Algebraic Geometry Homework

William A. Stein

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1 Solutions

Problem 1 (II.2.14(a)) *Let S be a graded ring. Show that $Proj\, S = \emptyset$ iff every element of S_+ is nilpotent.*

PROOF. This is equivalent to showing that the nilradical of S is equal to the intersection of all homogenous primes of S. Indeed, if every element of S_+ is nilpotent then every homogenous prime contains S_+ so $Proj\, S = \emptyset$. Conversely, if $Proj\, S = \emptyset$, then every homogenous prime contains S_+ so the nilradical of S contains S_+ so that every element of S_+ is nilpotent.

It remains to show that the intersection of all homogenous primes of S is the nilradical of S. Using Zorn's lemma we can show that if I is a proper homogenous ideal then there is at least one maximal homogenous ideal containing I. (The proof preceeds just as on page 2 of Matsumura except one notes that the union of a chain of *homogenous* ideals is homogeneous. If $I_1 \subset I_2 \subset \cdots$ is a chain of homogeneous ideals, then $\cup_{n=1}^{\infty} I_n$ is an ideal and if $x \in \cup_{n=1}^{\infty} I_n$ then $x \in I_n$ for some n, so the homogenous components of x are in I_n, so they are in $\cup_{n=1}^{\infty} I_n$, so $\cup_{n=1}^{\infty} I_n$ is homogeneous.)

Theorem 1 *Let T be a multiplicative set and I a homogeneous ideal disjoint from T; then there exists a homogeneous prime ideal containing I and disjoint from T.*

PROOF. Using Zorn's lemma we see that the set of homogeneous ideals disjoint from T and containing I contains a maximal element, say P. Then P is prime. For if $x \notin P$, $y \notin P$ are homogenous, then $P + (x)$ and $P + (y)$

are both homogenous and so they meet T, so their product also meets T. However,

$$(P + (x))(P + (y)) \subset P + (xy)$$

so $xy \notin P$ since P does not meet T.

Theorem 2 *The nilradical of S is the intersection of the homogeneous primes of S.*

PROOF. Suppose x is in the nilradical of S so that x is nilpotent, say $x^n = 0$. If I is a homogenous prime then $x^n = 0 \in I$ so, by induction, $x \in I$. Conversely, suppose x is not nilpotent. Then $T = \{1, x, x^2, \ldots\}$ is a multiplicative set disjoint from (0). So, by the above theorem, there is a homogeneous prime ideal containing (0) disjoint from T. Thus x is not in the intersection of all homogeneous prime ideals of S.

Problem 2 (II.2.14(b)) *Let $\varphi : S \to T$ be a graded homomorphism of graded rings. Let $U = \{p \in Proj T : p \not\supseteq \varphi(S_+)\}$. Show that U is an open subset of $Proj T$, and show that φ determines a natural morphism $f : U \to Proj S$.*

PROOF. U is open because $Proj T - U = \{p \in Proj T : p \supseteq \varphi(S_+)\} = \{p \in Proj T : p \supseteq T\varphi(S_+)\} = V(T\varphi(S_+))$ and the ideal $T\varphi(S_+)$ is homogenous because it is generated by the homogeneous elements $\{\varphi(f) : f \in S_+$ is homogeneous$\}$.

The natural morphism $f : U - Proj S$ is defined as follows. As a map on topological spaces we specify that, for $x \in U$, $f(x) = \varphi^{-1}(x)$. Because of the way U was chosen and since φ is homogeneous f maps U into $Proj S$, so f is well-defined. If $V(a)$ is a closed subset of $Proj S$ with a homogeneous, then $f^{-1}(V(a)) = \{p \in U : \varphi^{-1}(p) \supseteq a\} = \{p \in U : p \supseteq \varphi(a)\} = V(T\varphi(a))$. As above, $T\varphi(a)$ is homogeneous so $f^{-1}(V(a))$ is closed so f is continuous.

To define the associated map $f^{\#} : \mathcal{O}_{Proj S} \to f_*\mathcal{O}_U$ of sheaves let $V \subset Proj S$ be open. Then an element of $\mathcal{O}_{Proj S}(V)$ is a map $s : V \to \bigsqcup_{p \in V} S_{(p)}$ such that s is locally a quotient of elements of S. We specify that $f^{\#}(s) = \varphi \circ s \circ f : f^{-1}(V) \to \bigsqcup_{p \in f^{-1}(V)} T_{(p)}$. To see that $f^{\#}(s) \in f_*\mathcal{O}(V)$ we must check that it is locally a quotient. So suppose $p \in f^{-1}(V)$. Let $W \subset V$ be an open neighborhood of $f(p)$ on which s is represented as a quotient. Then $f^{\#}(s)$ is represented as a quotient on the open set $f^{-1}(W)$, as required. Since $f^{\#}$ respects the restriction maps we see that f is a morphism.

Problem 3 (II.2.14(c)) *f can be an isomorphism even when φ is not. For example, suppose that $\varphi_d : S_d \to T_d$ is an isomorphism for all $d \geq d_0$. Show that $U = ProjT$ and the morphism $f : ProjT \to ProjS$ is an isomorphism.*

PROOF. To see that $U = ProjT$ note that if $p \in ProjT$ but $p \notin U$ then $p \supseteq \varphi(S_+)$. In particular, $p \supseteq \bigoplus_{d \geq d_0} T_d$, so $p \supseteq (T_+)^d$ so, since p is prime, $p \supseteq T_+$, a contradiction.

Let $\{g_\alpha\}$ be a set of generators of T_+. Then $\cup_\alpha D_T(g_\alpha) = \cup_\alpha \{x \in ProjT : g_\alpha \notin x\} = ProjT$ since every prime in $ProjT$ must omit some g_α. Since $g_\alpha \notin x$ iff $g_\alpha^{d_0} \notin x$ for x prime, we may replace the g_α by elements of $T_{\geq d_0}$ and still have a cover of $ProjT$ by distinguished open sets.

Our strategy is as follows. We first show that $f|_{D_T(g_\alpha)} : D_T(g_\alpha) \to D_S(\varphi^{-1}(g_\alpha)$ is an isomorphism for each α and then show that the open sets $D_S(\varphi^{-1}(g_\alpha))$ cover $ProjS$. Then showing that f is injective completes the proof.

Let $g = g_\alpha$ be one of our g_α. By Proposition 2.5, $D_T(g) \cong SpecT_{(g)}$ and so $f' = f|_{D_T(g)}$ is a morphism of affine schemes $f' : SpecT_g \to SpecS_{(\varphi^{-1}(g))}$. This map is induced by $\overline{\varphi} : S_{(\varphi^{-1}(g))} \to T_{(g)}$ where $\overline{\varphi}$ is the localization of the ring homomorphism $\varphi : S \to T$. So we just need to verify that $\overline{\varphi}$ is an isomorphism. Suppose $\overline{\varphi}(a/b) = 0$. Then $\overline{\varphi}(a\varphi^{-1}(g)/b\varphi^{-1}(g)) = \overline{\varphi}(a/b) = 0$ so $\varphi(a\varphi^{-1}(g))/\varphi(b\varphi^{-1}(g)) = 0$ in $T_{(g)}$ so there is n such that $g^n\varphi(a\varphi^{-1}(g)) = 0$ in T so $\varphi(a\varphi^{-1}(g^{n+1}) = 0$. Thus $a\varphi^{-1}(g)^{n+1} = 0$ since φ is an isomorphism in high enough degree. Thus $a = 0$ in $S_{(\varphi^{-1}(g))}$, so $a/b = 0$ in $S_{(\varphi^{-1}(g))}$. This shows that $\overline{\varphi}$ is injective. To see that $\overline{\varphi}$ is surjective let $a/g^n \in T_{(g)}$. Then $\varphi^{-1}(ag)/\varphi^{-1}(g^{n+1})$ is a well-defined element of $S_{(\varphi^{-1}(g))}$ and $\overline{\varphi}(\varphi^{-1}(ag)/\varphi^{-1}(g^{n+1})) = ag/g^{n+1} = a/g^n$, which shows that $\overline{\varphi}$ is surjective.

Next we verify that $\cup_{g_\alpha} D_S(\varphi^{-1}(g_\alpha) = ProjS$. Suppose $x \notin D_S(\varphi^{-1}(g_\alpha)$ for all α. Then $\varphi^{-1}(g_\alpha) \in x$ for each α, so, since we may assume that the g_α generate $T_{\geq d_0}$, and x is prime, $\varphi^{-1}(T_{\geq d_0}) \subseteq x$, a contradiction since $S_+ \not\subseteq x$.

Next we show that the induced map $f : ProjT \to ProjS$ is injective. Let $p, q \in ProjT$ and suppose $f(p) = f(q)$. Then $\varphi^{-1}(p) = \varphi^{-1}(q)$ so, since φ is an isomorphism, for $d \geq d_0$ we see that $p \cap T_d = q \cap T_d$. So if $a \in p$ is homogeneous then $a^n \in p \cap T_d$ for some n and $d \geq d_0$. So $a^n \in q \cap T_d$ so $a^n \in q$ so $a \in q$. Likewise $a \in q$ implies $a \in p$. Thus $p = q$ so f is injective.

Finally, no solution is complete without an actual example of a map $\varphi : S \to T$ which satisfies the hypothesis of the theorem. Let $T = k[x, y]$ and let $S = T_0 + T_2 + \cdots$ and let $\varphi : S \hookrightarrow T$ be the inclusion map. Then φ

is graded and an isomorphism for $d \geq 2$. But φ is not an isomorphism.

Problem 4 (II.2.14(d)) *Let V be a projective variety with homogeneous coordinate ring S. Show that $t(V) \cong Proj\, S$.*

Proof. Define $f : t(V) \to Proj\, S$ by f takes a point $x \in t(V)$ to its homogeneous ideal $I(x) \in Proj\, S$. By exercise I.2.4 f is injective and surjective hence a bijection. Furthermore, $x \supseteq y$ iff $f(x) \subseteq f(y)$ so f and f^{-1} send closed sets to closed sets hence f is a homeomorphism.

To define $f^{\#}$ let U be an open subset of $Proj\, S$. Then we must define $f^{\#}$ so that (notation as in the proof of proposition 2.6) $f^{\#} : \mathcal{O}_{Proj\, S}(U) \to f_*\mathcal{O}_{t(V)}(U) = \mathcal{O}_{t(V)}(f^{-1}(U)) = \mathcal{O}_V(\alpha^{-1}(f^{-1}(U)))$. Let $s \in Proj_S(U)$. Then s can locally be represented in the form g/h where $g, h \in S$ have the same degree and h is nonzero on the appropriate subset of V. Thus s naturally defines an element of $\mathcal{O}_V(\alpha^{-1}(f^{-1}(U)))$ and any element of $\mathcal{O}_V(\alpha^{-1}(f^{-1}(U)))$ defines an element of $\mathcal{O}_{Proj\, S}(U)$. Thus $f^{\#}$ is an isomorphism. [I'm glossing over a lot of details!]

Problem 5 (II.2.16(a)) *Let X be a scheme, let $f \in \Gamma(X, \mathcal{O}_X)$, and define X_f to be the subset of points $x \in X$ such that the stalk f_x of f at x is not contained in the maximal ideal m_x of the local ring $\mathcal{O}_x$. (a) If $U = Spec B$ and $\overline{f} \in B = \Gamma(U, \mathcal{O}_{X|U})$ is the restriction of f, show that $U \cap X_f = D(\overline{f})$ so X_f is an open subset of X.*

Proof. Note that $D(\overline{f}) = \{x \in U : \overline{f} \notin x\} = \{x \in U : \overline{f}_x \notin m_x\} = \{x \in U : f_x \notin m_x\} = U \cap X_f$. Thus $U \cap X_f$ is an open subset of U. Now let $\{U_\alpha\}$ be an affine open over of X. Then $X_f = \cup_\alpha (U_\alpha \cap X_f)$ is the union of open sets, hence open.

Problem 6 (II.2.16(b)) *Assume that X is quasi-compact. Let $A = \Gamma(X, \mathcal{O}_X)$, and let $a \in A$ be an element whose restriction to X_f is 0. Show that for some $n > 0$, $f^n a = 0$.*

Proof. Using the fact that X is quasi-compact we can find a finite cover $\{U_i = Spec B_i\}_{i=1}^m$ of X by affine open sets. Since $a|_{X_f}$ is 0, the image of a in $\mathcal{O}_{X_f \cap U_i} = (B_i)_{\overline{f}}$ is 0. Thus there exists n_i such that $\overline{f}^{n_i}\overline{a} = 0$ in B_i. That is, $f^{n_i}a|_{X_f \cap U_i}$ is 0. Letting $n = \max\{n_1, \ldots, n_m\}$ we see that $f^n a|_{X_f \cap U_i}$ is 0 for each i whence $f^n a = 0$ in $A = \Gamma(X, \mathcal{O}_X)$.

Problem 7 (II.2.16(c)) *Assume X has a finite cover by open affines U_i such that each intersection $U_i \cap U_j$ is quasi-compact. Let $b \in \Gamma(X_f, \mathcal{O}_{X_f})$. Show that for some $n > 0$, $f^n b$ is the restriction of an element of A.*

Proof. Write $U_i = SpecB_i$. Then $b|_{U_i} \in (B_i)_f$ so, from the definition of $(B_i)_f$, there exists n_i such that $f^{n_i} b|_{U_i} \in B_i = \mathcal{O}_X(U_i)$. Let $N = \max\{n_i\}$ and let $g_i = f^N b|_{U_i} \in \mathcal{O}_X(U_i)$. Then $g_i|_{X_f \cap U_i \cap U_j} = f^N b|_{U_i \cap U_j \cap X_f} = g_j|_{X_f \cap U_i \cap U_j}$ so $(g_i - g_j)|_{(U_i \cap U_j)_f} = 0$. By part (b) since $U_i \cap U_j$ is quasi-compact there is an integer n_{ij} such that $f^{n_{ij}}(g_i - g_j) = 0$ in $\mathcal{O}_{U_i \cap U_j}$. Let $M = \max\{n_{ij}\}$, and let $h_i = f^M g_i$. Then $h_i \in \mathcal{O}(U_i)$ for each i and $h_i|_{U_i \cap U_j} = h_j|_{U_i \cap U_j}$ so we can find $h \in \Gamma(X, \mathcal{O}_X)$ such that $h|_{U_i} = h_i$. But, for each i, $h|_{X_f \cap U_i} = f^M g_i|_{X_f \cap U_i} = f^M f^N b|_{X_f \cap U_i} = f^{M+N} b|_{X_f \cap U_i}$ so by uniqueness (sheaf axiom iii), $h|_{X_f} = f^{M+N} b$, as desired.

Problem 8 (II.2.16(d)) *With the hypothesis of (c), conclude that $\Gamma(X_f, \mathcal{O}_{X_f}) \cong A_f$.*

Proof. Define a homomorphism $\varphi : A_f \to \Gamma(X_f, \mathcal{O}_{X_f})$ by $\varphi(a/f^n) = a|_{X_f}/f^n|_{X_f}$. This is well-defined because the stalk f_x is invertible in each local ring $\mathcal{O}_x$ for each $x \in X_f$. φ is a homomorphism since restriction is a homomorphism. Suppose $\varphi(a/f^n) = 0$, then $a|_{X_f}/f|^n_{X_f} = 0$ so $a|_{X_f} = 0$. By part (b) there exists m such that $f^m a = 0$, so a is 0 in A_f, whence φ is injective. Suppose $b \in \Gamma(X_f, \mathcal{O}_{X_f})$, then by part (c) there exists n such that $f^n b$ is the restriction of some $a \in \Gamma(X, \mathcal{O}_X)$ to X_f. Then $\varphi(a/f^n) = a|_{X_f}/f|^n_{X_f} = f^n b/f^n|_{X_f} = b$ so φ is surjective, which establishes the desired isomorphism.

Problem 9 (II.2.17(a)) *A Criterion for Affineness. Let $f : X \to Y$ be a morphism of schemes, and assume that Y can be covered by open sets U_i, such that for each i, the induced map $f^{-1}(U_i) \to U_i$ is an isomorphism. Then f is an isomorphism.*

Proof. Define a morphism $g : Y \to X$ as follows. On each open set U_i let g_i be the morphism inverse to $f|_{f^{-1}(U_i)} : f^{-1}(U_i) \to U_i$. Then $g_i|_{U_i \cap U_j} = f|_{f^{-1}(U_i) \cap f^{-1}(U_j)} = g_j|_{U_i \cap U_j}$ so, as in step 3 of the proof of Theorem 3.3, we can glue the morphisms g_i to obtain a morphism $g : Y \to X$ such that $g|_{U_i} = g_i$. As a map of spaces g is clearly inverse to f. Since each g_i is an isomorphism, the induced maps $g^{\#}$ on stalks are isomorphisms so the induced map on sheaves is an isomorphism. Thus g is an isomorphism.

Algebraic Geometry Homework
II.3 #'s 6,7,8,12,14
II.4 #'s 2, 4

William A. Stein

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Problem 1 (3.6) *Let X be an integral scheme. Show that the local ring $\mathcal{O}_\xi$ of the generic point ξ of X is a field. Call it $K(X)$. Show also that if $U = \operatorname{Spec} A$ is any open affine subset of X, then $K(X)$ is isomorphic to the quotient field of A.*

Proof. Let $U = \operatorname{Spec} A$ be any nonempty open affine subset of X. Then since the closure of a generic point of X is all of X, every open set must contain a generic point. Thus if ξ is a generic point, then $\xi \in U$. But A is an integral domain so (0) is the unique generic point of U, whence $\xi = (0)$. This shows the generic point is unique if it exists. Since X is integral it is irreducible so every open set intersects U. Thus every open set contains $(0) \in \operatorname{Spec} A = U$, so X actually contains a generic point $\xi = (0)$. Furthermore, $\mathcal{O}_\xi \cong A_{(0)}$ is the quotient field of A.

Problem 2 (3.7) *Let $f : X \to Y$ be a dominant, generically finite morphism of finite type of integral schemes. Show that there is an open dense subset $U \subseteq Y$ such that the induced morphism $f^{-1}(U) \to U$ is finite.*

Proof. Let $U = \operatorname{Spec} B$ be an open affine subset of Y which contains the generic point of Y. Let $V = \operatorname{Spec} A$ be an open affine subset of $f^{-1}(U)$. Since f is of finite type A is a finitely generated B-algebra. The generic point of X is in V since every open set contains the generic point. Let $\varphi : B \to A$ be the homomorphism corresponding to the induced morphism of affine schemes $f : V \to U$. Since f is dominant, we know that φ is injective. The induced map on stalks then gives an inclusion of function fields $K(Y) = B_{(0)} \hookrightarrow A_{(0)} = K(X)$. Since A is a finitely generated B-algebra, $K(X)$ is a finitely generated field extension of $K(Y)$. If this field extension is not of finite degree then $K(X)$ must contain an element which is transcendental over $K(Y)$. Thus A must contain an element t which is transcendental over B. But then infinitely many primes of A lie over (0). Indeed, since $K(X)$ is finitely generated over $K(Y)$, $K(X)$ is a finite algebraic extension of $k(t)$ for some field k. Then (since the algebraic closure of a field is infinite), there are infinitely many irreducible polynomials in $k[t]$. Since $K(X)$ is finite algebraic over $k(t)$ infinitely many of these must remain irreducible in A. Multiplying through denominators this gives infinitely many irreducible elements of A which generate prime ideals which lie over (0). This would contradict the fact that f is generically finite. Thus $K(X)$ is finite over $K(Y)$.

Let $x_1, \dots, x_n$ generate A as a B-algebra. Then, since $K(X)$ is finite over $K(Y)$ each x_i satisfies some polynomial f_i with coefficients in B. Let b be the product of all of the leading coefficients of the polynomials f_i. If b is a unit in B then so are all of the leading coefficients of the f_i so we can divide by them and hence assume the f_i are monic polynomials. If not, replace B by the localization B_b and repeat the whole argument with $U = \text{Spec } B_b$. In either case we may assume the f_i are monic from which we conclude that A is a finitely generated integral extension of B, thus A is a finite module over B.

Now let $U = \text{Spec } B$ be an open affine subset of Y which contains the generic point. Since f is of finite type we may write $f^{-1}(U) = \cup_{i=1,\dots,n} V_i$ where each $V_i = \text{Spec } A_i$ is finitely generated B-algebra. By the work above we may shrink U so that we can assume each A_i is actually a finitely generated B-module. To complete the proof we need to show that there is a distinguised open subset of U (which necessarily contains the generic point) whose inverse image under f is an open affine which is the spectrum of a finitely generated B-module. Let $\varphi_i : B \to A_i$ be the homomorphism which induces $f|_{V_i}$. Since f is dominant, each φ_i is an injection. Thus we may, for notational convenience, identify B with it's images in the various A_i. The morphism f is then induced by the inclusion map $B \hookrightarrow A_i$.

Since $\cap_{i=1,\dots,n} V_i$ is open we can, for each i, $1 \leq i \leq n-1$, find $\alpha_i \in A_i$ such that $\text{Spec } (A_i)_{\alpha_i} \subseteq \cap_{i=1,\dots,n} V_i$. Since A_i is a finite module over B there is an integral equation

$$\alpha_i^n + b_{n-1}\alpha_i^{n-1} + \cdots + b_0 = 0$$

where each $b_j \in B$ and $b_0 \neq 0$. Let $b = \prod_{i=1,\ldots,n-1} b_i$. Then any prime of A_i which contains α_i must also contain b_i and hence b. Therefore Spec $(A_i)_b \subseteq$ Spec $(A_i)_\alpha$. We then have that $g^{-1}(\text{Spec } B_b) = \cup_{i=1,\ldots,n-1}\text{Spec } (A_i)_b \cup \text{Spec } (A_n)_b =$ Spec $(A_n)_b$. The latter equality follows since, for $1 \leq i \leq n-1$, Spec $(A_i)_b \subseteq \cap_{i=1,\ldots,n} V_i \cap f^{-1}(U_b) \subseteq f^{-1}(U_b) \cap V_n =$ Spec $(A_n)_b$. We thus see that U_b is a dense open subset of Y such that the morphism $f : f^{-1}(U_b) \to U_b$ is finite (since $f^{-1}(U_b)$ is the affine scheme Spec $(A_n)_b$ which a finite B_b-module). This completes the proof.

Problem 3 (3.8) *Normalization. Let X be an integral scheme. For each open affine subset $U =$ Spec A, let $\tilde{A}$ be the integral closure of A in its quotient field, and let $\tilde{U} =$ Spec $\tilde{A}$. Show that one can glue the schemes $\tilde{U}$ to obtain a normal integral scheme $\tilde{X}$, called the normalization of X. Show also that there is a morphism $\tilde{X} \to X$, having the following universal property: for every normal integral scheme Z, and for every dominant morphism $f : Z \to X$, f factors uniquely through $\tilde{X}$. If X is of finite type over a field k, then the morphism $\tilde{X} \to X$ is a finite morphism.*

Proof.

We first verify the universal property for affine schemes where it is clear what the normalization is.

Proposition 1 *Suppose $X =$ Spec A, $\tilde{X} =$ Spec $\tilde{A}$ its normalization and $Z =$ Spec B is a normal integral scheme. Then every dominant morphism $f : Z \to X$ factors uniquely through $\tilde{X}$.*

Proof. Let $\varphi : A \to B$ be the homomorphism corresponding to f. Then, since f is dominant, φ is injective. Indeed, $f(Z) \subseteq V(\ker(\varphi))$ so if $\ker(\varphi) \neq 0$ then $f(Z)$ doesn't meet the nonempty open set $X - V(\ker(\varphi))$ (nonempty since A is a domain so (0) is prime so $(0) \in X - V(\ker(\varphi))$. So there is a unique extension of φ to a homomorphism from $\tilde{A} \to B$. Indeed, define $\overline{\varphi}(a/b) = \varphi(a)/\varphi(b)$. Then since φ is injective this is well-defined since $b \neq 0$ implies $\varphi(b) \neq 0$. Furthermore, if ψ is another possible extension of φ to $\tilde{A}$, then $\psi(b)\psi(a/b) = \psi(a) = \varphi(a) = \varphi(b)\overline{\varphi}(a/b) = \psi(b)\overline{\varphi}(a/b)$ so cancelling shows that $\psi(a/b) = \overline{\varphi}(a/b)$. Thus there is a unique morphism $f' : Z \to \tilde{X}$

whose composition with the natural map $\tilde{X} \to X$ equals f. (The natural map is induced by the inclusion $A \hookrightarrow \tilde{A}$.)

Now we prove the universal property holds when Z is an arbitrary normal integral scheme but X is still affine.

Proposition 2 *Let $X = \text{Spec } A$, $\tilde{X} = \text{Spec } \tilde{A}$ its normalization and Z be any normal integral scheme. Then every dominant morphism $f : Z \to X$ factors uniquely through $\tilde{X}$.*

PROOF. Let U_i be a cover of Z by open affines. If $U = U_i$ is any U_i then U is a normal integral affine scheme and $f|_U$ is a dominant morphism. Indeed, U is dense in Z since Z is irreducible (Proposition 3.1). Thus $f^{-1}(\overline{f(U)}) \supseteq \overline{U} = Z$ so $f^{-1}(\overline{f(U)}) = Z$ so $\overline{f(U)} \supseteq f(Z)$ so $\overline{f(U)} = \overline{f(Z)} = X$. We can thus apply the above proposition to find a unique morphism $g_i : U_i \to \tilde{X}$ such that $\psi \circ g_i = f|_U$ where $\psi : \tilde{X} \to X$. By uniqueness on a cover of $U_i \cap U_j$ by open affines, $g_i|_{U_i \cap U_j} = g_j|_{U_i \cap U_j}$. We can thus glue the morphism g_i to obtain a morphism $g : Z \to \tilde{X}$ such that $\psi \circ g = f$. The morphism g is evidently unique.

Now we can define the identification maps φ_{ij}. Let $\{U_i = \text{Spec } A_i\}$ be the open affine subsets of X. Let $\{\tilde{U}_i = \text{Spec } \tilde{A}_i\}$ be the associated normalizations. Let $\psi_i : \tilde{U}_i \to U_i$ be the morphism induced by the inclusion $A_i \hookrightarrow \tilde{A}_i$. Let $W_{ij} = \psi_i^{-1}(U_i \cap U_j)$. Then W_{ij} is an open subset of a normal scheme hence normal. $\psi_i : W_{ij} \to U_i \cap U_j \subseteq U_j$ so there is a unique morphism which we call $\varphi_{ij} : W_{ij} \to \tilde{U}_j$ such that $\psi_j|_{W_{ji}} \circ \varphi_{ij} = \psi_i|_{W_{ij}}$. By uniqueness we see that $\varphi_{ij} \circ \varphi_{ji} = id$ so $\varphi_{ij} = \varphi_{ji}^{-1}$. Furthermore, for each i, j, k, $\varphi_{ij}(W_{ij} \cap W_{ik}) = \psi_j^{-1}(\psi_i(W_{ij} \cap W_{ik})) = \psi_j^{-1}(\psi_i(W_{ij})) \cap \psi_j^{-1}(\psi_i(W_{ik})) = W_{ji} \cap W_{jk}$. By uniqueness, $\varphi_{jk} \circ \varphi_{ij} = \varphi_{ik}$ on $W_{ij} \cap W_{ik}$. So by the glueing lemma (Exercise 2.12) we may glue to obtain a scheme $\tilde{X}$. We can also glue the morphisms ψ_i to obtain a morphism $\psi : \tilde{X} \to X$.

Next, we must verify that the universal property holds in general. Let Z be an arbitrary normal integral scheme, and let X and $\tilde{X}$ be as above and suppose $f : Z \to X$ is a morphism. Cover X be open affines U_i. Then for each morphism $f|_{f^{-1}(U_i)}$ we can apply the above proposition to find a morphism g_i such that $\psi \circ g_i = f|_{f^{-1}(U_i)}$. By uniqueness we can glue these morphism to obtain the required morphism $g : Z \to \tilde{X}$.

Now we check that $\tilde{X}$ is a normal integral scheme. Note first that each $\tilde{U}_i$ is the spectrum of an integrally closed domain and is hence a normal integral scheme (since the localization of an integrally closed domain is integrally closed). Let $x \in \tilde{X}$. Then x is contained in some $\tilde{U}_i$. But the local ring of x in $\tilde{X}$ is the same as the local ring of x in $\tilde{U}_i$ which is integrally closed. This shows that $\tilde{X}$ is normal.

Since X is irreducible, every U_i intersects every U_j. Thus every $\tilde{U}_i$ intersects every $\tilde{U}_j$ after glueing. Since each $\tilde{U}_j$ is irreducible and they all overlap this implies $\tilde{X}$ is irreducible. Indeed, if $\tilde{X} = A \cup B$ with A and B closed, then every $\tilde{U}_i$ is either completely contained in A or in B. If they are not all contained in one of A or B then we can find an open subset U contained in A and an open subset V contained in B but not contained in A. Then $V = (U \cap V) \cup (U^c \cap V) = (A \cap V) \cup (U^c \cap V)$ which expresses V as a union of two proper closed subsets of V. $A \cap V$ is a proper subset of V since V is not contained in A and $U^c \cap V$ is a proper subset of V since $U \cap V \neq \emptyset$. This contradicts the fact that V is irreducible. Thus $\tilde{X} = A$ or $\tilde{X} = B$ whence $\tilde{X}$ is irreducible.

Now we check that the structure sheaf has no nilpotents. Let U be an open subset of $\tilde{X}$ and suppose $f \in \mathcal{O}_{\tilde{X}}(U)$ is nilpotent. Then since f is nonzero, there is some point $x \in \tilde{X}$ so that the stalk f_x of f at x in the local ring $\mathcal{O}_{\tilde{X}}$ is nonzero and nilpotent (use sheaf axiom (iii) and the definition of $\mathcal{O}_{\tilde{X}}$.) Let $\tilde{U}_i$ be some $\tilde{U}_i$ which contains x. Then the local ring at x is a localization of the integral domain $\tilde{A}_i$ so it can't contain any nilpotents. Thus the scheme $\tilde{X}$ is reduced.

Now we check that if X is of finite type over a field k, then the morphism $f : \tilde{X} \to X$ is a finite morphism. The U_i form an open cover of X and $f^{-1}(U_i) = \operatorname{Spec} \tilde{A}_i$ is affine for each i, so we just need to check that $\tilde{A}_i$ is a finite module over A_i. This follows from Theorem 3.9A of chapter I.

Problem 4 (3.12) *Closed Subschemes of* Proj S.

(a) Let $\varphi : S \to T$ be a surjective homomorphism of graded rings, preserving degrees. Show that the open set U of (Ex. 2.14) is equal to Proj T*, and the morphism $f : \operatorname{Proj} T \to \operatorname{Proj} S$ is a closed immersion.*

(b) If $I \subseteq S$ is a homogeneous ideal, take $T = S/I$ and let Y be the closed subscheme of $X = \operatorname{Proj} S$ defined as the image of the closed immersion Proj $S/I \to X$. *Show that different homogeneous ideals can give rise to the same closed subscheme. For example, let d_0 be an integer, and let $I' = \bigoplus_{d \geq d_0} I_d$. Show that I and I' determine the same closed subscheme.*

Proof. (a) Since φ is graded and surjective, $\varphi(S_+) = T_+$ from which it is immediate that $U = \operatorname{Proj} T$. By the first isomorphism theorem $T \cong$

$S/\ker(\varphi)$ so $f(\text{Proj } T) = f(\text{Proj } S/\ker(\varphi)) = V(\ker(\varphi))$ which is a closed subset of Proj S. (This is just the fact that there is a one to one correspondence between homogeneous ideals of $S/\ker(\varphi)$ and homogeneous ideals of S which contain $\ker(\varphi)$.) The map on the stalk corresponding to a point $x \in \text{Proj } T$ is the map $S_{(\varphi^{-1}(x))} \to T_{(x)}$ induced by φ. This map is surjective since φ is surjective. Thus the induced map on sheaves is surjective.

(b) Let $\varphi : S/I' \to S/I$ be the natural projection homomorphism. (This makes sense because S/I is a quotient of S/I'. Indeed, $S/I = (S/I')/\bigoplus_{0 \leq d < d_0} I_d$.) Then φ is a graded homomorphism of graded rings such that φ_d is the identity for $d \geq d_0$. So by (Exercise 2.14c) φ induces an isomorphism $f : \text{Proj } S/I \to \text{Proj } S/I'$. Since this is a morphism over Proj S (the corresponding triangle of homomorphisms commutes) it follows that I and I' give rise to the same closed subscheme.

Problem 5 (3.14) *If X is a scheme of finite type over a field, show that the closed points of X are dense. Give an example to show that this is not true for arbitrary schemes.*

PROOF. Since X is of finite type over k we can cover X with affine open sets $U_i = \text{Spec } A_i$ where each A_i is a finitely generated k-algebra. Let U be an open subset of X. We must show that U contains a closed point. Since the U_i cover X, U must intersect some U_i. Then $U \cap U_i$ contains a distinguised open subset of U_i. So, to show that every open set contains a closed point, it suffices to show that every nonempty distinguised open subset of each U_i contains a closed points of X. Since a distinguished open subset $(U_i)_x$ of a U_i is also the spectrum of a finitely generated k-algebra Spec $(A_i)_x$ we can just add it to our collection $\{U_i\}$. The problem thus reduces to showing that each U_i contains a closed point.

Proposition 3 *With the notation as above, if $x \in U_i$ is closed in U_i (here U_i has the subspace topology) then x is closed in X.*

PROOF. Suppose $x \in U_j$. There is a natural injection $U_i \cap U_j \hookrightarrow U_j$. Let Spec $(B_i)_f$ be a distinguished open subset of U_i contained in $U_i \cap U_j$ which contains x. Then we have a morphism Spec $(B_i)_f \hookrightarrow U_j = \text{Spec } B_j$. We thus get a ring homomorphism $\varphi : B_j \to (B_i)_f$ of Jacobson rings. Since it is induced by a restriction of the identity map $X \to X$ which is a morphism over k, φ is a k-algebra homomorphism. Since $(B_i)_f$ is a finitely generated

k-algebra, $(B_i)_f$ is also a finitely generated B_j-algebra. Since x is closed in Spec $(B_i)_f$, x is a maximal ideal of $(B_i)_f$. Thus by page 132 of Eisenbud's *Commutative Algebra* $\varphi^{-1}(x)$ is a maximal ideal of B_j. Thus x is also a closed point of U_j in the subspace topology on U_j. Thus $X - x = \cup_i(U_i - x)$ is a union of open subsets of X, hence open in X, so x is closed.

To finish we just need to know that U_i has a closed point.

Proposition 4 *Let $X =$ Spec A be an affine scheme with A a finitely generated k-algebra. Then any nonempty distinguished open subset of X contains a closed point.*

PROOF. The key observation is that A is a Jacobson algebra since it finitely generated over a field, so by page 131 of Eisenbud's *Commutative Algebra* the Jacobson radical of A equals the nilradical of A. Let $D(f)$ be a nonempty distinguised open subset of X. Then some prime omits f so f is not in the nilradical of A. Thus f is not in the Jacobson radical of A so there is some maximal ideal m so that $f \notin m$. Then $m \in D(f)$ and m is a closed point of X since m is maximal.

Strangely enough I never used the hypothesis that X is of finite type over k but just the weeker hypothesis that X is *locally* of finite type over k. Did I miss something?

Finally, we present a counterexample in the more general situation. Let $X =$ Spec $Z_{(2)}$. Then X contains precisely one closed point, the ideal (2). So the set of closed points in X is not dense in X. In fact, if X is the spectrum of any DVR we also get a counterexample.

Problem 6 (4.2) *Let S be a scheme, let X be a reduced scheme over S, and let Y be a seperated scheme over S. Let f_1 and f_2 be two S-morphisms of X to Y which agree on an open dense subset U of X. Show that $f_1 = f_2$. Give examples to show that this result fails if either (a) X is nonreduced, or (b) Y is nonseparated.*

PROOF. Let $g = (f_1, f_2)_S : X \to Y \times_S Y$ be the product of f_1 and f_2 over S. By hypothesis the diagonal $T = \Delta_Y(Y)$ is a closed subscheme of $Y \times_S Y$. Thus $Z = g^{-1}(T)$ is a closed subscheme of X. If $h : U \to Y$ is the common restriction of f_1 and f_2 to U, then, since $g|_U$ makes the correct diagram commute, the restriction of g to U is $g' = (h,h)_S$ and $g' = \Delta_Y \circ h$ since $\Delta_Y \circ h$ makes the correct diagram commute.

Thus $\Delta_Y^{-1}(T) = Y$ implies

$$g^{-1}(T) \supseteq (g')^{-1}(T) = h^{-1}(\Delta_Y^{-1}(T)) = h^{-1}(Y) = U.$$

Thus $g^{-1}(T)$ is a closed set which contains the dense set U. Thus $g^{-1}(T) = X$ so $g(X) \subseteq T$. So, by the proposition below, since X is reduced, g factors as $g = \Delta_Y \circ f$ where $f : X \to Y$. From the definition of Δ_Y, we see that $\pi_1 \circ \Delta_Y = \mathrm{id}_Y = \pi_2 \circ \Delta_Y$. Thus $f_1 = \pi_1 \circ g = \pi_1 \circ \Delta_Y \circ f = f$ and $f_2 = \pi_2 \circ g = \pi_2 \circ \Delta_Y \circ f = f$ so $f = f_1 = f_2$, as desired.

Proposition 5 *Let X be a reduced scheme, $f : X \to Y$ a morphism, Z a closed subscheme of Y, $j : Z \hookrightarrow Y$, such that $f(X) \subseteq j(Z)$. Then f factors uniquely as*

$$X \xrightarrow{g} Z \xhookrightarrow{j} Y.$$

Proof. First assume X and Y are affine, $X = \mathrm{Spec}\ A$, $Y = \mathrm{Spec}\ B$, $Z = \mathrm{Spec}\ B/I$. (Use exercise 3.11 to see that every closed subscheme Z of Y is of the form Spec B/I.) Let $\varphi : B \to A$ be the homomorphism which induces f. Since $f(X) \subseteq Z$, the inverse image of any prime of A contains I. Since A is reduced the intersection of all primes of A equals $\{0\}$. Thus

$$\ker(\varphi) = \varphi^{-1}(\{0\}) = \varphi^{-1}(\cap_{\mathrm{primes} p} p) \subseteq I$$

so φ factors uniquely through B/I.

$$B \xrightarrow{j^\#} B/I \xhookrightarrow{g^\#} A$$

This proves the proposition when X and Y are affine.

Now suppose X is an arbitrary reduced scheme. Cover X by open affines $X = \cup_i U_i$. For each i let g_i be the unique map which factors $f|_{U_i}$ through Z. By uniqueness $g_i|_{U_i \cap U_j} = g_j|_{U_i \cap U_j}$, so we can glue the g_i to obtain a morphism $g : X \to Z$ such that $j \circ g = f$. Now suppose both X and Y are arbitrary. Cover Y by open affines, take their inverse images in X, perform the construction locally for each one, use uniqueness and glue.

Counterexamples.

(a) Let $A = k[x, y]/(x^2, xy)$, let $X = Y = \mathrm{Spec}\ A$ and let $S = \mathrm{Spec}\ k$. Then Y is affine hence seperable over S, but X is not reduced. Let $f : X \to Y$ be the morphism induced by the identity homomorphism $\mathrm{id} : A \to A$. Let $g : X \to Y$ be the morphism induced by the homomorphism $\varphi : A \to A$:

$x \mapsto 0, y \mapsto y$. Let $U = D(y) = \text{Spec } A_y$. Then since $A_y \cong \text{Spec } k[y, y^{-1}]$, the localized homomorphisms agree, $\text{id}_y = \varphi_y$. Thus $f|_U = g|_U$. Now X is irreducible since A has just one minimal prime, namely (x), so U is dense in X. But, $f \neq g$. since $f^\# = \text{id} \neq \varphi = g^\#$.

(b) Let X be the affine line and Y the affine line with a doubled origin both thought of as schemes over $S = \text{Spec } k$. Let $f_1 : X \to Y$ be one of the inclusions of the affine line in Y and let $f_2 : X \to Y$ be the other one. Then f_1 and f_2 agree on X minus the origin but not on X.

[Reference, EGA, I.8.2.2.1.]

Problem 7 (4.4) *Let $f : X \to Y$ be a morphism of separated schemes of finite type over a noetherian scheme S. Let Z be a closed subscheme of X which is proper over S. Show that $f(Z)$ is closed in Y, and that $f(Z)$ with its image subscheme structure is proper over S.*

PROOF. First we show that since X, Y and Z are of finite type over S and S is noetherian, X, Y and Z are noetherian. Suppose $g : X \to S$ is the map from X to S. Cover S by finitely many Spec A_i, A_i noetherian. Then for each $f^{-1}(\text{Spec } A_i) = \cup_j \text{Spec } B_{ij}$, with B_{ij} a finitely generated A_i-algebra. Since A_i is noetherian each B_{ij} is noetherian (this is the Hilbert basis theorem). Since $X = \cup_{ij} \text{Spec } B_{ij}$, X is noetherian. One shows that Y and Z are noetherian in exactly the same way.

Since the following diagram commutes 4.8(e) implies $f|_Z$ is proper.

Thus $f|_Z(Z)$ is closed in Y. (I'm assuming Z is an S-subscheme of X so that the diagram must commute.)

We now have $f(Z) \hookrightarrow Y \to S$. We must show the composition $f(Z) \to S$ is proper. By Corollary 4.8a the closed immersion $f(Z) \hookrightarrow Y$ is proper. We are given that the map $Y \to S$ is seperated and of finite type. Since the composition of seperated morphisms is seperated and the composition of finite type morphisms is of finite type, the morphism $f(Z) \to S$ is seperated and of finite type. The hard part is to show that it is universally closed.

Since the morphism $Z \to S$ is proper it is closed and from above the morphism $Z \to f(Z)$ is closed so the morphism $f(Z) \to S$ is closed. Let W be an any scheme over S. We must show that the map $f(Z) \times_S W \to W$ is closed. We have the following diagram.

If we can show that g_1 is surjective we will be done. For then if A is closed subset of $f(Z) \times_S W$,

$$g_2(A) = g_3(g_1^{-1}(A)),$$

which, since g_3 is closed, is also closed. (We wouldn't have equality in the above expression if g_1 weren't surjective.)

In order to establish the surjectivity of g_1 we prove that the property of being a surjective morphism is preserved under base extension. It will then follow, since g_1 is a base extension of the surjective morphism $Z \to f(Z)$, that g_1 is surjective.

Proposition 6 *Let X and Y be schemes over S. Suppose $x \in X$ and $y \in Y$ both lie over the same point $s \in S$. Then there exists $\alpha \in X \times_S Y$ such that $p_X(\alpha) = x$ and $p_Y(\alpha) = y$.*

Proof. Let $g_1 : \text{Spec } (k(x)) \to X$ and $g_2 : \text{Spec } (k(y)) \to Y$ be the natural maps with $g_1((0)) = x$ and $g_2((0)) = y$. Let $Z = \text{Spec } (k(x)) \times_{\text{Spec } (k(s))} \text{Spec } (k(y)) = \text{Spec } (k(x) \otimes_{k(s)} k(y))$, and let π_1 be the projection to Spec $(k(x))$, π_2 the projection to Spec $(k(y))$. Let $g = g_1 \times_S g_2$ be the product of $g_1 \circ \pi_1$ with $g_2 \circ \pi_2$. So $g : Z \to X \times_S Y$. See the following diagram.

Since Z is the spectrum of the tensor product of two fields over a common base field, $Z \neq \emptyset$ so there is some $z \in Z$. By the definition of g, $g_1 \circ \pi_1 = p_x \circ g$ and $g_2 \circ \pi_2 = p_y \circ g$ so $x = g_1 \circ \pi_1(z) = p_x \circ g(z)$ and $y = g_2 \circ \pi_2(z) = p_y \circ g(z)$ so we may take $\alpha = g(z)$.

Proposition 7 *If $f : X \to Y$ is a surjective S-morphism then $f \times 1 : X \times_S S' \to Y \times_S S'$ is surjective.*

Proof. We have the following diagram.

Let $y' \in Y \times_S S'$. Then $q(y') \in Y = f(X)$ so there is $x \in X$ such that $f(x) = q(y')$. Then by the above proposition there is some $\alpha \in X \times_S S'$ such that $p(\alpha) = x$ and $(f \times 1)(\alpha) = y'$. Thus $f \times 1$ is surjective.

Algebraic Geometry Homework
II.5, 1,2,3,4,5,7,8

William A. Stein

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Problem 1 (II.5.1) *Let $(X, \mathcal{O}_X)$ be a ringed space, and let $\mathcal{E}$ be a locally free $\mathcal{O}_X$-module of finite rank. We define the dual of $\mathcal{E}$, denoted by $\check{\mathcal{E}}$, to be the sheaf $\mathcal{H}\mathrm{om}_{\mathcal{O}_X}(\mathcal{E}, \mathcal{O}_X)$.*

(a) Show that $(\check{\mathcal{E}})\check{} \cong \mathcal{E}$.

(b) For any $\mathcal{O}_X$-module $\mathcal{F}$, $\mathcal{H}\mathrm{om}_{\mathcal{O}_X}(\mathcal{E}, \mathcal{F}) \cong \check{\mathcal{E}} \otimes_{\mathcal{O}_X} \mathcal{F}$.

(c) For any $\mathcal{O}_X$-modules $\mathcal{F}, \mathcal{G}$, $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{E}\otimes\mathcal{F}, \mathcal{G}) \cong \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{H}\mathrm{om}_{\mathcal{O}_X}(\mathcal{E}, \mathcal{G}))$.

(d) (Projection Formula). If $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a morphism of ringed spaces, if $\mathcal{F}$ is an $\mathcal{O}_X$-module, and if $\mathcal{E}$ is a locally free $\mathcal{O}_Y$-module of finite rank, then there is a natural isomorphism $f_*(\mathcal{F} \otimes_{\mathcal{O}_X} f^*\mathcal{E}) \cong f_*(\mathcal{F}) \otimes_{\mathcal{O}_Y} \mathcal{E}$.

Proof. (a) A free module of finite rank is *canonically* isomorphic to its double-dual via $\check{m}(\lambda) = \lambda(m)$ where $m \in M$, $\lambda \in \check{M}$, and $\check{m} \in (\check{M})\check{}$. Let U be an open set on which $\mathcal{E}|U$ is a free $\mathcal{O}_X$-module of finite rank. Define a map $\varphi : \mathcal{E}|_U \to (\check{\mathcal{E}})\check{}|_U$ by, for all $V \subseteq U$, $\mathcal{E}(V) \to (\check{\mathcal{E}})\check{}(V)$ is the isomorphism described above. Since the isomorphisms are canonical, we can patch on intersections and define a global isomorphism.

(b) As above we may assume $\mathcal{E}$ is a free $\mathcal{O}_X$-module. Let $e_1, \ldots, e_n$ be a basis for $\mathcal{F}$ and let $e_1^*, \ldots, e_n^*$ be the corresponding dual basis. Let U be an open subset of X. Define $\varphi_U : \mathcal{H}\mathrm{om}_{\mathcal{O}_X}(\mathcal{E}, \mathcal{F})|_U \to (\mathrm{Hom}(\mathcal{E}, \mathcal{O}_X) \otimes_{\mathcal{O}_X} \mathcal{F})|_U$ by $f \mapsto \sum_{i=1}^n e_i^* \otimes f(e_i)$. Define $\psi_U : (\mathrm{Hom}(\mathcal{E}, \mathcal{O}_X) \otimes_{\mathcal{O}_X} \mathcal{F})|_U \to \mathcal{H}\mathrm{om}_{\mathcal{O}_X}(\mathcal{E}, \mathcal{F})|_U$ by $f \otimes a \mapsto (x \mapsto f(x)a)$. For convenience of notation write $\varphi = \varphi_U$ and $\psi = \psi_U$. Let $f \otimes a \in (\mathrm{Hom}(\mathcal{E}, \mathcal{O}_X) \otimes_{\mathcal{O}_X} \mathcal{F})|_U$. Then $\varphi \circ \psi(f \otimes a) = \varphi(x \mapsto f(x)a) = \sum_{i=1}^n e_i^* \otimes f(e_i)a = \sum_{i=1}^n e_i^* f(e_i) \otimes a = f \otimes a$. Let $f \in \mathcal{H}\mathrm{om}_{\mathcal{O}_X}(\mathcal{E}, \mathcal{F})$, then $\psi \circ \varphi(f) = \psi(\sum_{i=1}^n e_i^* \otimes f(e_i)) = (x \mapsto \sum_{i=1}^n e_i^*(x) f(e_i)) = (x \mapsto$

$f(\sum_{i=1}^n e_i^*(x)e_i)) = (x \mapsto f(x))$. Thus φ and ψ are inverse bijective homomorphisms, hence ring isomorphisms, and since they respect the restriction maps we see that the corresponding sheaves are isomorphic.

(c) On each open set define $\varphi|_U : \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{E}\otimes\mathcal{F}, \mathcal{G}) \to \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{E}, sg))$ by $f \mapsto (a \mapsto (e \mapsto f(e\otimes a)))$. (For notational convenience we omit the sheaf restrictions.) If $\varphi(f) = 0$ then the map $(a \mapsto (e \mapsto f(e \otimes a)))$ is 0 so f is the zero map, hence φ is injective. Let $f \in \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{E}, sg))$. Define $g \in \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{E} \otimes \mathcal{F}, \mathcal{G})$ by $g(a \otimes b) = (f(b))(a)$. Then $\varphi(g) = (a \mapsto (e \mapsto g(e \otimes a))) = (a \mapsto (e \mapsto (f(a))(e)) = (a \mapsto f(a)) = f$ so φ is surjective. Thus φ is the desired isomorphism which, since φ evidently commutes with the restriction maps, induces an isomorphism of sheaves.

(d) First we consider the case when $\mathcal{E} \cong \mathcal{O}_Y^n$. One one hand,

$$\begin{aligned} f_*(\mathcal{F}) \otimes_{\mathcal{O}_Y} \mathcal{E} &\cong f_*(\mathcal{F}) \otimes_{\mathcal{O}_Y} \mathcal{O}_Y^n \\ &\cong \oplus_{i=1}^n (f_*(\mathcal{F}) \otimes_{\mathcal{O}_Y} \mathcal{O}_Y) \\ &\cong \oplus_{i=1}^n f_*(\mathcal{F}). \end{aligned}$$

On the other hand,

$$\begin{aligned} \mathcal{F} \otimes_{\mathcal{O}_X} f^*\mathcal{E} &= \mathcal{F} \otimes_{\mathcal{O}_X} f^*(\mathcal{O}_Y^n) \\ &\cong \mathcal{F} \otimes_{\mathcal{O}_X} (f^{-1}(\mathcal{O}_Y^n) \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X) \\ &\cong \mathcal{F} \otimes_{\mathcal{O}_X} ((f^{-1}(\mathcal{O}_Y)^n \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X) \\ &\cong \mathcal{F} \otimes_{\mathcal{O}_X} (\sum_{i=1}^n f^{-1}\mathcal{O}_Y \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X) \\ &\cong \mathcal{F} \otimes_{\mathcal{O}_X} (\mathcal{O}_X^n) \\ &\cong (\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X)^n = \mathcal{F}^n \end{aligned}$$

where $f^{-1}(\mathcal{O}_Y^n) \cong f^{-1}(\mathcal{O}_Y)^n$ since f^{-1} is a left adjoint functor hence commutes with direct sums (which are a right universal construction).

Putting this together we have that

$$f_*(\mathcal{F} \otimes_{\mathcal{O}_Y} f^*(\mathcal{E})) = f_*(\mathcal{F}^n) = f_*(\oplus_{i=1}^n \mathcal{F}) = \oplus_{i=1}^n f_*(\mathcal{F}).$$

In general, we construct isomorphisms as above on an open cover then, since all of the isomorphisms are canonical, the isomorphisms match up on the intersections so we can glue to obtain an isomorphism.

Problem 2 (II.5.2) *Let R be a discrete valuation ring with quotient field K, and let $X = SpecR$.*

(a) To give an $\mathcal{O}_X$-module $\mathcal{F}$ is equivalent to giving an R-module M, a K-vector space L, and a homomorphism $\rho : M \otimes_R K \to L$.

(b) That $\mathcal{O}_X$-module is quasi-coherent if and only if ρ is an isomorphism.

Proof. (a) First suppose we are given an $\mathcal{O}_X$-module $\mathcal{F}$. Since R is a DVR, X has exactly two nonempty open sets, X and the set consisting of the generic point, $\{\xi\}$. Let $M = \Gamma(\mathcal{F}, X)$ and let $L = \Gamma(\mathcal{F}, \{\xi\})$. Since $\Gamma(\mathcal{O}_X, X) = R$ and $\Gamma(\mathcal{O}_X, \{\xi\}) = K$, M is an R-module and L is a K-vector space. Let $g : M \to L$ be the restriction map. Define ρ by $\rho(m \otimes \alpha) = \alpha \cdot g(m)$. Since g is a homomorphism, so is ρ.

Now suppose we are given an R-module M, a K-vector space L, and a homomorphism $\rho : M \otimes_R K \to L$. Define an $\mathcal{O}_X$-module $\mathcal{F}$ as follows. Let $\Gamma(\mathcal{F}, X) = M$ and $\Gamma(\mathcal{F}, \{\xi\}) = L$. Define the restriction map $g : M \to L$ by $g(m) = \rho(m \otimes 1)$. We just need to check that g is a valid restriction map. Let $r \in R, m \in M$, then g is a valid restriction map iff $r \cdot g(m) = g(rm)$, that is, when $r \cdot \rho(m \otimes 1) = \rho(rm \otimes 1) = \rho(m \otimes r)$. So we must verify that the given homomorphism ρ is R-linear. But there is absolutely no reason why this should be the case! (For example, let $M = L = K$, then $\rho : K \to K$ and it is easy to construct nontrivial homomorphisms of the additive group of a field). I think the problem is imprecisely stated. It should be assumed throughout that ρ is K-linear.

(b) First suppose $\mathcal{F}$ is quasi-coherent. Then $\mathcal{F} = \tilde{M}$ so Proposition 5.1 implies that $L = \Gamma(\tilde{M}, \{\xi\}) = (R - 0)^{-1}M \cong M \otimes_R K$. Thus ρ must be an isomorphism. Conversely, if ρ is an isomorphism, we see that $\mathcal{F} \cong \tilde{M}$ since they are the same on each open set and the restriction map is the same.

Problem 3 (II.5.3) *Let $X = SpecA$ be an affine scheme. Show that the functors $\tilde{}$ and Γ are adjoint, in the following sense: for any A-module M, and for any sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$, there is a natural isomorphism*

$$\mathrm{Hom}_A(M, \Gamma(X, \mathcal{F})) \cong \mathrm{Hom}_{\mathcal{O}_X}(\tilde{M}, \mathcal{F}).$$

Proof.

Define a homomorphism $F : \mathrm{Hom}_A(M, \Gamma(X, \mathcal{F})) \to \mathrm{Hom}_{\mathcal{O}_X}(\tilde{M}, \mathcal{F})$ as follows. Send a ring homomorphism $\varphi : M \to \Gamma(X, \mathcal{F})$ to the morphism of sheaves $F(\varphi) : \tilde{M} \to \mathcal{F}$. It suffices to define $F(\varphi)$ on distinguished open sets (Eisenbud & Harris, page 13). For $f \in A$ let $F(\varphi)_{D(f)}$ be the map

$$\frac{m}{f^n} \mapsto \frac{1}{f^n} \cdot \mathrm{res}_{X,D(f)}(\varphi(m))$$

where $\mathrm{res}_{X,D(f)} : \mathcal{F}(X) \to \mathcal{F}(D(f))$ is the restriction map of $\mathcal{F}$. $F(\varphi)_{D(f)}$ is a well-defined homomorphism since both φ and $\mathrm{res}_{X,D(f)}$ are homomorphisms and φ is an A-module homomorphism. Next note that $F(\varphi)$ commutes with the restriction maps since each $\mathrm{res}_{X,D(f)}$ does. To see that F is injective suppose φ and ψ are two homomorphisms $M \to \Gamma(X, \mathcal{F})$. If $F(\varphi) = F(\psi)$ then, in particular, $\varphi = F(\varphi)_X = F(\psi)_X = \psi$. To see that F is surjective, let $\varphi \in \mathrm{Hom}_{\mathcal{O}_X}(\tilde{M}, \mathcal{F})$. Define $\psi : M \to \Gamma(X, \mathcal{F})$ by letting $\psi = \varphi_X$, that is, by taking the induced map on global sections. Then, for $f \in A$, $F(\psi)_{D(f)} : \tilde{M}(D(f)) \to \mathcal{F}(D(f))$ is the map $(\frac{m}{f^n} \mapsto \frac{1}{f^n}\mathrm{res}_{X,D(f)} \circ \varphi_X(m))$ which, since φ is a morphism of sheaves, equals $= (\frac{m}{f^n} \mapsto \frac{1}{f^n}\varphi_{D(f)}(m) = \varphi_{D(f)}(\frac{m}{f^n}) = \varphi_{D(f)}$. (We are just using the fact that φ commutes with the appropriate restriction maps.) Thus $F(\psi)$ agrees with φ on a basis for X hence $F(\psi) = \varphi$. This shows that F is surjective. So F is an isomorphism, as required.

Problem 4 (II.5.4) *Show that a sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ on a scheme X is quasi-coherent if and only if every point of X has a neighborhood U, such that $\mathcal{F}|_U$ is isomorphic to a cokernel of a morphism of free sheaves on U. If X is noetherian, then $\mathcal{F}$ is coherent iff it is locally a cokernel of a morphism of free sheaves of finite rank.*

Proof. Suppose first that $\mathcal{F}$ is quasi-coherent. Let $x \in X$. Then there is an affine open neighborhood $U = Spec A$ of x such that $\mathcal{F}|_U \cong \tilde{M}$, M an A-module. It suffices to show that M is isomorphic to a cokernel of a morphism of finitely generated A-algebras. Indeed, if $\varphi : A^{(I)} \to A^{(J)}$ then $\mathrm{Coker}(\tilde{\varphi}) = \tilde{A^{(J)}}/\varphi(A^{(I)})\tilde{} = (A^{(J)}/\varphi(A^{(I)}))\tilde{}$ since, for all $f \in A$, $(A^{(J)})_f/\varphi(A^{(I)})_f = (A^{(J)}/\varphi(A^{(I)}))_f$ so that they agree on a basis.

Let $A^{|M|}$ be the free A-module on the elements of M. Let $\varphi : A^{|M|} \to M$ be the natural map. Similiary, let $\psi : A^{|\ker(\varphi)|} \to \ker(\varphi) \subseteq A^{|M|}$ be the natural map. Then $\mathrm{Coker}(\psi) \cong A^{|M|}/\ker(\varphi) \cong M$, as required.

Now assume the $\mathcal{F}$ is coherent. We proceed as above but now M is a finitely generated A-module, generated by $e_1, \ldots, e_n$, say, and we must show that M is the cokernel of a morphism of free modules of finite rank. Let

$\varphi : A^n \to M$ be the map which takes the ith generator $(0, \dots, 1, \dots, 0)$ of A^n to $e_i \in M$. Then $\ker(\varphi)$ is a submodule of M so, since M is neotherian (any finitely generated module over a noetherian ring is noetherian), $\ker(\varphi)$ is finitely generated. Let $f_1, \dots, f_m$ be a generating set. Let $\psi : A^m \to \ker(\varphi)$ be the surjection defined by sending the ith basis element of A^m to f_i. Then $\mathrm{Coker}(\psi) \cong A^n/\psi(A^m) \cong A^n/\ker(\varphi) \cong M$. Thus M is isomorphic to a cokernel of a morphism of free sheaves of finite rank.

Problem 5 (II.5.5) *Let $f : X \to Y$ be a morphism of schemes.*

(a) Show by example that if $\mathcal{F}$ is coherent on X, then $f_*\mathcal{F}$ need not be coherent on Y, even if X and Y are varieties over a field k.

(b) Show that a closed immersion is a finite morphism.

(c) If f is a finite morphism of neotherian schemes, and if $\mathcal{F}$ is coherent on X, then $f_*\mathcal{F}$ is coherent on Y.

Proof. (a) Lex k be a field, let $X = Spec(k[x]_x)$, $Y = Spec(k[x])$ and $\mathcal{F} = \mathcal{O}_X$. Then $f_*(\mathcal{F})(U) = \mathcal{F}(f^{-1}(U))$, so $f_*(\mathcal{F}) = (k[x]_x)\tilde{}^Y$. But $(k[x]_x)\tilde{}^Y$ is not a coherent sheaf of $\mathcal{O}_Y$-modules. Indeed, if it were, there would be a distinguished neighborhood $D(f)$ of 0 so that $(k[x]_x)\tilde{}^Y|_{D(f)}$ is a finitely generated module over $k[x]$ (here I'm using the fact that over open set is distinguished in the topology of $Spec(k[x])$). But, $k[x]_f$ is never a finitely generated module over $k[x]$ for any f of degree at least 1. For $k[x]_f$ contains elements of arbitrarily small degree whereas the degrees of elements of $k[x]\{\alpha_1, \dots, \alpha_n\}$ are bounded below. ($k[x]\{\alpha_1, \dots, \alpha_n\}$ is the $k[x]$-module generated by $\alpha_1, \dots, \alpha_n$.)

(b) Let $f : Y \to X$ be a closed immersion. Let $U = Spec(A) \subseteq X$ and let $W = f^{-1}(U) \subseteq Y$. Then W is a a scheme (give it the induced scheme structure as an open subset of Y). Furthermore, $f(W) = U \cap f(Y)$ is a relatively closed subset of U, that is, a closed subset of $Spec(A)$. The map $f^\# : \mathcal{O}_U \to \mathcal{O}_W$ is surjective since $f^\# : \mathcal{O}_X \to \mathcal{O}_Y$ is surjective and surjectivety is a local property. Thus W is a closed subscheme of $Spec(A)$ so, by Corollary 5.10, $W \cong Spec(A/I)$ for some ideal I of A. Since A/I is a finitely generated A module (generated by $1 + I$), f is a finite morphism.

(c) Let $U = Spec(A) \subseteq Y$ be an affine open subset of Y. Then, since f is finite, $f^{-1}(U) = Spec(B)$ is affine with B is a finitely generated A-module. By Proposition 5.4 it suffices to show that $f_*(\mathcal{F})|_U$ is $\tilde{M}$ for some finitely generated A-module M. Now by Proposition 5.4, since $f^{-1}(U)$ is affine and B is noetherian, $\mathcal{F}|_{f^{-1}(U)} = \tilde{M}$ for some finitely generated B module M.

But $f_*(\mathcal{F})|_U = (f|_{f^{-1}(U)})_*\mathcal{F}|_{f^{-1}(U)} = (f|_{f^{-1}(U)})_*(\tilde{M}) = (\tilde{A^M})$ where the last equality follows from Proposition 5.2(d). Since B is a finite module over A and M is a finite module over B, it follows that M is a finite module over A which completes the proof.

Problem 6 (II.5.7) *Let X be a noetherian scheme, and let sf be a coherent sheaf.*

(a) If the stalk $\mathcal{F}_x$ is a free $\mathcal{O}_X$-module for some point $x \in X$, then there is a neighborhood U of x such that $\mathcal{F}|_U$ is free.

(b) $\mathcal{F}$ is locally free iff its stalks $\mathcal{F}_x$ are free $\mathcal{O}_X$-modules for all $x \in X$.

(c) $\mathcal{F}$ is invertible iff there is a coherent sheaf $\mathcal{G}$ such that $\mathcal{F} \otimes \mathcal{G} \cong \mathcal{O}_X$.

Proof. (a) Let $U = Spec(A)$ be a neighborhood of x so that $\mathcal{F}|_U = \tilde{M}$, M a finitely generated A-module. Then $\mathcal{F}_x = M_x$ so we have reduced the problem to the following purely algebraic result.

Proposition 1 *If M is a finitely generated free A module and there is a prime $\wp$ of A such that $M_\wp$ is a free $A_\wp$-module, then there exists $f \in A$ such that $f \notin \wp$ and M_f is a free A_f-module.*

Once we have proven this we will know that $\mathcal{F}|_{D_U(f)}$ is free since Proposition 5.1 asserts that $\tilde{M}(D(f)) \cong M_f$.

Proof. (of Proposition) Let $a_1, \ldots, a_n \in M$ be a free basis of $M_\wp$ over $A_\wp$ (we can clear denominators so that we may assume all a_i lie in M.) Let $b_1, \ldots, b_m \in M$ be a generating set for M over A. For each i we can write b_i as an $A_\wp$-linear combination of the a_i. Clearing denominators we see that $d_i b_i \in A\{a_1, \ldots, a_n\}$ for some $d_i \notin \wp$. Let $f = \prod d_i$, then $f \notin \wp$ and $a_1, \ldots, a_n$ have A_f-span including all of the b_i, and thus including M, and thus including M_f. But $a_1, \ldots, a_n$ is free over $A_\wp$ hence over A_f since $A_f \subseteq A_\wp$. [This proposition is in Bourbaki, *Commutative Algebra*, II.5.1, although the proof is more abstract than mine.]

(b) ($\Longrightarrow$) Let $x \in X$ and let $U = Spec(A)$ be an open neighborhood of x such that $\mathcal{F}|_U = \tilde{M}$ with M a free A module. Suppose $\wp$ is the prime of A corresponding to x. Then $\mathcal{F}_x = M_\wp$ which is a free $A_\wp$-module. Indeed, if $e_1, \ldots, e_n$ is a free A-basis for M, then it is also a free $A_\wp$-basis for $M_\wp$. For if $\frac{a_1}{b_1}e_1 + \cdots + \frac{a_n}{b_n}e_n = 0, b_i \notin \wp$, then $a_1\frac{b}{b_1}e_1 + \cdots + a_n\frac{b}{b_n}e_n = 0, b = \prod b_i$ so $b\frac{a_i}{b_i} = 0$ for each i, so $\frac{a_i}{b_i} = 0$ in the localization $A_\wp$ since $b \notin \wp$.

($\Longleftarrow$) By part (a) every point has a neighborhood on which $\mathcal{F}$ is free. Therefore X can be covered by open affines on which $\mathcal{F}$ is free so $\mathcal{F}$ is locally free.

(c) ($\Longrightarrow$) Let $\mathcal{G} = \check{\mathcal{F}} = \mathcal{H}\mathrm{om}(\mathcal{F}, \mathcal{O}_X)$, we will show that $\mathcal{F} \otimes \mathcal{G} \cong \mathcal{O}_X$. To define a morphism $\varphi : \mathcal{F} \otimes \mathcal{H}\mathrm{om}(\mathcal{F}, \mathcal{O}_X) \to \mathcal{O}_X$ it is enough to define φ on the presheaf $(U \mapsto \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{H}\mathrm{om}(\mathcal{F}(U), \mathcal{O}_X(U))$. Define φ by $a \otimes f \mapsto f(a)$. Thus φ commutes with the restrictions so φ defines a valid morphism of sheaves.

Let U be an open affine subset of X such that $\mathcal{F}(U)$ is a free $\mathcal{O}_X(U)$-module of rank 1 with basis e_0. Then $\mathcal{H}\mathrm{om}(\mathcal{F}(U), \mathcal{O}_X(U))$ has basis e_0^* where $e_0^*(e_0) = 1$. Thus $\mathcal{H}\mathrm{om}(\mathcal{F}(U), \mathcal{O}_X(U))$ is a free $\mathcal{O}_X(U)$-module of rank 1. Thus every element of $\mathcal{F}(U) \otimes \mathrm{Hom}(\mathcal{F}(U), \mathcal{O}_X(U))$ can be written in the form $a \otimes f$ (as opposed to as a sum of such products). Now $\varphi_U(a \otimes f) = 0$ implies $f(a) = 0$ which implies $a = 0$ or $f = 0$ so $a \otimes f = 0$, so φ_U is injective. Since $\varphi_U(ae_0 \otimes e_0^*) = a$, φ_U is surjective. Thus φ_U is an isomorphism.

Use the definition of locally free of rank 1 to cover X by affine open sets U such that $\mathcal{F}|_U$ is a free $\mathcal{O}_X|_U$ module of rank 1. Then, by Proposition 5.1 (c) and an argument like that for (b) above, any distinguished open subset of U has a $\mathcal{F}$-sections free of rank 1. Since φ is an isomorphism on each of these distinguished open sets (use the argument in the above paragraph) and these distinguished open sets form a basis for the topology on X it follows that φ must be an isomorphism.

($\Longleftarrow$) Because of part (b) above it suffices to show that $\mathcal{F}_x$ is free of rank one for each $x \in X$. Since $\mathcal{F}_x \otimes_{\mathcal{O}_x} \mathcal{G}_x = (\mathcal{F} \otimes \mathcal{G})_x \cong \mathcal{O}_{X,x}$ the problem is reduced to the following purely algebraic statement.

Proposition 2 *Let M and N be finitely generated modules over a local ring (A, m) and suppose that $M \otimes_A N \cong A$. Then M is free of rank 1.*

Proof. Let $k = A/m$. Let $\nu : M \otimes_A N \to A$ be the given isomorphism. Then, taking the product with the identity, we get an isomorphism $\nu \otimes 1_k : (M \otimes_A N) \otimes_A k \to A \otimes_A k \cong k$ (it is obvious that $\nu \otimes 1_k$ is surjective, but it is not at all obvious that it is injective, for this see the Bourbaki reference below.) Thus $k \cong M \otimes_A N \otimes_A k = M \otimes_A (k \otimes_A k) \otimes_A N = (M \otimes_A k) \otimes_A (N \otimes_A k) = (M \otimes_A k) \otimes_k (N \otimes_A k) = (M/mM) \otimes_k (N/mN)$ so, since the k-rank of $(M/mM) \otimes_k (N/mN)$ is 1 and is the product of the ranks of M/mM and N/mN, each has rank 1. In particular, M/mM is monogeneous (generated by one element) as an A/m-module and hence as an A-module so, by Nakayama's lemma, M is monogeneous as an A-module. Since $\mathrm{Ann}_A(M)$ anihilates $M \otimes_A N = A$ as well, it is 0 (any element of $\mathrm{Ann}_A(M)$ would have to annihilate the identity of A and hence be 0). Thus

M is a free module of rank one over A. [See Bourbaki, *Commutative Algebra*, II.5.4, for a more general theorem.]

Problem 7 (II.5.8) *Again let X be a noetherian scheme, and $\mathcal{F}$ a coherent sheaf on X. We will consider the function*

$$\varphi(x) = \dim_{k(x)} \mathcal{F}_x \otimes_{\mathcal{O}_X} k(x)$$

where $k(x) = \mathcal{O}_X/m_x$ is the residue field at the point x. Use Nakayama's lemma to prove the following results.

(a) The function φ is upper semi-continuous, i.e., for any $n \in \boldsymbol{Z}$, the set $\{x \in X : \varphi(x) \geq n\}$ is closed.

(b) If $\mathcal{F}$ is locally free, and X is connected, then φ is a constant function.

(c) Conversely, if X is reduced, and φ is constant, then $\mathcal{F}$ is locally free.

Proof.

(a) We must show that $\{x : \varphi(x) \geq n\}$ is closed. A good way to do this is by showing that $\{x : \varphi(x) < n\}$ is open. To do this we show that if $\varphi(x) = m$ then there is an open neighborhood U of x so that, for all $y \in U$, $\varphi(y) \leq m$. Since we need only look locally, we can assume that $X = Spec A$, $\mathcal{F} = \tilde{M}$, M a finitely generated A-module. Note that $\mathcal{F}_x \otimes_{\mathcal{O}_X} k(x) = M_\wp \otimes_{A_\wp} A_\wp/\wp A_\wp = M_\wp/\wp M_\wp$. Let $s_1, \ldots, s_m \in M$ be elements whose images form a basis for the vector space $M_\wp/\wp M_\wp$ over $A_\wp/\wp A_\wp$ (to do this choose a basis for $M_\wp/\wp M_\wp$ then clear fractions). Note that the images of the s_i in fact generate $M_\wp/\wp M_\wp$ as an $A_\wp$-module. By Nakayama's lemma the s_i generate $M_\wp$ as an $A_\wp$-module. Let $m_1, \ldots, m_k$ be a generating set for M over A. Write $m_j = \sum \frac{a_i}{b_i} s_i$, $b_i \notin \wp$, then, if $c_j = \prod b_i$, $c_j m_j$ is in the A-span of the s_i. Let $f = \prod c_j$. Then $\wp \in D(f)$ and if $q \in D(f)$, then $m_1, \ldots, m_k$ all lie in the A_q-span of $s_1, \ldots, s_m$ (since $c_j m_j$ is in the A-span of the s_i and c_j is inverted in A_q. Thus M is spanned by the s_i over A_q, so M_q is spanned by the s_i over A_q. It follows that $\varphi(q) = \dim M_q/qM_q \leq m$ since the images of the s_i generate M_q/qM_q as a vector space over $A_q/qM_q = k(q)$. Taking $D(f)$ as our open neighborhood completes the proof.

(b) Choose n so that some section of $\mathcal{F}$ has rank n. Let U be the union of all open sets W such that $\mathcal{F}|_W \cong \mathcal{O}_X^n|_W$. Then U is nonempty. Let V be the union of all open sets W such that $\mathcal{F}|_W \cong \mathcal{O}_X^m|_W$, $m \neq n$. Since $\mathcal{F}$ is locally free, $U \cup V = X$. Suppose $x \in U \cap V$, then $\mathcal{F}_x$ has rank n and rank $m \neq n$ (since rank is preserved under localization), a contradiction. Thus $U \cap V = \emptyset$. Since U is nonempty and open, $X - U = V$ is open and X

is connected, thus we conclude that $V = X - U = \emptyset$. Thus every point is contained in an open set W such that $\mathcal{F}|_W \cong \mathcal{O}_X^n|_W$.

Let $x \in X$ and let $U = Spec(A)$ be an affine open set containing x such that $\mathcal{F}|_U \cong \tilde{M}$. By the above argument, M is a free A-module of rank n. Thus $\varphi(x) = \dim_k M_x \otimes_{A_x} A_x/m_x = \dim_k A_x^n \otimes_k A_x/m_x = \dim_k (A_x/m_x)^n = \dim_k k^n = n$, as desired.

(c) Let $x \in X$. By exercise 5.7b it suffices to show that the stalk $\mathcal{F}_x$ is free. Since $\mathcal{F}$ is coherent we can find an affine open set $U = Spec(A)$ such that $\mathcal{F}|_U = \tilde{M}$ for some finitely generated A-module, $x \in U$, and A_f is reduced for each $f \in A$. Let $\wp$ be the prime of A corresponding to x. We must show that $M_\wp$ is free over $A_\wp$. Let $s_1, \ldots, s_n \in M$ be preimages of a basis of $M_\wp/\wp M_\wp$ over $k(x) = A_\wp/\wp A_\wp$ (find these as in part (a)). Then, by Nakayama's lemma, the s_i generate $M_\wp$ over $A_\wp$.

We must show that the s_i are linearly independent over $A_\wp$. It will then follow that $M_\wp$ is free of rank n over $A_\wp$. So suppose

$$\frac{a_1}{b_1}s_1 + \cdots + \frac{a_n}{b_n}s_n = 0$$

in $M_\wp$ with $\frac{a_i}{b_i} \in A_\wp$. Then for each i, $b_i \notin \wp$ and $a_i \in A$. Since the s_i are linearly independent over $A_\wp/\wp A_\wp$, for each i there exists $c_i \notin \wp$ such that $\frac{c_i a_i}{b_i} \in \wp$. Thus $c_i a_i \in \wp$ so $a_i \in \wp$. Let r be as in part (a) so that $q \in D(r)$ implies the s_i generate at least M over A_q. By definition there exists $c \in A$ such that

$$c(b_2 \cdots b_n a_1 s_1 + \cdots + b_1 \cdots b_{n-1} a_n s_n) = 0$$

in A. Let $f = rc\prod b_i$, then if $q \in D(f)$, then $s_1, \ldots, s_n$ generate M_q/qM_q over A_q and, since M_q/qM_q has dimension n (since φ is constant), the s_i are actually a basis for M_q/qM_q over A_q/qA_q.

Since $c|f$, $c \notin q$ so, as above, $\frac{a_1}{b_1}s_1 + \cdots + \frac{a_n}{b_n}s_n = 0$ in M_q, so, as above, $a_i \in q$ for each i. Thus, for all $q \in D(f)$, $a_i \in q$, so a_i lies in the nilradical of A_f which, since A_f is reduced, means that $a_i = 0$ in A_f. So a_i maps to 0 under the map $A_f \to A_\wp$. Thus $s_1, \ldots, s_n$ are linearly independent over $A_\wp$ so $M_\wp$ is free of rank n over $A_\wp$. Applying exercise (5.7b) then completes the proof.

Homework 2, MAT256B
Chapter III, 4.8, 4.9, 5.6

William A. Stein

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1 Homework

Exercise 1.1. **(4.8)** *Cohomological Dimension.* Let X be a neotherian separated scheme. We define the *cohomological dimension* of X, denoted $\mathrm{cd}(X)$, to be the least integer n such that $H^i(X, \mathcal{F}) = 0$ for all quasi-coherent sheaves $\mathcal{F}$ and all $i > n$. Thus for example, Serre's theorem (3.7) says that $\mathrm{cd}(X) = 0$ if and only if X is affine. Grothendieck's theorem (2.7) implies that $\mathrm{cd}(X) \leq \dim X$.

(a) In the definition of $\mathrm{cd}(X)$, show that it is sufficient to consider only coherent sheaves on X.

(b) If X is quasi-projective over a field k, then it is even sufficient to consider only locally free coherent sheaves on X.

(c) Suppose X has a covering by $r+1$ open affine subsets. Use Čech cohomology to show that $\mathrm{cd}(X) \leq r$.

(d) If X is a quasi-projective variety of dimension r over a field k, then X can be covered by $r+1$ open affine subsets. Conclude that $\mathrm{cd}(X) \leq \dim X$.

(e) Let Y be a set-theoretic complete intersection of codimension r in $X = \mathbf{P}^n_k$. Show that $\mathrm{cd}(X - Y) \leq r - 1$.

Proof. (a) It suffices to show that if, for some i, $H^i(X, \mathcal{F}) = 0$ for all coherent sheaves $\mathcal{F}$, then $H^i(X, \mathcal{F}) = 0$ for all quasi-coherent sheaves $\mathcal{F}$. Thus suppose the ith cohomology of all coherent sheaves on X vanishes and let $\mathcal{F}$ be quasi-coherent. Let $(\mathcal{F}_\alpha)$ be the collection of coherent subsheaves of $\mathcal{F}$, ordered by inclusion. Then by (II, Ex. 5.15e) $\varinjlim \mathcal{F}_\alpha = \mathcal{F}$, so by (2.9)

$$H^i(X, \mathcal{F}) = H^i(X, \varinjlim \mathcal{F}_\alpha) = \varinjlim H^i(X, \mathcal{F}_\alpha) = 0.$$

(b) Suppose n is an integer and $H^i(X, \mathcal{F}) = 0$ for all coherent locally free sheaves $\mathcal{F}$ and integers $i > n$. We must show $H^i(X, \mathcal{F}) = 0$ for all coherent $\mathcal{F}$ and all $i > n$, then applying (a) gives the desired result. Since X is quasiprojective there is an open immersion

$$i : X \hookrightarrow Y \subset \mathbf{P}^n_k$$

with Y a closed subscheme of $\mathbf{P}^n_k$ and $i(X)$ open in Y. By (II, Ex. 5.5c) the sheaf $\mathcal{F}$ on X pushes forward to a coherent sheaf on $\mathcal{F}' = i_*\mathcal{F}$ on Y. By (II, 5.18) we may write $\mathcal{F}'$ as a quotient of a locally free coherent sheaf $\mathcal{E}'$ on Y. Letting $\mathbf{R}'$ be the kernel gives an exact sequence

$$0 \to \mathbf{R}' \to \mathcal{E}' \to \mathcal{F}' \to 0$$

with R' coherent (it's the quotient of coherent sheaves). Pulling back via i to X gives an exact sequence

$$0 \to \mathbf{R} \to \mathcal{E} \to \mathcal{F} \to 0$$

of coherent sheaves on X with E locally free. The long exact sequence of cohomology shows that for $i > n$, there is an exact sequence

$$0 = H^i(X, \mathcal{E}) \to H^i(X, \mathcal{F}) \to H^{i+1}(X, \mathbf{R}) \to H^{i+1}(X, \mathcal{E}) = 0.$$

$H^i(X, \mathcal{E}) = H^{i+1}(X, \mathcal{E}) = 0$ because we have assumed that, for $i > n$, cohomology vanishes on locally free coherent sheaves. Thus $H^i(X, \mathcal{F}) \cong H^{i+1}(X, \mathbf{R})$. But if $k = \dim X$, then Grothendieck vanishing (2.7) implies that $H^{k+1}(X, \mathbf{R}) = 0$ whence $H^k(X, \mathcal{F}) = 0$. But then applying the above argument with $\mathcal{F}$ replaced by $\mathbf{R}$ shows that $H^k(X, \mathbf{R}) = 0$ which implies $H^{k-1}(X, \mathcal{F}) = 0$ (so long as $k - 1 > n$). Again, apply the entire argument with $\mathcal{F}$ replaced by $\mathbf{R}$ to see that $H^{k-1}(X, \mathbf{R}) = 0$. We can continue this descent and hence show that $H^i(X, \mathcal{F}) = 0$ for all $i > n$.

(c) By (4.5) we can compute cohomology by using the Čech complex resulting from the cover $\mathfrak{U}$ of X by $r + 1$ open affines. By definition $\mathcal{C}^p = 0$ for all $p > r$ since there are no intersections of $p + 1 \geq r + 2$ distinct open sets in our collection of $r + 1$ open sets. The Čech complex is

$$\mathcal{C}^0 \to \mathcal{C}^1 \to \cdots \to \mathcal{C}^r \to \mathcal{C}^{r+1} = 0 \to 0 \to 0 \to \cdots .$$

Thus if $\mathcal{F}$ is quasicoherent then $\check{\mathrm{H}}^p(\mathfrak{U}, \mathcal{F}) = 0$ for any $p > r$ which implies that $\mathrm{cd}(X) \leq r$.

(d) I will first present my solution in the special case that X is projective. The more general case when X is quasi-projective is similiar, but more complicated, and will be presented next. Suppose $X \subset \mathbf{P}^n$ is a projective variety of dimension r. We must cover X with $r+1$ open affines. Let U be nonempty open affine subset of X. Since X is irreducible, the irreducible components of $X - U$ all have codimension at least one in X. Now pick a hyperplane H which doesn't completely contain any irreducible component of $X - U$. We can do this by choosing one point P_i in each of the finitely many irreducible components of $X - U$ and choosing a hyperplane which avoids all the P_i. This can be done because the field is infinite (varieties are only defined over algebraically closed fields) so we can always choose a vector not orthogonal to any of a finite set of vectors. Since X is closed in $\mathbf{P}^n$ and $\mathbf{P}^n - H$ is affine, $(\mathbf{P}^n - H) \cap X$ is an open affine subset of X. Because of our choice of H, $U \cup ((\mathbf{P}^n - H) \cap X)$ is only missing codimension two closed subsets of X. Let $H_1 = H$ and choose another hyperplane H_2 so it doesn't completely contain any of the (codimension two) irreducible components of $X - U - (\mathbf{P}^n - H_1)$. Then $(\mathbf{P}^n - H_2) \cap X$ is open affine and $U \cup ((\mathbf{P}^n - H_1) \cap X) \cup ((\mathbf{P}^n - H2) \cap X)$ is only missing codimension three closed subsets of X. Repeating this process a few more times yields hyperplanes $H_1, \cdots, H_r$ so that

$$U, (\mathbf{P}^n - H_1) \cap X, \ldots, (\mathbf{P}^n - H_r) \cap X$$

form an open affine cover of X, as desired.

Now for the quasi-projective case. Suppose $X \subset \mathbf{P}^n$ is quasi-projective. From (I, Ex. 3.5) we know that $\mathbf{P}^n$ minus a hypersurface H is affine. Note that the same proof works even if H is a union of hypersurfaces. We now proceed with the same sort of construction as in the projective case, but we must choose H more cleverly to insure that $(\mathbf{P}^n - H) \cap X$ is affine. Let U be a nonempty affine open subset of X. As before pick a hyperplane which doesn't completey contain any irreducible component of $X - U$. Since X is only quasi-projective we can't conclude that $(\mathbf{P}^n - H) \cap X$ is affine. But we do know that $(\mathbf{P}^n - H) \cap \overline{X}$ is affine. Our strategy is to add some hypersurfaces to H to get a union of hypersurfaces S so that

$$(\mathbf{P}^n - S) \cap \overline{X} = (\mathbf{P}^n - S) \cap X.$$

But, we must be careful to add these hypersurfaces in such a way that $((\mathbf{P}^n - S) \cap X) \cup U$ is missing only codimension two or greater subsets of X. We do this as follows. For each irreducible component Y of $\overline{X} - X$ choose a hypersurface H' which completely contains Y but which does not completely contain any irreducible component of $X - U$. That this can be done is the content of a lemma which will be proved later (just pick a point in each irreducible component and avoid it). Let S by the union of all of the H' along with H. Then $\mathbf{P}^n - S$ is affine and so

$$(\mathbf{P}^n - S) \cap X = (\mathbf{P}^n - S) \cap \overline{X}$$

is affine. Furthermore, S properly intersects all irreducible components of $X - U$, so $((\mathbf{P}^n - S) \cap X) \cup U$ is missing only codimension two or greater subsets of X. Repeating this process as above several times yields the desired result because after each repetition the codimension of the resulting pieces is reduced by 1.

Lemma 1.2. *If Y is a projective variety and $p_1, \ldots, p_n$ is a finite collection of points not on Y, then there exists a (possibly reducible) hypersurface H containing Y but not containing any of the p_i.*

By a possibly reducible hypersurface I mean a union of irreducible hypersurfaces, not a hypersurface union higher codimension varieties.

Proof. This is obviously true and I have a proof, but I think there is probably a more algebraic proof. Note that k is infinite since we only talk about varieties over algebraically closed fields. Let $f_1, \cdots, f_m$ be defining equations for Y. Thus Y is the common zero locus of the f_i and not all f_i vanish on any p_i. I claim that we can find a linear combination $\sum a_i f_i$ of the f_i which doesn't vanish on any p_i. Since k is infinite and not all f_i vanish on p_1, we can easily find a_i so that $\sum a_i f_i(p_1) \neq 0$ and all the $a_i \neq 0$. If $\sum a_i f_i(p_2) = 0$ then, once again since k is infinite, we can easily "jiggle" the a_i so that $\sum a_i f_i(p_2) \neq 0$ and $\sum a_i f_i(p_1)$ is still nonzero. Repeating this same argument for each of the finitely many points p_i gives a polynomial $f = \sum a_i f_i$ which doesn't vanish on any p_i. Of course I want to use f to define our hypersurface, but I can't because f might not be homogeneous. Fortunately, this is easily dealt with by suitably multiplying the various f_i by the defining equation of a hyperplane not passing through any p_i, then repeating the above argument. Now let H be the hypersurface defined by $f = \sum a_i f_i$. Then by construction H contains Y and H doesn't contain any p_i. □

(e) Suppose Y is a set-theoretic complete intersection of codimension r in $X = \mathbf{P}^n_k$. Then Y is the intersection of r hypersurfaces, so we can write $Y = H_1 \cap \cdots \cap H_r$ where each H_i is a hypersurface. By (I, Ex. 3.5) $X - H_i$ is affine for each i, thus

$$X - Y = (X - H_1) \cup \cdots \cup (X - H_r)$$

can be covered by r open affine subsets. By (c) this implies $\operatorname{cd}(X - Y) \leq r - 1$ which completes the proof. □

Exercise 1.3. **(4.9)** Let $X = \operatorname{Spec} k[x_1, x_2, x_3, x_4]$ be affine four-space over a field k. Let Y_1 be the plane $x_1 = x_2 = 0$ and let Y_2 be the plane $x_3 = x_4 = 0$. Show that $Y = Y_1 \cup Y_2$ is not a set-theoretic complete intersection in X. Therefore the projective closure $\overline{Y}$ in $\mathbf{P}^4_k$ is not a set-theoretic complete intersection.

Proof. By (Ex. 4.8e) it suffices to show that $H^2(X-Y,\mathcal{O}_{X-Y}) \neq 0$. Suppose Z is a closed subset of X, then by (Ex. 2.3d), for any $i \geq 1$, there is an exact sequence

$$H^i(X,\mathcal{O}_X) \to H^i(X-Z,\mathcal{O}_{X-Z}) \to H^{i+1}_Z(X,\mathcal{O}_X) \to H^{i+1}(X,\mathcal{O}_X).$$

By (3.8), $H^i(X,\mathcal{O}_X) = H^{i+1}(X,\mathcal{O}_X) = 0$ so $H^i(X-Z,\mathcal{O}_{X-Z}) = H^{i+1}_Z(X,\mathcal{O}_X)$. Applying this with $Z = Y$ and $i = 2$ shows that

$$H^2(X-Y,\mathcal{O}_{X-Y}) = H^3_Y(X,\mathcal{O}_X).$$

Thus we just need to show that $H^3_Y(X,\mathcal{O}_X) \neq 0$.

Mayer-Vietoris (Ex. 2.4) yields an exact sequence

$$\begin{gathered} H^3_{Y_1}(X,\mathcal{O}_X) \oplus H^3_{Y_2}(X,\mathcal{O}_X) \to H^3_Y(X,\mathcal{O}_X) \to \\ H^4_{Y_1\cap Y_2}(X,\mathcal{O}_X) \to H^4_{Y_1}(X,\mathcal{O}_X) \oplus H^4_{Y_2}(X,\mathcal{O}_X) \end{gathered}$$

As above, $H^3_{Y_1}(X,\mathcal{O}_X) = H^2(X-Y_1,\mathcal{O}_{X-Y_1})$. But $X-Y_1$ is a set-theoretic complete intersection of codimension 2 so $\operatorname{cd}(X-Y_1) \leq 1$, whence $H^2(X-Y_1,\mathcal{O}_{X-Y_1}) = 0$. Similiarly

$$H^2(X-Y_2,\mathcal{O}_{X-Y_2}) = H^3(X-Y_1,\mathcal{O}_{X-Y_1}) = H^3(X-Y_2,\mathcal{O}_{X-Y_2}) = 0.$$

Thus from the above exact sequence we see that $H^3_Y(X,\mathcal{O}_X) = H^4_{Y_1\cap Y_2}(X,\mathcal{O}_X)$.

Let $P = Y_1 \cap Y_2 = \{(0,0,0,0)\}$. We have reduced to showing that $H^4_P(X,\mathcal{O}_X)$ is nonzero. Since $H^4_P(X,\mathcal{O}_X) = H^3(X-P,\mathcal{O}_{X-P})$ we can do this by a direct computation of $H^3(X-P,\mathcal{O}_{X-P})$ using Čech cohomology. Cover $X-P$ by the affine open sets $U_i = \{x_i \neq 0\}$. Then the Čech complex is

$$\begin{aligned} & k[x_1,x_2,x_3,x_4,x_1^{-1}] \oplus \cdots \oplus k[x_1,x_2,x_3,x_4,x_4^{-1}] \xrightarrow{d_0} \\ & k[x_1,x_2,x_3,x_4,x_1^{-1},x_2^{-1}] \oplus \cdots \oplus k[x_1,x_2,x_3,x_4,x_3^{-1},x_4^{-1}] \xrightarrow{d_1} \\ & k[x_1,x_2,x_3,x_4,x_1^{-1},x_2^{-1},x_3^{-1}] \oplus \cdots \oplus k[x_1,x_2,x_3,x_4,x_2^{-1},x_3^{-1},x_4^{-1}] \xrightarrow{d_2} \\ & k[x_1,x_2,x_3,x_4,x_1^{-1},x_2^{-1},x_3^{-1},x_4^{-1}] \end{aligned}$$

Thus

$$H^3(X-P,\mathcal{O}_{X-P}) = \{x_1^i x_2^j x_3^k x_4^\ell : i,j,k,\ell < 0\} \neq 0.$$

□

Exercise 1.4. **(5.6)** *Curves on a Nonsingular Quadric Surface.* Let Q be the nonsingular quadric surface $xy = zw$ in $X = \mathbf{P}^3_k$ over a field k. We will consider locally principal closed subschemes Y of Q. These correspond to Cartier divisors on Q by (II, 6.17.1). On the other hand, we know that $\operatorname{Pic} Q \cong \mathbf{Z} \oplus \mathbf{Z}$, so we can talk about the *type* (a,b) of Y (II, 6.16) and (II, 6.6.1). Let us denote the invertible sheaf $\mathcal{L}(Y)$ by $\mathcal{O}_Q(a,b)$. Thus for any $n \in \mathbf{Z}$, $\mathcal{O}_Q(n) = \mathcal{O}_Q(n,n)$.

[**Comment!** In my solution, a subscheme Y of type (a,b) corresponds to the invertible sheaf $\mathcal{O}_Q(-a,-b)$. I think this is reasonable since then $\mathcal{O}_Q(-a,-b) = \mathcal{L}(-Y) = \mathcal{I}_Y$. The correspondence is not clearly stated in the problem, but this choice works.]

(a) Use the special case $(q,0)$ and $(0,q)$, with $q > 0$, when Y is a disjoint union of q lines $\mathbf{P}^1$ in Q, to show:

1. if $|a-b| \leq 1$, then $H^1(Q,\mathcal{O}_Q(a,b)) = 0$;
2. if $a,b < 0$, then $H^1(Q,\mathcal{O}_Q(a,b)) = 0$;

3. if $a \leq -2$, then $H^1(Q, \mathcal{O}_Q(a,0)) \neq 0$).

Solution. First I will prove a big lemma in which I explicitely calculate $H^1(Q, \mathcal{O}_Q(0,-q))$ and some other things which will come in useful later. Next I give an independent computation of the other cohomology groups (1), (2).

Lemma 1.5. *Let $q > 0$, then*

$$\dim_k H^1(Q, \mathcal{O}_Q(-q,0)) = H^1(Q, \mathcal{O}_Q(0,-q)) = q - 1.$$

Furthermore, we know all terms in the long exact sequence of cohomology associated with the short exact sequence

$$0 \to \mathcal{O}_Q(-q,0) \to \mathcal{O}_Q \to \mathcal{O}_Y \to 0.$$

Proof. We prove the lemma only for $\mathcal{O}_Q(-q,0)$, since the argument for $\mathcal{O}_Q(0,-q)$ is exactly the same. Suppose Y is the disjoint union of q lines $\mathbf{P}^1$ in Q so $\mathcal{I}_Y = \mathcal{O}_Q(-q,0)$. The sequence

$$0 \to \mathcal{O}_Q(-q,0) \to \mathcal{O}_Q \to \mathcal{O}_Y \to 0$$

is exact. The associated long exact sequence of cohomology is

$$\begin{aligned} 0 \to &\Gamma(Q, \mathcal{O}_Q(-q,0)) \to \Gamma(Q, \mathcal{O}_Q) \to \Gamma(Q, \mathcal{O}_Y) \\ \to &H^1(Q, \mathcal{O}_Q(-q,0)) \to H^1(Q, \mathcal{O}_Q) \to H^1(Q, \mathcal{O}_Y) \\ \to &H^2(Q, \mathcal{O}_Q(-q,0)) \to H^2(Q, \mathcal{O}_Q) \to H^2(Q, \mathcal{O}_Y) \to 0 \end{aligned}$$

We can compute all of the terms in this long exact sequence. For the purposes at hand it suffices to view the summands as k-vector spaces so we systematically do this throughout. Since $\mathcal{O}_Q(-q,0) = \mathcal{I}_Y$ is the ideal sheaf of Y, its global sections must vanish on Y. But $\mathcal{I}_Y$ is a subsheaf of $\mathcal{O}_Q$ whose global sections are the constants. Since the only constant which vanish on Y is 0, $\Gamma(Q, \mathcal{O}_Q(-q,0)) = 0$. By (I, 3.4), $\Gamma(Q, \mathcal{O}_Q) = k$. Since Y is the disjoint union of q copies of $\mathbf{P}^1$ and each copy has global sections k, $\Gamma(Q, \mathcal{O}_Y) = k^{\oplus q}$. Since Q is a complete intersection of dimension 2, (Ex. 5.5 b) implies $H^1(Q, \mathcal{O}_Q) = 0$. Because Y is isomorphic to several copies of $\mathbf{P}^1$, the general result (proved in class, but not in the book) that $H^n_*(\mathcal{O}_{\mathbf{P}^n}) = \{\sum a_I X_I : \text{entries in } I \text{ negative}\}$ implies $H^1(Q, \mathcal{O}_Y) = H^1(Y, \mathcal{O}_Y) = 0$. Since Q is a hypersurface of degree 2 in $\mathbf{P}^3$, (I, Ex. 7.2(c)) implies $p_a(Q) = 0$. Thus by (Ex. 5.5c) we see that $H^2(Q, \mathcal{O}_Q) = 0$. Putting together the above facts and some basic properties of exact sequences show that $H^1(Q, \mathcal{O}_Q(-q,0)) = k^{\oplus(q-1)}$, $H^2(Q, \mathcal{O}_Q(-q,0)) = 0$ and $H^2(Q, \mathcal{O}_Y) = 0$. Our long exact sequence is now

$$\begin{aligned} 0 \to &\Gamma(Q, \mathcal{O}_Q(-q,0)) = 0 \to \Gamma(Q, \mathcal{O}_Q) = k \to \Gamma(Q, \mathcal{O}_Y) = k^{\oplus q} \\ \to &H^1(Q, \mathcal{O}_Q(-q,0)) = k^{\oplus(q-1)} \to H^1(Q, \mathcal{O}_Q) = 0 \to H^1(Q, \mathcal{O}_Y) = 0 \\ \to &H^2(Q, \mathcal{O}_Q(-q,0)) = 0 \to H^2(Q, \mathcal{O}_Q) = 0 \to H^2(Q, \mathcal{O}_Y) = 0 \to 0 \end{aligned}$$

□

Number (3) now follows immediately from the lemma because

$$H^1(Q, \mathcal{O}_Q(a,0)) = k^{\oplus(-a-1)} \neq 0$$

for $a \leq -2$.

Now we compute (1) and (2). Let a be an arbitrary integer. First we show that $\mathcal{O}_Q(a,a) = 0$. We have an exact sequence

$$0 \to \mathcal{O}_{\mathbf{P}^3}(-2) \to \mathcal{O}_{\mathbf{P}^3} \to \mathcal{O}_Q \to 0$$

where the first map is multiplication by $xy - zw$. Twisting by a gives an exact sequence

$$0 \to \mathcal{O}_{\mathbf{P}^3}(-2+a) \to \mathcal{O}_{\mathbf{P}^3}(a) \to \mathcal{O}_Q(a) \to 0.$$

The long exact sequence of cohomology yields an exact sequence

$$\cdots \to H^1(\mathcal{O}_{\mathbf{P}^3}(a)) \to H^1(\mathcal{O}_Q(a)) \to H^2(\mathcal{O}_{\mathbf{P}^3}(-2+a)) \to \cdots$$

But from the explicit computations of projective space (5.1) it follows that $H^1(\mathcal{O}_{\mathbf{P}^3}(a)) = 0$ and $H^2(\mathcal{O}_{\mathbf{P}^3}(-2+a)) = 0$ from which we conclude that $H^1(\mathcal{O}_Q(a)) = 0$.

Next we show that $\mathcal{O}_Q(a-1,a) = 0$. Let Y be a single copy of $\mathbf{P}^1$ sitting in Q so that Y has type $(1,0)$. Then we have an exact sequence

$$0 \to \mathcal{I}_Y \to \mathcal{O}_Q \to \mathcal{O}_Y \to 0.$$

But $\mathcal{I}_Y = \mathcal{O}_Q(-1,0)$ so this becomes

$$0 \to \mathcal{O}_Q(-1,0) \to \mathcal{O}_Q \to \mathcal{O}_Y \to 0.$$

Now twisting by a yields the exact sequence

$$0 \to \mathcal{O}_Q(a-1,a) \to \mathcal{O}_Q(a) \to \mathcal{O}_Y(a) \to 0.$$

The long exact sequence of cohomology gives an exact sequence

$$\cdots \to \Gamma(\mathcal{O}_Q(a)) \to \Gamma(\mathcal{O}_Y(a)) \to H^1(\mathcal{O}_Q(a-1,a)) \to H^1(\mathcal{O}_Q(a)) \to \cdots$$

We just showed that $H^1(\mathcal{O}_Q(a)) = 0$, so to see that $H^1(\mathcal{O}_Q(a-1,a)) = 0$ it suffices to note that the map $\Gamma(\mathcal{O}_Q(a)) \to \Gamma(\mathcal{O}_Y(a))$ is surjective. This can be seen by writing $Q = \mathrm{Proj}(k[x,y,z,w]/(xy-zw))$ and (w.l.o.g.) $Y = \mathrm{Proj}(k[x,y,z,w]/(xy-zw,x,z))$ and noting that the degree a part of $k[x,y,z,w]/(xy-zw)$ surjects onto the degree a part of $k[x,y,z,w]/(xy-zw,x,z)$. Thus $H^1(\mathcal{O}_Q(a-1,a)) = 0$ and exactly the same argument shows $H^1(\mathcal{O}_Q(a,a-1)) = 0$. This gives (1).

For (2) it suffices to show that for $a > 0$,

$$H^1(\mathcal{O}_Q(-a,-a-n)) = H^1(\mathcal{O}_Q(-a-n,-a)) = 0$$

for all $n > 0$. Thus let $n > 0$ and suppose Y is a disjoint union of n copies of $\mathbf{P}^1$ in such a way that $\mathcal{I}_Y = \mathcal{O}_Q(0,-n)$. Then we have an exact sequence

$$0 \to \mathcal{O}_Q(0,-n) \to \mathcal{O}_Q \to \mathcal{O}_Y \to 0.$$

Twisting by $-a$ yields the exact sequence

$$0 \to \mathcal{O}_Q(-a,-a-n) \to \mathcal{O}_Q(-a) \to \mathcal{O}_Y(-a) \to 0.$$

The long exact sequence of cohomology then gives an exact sequence

$$\cdots \to \Gamma(\mathcal{O}_Y(-a)) \to H^1(\mathcal{O}_Q(-a,-a-n)) \to H^1(\mathcal{O}_Q(-a)) \to \cdots$$

As everyone knows, since Y is just several copies of $\mathbf{P}^1$ and $-a < 0$, $\Gamma(\mathcal{O}_Y(-a)) = 0$. Because of our computations above, $H^1(\mathcal{O}_Q(-a)) = 0$. Thus $H^1(\mathcal{O}_Q(-a,-a-n)) = 0$, as desired. Showing that $H^1(\mathcal{O}_Q(-a-n,-a)) = 0$ is exactly the same.

(b) Now use these results to show:

1. If Y is a locally principal closed subscheme of type (a, b) with $a, b > 0$, then Y is connected.

   Proof. Computing the long exact sequence associated to the short exact sequence

   $$0 \to \mathcal{I}_Y \to \mathcal{O}_Q \to \mathcal{O}_Y \to 0$$

   gives the exact sequence

   $$0 \to \Gamma(Q, \mathcal{I}_Y) \to \Gamma(Q, \mathcal{O}_Q) \to \Gamma(Q, \mathcal{O}_Y) \to H^1(Q, \mathcal{I}_Y) \to \cdots$$

   But, $\Gamma(\mathcal{I}_Y) = 0$, $\Gamma(Q, \mathcal{O}_Q) = k$, and by (a)2 above $H^1(Q, \mathcal{I}_Y) = H^1(Q, \mathcal{O}_Q(-a, -b)) = 0$. Thus we have an exact sequence

   $$0 \to 0 \to k \to \Gamma(\mathcal{O}_Y) \to 0 \to \cdots$$

   from which we conclude that $\Gamma(\mathcal{O}_Y) = k$ which implies Y is connected. $\square$

2. now assume k is algebraically closed. Then for any $a, b > 0$, there exists an irreducible nonsingular curve Y of type (a, b). Use (II, 7.6.2) and (II, 8.18).

   Proof. Given (a, b), (II, 7.6.2) gives a closed immersion

   $$Q = \mathbf{P}^1 \times \mathbf{P}^1 \to \mathbf{P}^a \times \mathbf{P}^b \to \mathbf{P}^n$$

   which corresponds to the invertible sheaf $\mathcal{O}_Q(-a, -b)$ of type (a, b). By Bertini's theorem (II, 8.18) there is a hyperplane H in $\mathbf{P}^n$ such that the hyperplane section of the (a, b) embedding of Q in $\mathbf{P}^n$ is nonsingular. Pull this hyperplane section back to a nonsingular curve Y of type (a, b) on Q in $\mathbf{P}^3$. By the previous problem, Y is connected. Since Y comes from a hyperplane section this implies Y is irreducible (see the remark in the statement of Bertini's theorem). $\square$

3. an irreducible nonsingular curve Y of type (a, b), $a, b > 0$ on Q is projectively normal (II, Ex. 5.14) if and only if $|a - b| \leq 1$. In particular, this gives lots of examples of nonsingular, but not projectively normal curves in $\mathbf{P}^3$. The simplest is the one of type $(1, 3)$ which is just the rational quartic curve (I, Ex. 3.18).

   Proof. Let Y be an irreducible nonsingular curve of type (a, b). The criterion we apply comes from (II, Ex 5.14d) which asserts that the maps

   $$\Gamma(\mathbf{P}^3, \mathcal{O}_{\mathbf{P}^3}(n)) \to \Gamma(Y, \mathcal{O}_Y(n))$$

   are surjective for all $n \geq 0$ if and only if Y is projectively normal. To determine when this occurs we have to replace $\Gamma(\mathbf{P}^3, \mathcal{O}_{\mathbf{P}^3}(n))$ with $\Gamma(Q, \mathcal{O}_Q(n))$. It is easy to see that the above criterion implies we can make this replacement if Q is projectively normal. Since $Q \cong \mathbf{P}^1 \times \mathbf{P}^1$ is locally isomorphic to $\mathbf{A}^1 \times \mathbf{A}^1 \cong \mathbf{A}^2$ which is normal, we see that Q is normal. Then since Q is a complete intersection which is normal, (II, 8.4b) implies Q is projectively normal.

   Consider the exact sequence

   $$0 \to \mathcal{I}_Y \to \mathcal{O}_Q \to \mathcal{O}_Y.$$

Twisting by n gives an exact sequence

$$0 \to \mathcal{I}_Y(n) \to \mathcal{O}_Q(n) \to \mathcal{O}_Y(n).$$

Taking cohomology yields the exact sequence

$$\cdots \to \Gamma(Q, \mathcal{O}_Q(n)) \to \Gamma(Q, \mathcal{O}_Y(n)) \to H^1(Q, \mathcal{I}_Y(n)) \to \cdots$$

Thus Y is projectively normal precisely if $H^1(Q, \mathcal{I}_Y(n)) = 0$ for all $n \geq 0$. When can this happen? We apply our computations from part (a). Since $\mathcal{O}_Q(n) = \mathcal{O}_Q(n,n)$,

$$\mathcal{I}_Y(n) = \mathcal{O}_Q(-a,-b)(n) = \mathcal{O}_Q(-a,-b) \otimes_{\mathcal{O}_Q} \mathcal{O}_Q(n,n) = \mathcal{O}_Q(n-a, n-b)$$

If $|a-b| \leq 1$ then $|(n-a)-(n-b)| \leq 1$ for all n so

$$H^1(Q, \mathcal{O}_Q(-a,-b)(n)) = 0$$

for all n which implies Y is projectively normal. On the other hand, if $|a-b| > 1$ let n be the minimum of a and b, without loss assume b is the minimum, so $n = b$. Then from (a) we see that

$$\mathcal{O}_Q(-a,-b)(n) = \mathcal{O}_Q(-a,-b)(b) = \mathcal{O}_Q(-a+b, 0) \neq 0$$

since $-a+b \leq -2$. $\square$

(c) If Y is a locally principal subscheme of type (a,b) in Q, show that $p_a(Y) = ab-a-b+1$. [Hint: Calculate the Hilbert polynomials of suitable sheaves, and again use the special case (q,0) which is a disjoint union of q copies of $\mathbf{P}^1$.]

Proof. The sequence

$$0 \to \mathcal{O}_Q(-a,-b) \to \mathcal{O}_Q \to \mathcal{O}_Y \to 0$$

is exact so

$$\chi(\mathcal{O}_Y) = \chi(\mathcal{O}_Q) - \chi(\mathcal{O}_Q(-a,-b)) = 1 - \chi(\mathcal{O}_Q(-a,-b)).$$

Thus

$$p_a(Y) = 1 - \chi(\mathcal{O}_Y) = \chi(\mathcal{O}_Q(-a,-b)).$$

The problem is thus reduced to computing $\chi(\mathcal{O}_Q(-a,-b))$.

Assume first that $a, b < 0$. To compute $\chi(\mathcal{O}_Q(-a,-b))$ assume $Y = Y_1 \cup Y_2$ where $\mathcal{I}_{Y_1} = \mathcal{O}_Q(-a,0)$ and $\mathcal{I}_{Y_2} = \mathcal{O}_Q(0,-b)$. Thus we could take Y_1 to be a copies of $\mathbf{P}^1$ in one family of lines and Y_2 to be b copies of $\mathbf{P}^1$ in the other family. Tensoring the exact sequence

$$0 \to \mathcal{I}_{Y_1} \to \mathcal{O}_Q \to \mathcal{O}_{Y_1} \to 0$$

by the flat module $\mathcal{I}_{Y_2}$ yields an exact sequence

$$0 \to \mathcal{I}_{Y_1} \otimes \mathcal{I}_{Y_2} \to \mathcal{I}_{Y_2} \to \mathcal{O}_{Y_1} \otimes \mathcal{I}_{Y_2}$$

[Note: I use the fact that $\mathcal{I}_{Y_2}$ is flat. This follows from a proposition in section 9 which we haven't yet reached, but I'm going to use it anyways. Since Y_2 is locally principal, $\mathcal{I}_{Y_2}$ is generated locally by a single element and since Q is a variety it is integral. Thus $\mathcal{I}_{Y_2}$ is locally free so by (9.2) $\mathcal{I}_{Y_2}$ is flat.] This exact sequence can also be written as

$$0 \to \mathcal{O}_Q(-a,-b) \to \mathcal{O}_Q(0,-b) \to \mathcal{O}_Y \otimes \mathcal{O}_Q(0,-b) \to 0.$$

The associated long exact sequence of cohomology is

$$\begin{aligned}
0 \to &\Gamma(Q, \mathcal{O}_Q(-a,-b)) \to \Gamma(Q, \mathcal{O}_Q(0,-b)) \to \Gamma(Q, \mathcal{O}_{Y_1} \otimes \mathcal{O}_Q(0,-b)) \\
\to &H^1(Q, \mathcal{O}_Q(-a,-b)) \to H^1(Q, \mathcal{O}_Q(0,-b)) \to H^1(Q, \mathcal{O}_{Y_1} \otimes \mathcal{O}_Q(0,-b)) \\
\to &H^2(Q, \mathcal{O}_Q(-a,-b)) \to H^2(Q, \mathcal{O}_Q(0,-b)) \to H^2(Q, \mathcal{O}_{Y_1} \otimes \mathcal{O}_Q(0,-b)) \to 0
\end{aligned}$$

The first three groups of global sections are 0. Since $a, b < 0$, (a) implies $H^1(Q, \mathcal{O}_Q(-a,-b)) = 0$. From the lemma we know that $H^1(Q, \mathcal{O}_Q(0,-b)) = k^{\oplus(b-1)}$. Also by the lemma we know that $H^2(Q, \mathcal{O}_Q(0,-b)) = 0$. Since $\mathcal{O}_{Y_1} \otimes \mathcal{O}_Q(0,-b)$ is isomorphic to the ideal sheaf of $b-1$ points in each line of Y_1, a similiar proof as that used in the lemma shows that

$$H^1(Q, \mathcal{O}_Y \otimes \mathcal{O}_Q(0,-b)) = k^{\oplus a(b-1)}.$$

Plugging all of this information back in yields the exact sequence

$$\begin{aligned}
0 \to &\Gamma(Q, \mathcal{O}_Q(-a,-b)) = 0 \to \Gamma(Q, \mathcal{O}_Q(0,-b)) = 0 \to \Gamma(Q, \mathcal{O}_{Y_1} \otimes \mathcal{O}_Q(0,-b)) = 0 \\
\to &H^1(Q, \mathcal{O}_Q(-a,-b)) = 0 \to H^1(Q, \mathcal{O}_Q(0,-b)) = k^{\oplus(b-1)} \\
&\qquad \to H^1(Q, \mathcal{O}_{Y_1} \otimes \mathcal{O}_Q(0,-b)) = k^{\oplus a(b-1)} \\
\to &H^2(Q, \mathcal{O}_Q(-a,-b)) \to H^2(Q, \mathcal{O}_Q(0,-b)) = 0 \\
&\qquad \to H^2(Q, \mathcal{O}_{Y_1} \otimes \mathcal{O}_Q(0,-b)) = 0 \to 0
\end{aligned}$$

From this we conclude that

$$\chi(\mathcal{O}_Q(-a,-b)) = 0 + 0 + h^2(Q, \mathcal{O}_Q(-a,-b)) = a(b-1) - (b-1) = ab - a - b + 1$$

which is the desired result.

Now we deal with the remaining case, when Y is a disjoint copies of $\mathbf{P}^1$. We have

$$p_a(Y) = 1 - \chi(\mathcal{O}_Y) = 1 - \chi(\mathcal{O}_{\mathbf{P}^1}^{\oplus a}) = 1 - a\chi(\mathcal{O}_{\mathbf{P}^1}) = 1 - a$$

which completes the proof. □

Homework 2, MAT256B
II.8.4, III.6.8, III.7.1, III.7.3

William A. Stein

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1 Exercise II.8.4

Complete Intersections in $\mathbf{P}^n$. *A closed subscheme* Y *of* $\mathbf{P}^n_k$ *is called a* (strict, global) complete intersection *if the homogenous ideal* I *of* Y *in* $S = k[x_0, \ldots, x_n]$ *can be generated by* r *elements where* $r = \operatorname{codim}(Y, \mathbf{P}^n)$.

(a) Let Y *be a closed subscheme of codimension* r *in* $\mathbf{P}^n$. *Then* Y *is a complete intersection iff there are hypersurfaces (i.e., locally principal subschemes of codimension* 1*)* $H_1, \ldots, H_r$*, such that* $Y = H_1 \cap \cdots \cap H_r$ as schemes, *i.e.,* $\mathcal{I}_Y = \mathcal{I}_{H_1} + \cdots + \mathcal{I}_{H_r}$.

($\Rightarrow$) By (II, Ex 5.14) I is defined to be $\Gamma_*(\mathcal{I}_Y)$. By (II, 5.15), $\tilde{I} \cong \mathcal{I}_Y$. Write $I = (f_1, \ldots, f_r)$, then since localization commutes with taking sums,

$$\mathcal{I}_Y = (f_1, \ldots, f_r)\tilde{} = ((f_1) + \cdots + (f_r))\tilde{} = (f_1)\tilde{} + \cdots + (f_r)\tilde{}.$$

Let H_i be the locally principal closed subscheme of codimension 1 determined by the ideal sheaf $(f_i)\tilde{}$. Then Y is the intersection of the H_i.

($\Leftarrow$) Someone suggested I should apply unmixedness and primary decomposition to some ideal somewhere and use the fact that a saturated ideal doesn't have primary components corresponding to the irrelevant ideal or something like that. NOT DONE.

(b) If Y *is a complete intersection of dimension* ≥ 1 *in* $\mathbf{P}^n$*, and if* Y *is normal, then* Y *is projectively normal (Ex. 5.14).*

Let Z be the cone over Y, then $A(Z) = S/I(Y)$. By (I, Ex. 3.17d), $A(Z)$ is integrally closed iff Z is normal. By definition $A(Z)$ is integrally closed iff Y is projectively normal. Thus we must show that Z is normal. Since Y is a complete intersection, $I(Y) = (f_1, \ldots, f_r)$ so Z is a complete intersection subscheme of $\mathbf{A}^{n+1}$. By (II, 8.23) Z is normal iff Z is regular in codimension 1. Also by (II, 8.23) Y is regular in codimension 1 because we have assumed Y is normal. But Y regular in codimension 1 implies Z regular in codimension 1. [We used this last semester in (II, Ex. 6.3d). Intuitively, the only singularity in Z not in Y is the cone point which has codimension > 1. This is because Z is locally $U_i \times \mathbf{A}^1$.]

(c) With the same hypothesis as in (b), conclude that for all $\ell \geq 0$*, the natural map* $\Gamma(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(\ell)) \to \Gamma(Y, \mathcal{O}_Y(\ell))$ *is surjective. In particular, taking* $\ell = 0$*, show that* Y *is connected.*

That the map $\Gamma(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(\ell)) \to \Gamma(Y, \mathcal{O}_Y(\ell))$ is surjective is just the statement of (II, Ex. 5.14d). When $\ell = 0$ this says that $k = \Gamma(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(\ell))$ surjects onto $\Gamma(Y, \mathcal{O}_Y(\ell))$ so $\dim \Gamma(Y, \mathcal{O}_Y) \leq 1$ and hence Y is connected. [If Y were not connected then $\Gamma(Y, \mathcal{O}_Y) = k \oplus \cdots \oplus k$ where the number of direct summands equals the number of components of Y.]

(d) Now suppose given integers $d_1, \ldots, d_r \geq 1$*, with* $r < n$*. Use Bertini's theorem (8.18) to show that there exists nonsingular hypersurfaces* $H_1, \ldots, H_r$ *in* $\mathbf{P}^n$*, with* $\deg H_i = d_i$*, such that the scheme* $Y = H_1 \cap \cdots \cap H_r$ *is irreducible and nonsingular in codimension* r *in* $\mathbf{P}^n$.

[To apply Bertini's theorem we must assume k is algebraically closed. I'm going to make this assumption now. Maybe there is a way around this?]

Let $\mathbf{P}^n_k \hookrightarrow \mathbf{P}^{d_1\text{-uple}}_k$ be the d_1-uple embedding of $\mathbf{P}^n_k$. Use Bertini's theorem to choose a hyperplane in $\mathbf{P}^{d_1\text{-uple}}_k$ which has nonsingular intersection with the image of $\mathbf{P}^n_k$. It pulls back to a degree d_1 nonsingular hypersurface H_1 in $\mathbf{P}^n_k$. If $r > 1$ consider the d_2-uple embedding $\mathbf{P}^n_k \hookrightarrow \mathbf{P}^{d_2\text{-uple}}_k$. The image of H_1 is a nonsingular variety in $\mathbf{P}^{d_2\text{-}uple}_k$ of dimension ≥ 2. By Bertinni's theorem there is a hyperplane in $\mathbf{P}^{d_2-uple}_k$ whose intersection with the image of H_1 is nonsingular and of dimension one less than H_1. Pulling back we obtain a hypersurface H_2 such that $H_1 \cap H_2$ is nonsingular and H_2 has degree d_2. Continuing inductively in this way and noting that $\dim H_1 \cap \cdots \cap H_{r-1} \geq 2$ (since $r < n$) completes the proof.

(e) If Y is a nonsingular complete intersection as in (d) show that $\omega_Y \cong \mathcal{O}_Y(\sum d_i - n - 1)$.

By (III, 8.20) $\omega_{H_1} \cong \omega_{\mathbf{P}^n} \otimes \mathcal{L}(H_1) \otimes \mathcal{O}_{H_1}$. By the explicit computation of $Cl\mathbf{P}^n$ (II, 6.17) we know that $\mathcal{L}(H_1) \cong \mathcal{O}_{\mathbf{P}^n}(d_1)$. Thus

$$\omega_{H_1} \cong \mathcal{O}_{\mathbf{P}^n}(-n-1) \otimes \mathcal{O}_{\mathbf{P}^n}(d_1) \otimes \mathcal{O}_{H_1} \cong \mathcal{O}_{H_1}(d_1 - n - 1).$$

By (8.20) we have that

$$\omega_{H_1 \cap H_2} \cong \omega_{H_1} \otimes \mathcal{L}(H_2.H_1) \otimes \mathcal{O}_{H_1 \cap H_2}.$$

We know that $H_2 \sim d_2\mathbf{P}^{n-1}$ (linear equivalence) so by (II, 6.2b) this implies $H_2.H_1 \sim d_2\mathbf{P}^{n-1}.H_1$. But $d_2\mathbf{P}^{n-1}.H_1$ corresponds to the invertible sheaf (see (II, Ex 6.8c) $\mathcal{O}_{H_1}(d_2)$. Thus

$$\omega_{H_1 \cap H_2} \cong \mathcal{O}_{H_1}(d_1 - n - 1) \otimes \mathcal{O}_{H_1}(d_2) \otimes \mathcal{O}_{H_1 \cap H_2} \cong \mathcal{O}_{H_1 \cap H_2}(d_1 + d_2 - n - 1).$$

Repeating this argument inductively yields the desired isomorphism.

(f) If Y is a nonsingular hypersurface of degree d in $\mathbf{P}^n$, use (c) and (e) above to show that $p_g(Y) = \binom{d-1}{n}$. Thus $p_g(Y) = p_a(Y)$ (I, Ex. 7.2).

By definition $p_g(Y) = \dim_k \Gamma(Y, \omega_Y)$. By (e), $\omega_Y \cong \mathcal{O}(d - n - 1)$ and by (c) the natural map

$$\Gamma(X, \mathcal{O}_X(d - n - 1)) \to \Gamma(Y, \mathcal{O}_Y(d - n - 1))$$

is surjective. We show that it is also injective. By (III, 5.5a) an element $f \in \Gamma(X, \mathcal{O}_X(d - n - 1))$ can be represented as a homogeneous polynomial of degree $d - n - 1$. Now f maps to 0 in $\Gamma(Y, \mathcal{O}_Y(d - n - 1))$ iff f vanishes on Y, that is to say, $Y \subset Z(f)$. But $\deg Y = d > d - n - 1 = \deg f$ so Y can not be contained in the hypersurface $Z(f)$ unless $f = 0$. [Proof: $Y = Z(g) \subset Z(f)$ implies $(f) \subset (g)$ so f is a multiple of g, but g has degree strictly greater than f so must be 0.] Thus

$$\dim_k \Gamma(Y, \mathcal{O}_Y(d - n - 1)) = \dim_k \Gamma(X, \mathcal{O}_X(d - n - 1)) = \binom{d-1}{n}$$

since the number of monomoials in $k[x_0, \ldots, x_n]$ of degree $d - n - 1$ is $\binom{d-1}{n}$ as desired.

(g) If Y is a nonsingular curve in $\mathbf{P}^3$, which is a complete intersection of nonsingular surfaces of degrees d, e, then $p_g(Y) = \frac{1}{2}de(d + e - 4) + 1$. Again the geometric genus is the same as the arithmetic genus (I, Ex. 7.2).

Let H be the hypersurface of degree d. There is an exact sequence

$$0 \to \mathcal{O}_{\mathbf{P}^n}(-d) \to \mathcal{O}_{\mathbf{P}^n} \to \mathcal{O}_H \to 0.$$

Twisting by a and computing dimensions we see that

$$\dim \mathcal{O}_H(a) = \dim \mathcal{O}_{\mathbf{P}^n}(a) - \dim \mathcal{O}_{\mathbf{P}^n}(a-d) = \binom{3+a}{3} - \binom{3+a-d}{3}.$$

Using reasoning like that in (e) we obtain an exact sequence

$$0 \to \mathcal{O}_H(-e) \to \mathcal{O}_H \to \mathcal{O}_Y \to 0.$$

Twisting by $e+d-4$ yields the exact sequence

$$0 \to \mathcal{O}_H(d-4) \to \mathcal{O}_H(e+d-4) \to \mathcal{O}_Y(e+d-4) \to 0.$$

Applying the above explicit computation of $\dim \mathcal{O}_H(a)$ we see that

$$\dim \mathcal{O}_Y(e+d-4) = \binom{e+d-1}{3} - \binom{e-1}{3} - \binom{d-1}{3} + chose{-}13.$$

After some algebra the latter expression becomes $\frac{1}{2}ed(e+d-4)+1$, as desired.

[Comment 1: We could have also solved (f) using this method.]

[Comment 2: Serre duality gives another solution. By (III, 7.12.4) $p_g(Y) = \dim H^0(Y, \omega_Y) = \dim H^1(Y, \mathcal{O}_Y) = p_a(Y)$. But by (I, Ex. 7.2d) $p_a(Y) = \frac{1}{2}de(d+e-4)+1$.]

2 Exercise III.6.8

Prove the following theorem of Kleiman: if X is a noetherian, integral, seperated, locally factorial scheme, then every coherent sheaf on X is a quotient of a locally free sheaf (of finite rank).

(a) First show that open sets of the form X_s, for various $s \in \Gamma(X, \mathcal{L})$ and various invertible sheaves $\mathcal{L}$ on X, form a base for the topology of X.

Let $x \in U \subset X$ with U open.

Case 1. $W = X - U$ is irreducible. Since $x \notin W$, $\mathcal{O}_x \not\subset \mathcal{O}_W$. [This assertion is a matter of some difficulty among the others working on this problem. It is not hard to see when X is a variety in the classical sense. But in the more general situation it isn't at all clear and may use the hypothesis that X is seperated in an essential way. For example, the affine line with a doubled origin has two different local rings which are equal. I'm not sure how to resolve this but there was some talk of using the valuative criterion for seperatedness. PUT CORRECT SOLUTION HERE AFTERWARDS.] Thus let $h \in K$ be a rational function such that $h \notin \mathcal{O}_W$ but $h \in \mathcal{O}_x$. Let $(h) = D_1 - D_2$ with D_1=zeros of h and D_2=poles of h. Since $h \in \mathcal{O}_x$, we have $x \notin \mathrm{Var}(D_2)$=the underlying scheme of the effective divisor D_2. (This is because h can't have a pole at x.) Furthermore $y \in W$ implies $\mathcal{O}_y \subset \mathcal{O}_W$ so $h \notin \mathcal{O}_y$ thus $y \in \mathrm{Var}(D_2)$ (this is because X is factorial so $\mathcal{O}_y$ is integrally closed so $v_y(h) < 0$ iff $h \notin \mathcal{O}_y$.) Thus $W \subset \mathrm{Var}(D_2)$. Since X is factorial and D_2 is effective (II, 6.11) implies D_2 corresponds to an effective Cartier divisor and hence there exists an open cover $\mathfrak{U} = (U_i)$ of X and rational functions $h_i \in K$ such that $h_i|U_i \in \mathcal{O}_{U_i}$ and $(h_i) = D_2$ on U_i. Since $\frac{h_i}{h_j} \in \mathcal{O}_x(U_i \cap U_j)^*$ and X is normal, $(\frac{h_i}{h_j}) = 0$. Let $\mathcal{L}$ be the locally free invertible sheaf represented by the Cartier divisor (U_i, h_i) (so $\mathcal{L}$ is locally generated by $1/h_i$ on U_i), and let $u_i : \mathcal{L}|U_i \to \mathcal{O}_x|U_i$ be the isomorphism given by multiplication by h_i. Define $s(y) = u_i^{-1}(h_i(y))$ for $y \in U_i$. By this we mean u_i^{-1} of the map $y \mapsto h_i(y)$, i.e., s is the glueing of the inverse images of the $h_i \in \mathcal{O}_x(u_i)$. Thus s is a section of $\mathcal{L}$ such that $X_s \cap U_i = U_i - \mathrm{Var}(D_2)$. Thus $X_s = X - \mathrm{Var}(D_2) \subset U$.

Case 2. $W = X - U$ is reducible. Using the fact that X is noetherian write $W = Z_1 \cup \cdots \cup Z_n$. From case 1 we know that there exists invertible sheaves $\mathcal{L}_1, \ldots, \mathcal{L}_n$ and sections $s_i \in \Gamma(X, \mathcal{L}_i)$, $i = 1, \ldots, n$ such that $x \in X_{s_i} \subset X - Z_i$. Let $s = s_1 \otimes \cdots \otimes s_n \in \Gamma(X, \mathcal{L}_1 \otimes \cdots \otimes \mathcal{L}_n)$. Then $X_s = \cap_{i=1}^n X_{s_i}$ hence $x \in X_s \subset U$.

[[This proof was copied from Borelli's paper with little modification. One danger is that the corresponding theorem in Borelli's paper assumes X to be a factorial *variety*, not a more general scheme as above. Part (b) below was not in Borelli.]]

(b) Now use (II, 5.14) to show that any coherent sheaf is a quotient of a direct sum $\oplus \mathcal{L}_i^{n_i}$ for various invertible sheaves $\mathcal{L}_i$ and various integers n_i.

Let $\mathcal{F}$ be a coherent sheaf on X. Let U be an open set on which $\mathcal{F}_{|U} \cong \tilde{M}$. Suppose $X_f \subset U$ where f is a global section of some invertible sheaf $\mathcal{L}$. Our strategy is to construct an appropriate map $\oplus \mathcal{L}_i^{n_i} \to \mathcal{F}$ which is surjective when restricted to X_f, then use the fact that X is noetherian and that the X_f form a basis for the topology on X to cover X which such X_f and then take the sum of all the resulting maps.

Let $m_1, \ldots, m_r$ generate M. Let $t_1, \ldots, t_r$ be the restrictions of the m_i to X_f. By (II, 5.14b) there exists n so that

$$t_1 f^n, \ldots, t_r f^n \in \Gamma(X_{f^n}, \mathcal{F} \otimes \mathcal{L}^{\otimes n})$$

extend to global sections $s_1, \ldots, s_n$ of $\Gamma(X, \mathcal{F} \otimes \mathcal{L}^{\otimes n})$. Define a map

$$\oplus_{i=1}^n \mathcal{O}_X \to \mathcal{F} \otimes \mathcal{L}^{\otimes n}$$

by sending $(0, \ldots, 0, 1, 0, \ldots, 0)$ (1 in the ith position only) to s_i. Then tensoring with $(\mathcal{L}^{\otimes n})^{-1} = (\mathcal{L}^{-1})^{\otimes n}$ we obtain a map

$$\Theta : \oplus_{i=1}^n (\mathcal{L}^{-1})^{\otimes n} \to \mathcal{F}.$$

The map Θ is surjective when restricted to X_f. To see this let p be a point of X_f. The stalk of $\mathcal{F}$ at p is generated by the stalks of $m_1, \ldots, m_n$ at p. Since the t_i are all in the image of the map Θ and the stalks of the t_i at p are the same as the stalks of the m_i at p it follows that the stalks of the m_i are all in the image under Θ of the stalk of $\oplus_{i=1}^n (\mathcal{L}^{-1})^{\otimes n}$ at p.

Take the direct sum of all such maps over a suitable open cover (U) of X and suitable open covers (X_f) of each U. Since X is noetherian we can arrange it so this sum is finite.

3 Exercise III.7.1

Let X be an integral projective scheme of dimension ≥ 1 over a field k, and let $\mathcal{L}$ be an ample invertible sheaf X. Then $H^0(X, \mathcal{L}^{-1}) = 0$.

Lemma 3.1. *If $\mathcal{M} \neq \mathcal{O}_X$ is an invertible sheaf which is generated by its global sections then $H^0(X, \mathcal{M}^{-1}) = 0$.*

Proof. By the proof of (II 6.12) $\mathcal{M}^{-1} = \mathcal{M}^\vee = \mathcal{H}\mathrm{om}(\mathcal{M}, \mathcal{O}_X)$. Thus we must show that $\Gamma(X, \mathcal{H}\mathrm{om}(\mathcal{M}, \mathcal{O}_X)) = 0$, i.e., that $\mathrm{hom}_{\mathcal{O}_X}(\mathcal{M}, \mathcal{O}_X) = 0$. Since $\mathcal{M}$ is generated by global sections (m_i) to give a morphism $f : \mathcal{M} \to \mathcal{O}_X$ is the same as to give the images $\alpha_i = f(m_i) \in \Gamma(X, \mathcal{O}_X)$ of the m_i. Since X is integral and projective $\Gamma(X, \mathcal{O}_X) = k$ so the α_i all lie in k. Thus if f is nonzero then some $\alpha_i \neq 0$ so $\frac{1}{\alpha_i} m_i \mapsto 1$. Let $t = \frac{1}{\alpha_i} m_i \in \Gamma(X, \mathcal{M})$. Let p be any point of X. The map $f_p : \mathcal{M}_p \to \mathcal{O}_{X,p}$ sends t_p to 1 so it is surjective being a map of free $\mathcal{O}_{X,p}$-modules (and since $\mathcal{O}_{X,p}$ is generated by 1 as an $\mathcal{O}_{X,p}$-module). On the other hand $\mathcal{M}_p$ is free of rank 1 over the integral domain $\mathcal{O}_{X,p}$ so f must be injective.

Indeed, if $\mathcal{M}_p \cong \mathcal{O}_{X,p} \cdot g$ for some g and $ag \mapsto 0$ then $af(g) = 0$ so since $\mathcal{O}_{X,p}$ is a domain, $a = 0$ or $f(g) = 0$. But $f(g) \neq 0$ since f is surjective so $a = 0$ and so $ag = 0$ whence f is injective. Therefore f is an isomorphism since it is an isomorphism on stalks. Thus $\mathcal{M} \cong \mathcal{O}_X$ contrary to our assumption that $\mathcal{M} \not\cong \mathcal{O}_X$ so there can be no nonzero f in $\hom_{\mathcal{O}_X}(\mathcal{M}, \mathcal{O}_X) = \Gamma(X, \mathcal{M}^{-1})$, as desired. □

Suppose that $\mathcal{L}$ is ample. If $\mathcal{L} = \mathcal{O}_X$ then $\mathcal{L}$ can not be ample, for if $\mathcal{O}_X$ is ample then since $\mathcal{O}_X^{\otimes n} = \mathcal{O}_X$ for any $n \geq 1$ it follows by (II.7.5) that $\mathcal{O}_X$ is very ample. This means that there is an immersion $i : X \hookrightarrow \mathbf{P}_k^n$ where $n = \dim \Gamma(X, \mathcal{O}_X) - 1 = 0$ which is impossible because X has dimension at least 1.

Thus we may assume $\mathcal{L} \not\cong \mathcal{O}_X$ and apply the above lemma. There is an n so that $\mathcal{L}^{\otimes n}$ is generated by its global sections. By the above lemma $H^0(X, (\mathcal{L}^{\otimes n})^\vee) = 0$. Since the collection of invertible sheaves forms a group and $\vee$ is the inverse operation it follows trivially that $(\mathcal{L}^{\otimes n})^\vee \cong (\mathcal{L}^\vee)^{\otimes n}$ and hence $\Gamma(X, (\mathcal{L}^\vee)^{\otimes n}) = 0$. Suppose $\mathcal{L}^\vee$ has a nonzero global section s. Let $p \in X$ be a point so that $s_p \neq 0$. It follows that $s \otimes \cdots \otimes s \neq 0$ in $(\mathcal{L}_p^\vee)^{\otimes n}$. Thus s defines a nonzero global section $s \otimes \cdots \otimes s$ of $(\mathcal{L}^\vee)^{\otimes n}$. [This last statement is a bit subtle because the tensor product is the sheaf associated to a certain presheaf so we don't know, *a priori*, that $s \otimes \cdots \otimes s$ maps to something nonzero under the θ of (II, Defn 1.2). But if $\theta(s \otimes \cdots \otimes s) = 0$ then $0 = \theta(s \otimes \cdots \otimes s)_p = (s \otimes \cdots \otimes s)_p$ so θ is not injective on stalks contradicting the comment after (II, Defn 1.2).] Thus if $H^0(X, \mathcal{L}^\vee) \neq 0$ then $H^0(X, (\mathcal{L}^{\otimes n})^\vee) \neq 0$, a contradiction. It follows that $H^0(X, \mathcal{L}^\vee) = 0$, as desired.

4 Exercise III.7.3

Let $X = \mathbf{P}_k^n$. Show that $H^q(X, \Omega_X^p) = 0$ for $p \neq q$, k for $p = q$, $0 \leq p, q \leq n$.

Our strategy is to use the exact sequence

$$0 \to \Omega_X \to \mathcal{O}_X(-1)^{\oplus n+1} \to \mathcal{O}_X \to 0$$

of (II, 8.13) along with (II, Ex 5.16 d) to reduce the computation of the cohomology of Ω_X^p to the computation of the cohomology of $\Lambda^p \mathcal{O}_X(-1)^{\oplus n+1}$. We then show inductively that the cohomology of $\Lambda^p \mathcal{O}_X(-1)^{\oplus n+1}$ vanishes for $p \geq 1$ thus completing the proof.

We compute the cohomology of Ω^r inductively on r.

Step 1, $r = 0$. Suppose $r = 0$ so $\Omega^r = \mathcal{O}_X$. Then by (III, 5.5) $H^0(X, \mathcal{O}_X) = k$ and $H^i(X, \mathcal{O}_X) = 0$ for $i \geq 1$. [Part (a) of (III, 5.5) gives $H^0(X, \mathcal{O}_X) = k$, part (b) gives $H^i(X, \mathcal{O}_X) = 0$ for $0 < i < n$ and part (d) gives $H^n(X, \mathcal{O}_X) \cong H^0(X, \mathcal{O}_X(-n-1))^\vee = 0$.]

Step 2 Show that

$$H^i(\Lambda^r \mathcal{O}_X(-1)^{\oplus n+1}) = 0$$

for $r \geq 1$.

[Matt Baker pointed out to me that

$$\Lambda^r \mathcal{O}_X(-1)^{\oplus n+1} \cong \mathcal{O}_X(-1)^{\oplus \binom{n+1}{r}}.$$

This is reasonable since it is true on stalks. It immediately implies the vanishing of the cohomology groups. My original more complicated proof of step 2 is included next anyways.]

Step 2a, $r = 1$. We treat $r = 1$ as a special case. We must show that $H^i(X, \mathcal{O}_X(-1)^{\oplus n+1} = 0$ or equivalently that $H^i(X, \mathcal{O}_X(-1)) = 0$. This is immediate from the explicit computations of (III, 5.5). The argument proceeds exactly as in step 1.

Step 2b, $r \geq 2$. We now assume $r \geq 2$ and proceed inductively on n. Since $r \geq 2$ there is an exact sequence

$$0 \to \Lambda^{r-1}\mathcal{O}_X(-1)^{\oplus n} \to \Lambda^r \mathcal{O}_X(-1)^{\oplus n+1} \to \Lambda^r \mathcal{O}_X(-1)^{\oplus n} \to 0.$$

I obtained the map

$$\Lambda^r \mathcal{O}_X(-1)^{\otimes n+1} \to \Lambda^r \mathcal{O}_X(-1)^{\otimes n}$$

by carefully applying (II, Ex 5.16d) to the map $\mathcal{O}_X(-1)^{\oplus n+1} \to \mathcal{O}_X(-1)^{\oplus n}$. But this map just turns out to be locally defined by $x_n \mapsto 0$ where $x_0, \ldots, x_n$ are local coordinates for $\mathcal{O}_X(-1)$. Then $x_{i_0} \wedge \cdots \wedge x_{i_r}$ maps to 0 if some $i_k = n$ and itself otherwise. The map

$$\Lambda^{r-1}\mathcal{O}_X(-1)^{\oplus n} \to \Lambda^r \mathcal{O}_X(-1)^{\oplus n+1}$$

identifies $\Lambda^{r-1}\mathcal{O}_X(-1)^{\oplus n}$ with the kernel of the next map. The kernel of the next map is locally generated by all "monomials" which contain an x_n. Since $r \geq 2$ we can identify $\Lambda^{r-1}\mathcal{O}_X(-1)^{\oplus n}$ with this kernel by just removing the x_n off of the wedge product. [This is not rigorous enough!]

By induction on n we have that

$$H^i(\Lambda^{r-1}\mathcal{O}_X(-1)^{\oplus n}) = H^i(\Lambda^r \mathcal{O}_X(-1)^{\oplus n}) = 0$$

for all i (we will do the base case $n = 1$ in just a moment). Thus, by the long exact sequence of cohomology we see that $H^i(\Lambda^r \mathcal{O}_X(-1)^{\oplus n+1}) = 0$ for all i. For $n = 1$, since $r \geq 2$ it follows that $\Lambda^{r-1}\mathcal{O}_X(-1) = \mathcal{O}_X(-1)$ or 0 and $\Lambda^r \mathcal{O}_X(-1) = 0$ and these both have trivial cohomology as computed above.

Step 3. The final step is to obtain the long exact sequence

$$\cdots H^i(\Omega^r) \to H^i(\Lambda^r \mathcal{O}_X(-1)^{\oplus n+1}) \to H^i(\Omega^{r-1}) \to \cdots$$

then apply step 2 and the induction hypothesis (we are inducting on r, the base case was established in step 1) to calculate $H^i(\Omega^r)$ for all i.

Suppose $r \geq 1$, then by (II, 8.13) we have an exact sequence

$$0 \to \Omega_X \to \mathcal{O}_X(-1)^{\oplus n+1} \to \mathcal{O}_X \to 0.$$

By (II, Ex. 5.16d), $\Lambda^r \mathcal{O}_X(-1)^{\oplus n+1}$ has a filtration

$$\Lambda^r \mathcal{O}_X(-1)^{\oplus n+1} = F^0 \supseteq F^1 \supseteq \cdots \supseteq F^r \supseteq F^{r+1} = 0$$

with quotients

$$F^p/F^{p+1} \cong \Omega^p \otimes \Lambda^{r-p}\mathcal{O}_X = \begin{cases} 0 & \text{if } r - p \geq 2 \\ \Omega^p & \text{if } r - p \text{ is 0 or 1} \end{cases}$$

Thus

$$\Lambda^r \mathcal{O}_X(-1)^{\oplus n+1} = F^0 = \cdots = F^{r-1}$$

and the filtration becomes

$$\Lambda^r \mathcal{O}_X(-1)^{\oplus n+1} \supset F^r \supset F^{r+1} = 0$$

with $\Lambda^r \mathcal{O}_X(-1)^{\oplus n+1}/F^r \cong \Omega^{r-1}$ and $F^r \cong \Omega^r$. This gives an exact sequence

$$0 \to \Omega^r \to \Lambda^r \mathcal{O}_X(-1)^{\oplus n+1} \to \Omega^{r-1} \to 0.$$

The associated long exact sequence of cohomology gives for each i an exact sequence

$$H^i(\Lambda^r \mathcal{O}_X(-1)^{\oplus n+1}) \to H^i(\Omega^{r-1}) \to H^{i+1}(\Omega^r) \to H^{i+1}(\Lambda^r \mathcal{O}_X(-1)^{\oplus n+1}).$$

But by step 2 the groups $H^i(\Lambda^r \mathcal{O}_X(-1)^{\oplus n+1})$ all vanish. Thus $H^i(\Omega^{r-1}) \cong H^{i+1}(\Omega^r)$. By induction on r this shows that

$$H^i(\Omega^r) = \begin{cases} 0 & \text{if } i \neq r \\ k & \text{if } i = r \end{cases}.$$

This completes the proof.