \longrightarrow H^q(X, \mathscr{F} \otimes \mathscr{L}^n) \longrightarrow H^q(X, \mathscr{G} \otimes \mathscr{L}^n) \longrightarrow \cdots.$$

The terms on the outside can be computed over Z and Y (using $f^* \mathscr{L}$ and $g^* \mathscr{L}$ instead of $\mathscr{L}$), respectively (Lemma 2.10), so they vanish for $q > 0$ and $n \geq n_0$. Hence the same is true for the middle term, and we conclude that $\mathscr{L}$ is ample on X.

(d) The proof of (a) shows that if $\mathscr{L}$ is ample on Y, then $f^* \mathscr{L}$ is ample on X.

Suppose conversely that $f^* \mathscr{L}$ is ample on X. By (b) and (c), we may assume that X and Y are integral schemes: f finite implies that it is closed, so an irreducible component of X maps to a closed subset of Y. In fact, this closed subset of Y is irreducible because f is also surjective. Let $\mathscr{F} \in \mathbf{Coh}(Y)$. Then there exists $\mathscr{G} \in \mathbf{Coh}(X)$ and a morphism

$\beta\colon f_*\mathscr{G} \to \mathscr{F}^r$ where r is the dimension of $K(X)$ over $K(Y)$ and β is an isomorphism at the generic point (Ex. 4.2(b)). Then as in (a),

$$H^q(X, \mathscr{G} \otimes (f^*\mathscr{L})^n) \cong H^q(Y, f_*\mathscr{G} \otimes \mathscr{L}^n),$$

and in particular, there exists n_0 such that both sides vanish for $n \geq n_0$ and $q > 0$ (Proposition 5.3). Since $f_*\mathscr{G} \in \mathbf{Coh}(Y)$ (Ex. II.5.5(c)), we conclude that $\ker\beta$, $f_*\mathscr{G}/\ker\beta$, and $\operatorname{coker}\beta$ are coherent sheaves on Y (Proposition II.5.7). Since β is an isomorphism at the generic point, it an isomorphism on some open set containing the generic point, so $\ker\beta$ and $\operatorname{coker}\beta$ are supported on proper closed subsets. By Noetherian induction and using the long exact sequence on cohomology of the two exact sequences

$$0 \longrightarrow \ker\beta \longrightarrow f_*\mathscr{G} \longrightarrow \operatorname{image}\beta \longrightarrow 0$$

and

$$0 \longrightarrow \operatorname{image}\beta \longrightarrow \mathscr{F}^r \longrightarrow \operatorname{coker}\beta \longrightarrow 0\ ,$$

we conclude that

$$H^q(Y, \mathscr{F}^r \otimes \mathscr{L}^n) = 0$$

for $n \geq n_0$ and $q > 0$. Finally, we use that tensor product and cohomology commute with direct sums (Proposition 2.9) to get

$$H^q(Y, \mathscr{F} \otimes \mathscr{L}^n) = 0$$

for $n \geq n_0$ and $q > 0$. Therefore, $\mathscr{L}$ is ample on Y (Proposition 5.3).

10. For each i, we get a short exact sequence

$$0 \longrightarrow \operatorname{image}(F^{i-1} \to F^i) \longrightarrow F^i \longrightarrow F^{i+1}/\operatorname{image}(F^i \to F^{i+1}) \longrightarrow 0$$

where we say that $F^0 = F^{r+1} = 0$. By Theorem 5.2(b), we can find an n_0 such that for all $n \geq n_0$, higher cohomology of the nth twists of each term above vanishes. Thus,

$$\Gamma(X, \mathscr{F}^1(n)) \longrightarrow \Gamma(X, \mathscr{F}^2(n)) \longrightarrow \cdots \longrightarrow \Gamma(X, \mathscr{F}^r(n))$$

is exact for all $n \geq n_0$.

6 Ext Groups and Sheaves

4. Choose $\mathscr{F} \in \mathbf{Coh}(X)$. By assumption, there exists a surjection $\mathscr{L} \to \mathscr{F}$ where $\mathscr{L}$ is a locally free sheaf on X. Then

$$\mathscr{E}xt^i(\mathscr{L}, \mathscr{G})_x \cong \operatorname{Ext}^i_{\mathcal{O}_{X,x}}(\mathscr{L}_x, \mathscr{G}_x)$$

for all $x \in X$ and $i > 0$ (Proposition 6.8). The right hand side is 0 since $\mathscr{L}_x$ is a free $\mathcal{O}_{X,x}$-module (Ex. II.5.7(b)). So $\mathscr{E}xt^i(\mathscr{L}, \mathscr{G}) = 0$ for all $i > 0$. Hence $\mathscr{E}xt^i(-, \mathscr{G})$ is coeffaceable for all $i > 0$, which means that $(\mathscr{E}xt^i(-, \mathscr{G}))_{i \geq 0}$ is a universal δ-functor $\mathbf{Coh}(X)^{\mathrm{op}} \to \mathbf{Mod}(X)$.

7. We first show that the functors $\operatorname{Ext}^i_X(\widetilde{M}, \widetilde{-})$ and $\operatorname{Ext}^i_A(M, -)$ agree. For $i = 0$, this follows from Corollary II.5.5. Given an injective A-module I, the quasi-coherent sheaf $\widetilde{I}$ is injective in the

category **Qco**(X). Hence $\mathscr{E}xt^i_X(\widetilde{M},\widetilde{I}) = 0$ for $i > 0$, which means the first functor is effaceable, so we get the desired claim. In particular, this implies that for M and N finitely generated,

$$\operatorname{Ext}^i_X(\widetilde{M},\widetilde{N}) \cong \operatorname{Ext}^i_A(M,N).$$

Now we consider the functors $\mathscr{E}xt^i_X(\widetilde{M},\widetilde{-})$ and $\operatorname{Ext}^i_A(M,-)^\sim$. Again for $i = 0$, both are isomorphic to $\operatorname{Hom}_A(M,-)^\sim$ (Corollary II.5.5). The second functor is effaceable since $\operatorname{Ext}^i_A(M,I) = 0$ for $i > 0$ whenever I is injective. Using Lemma 6.1, $\mathscr{E}xt^i_X(\widetilde{M},\widetilde{I}) = 0$ for $i > 0$ and I injective, so the first functor is also effaceable. So we get an isomorphism

$$\mathscr{E}xt^i_X(\widetilde{M},\widetilde{N}) \cong \operatorname{Ext}^i_A(M,N)^\sim.$$

7 The Serre Duality Theorem

3. If $r = 0$, then the statement is true because the higher cohomology of the structure sheaf vanishes. For $n > r > 0$, first consider the short exact sequence of Theorem II.8.13:

$$0 \longrightarrow \Omega_{X/k} \longrightarrow \mathcal{O}_X(-1)^{n+1} \longrightarrow \mathcal{O}_X \longrightarrow 0\ .$$

Taking the rth exterior power we get a filtration (Ex. II.5.16(d))

$$\bigwedge^r \mathcal{O}_X(-1)^{n+1} = F^0 \supseteq F^1 \supseteq \cdots \supseteq F^r \supseteq F^{r+1} = 0$$

whose quotients are

$$F^p/F^{p+1} \cong \bigwedge^p \Omega_{X/k} \otimes \bigwedge^{r-p} \mathcal{O}_X.$$

Since $\mathcal{O}_X$ is rank 1, this quotient is 0 when $r - p \geq 2$, or equivalently, $p \leq r-2$. Hence the filtration looks like

$$\bigwedge^r \mathcal{O}_X(-1)^{n+1} = F^0 = F^1 = \cdots = F^{r-1} \supseteq F^r \supseteq F^{r+1} = 0,$$

so we have a short exact sequence

$$0 \longrightarrow F^r \longrightarrow F^{r-1} \longrightarrow F^{r-1}/F^r \longrightarrow 0$$

which translates to

$$0 \longrightarrow \Omega^r_{X/k} \longrightarrow \bigwedge^r \mathcal{O}_X(-1)^{n+1} \longrightarrow \Omega^{r-1}_{X/k} \longrightarrow 0\ .$$

The term in the middle has vanishing ith cohomology for $0 < i \leq n$ (Theorem 5.1(b,c)) and also for $i = 0$ because it has no global sections, so by considering the long exact sequence on cohomology, we get

$$H^{q-1}(X,\Omega^{r-1}_{X/k}) \cong H^q(X,\Omega^r_{X/k})$$

for all $1 \leq q < n$. So by induction, $H^q(X,\Omega^r_{X/k})$ is 0 if $r \neq q$ and k if $r = q$ for $0 \leq q \leq n$.

8 Higher Direct Images of Sheaves

2. By Corollary 3.6, we can embed $\mathscr{F}$ into a flasque, quasi-coherent sheaf $\mathscr{G}$. The short exact sequence
$$0 \longrightarrow \mathscr{F} \longrightarrow \mathscr{G} \longrightarrow \mathscr{G}/\mathscr{F} \longrightarrow 0$$
gives the long exact sequence
$$0 \longrightarrow f_*\mathscr{F} \longrightarrow f_*\mathscr{G} \longrightarrow f_*(\mathscr{G}/\mathscr{F}) \longrightarrow R^1 f_*\mathscr{F} \longrightarrow 0$$
and that $R^i f_*\mathscr{F} \cong R^{i-1} f_*(\mathscr{G}/\mathscr{F})$ for all $i > 1$ (Corollary 8.3). Since $\mathscr{G}/\mathscr{F}$ is quasi-coherent, by induction it will suffice to show that $R^1 f_*\mathscr{F} = 0$, or equivalently, that $f_*\mathscr{G} \to f_*(\mathscr{G}/\mathscr{F})$ is surjective. Pick an open set $U \subseteq Y$. Then to check surjectivity, we need only check local surjectivity in the sense of (Ex. II.1.3(a)), so we may as well assume that U is affine. Then the map $\Gamma(U, f_*\mathscr{G}) \to \Gamma(U, f_*(\mathscr{G}/\mathscr{F}))$ can be rewritten as $\Gamma(f^{-1}(U), \mathscr{G}) \to \Gamma(f^{-1}(U), \mathscr{G}/\mathscr{F})$. By assumption, $f^{-1}(U) = \operatorname{Spec} A$ is affine, so surjectivity of this map follows from Corollary II.5.5 and Proposition II.5.6 because it is equivalent to the projection of modules $M \to M/N$ where M and N are A-modules such that $\widetilde{M} = \mathscr{G}|_{f^{-1}(U)}$ and $\widetilde{N} = \mathscr{F}|_{f^{-1}(U)}$.

3. We consider the functors $R^i f_*(- \otimes f^*\mathscr{E})$ and $R^i f_*(-) \otimes \mathscr{E}$. That the two agree for $i = 0$ is the content of (Ex. II.5.1(d)). Pick an open affine $U \subseteq Y$ such that $\mathscr{E}|_U$ is free. It is enough to find natural isomorphisms on U (replacing X with $f^{-1}(U)$) and then glue them together to get an isomorphism on Y. So we may assume that $\mathscr{E} = \mathcal{O}_Y^n$ for some n. Then $f^*\mathscr{E} = f^*\mathcal{O}_Y^n = \mathcal{O}_X^n$. Tensoring with $\mathcal{O}_Y^n$ is exact, so $R^i f_*(-) \otimes \mathscr{E}$ is a δ-functor. Also, the first functor becomes $\mathscr{F} \mapsto R^i f_*(\mathscr{F}^n)$, so is also a δ-functor because $\mathscr{F} \mapsto \mathscr{F}^n$ is an exact functor. If $\mathscr{F}$ is flasque, then $R^i(\mathscr{F}) \otimes \mathscr{E} = 0$ for $i > 0$ (Corollary 8.3), and $R^i f_*(\mathscr{F}^n) = 0$ for $i > 0$ because $\mathscr{F}^n$ will also be flasque. Hence both functors are effaceable, so are isomorphic functors. Thus we conclude the projection formula
$$R^i f_*(\mathscr{F} \otimes f^*\mathscr{E}) \cong R^i f_*(\mathscr{F}) \otimes \mathscr{E}$$
on U, and since this isomorphism is natural (given by natural isomorphism of functors), it globalizes to all of Y.

9 Flat Morphisms

1. Let U be an open set in X. Then $f(U)$ is a constructible set in Y (Ex. II.3.19). To see that $f(U)$ is open, it is enough to show that it is stable under generization (Ex. II.3.18(c)), i.e., if $y \in f(U)$ and $y \in \overline{\{y'\}}$, then $y' \in f(U)$ where $\overline{Z}$ denotes the closure of Z. A generization y' of y is nothing more than a prime ideal in $\operatorname{Spec} \mathcal{O}_{Y,y}$ since $\overline{\{y'\}} = V(y')$ in the affine case. Pick $x \in f^{-1}(y)$, then to see that $f(U)$ is closed under generization, it is enough to show that there exists $x' \in \operatorname{Spec} \mathcal{O}_{X,x}$ such that $f(x') = y'$. So we may assume that $X = \operatorname{Spec} \mathcal{O}_{X,x}$ and $Y = \operatorname{Spec} \mathcal{O}_{Y,y}$, and the surjectivity of $\operatorname{Spec} \mathcal{O}_{X,x} \to \operatorname{Spec} \mathcal{O}_{Y,y}$ is a consequence of the going-down property of flat maps.

10. (a) Let $X = \mathbf{P}^1_k$ and $\mathscr{T}_X = \mathscr{H}om_{\mathcal{O}_X}(\Omega^1_{X/k}, \mathcal{O}_X)$ be the tangent sheaf on X. We have an exact sequence
$$0 \longrightarrow \Omega^1_{X/k} \longrightarrow \mathcal{O}_X(-1) \oplus \mathcal{O}_X(-1) \longrightarrow \mathcal{O}_X \longrightarrow 0$$
(Theorem II.8.13), so applying the functor $\mathscr{H}om_{\mathcal{O}_X}(-, \mathcal{O}_X)$, we get a short exact sequence
$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X(1) \oplus \mathcal{O}_X(1) \longrightarrow \mathscr{T}_X \longrightarrow 0\,.$$

By Grothendieck vanishing (Theorem 2.7), the long exact sequence on cohomology gives a surjection
$$H^1(X, \mathcal{O}_X(1) \oplus \mathcal{O}_X(1)) \to H^1(X, \mathscr{T}_X).$$
We claim that $H^1(X, \mathcal{O}_X(1) \oplus \mathcal{O}_X(1)) = 0$. Since cohomology commutes with direct sums (Proposition 2.9), it is enough to compute $H^1(X, \mathcal{O}_X(1))$. Using the Čech complex, one can deduce in general that $H^1(X, \mathcal{O}_X(n))$ is the space of monomials of $S_{x,y}$ (the ring S with x and y inverted, where $S = k[x, y]$) of degree n modulo the monomials $x^a y^b$ where $a + b = n$ and either $a \geq 0$ or $b \geq 0$. In the case $n = 1$, we get $H^1(X, \mathcal{O}_X(1)) = 0$. Hence $H^1(X, \mathscr{T}_X) = 0$, so $\mathbf{P}^1_k$ has no infinitesimal deformations (Example 9.13.2), so is rigid.

(b)

(c)

Solutions to *Algebraic Geometry*[*]

Steven V Sam

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Contents

1 Riemann–Roch Theorem

1. Let g be the genus of X. If $n > 2g-2$, then $\dim_k H^0(X, \mathscr{L}(nP)) = n+1-g$, so for n sufficiently large, $H^0(X, \mathscr{L}(nP))$ has nonzero sections. Also,

$$H^0(X, \mathscr{L}(nP)) = \{f \in K(X) \mid v_P(f) \geq -n,\ v_Q(f) \geq 0 \text{ for } Q \neq P\},$$

which can be seen from the bijection between effective divisors linearly equivalent to nP and the projective space on $H^0(X, \mathscr{L}(D))$ (cf. Proposition II.7.7). So we can pick nonconstant $f \in H^0(X, \mathscr{L}(nP))$ so that f has a pole at P and poles nowhere else. To ensure that f is nonconstant, we can take n so that $\dim_k H^0(X, \mathscr{L}(nP)) > 1$.

2. For each P_i, by (Ex. 1.1), we can find a nonconstant rational function $f_i \in K(X)$ which is regular everywhere except at P, i.e., for every point $Q \neq P$, we have $v_Q(f_i) \geq 0$ and $v_P(f_i) < 0$ where v_Q denotes the discrete valuation of local ring at Q. Let $f = f_1 + \cdots + f_r$. For any $Q \notin \{P_1, \dots, P_r\}$, we have $v_Q(f) \geq \min\{v_Q(f_1), \dots, v_Q(f_r)\} \geq 0$. Suppose that $v_{P_i}(f) \geq 0$ for some i, i.e., $f \in \mathcal{O}_{X,P_i}$. By assumption, $f_j \in \mathcal{O}_{X,P_i}$ for $j \neq i$, so subtracting them from f implies that $f_i \in \mathcal{O}_{X,P_i}$, which is a contradiction. So we see that $v_{P_i}(f) < 0$ for all i.

3. Embed X as an open set in a proper curve $\overline{X}$. For example, we know by Remark II.4.10.2(e) that X can be embedded as an open subset of a complete variety, so Proposition I.6.7 and Proposition I.6.9 show that X can be embedded as an open subset of a complete curve, which we call $\overline{X}$. The complement of X in $\overline{X}$ is closed, and hence a finite set of points, call them $P_1, \dots, P_r$. By (Ex. 1.2), we can find a rational function $f \in K(\overline{Y})$ such that f has poles at each P_i and is regular

[*]by Robin Hartshorne

elsewhere. This function gives a morphism $f\colon \overline{X} \to \mathbf{P}^1_k$ such that $f^{-1}(\{\infty\}) = \{P_1, \dots, P_r\}$. Hence $f^{-1}(\mathbf{A}^1_k) = X$. By Proposition II.6.8, f is a finite morphism, so X is affine.

5. By definition, $\dim |D| = \ell(D) - 1$. So by Riemann–Roch,

$$\dim |D| - \ell(K - D) = \deg D - g.$$

We wish to show that

$$\ell(K - D) \leq g = \dim_k H^0(X, \mathscr{L}(K)).$$

By Proposition II.6.13(b), we know that

$$\ell(K - D) = \dim_k H^0(X, \mathscr{L}(K - D)) = \dim_k H^0(X, \mathscr{L}(K) \otimes \mathscr{L}(-D)).$$

Since D is effective, $\mathscr{L}(-D)$ is a subsheaf of $\mathcal{O}_X$ (cf. Proposition II.6.18). Tensoring with $\mathscr{L}(K)$ is an exact functor because $\mathscr{L}(K) \cong \omega_X$ is an invertible sheaf (this can be reduced to checking locally, in which case this is the statement that free modules are flat). Hence $\mathscr{L}(K) \otimes \mathscr{L}(-D)$ is a subsheaf of $\mathscr{L}(K) \otimes \mathcal{O}_X \cong \mathscr{L}(K)$, which means that

$$\dim_k H^0(X, \mathscr{L}(K) \otimes \mathscr{L}(-D)) \leq \dim_k H^0(X, \mathscr{L}(K)) = g.$$

Thus the inequality $\dim |D| \leq \deg D$ holds.

Equality holds if and only if $\ell(K - D) = g$. Suppose this equality holds. In the case that $g > 0$, we see that $\mathscr{L}(K) \otimes \mathscr{L}(-D) = \mathscr{L}(K)$, which means that $\mathscr{L}(-D) \cong \mathcal{O}_X$. We see that $D \sim 0$ (Corollary II.6.14), and since D is effective, we conclude that $D = 0$.

Conversely, if $D = 0$, then $\ell(K - D) = \ell(K) = g$ by definition. If $g = 0$, then $\deg(K - D) = -2 - \deg D$ (Example 1.3.3). Since D is effective, $\deg D \geq 0$, so $\deg(K - D) < 0$, which implies $\ell(K - D) = 0 = g$ (Lemma 1.2).

6. Pick $g + 1$ distinct points $P_1, \dots, P_{g+1}$, and let $D = P_1 + P_2 + \cdots + P_{g+1}$. By Riemann–Roch, we have

$$\ell(D) = \deg D + 1 - g + \ell(K - D) = 2 + \ell(K - D),$$

so $\ell(D) \geq 2$, which means that there is some nonconstant rational function $f \in \Gamma(X, \mathscr{L}(D))$ such that f has poles at some nonempty subset of $\{P_1, \dots, P_{g+1}\}$, and is regular elsewhere. Then f defines a nonconstant morphism $X \to \mathbf{P}^1_k$, which is finite by Proposition II.6.8. So the preimage of ∞ in $\mathbf{P}^1_k$ has $\leq g + 1$ points, which means $\deg f \leq g + 1$.

7. (a) We know that $\deg K = 2g - 2 = 2$ (Example 1.3.3) and $\dim |K| = \ell(K) - 1 = g - 1 = 1$ by definition. Suppose that K is not base point free. Let P be a point for which $s_P \in \mathfrak{m}_P \omega_P$ for all $s \in \Gamma(X, \omega)$. Let $\omega(-P)$ be the kernel of the surjection $\omega \to \omega_P / \mathfrak{m}_P \omega_P$. We get an exact sequence

$$0 \longrightarrow \Gamma(X, \omega(-P)) \longrightarrow \Gamma(X, \omega) \longrightarrow \omega_P / \mathfrak{m}_P \omega_P\ ,$$

and by assumption on P, $\Gamma(X, \omega(-P)) = \Gamma(X, \omega)$, so has dimension $g = 2$. Being a nonzero subsheaf of ω, $\omega(-P)$ is an invertible sheaf. Let $s, t \in \Gamma(X, \omega(-P))$ be linearly independent sections. Both s and t define divisors of zeroes of degree 1 since

$$\deg \omega(-P) = \deg \omega - \deg \omega_P / \mathfrak{m}_P \omega_P = 1$$

(Ex. II.6.12 and Proposition II.7.7). In particular, s and t both vanish at a single point, and these points are different because we have assumed that they are linearly independent. Thus, $\omega(-P)$ is base point free, so defines a morphism $\varphi\colon X \to \mathbf{P}^1_k$ (Remark II.7.8.1). By (Ex. II.6.8), $\varphi^*\mathscr{L}(D) = \mathscr{L}(\varphi^* D)$ for a divisor D on $\mathbf{P}^1_k$, so

$$\deg\varphi \cdot \deg\mathcal{O}(1) = \deg\omega(-P),$$

hence $\deg\varphi = 1$ (Proposition II.6.9). But this means that the function field $K(X)$ is a degree 1 extension of the function field of $\mathbf{P}^1_k$, i.e., they are the same. This implies that $X \cong \mathbf{P}^1_k$ (Corollary I.6.12). But genus is a birational invariant, and the genus of $\mathbf{P}^1_k$ is 0, so this is a contradiction. Thus, K is base point free.

Since $\dim_k H^0(X, \omega_X) = 2$, we can give two vectors that span it, and hence a morphism $f\colon X \to \mathbf{P}^1_k$. In fact, f is finite (Proposition II.6.8). By (Ex. II.6.8), $f^*\mathscr{L}(D) = \mathscr{L}(f^*D)$ for a divisor D on $\mathbf{P}^1_k$, so

$$\deg f \cdot \deg\mathcal{O}(1) = \deg\omega_X$$

(Proposition II.6.9). Since $\mathcal{O}(1)$ has degree 1 and we have just shown that $\deg\omega_X = 2$, we conclude that $\deg f = 2$, and hence X is hyperelliptic.

(b) Let $Q \cong \mathbf{P}^1_k \times \mathbf{P}^1_k$ denote the quadric surface $xy = zw$ in $\mathbf{P}^3_k$, and let p_1 and p_2 be projections onto the two components. As in Example II.6.6.1, they induce homomorphisms $p_1^*, p_2^*\colon \operatorname{Cl}\mathbf{P}^1_k \to \operatorname{Cl} Q$ where the maps are given by $D = \sum n_i Y_i \mapsto \sum n_i p_j^{-1}(Y_i)$. So if we take a curve X whose type is $(g+1, 2)$ in Q and restrict p_2 to X, then $p_2^*(pt)$ is a divisor of degree 2. The map not constant and hence finite (Proposition II.6.8). The preimage of a point in $\mathbf{P}^1_k$ under p_2 is then two points, so p_2 is a morphism of degree 2.

8. (a) Let $f\colon \widetilde{X} \to X$ be the normalization map. We start with the exact sequence

$$0 \longrightarrow \mathcal{O}_X \longrightarrow f_*\mathcal{O}_{\widetilde{X}} \longrightarrow \textstyle\sum_{P\in X} \widetilde{\mathcal{O}}_P/\mathcal{O}_P \longrightarrow 0$$

where $\widetilde{\mathcal{O}}_P/\mathcal{O}_P$ denotes the skyscraper sheaf at P with value $\widetilde{\mathcal{O}}_P/\mathcal{O}_P$. Then apply the functor Γ to get the long exact sequence

$$\begin{tikzcd} 0 \arrow[r] & \Gamma(X,\mathcal{O}_X) \arrow[r] & \Gamma(X, f_*\mathcal{O}_{\widetilde{X}}) \arrow[r] & \Gamma(X, \sum\widetilde{\mathcal{O}}_P/\mathcal{O}_P) \arrow[dlll, rounded corners] \\ H^1(X,\mathcal{O}_X) \arrow[r] & H^1(X, f_*\mathcal{O}_{\widetilde{X}}) \arrow[r] & H^1(X, \sum\widetilde{\mathcal{O}}_P/\mathcal{O}_P) \arrow[r] & 0 \end{tikzcd}$$

of $\mathcal{O}(X)$-modules. Skyscraper sheaves are flasque, so $H^1(X, \sum\widetilde{\mathcal{O}}_P/\mathcal{O}_P) = 0$ (Proposition III.2.5). Since X is integral, $\Gamma(X,\mathcal{O}_X)$ is a finite integral k-algebra, and hence a finite field extension of k since an integral Artinian ring is a field. So because k is algebraically closed, $\Gamma(X,\mathcal{O}_X) = k$. Also,

$$\Gamma(X, f_*\mathcal{O}_{\widetilde{X}}) = \Gamma(f^{-1}(X), \mathcal{O}_{\widetilde{X}}) = \Gamma(\widetilde{X}, \mathcal{O}_{\widetilde{X}}) = k$$

for similar reasons. Since f is a finite morphism (Ex. II.3.8), $f_*\mathcal{O}_{\widetilde{X}}$ is a coherent $\mathcal{O}_X$-module (Ex. II.5.5(c)), and so the cokernel $\sum\widetilde{\mathcal{O}}_P/\mathcal{O}_P$ is also coherent (Proposition II.5.7). Hence the long exact sequence is of k-vector spaces. The alternating sum of the dimensions on this long exact sequence is zero:

$$1 - 1 + \dim_k H^0(X, \sum\widetilde{\mathcal{O}}_P/\mathcal{O}_P) - \dim_k H^1(X,\mathcal{O}_X) + \dim_k H^1(X, f_*\mathcal{O}_{\widetilde{X}}) = 0. \quad (1)$$

The map $f\colon \widetilde{X} \to X$ is an affine morphism of Noetherian separated schemes, so we have

$$H^1(\widetilde{X}, \mathcal{O}_{\widetilde{X}}) \cong H^1(X, f_*\mathcal{O}_{\widetilde{X}})$$

(Ex. III.4.1). So

$$p_a(X) = \dim_k H^1(X, \mathcal{O}_X)$$

and

$$p_a(\widetilde{X}) = \dim_k H^1(X, f_*\mathcal{O}_{\widetilde{X}})$$

(Ex. III.5.3(a)), which turns (1) into

$$p_a(X) = p_a(\widetilde{X}) + \dim_k H^0(\textstyle\sum \widetilde{\mathcal{O}}_P/\mathcal{O}_P).$$

Finally, note that $\dim_k H^0(\sum \widetilde{\mathcal{O}}_P/\mathcal{O}_P) = \sum \delta_P$, and we're done.

(b) Suppose that $p_a(X) = 0$. Since $p_a(\widetilde{X}) = \dim_k H^1(\widetilde{X}, \mathcal{O}_{\widetilde{X}}) \geq 0$, and $\delta_P \geq 0$ for all $P \in X$, we conclude that $p_a(\widetilde{X}) = 0$ and $\delta_P = 0$ for all $P \in X$ by part (a). This implies that $\widetilde{\mathcal{O}}_P = \mathcal{O}_P$ for all $P \in X$, so each local ring of X is integrally closed. In dimension one, a Noetherian local ring is integrally closed if and only if it is regular, so X is nonsingular. Thus X is a curve, so $p_g(X) = p_a(X) = 0$ (Proposition 1.1), which means $X \cong \mathbf{P}^1_k$ (Example 1.3.5).

2 Hurwitz's Theorem

2. (a) Let $f\colon X \to \mathbf{P}^1_k$ be the finite morphism determined by $|K|$. The genus of $\mathbf{P}^1_k$ is 0, so plugging in the values for Hurwitz's theorem (Corollary 2.4) gives

$$2 \cdot 2 - 2 = 2 \cdot (-2) + \sum_{P \in X} \operatorname{length}(\Omega_{X/\mathbf{P}^1_k})_P \cdot P,$$

which simplifies to

$$\sum_{P \in X} \operatorname{length}(\Omega_{X/\mathbf{P}^1_k})_P \cdot P = 6. \tag{2}$$

If $Q \in \mathbf{P}^1_k$ is a closed point, then $\deg f^*Q = 2$ where $f^*Q = \sum_{f(P)=Q} v_P(t) \cdot P$ and t is a uniformizer for $\mathcal{O}_Q$ (Proposition II.6.9). The valuations $v_P(t)$ are nonnegative because f^* is a map of local rings, so either Q has two points in its preimage, or it has one which is ramified with ramification index 2. Combining this with (2) shows that f has exactly 6 ramification points each with ramification index 2.

(b) The field extension $K/k(x)$ is Galois of degree 2, so the corresponding morphism of curves $f\colon X \to \mathbf{P}^1_k$ has degree 2. We can write $K = k(x)[z]/(z^2 - (x - \alpha_1) \cdots (x - \alpha_6))$. Since f has degree 2, either a point in $\mathbf{P}^1_k$ has two points in its preimage, in which case it is unramified, or it has one point in its preimage with ramification index 2. Restricting to $\mathbf{A}^1_k$, the points with 1 point in its preimage correspond to the values of x for which $z^2 - (x-\alpha_1) \cdots (x-\alpha_6)$ has a double root. In particular, this happens at $x = \alpha_i$. To check the point at infinity, we do a change of coordinates $x \mapsto 1/x$ to get $z^2 = (1/x-\alpha_1) \cdots (1/x-\alpha_6)$, or $x^6 z^2 = (1 - \alpha_1 x) \cdots (1 - \alpha_6 x)$. Then $x = 0$ has two solutions for z, so this point is unramified. Using Hurwitz's theorem gives (we have tame ramification because $\operatorname{char} k \neq 2$)

$$2g_X - 2 = 2(-2) + 6 = 2,$$

so $g_X = 2$.

(c) By (Ex. I.6.6(a)), we need only find a linear fractional transformation sending P_1, P_2, P_3 to $0, 1, \infty$, respectively. The function

$$\varphi(z) = \begin{cases} \dfrac{z - P_1}{z - P_3} \cdot \dfrac{P_2 - P_3}{P_2 - P_1} & \text{if } P_1, P_2, P_3 \neq \infty \\ \dfrac{P_2 - P_3}{z - P_3} & \text{if } P_1 = \infty \\ \dfrac{z - P_1}{z - P_3} & \text{if } P_2 = \infty \\ \dfrac{z - P_1}{P_2 - P_3} & \text{if } P_3 = \infty \end{cases}$$

does exactly this.

(d) To check that this is a group action, pick $g, h \in \mathfrak{S}_6$. There are three indices for which those elements are mapped to $0, 1, \infty$, respectively. Whether we apply g and then normalize and apply h and normalize, or just apply hg and normalize, these three indices are the same at the end. Thus, by the uniqueness of linear fractional transformations (they are determined by three values), both of these actions are the same.

(e) By (a) and (b), the map $f\colon X \to \mathbf{P}^1_k$ is determined by its canonical linear system $|K|$. These are in bijection with triples of distinct elements $\beta_1, \beta_2, \beta_3$ of k, $\neq 0, 1$, modulo the action of Σ_6, by (c) and (d).

4. To show that X is nonsingular, we cover $\mathbf{P}^3_k$ by the affines given by $x = 1$, $y = 1$, and $z = 1$ and show that in each affine, the Jacobian of X has rank $2 - 1 = 1$. These cases are symmetric, so we just treat $z = 1$. Then our polynomial is $f(x, y) = x^3y + y^3 + x$, and $\partial f/\partial x = 1$ while $\partial f/\partial y = x^3$. At every point in the affine given by $x = 1$, this has rank 1, so X is nonsingular. From the computations of the first partial derivatives, it is immediate that all of the second partial derivatives vanish, so every point of X is an inflection point.

Now we return to $\mathbf{P}^3_k$. The tangent line to X at (x_0, y_0, z_0) is

$$\frac{\partial f}{\partial x}(x - x_0) + \frac{\partial f}{\partial y}(y - y_0) + \frac{\partial f}{\partial z}(z - z_0) = 0$$

where $f(x, y, z) = x^3y + y^3z + z^3x$. We compute these: $\partial f/\partial x = z^3$, $\partial f/\partial y = x^3$, and $\partial f/\partial z = x^3$, so the above equation becomes

$$z_0^3(x - x_0) + x_0^3(y - y_0) + y_0^3(z - z_0) = 0,$$

but $z_0^3x_0 + x_0^3y_0 + y_0^3z_0 = 0$ because (x_0, y_0, z_0) is a point on X. We see that the natural map $X \to X^*$ given by $P \mapsto T_P(X)$ can be described as $(x, y, z) \mapsto (z^3, x^3, y^3)$. The function fields of X and X^* are the quotient fields of $k[x, y, z]/(x^3y + y^3z + z^3x)$, and this map induces a Frobenius map on the function fields up to a permutation of the variables, and hence is purely inseparable. The Frobenius map is finite, so by Proposition 2.5, X and X^* are isomorphic (though not by the natural map).

5. (a) The field extension $K(X)/L$ is Galois since L is defined as a fixed field of a finite group. In particular, it is separable, so there is some element α such that $K(X) = L(\alpha)$. Hence we can write $K(X) = L[z]/p(z)$ for some irreducible polynomial p of degree n. Also, L is a finite field extension of $k(x)$, so p also involves the variable x. If $P \in X$ is a ramification point and $e_P = r$, then this means that plugging in $f(P)$ as x into p gives a root of z with

multiplicity r. In fact, all roots must have multiplicity r because G permutes them. Hence any point $Q \in f^{-1}(f(P))$ has ramification index r, which means that $f^{-1}(f(P))$ is a finite set consisting of n/r points. With this fact, we know that

$$\sum_{P \in X}(e_P - 1) = \sum_{i=1}^{s} \frac{n}{r_i}(r_i - 1) = \sum_{i=1}^{s}(n - n/r_i),$$

so Hurwitz's theorem implies (characteristic 0 implies all ramification is tame)

$$\frac{2g-2}{n} = 2g(Y) - 2 + \frac{1}{n}\sum_{i=1}^{s}(n - n/r_i) = 2g(Y) - 2 + \sum_{i=1}^{s}(1 - 1/r_i).$$

(b) Suppose we have

$$2g(Y) - 2 + \sum_{i=1}^{s}(1 - 1/r_i) > 0.$$

If $g(Y) > 1$, then the left hand side is > 1 and hence at least $1/42$. If $g(Y) = 1$, then the smallest the sum $\sum_{i=1}^{s}(1 - 1/r_i)$ can be is $1/2$ since $r_i \geq 2$. In the case $g(Y) = 0$, we must have $s > 2$. If we are to minimize the left hand side, we need to minimize the sum, and since $1 - 1/r_i > 0$, we need only to consider $s = 3$.

In this case, the left hand side is $1 - 1/r_1 - 1/r_2 - 1/r_3$. So we are interested in positive integers r_1, r_2, r_3 such that the sum $1/r_1 + 1/r_2 + 1/r_3 < 1$ is maximized. We see that if all $r_i > 3$, then at best this sum is $3/4$, and one can do better, and if all $r_i < 3$, then the sum is $3/2 > 1$, so we may assume $r_1 = 3$. If $r_2 > 2$ and $r_3 > 2$, then at best the sum is $1/3 + 1/3 + 1/4 = 11/12$. If we set $r_2 = 2$ and $r_3 = 7$, then the sum is $1/3 + 1/2 + 1/7 = 41/42 > 11/12$, so the maximum must have $r_2 = 2$. But then $1/r_3 < 1/6$, so we see that $r_3 = 7$ is optimal. Hence, we conclude that the left hand side is at least $1/42$. Putting this all together gives

$$\frac{2g-2}{n} \geq \frac{1}{42},$$

which translates to $n \leq 84(g-1)$.

3 Embeddings in Projective Space

1. Since linearly equivalent divisors give rise to isomorphic invertible sheaves, we assume without loss of generality that D is effective. The Riemann–Roch formula gives $\dim|D| - \dim|K - D| = \deg D - 1$. If $\deg D < 5$, we show in each case that D cannot be a very ample divisor. In the case that $\deg D = 0$, $\dim|D| = 0$, so the induced map to projective space is $X \to \mathbf{P}^0_k$, which cannot be a closed immersion. If $\deg D = 1$, then by (Ex. 1.5), $\dim|D| < 1$ and hence we are in the same situation.

   In the case $\deg D = 2$, $\deg(K - D) = 0$, so $\dim|K - D| = 0$, and so by Riemann–Roch, $\dim|D| = 1$ and the induced map $X \to \mathbf{P}^1_k$ cannot be a closed immersion because X is a curve of genus 2. If $\deg D = 3$, then $\deg(K - D) = -1$, so Riemann–Roch gives $\dim|D| = 1$, and we are in the same situation. Finally, for $\deg D = 4$, choose two points P and Q. Then $\deg(D - P - Q) = 2$, so $\dim|D - P - Q| = 1$ from above. By Proposition 3.1(b), we see that D cannot be very ample.

   Of course if $\deg D \geq 5$, we know already that D is very ample by Corollary 3.2(b).

3. Let $f_1, \ldots, f_{n-1}$ be homogeneous polynomials defining hypersurfaces $H_1, \ldots, H_{n-1} \subset \mathbf{P}^n$ such that X is their scheme-theoretic intersection, and let $d_1, \ldots, d_{n-1}$ be their respective degrees. Let $r = \sum d_i - n - 1$. By (Ex. II.8.4(e)), $\omega_X \cong \mathcal{O}_X(r)$, and we know that $r \geq -2$ because $d_i > 0$ for all i. If $r \leq 0$, then $g = \dim H^0(X, \omega_X) = \dim H^0(X, \mathcal{O}(r)) \leq 1$. Since we are assuming $g \geq 2$, we must have $r > 0$. Note that $\omega_X \cong \mathcal{O}_X(r)$ is very ample via the inclusion of X in $\mathbf{P}^n$ followed by the r-uple embedding of $\mathbf{P}^n$ (cf. (Ex. II.5.13)). Finally, the canonical divisor K has degree $2g - 2 = 2$, so is not very ample by (Ex. 3.1). Hence a genus 2 curve cannot be a complete intersection in any $\mathbf{P}^n$.

6. (a) Recall from (Ex. 1.5) that if D is an effective divisor, then $\dim |D| \leq \deg D$, and equality holds if and only if $g = 0$ or $D = 0$. In our case, take D to be $\mathcal{O}(1)$, the hyperplane section of X. Then $\dim |D| \leq 4$, and in the case $g = 0$, we see that $\dim |D| = 4$, so $X \subset \mathbf{P}^4$. If X is not contained in a $\mathbf{P}^3$ via this embedding, then X is the rational normal quartic up to an automorphism of $\mathbf{P}^4$ (Ex. 3.4(b)). Otherwise, if X is contained in some $\mathbf{P}^3$, then it is a rational quartic curve.

   If $g > 0$, then $X \subset \mathbf{P}^3$. If X is contained in a plane, then $g = \frac{1}{2}(d-1)(d-2) = 3$. Otherwise, $g < 3$ (Ex. 3.5(b)). Since D is very ample, (Ex. 3.1) says that $\deg D \geq 5$ if $g = 2$, so this is not a possibility. Hence $g = 1$ in this case.

   (b) The exact sequence

$$0 \longrightarrow \mathscr{I}_X \longrightarrow \mathcal{O}_{\mathbf{P}^3} \longrightarrow \mathcal{O}_X \longrightarrow 0$$

   gives rise to a long exact sequence by twisting by 2 and taking cohomology:

$$0 \longrightarrow H^0(\mathbf{P}^3, \mathscr{I}_X(2)) \longrightarrow H^0(\mathbf{P}^3, \mathcal{O}_{\mathbf{P}^3}(2)) \longrightarrow H^0(\mathbf{P}^3, \mathcal{O}_X(2)) \longrightarrow \cdots.$$

   We know that $\dim H^0(\mathbf{P}^3, \mathcal{O}_{\mathbf{P}^3}(2)) = \binom{5}{2} = 10$, and $\dim H^0(\mathbf{P}^3, \mathcal{O}_X(2)) = \dim |2D| + 1 < 8 + 1$ (Ex. 1.5). So $\dim H^0(\mathbf{P}^3, \mathscr{I}_X(2)) \geq 2$, which means that X is contained in at least two irreducible quadric surfaces. The intersection of two quadrics is a variety of degree 4, so we conclude that X is the complete intersection of two irreducible quadric surfaces in the case $g = 1$.

11. (a) As in the proof of Proposition 3.4, one can show that projection from a point O onto $\mathbf{P}^{n-1}$ is a closed immersion if and only if O is not on any secant line of X and O is not on any tangent line of X. Locally, the secant variety of X can be seen as the image of the morphism $(X \times X \setminus \Delta) \times \mathbf{P}^1 \to \mathbf{P}^n$ which sends (P, Q, t) to the point t on the secant line of P and Q. This has dimension $\leq r + r + 1 < n$, so is not all of $\mathbf{P}^n$. Similarly, the tangent variety has dimension $\leq r + 1 < n$, so we can find an O not on any secant or tangent line.

   (b)

4 Elliptic Curves

7. (a) The homomorphism $f^*\colon \operatorname{Pic} X' \to \operatorname{Pic} X$ preserves degree, so induces a map $f^*\colon \operatorname{Pic}^\circ X' \to \operatorname{Pic}^\circ X$. By Theorem 4.11, we can identify (X, P_0) and (X', P_0') with their Jacobian varieties, and by Remark 4.10.4, the closed points of X and X' can be identified (as groups) with $\operatorname{Pic}^\circ X$ and $\operatorname{Pic}^\circ X'$, respectively. Hence we get a dual morphism $\hat{f}\colon (X', P_0') \to (X, P_0)$.

   (b) The equality $(g \circ f)\hat{} = \hat{f} \circ \hat{g}$ follows from (a) and the functoriality of pullback.

   (c)

(d)

(e) The morphism n_X is the n-fold sum of the identity morphism on X, so $\hat{n}_X = n_X$ follows from (d) by taking $X' = X$. If m is the degree of n_X, then $n_X \circ n_X = m_X$ by (c), but $n_X \circ n_X$ is n^2 iterations of the $\mu(P) = P + P$, so $m = n^2$.

(f) From (c), we get that $\hat{f} \circ f = n_X$ where $n = \deg f$. Since degree is multiplicative, this means that $\deg \hat{f} \cdot \deg f = (\deg f)^2$ by (e), so $\deg \hat{f} = \deg f$ as long as $\deg f \neq 0$. In the case $\deg f = 0$, f is an isomorphism, so $\hat{f}$ is as well, and $\deg \hat{f} = 0$.

14. Let $f(x, y, z) = x^3 + y^3 - z^3$. The Hasse invariant of $X_{(p)}$ is 0 if and only if the coefficient of $(xyz)^{p-1}$ in f^{p-1} is 0 (Proposition 4.21). It is immediate that if $p \equiv 2 \pmod 3$, then the coefficient of $(xyz)^{p-1}$ in f^{p-1} must be 0. Otherwise, for $p \equiv 1 \pmod 3$, an application of the binomial expansion shows that the coefficient of $(xyz)^{p-1}$ in f^{p-1} is $\binom{p-1}{k}\binom{p-1-k}{k}$ where $k = (p-1)/3$. It is clear that this coefficient is not divisible by p, hence is nonzero. So the Hasse invariant of $X_{(p)}$ is 0 if and only if $p \equiv 2 \pmod 3$, and appealing to Dirichlet's theorem for arithmetic progressions, the set $\mathfrak{P}$ has density 1/2.

6 Classification of curves in $\mathbf{P}^3$

1. Let X be a rational curve of degree 4 in $\mathbf{P}^3$. Then we the short exact sequence

$$0 \longrightarrow \mathscr{I}_X \longrightarrow \mathcal{O}_{\mathbf{P}^3} \longrightarrow \mathcal{O}_X \longrightarrow 0$$

gives rise to a long exact sequence

$$0 \longrightarrow H^0(\mathscr{I}_X(2)) \longrightarrow H^0(\mathcal{O}_{\mathbf{P}^3}(2)) \longrightarrow H^0(\mathcal{O}_X(2)) \longrightarrow \cdots.$$

The dimension of $H^0(\mathcal{O}_{\mathbf{P}^3}(2))$ is 10, and the dimension of $H^0(\mathcal{O}_X(2))$ is $\dim|2D| + 1$ where D is the hyperplane section corresponding to $\mathcal{O}(1)$, and this latter number is 9 because $g = 0$ (Ex. 1.5). Hence $\dim H^0(\mathscr{I}_X(2)) \geq 1$, so X is contained in at least one quadric surface. If X is contained in two distinct quadric surfaces, then it is contained in the complete intersection of them, which has degree 4 and genus 1 (Ex. II.8.4(g)). But this is impossible, so X is contained in a unique quadric surface.

Up to isomorphism, there is one singular quadric surface in $\mathbf{P}^3$, which is the quadric cone. By Remark 6.4.1(d), we see that if X lies on the quadric cone, then the genus of X must be 1, so this possibility is ruled out. Hence the quadric surface is nonsingular.

6. Let X be a projectively normal curve in $\mathbf{P}^3$ which is not contained in any plane. This means that the natural map $H^0(\mathcal{O}_{\mathbf{P}^3}(k)) \to H^0(\mathcal{O}_X(k))$ is surjective for all $k \geq 0$ (Ex. II.8.4(c)). Let D be the hyperplane section of X.

Suppose $d = 6$. If $\mathcal{O}(1)$ is special, then $g = 4$ (Proposition 6.3). Otherwise, if $\mathcal{O}(1)$ is nonspecial, then $\dim H^0(\mathcal{O}_X(1)) = \deg D - g + 1 = 7 - g$. By surjectivity, $7 - g \leq 4$, which means $g \geq 3$. Furthermore, $g \leq 4$ by Theorem 6.4.

Now suppose $d = 7$. We know that $g \leq 6$ by Theorem 6.4. If $\mathcal{O}(1)$ is special, then $g \geq 5$. If $\mathcal{O}(1)$ is nonspecial, then as before, we can show that $8 - g \leq 4$; equivalently, $g \geq 4$. But if $g = 4$, then since $\mathcal{O}(2)$ is nonspecial, we have $\dim H^0(\mathcal{O}_X(2)) = \dim|2D| + 1 = 2\deg D - g + 1 = 11$. However, $\dim H^0(\mathcal{O}_{\mathbf{P}^3}(2)) = 10$, so this contradicts surjectivity. Hence $g = 5$ or $g = 6$.