

INI KA2W03: Mathematical physics: algebraic cycles, strings and amplitudes

Some sketchy notes

Table of Contents

Contents

Table of Contents	2
1 Preface	4
2 Talk 1: Tyler Kelly, Open Saito-Givental Invariants and Mirror Symmetry (Tuesday, July 19)	4
2.1 Saito-Givental Theory in dimension 1	5
2.2 Dimension 2	6
3 Talk 2: Man Wai Cheung, Cluster dualities and superpotentials for Grassmannian (Tuesday, July 19)	7
4 Talk 3: Dani Kaufman, Cluster varieties, amplitude symbols, and mutation invariants (Tuesday, July 19)	10
4.1 Cluster Varieties	10
4.2 Amplitudes	12
4.3 Cluster modular groups	12
4.4 Surface cluster ensembles	12
5 Talk 4: Goncalo Tabuada, Noncommutative generalized Riemann hypothesis (Tuesday, July 19)	13
5.1 GRH	13
5.2 Noncommutative geometry	14
5.3 Applications to commutative geometry	16
6 Herbert Gangl, The Aomoto polylogarithm via iterated integrals (Wednesday, July 20)	17
6.1 Classical Polylogarithms	17
6.2 Aomoto Polylogarithms	18
7 Adriana Salerno and Ursula Whitcher, Diagonal pencils and Hasse-Witt invariants (Wednesday, July 20)	19
8 Daniel Kaplan, Quiver varieties and symplectic resolutions of singularities (Wednesday, July 20)	20
8.1 Quivers	21
8.2 Local to global obstructions	23
9 Jonathan Rosenberg, Twisted derived equivalences and string theory dualities with B-field (Wednesday, July 20)	23
9.1 Generalizations	25

10 Andrew Harder, Hyperelliptic curves and planar 2-loop Feynman graphs (Wednesday, July 20)	25
10.1 Introductory	25
10.2 Extensions to 2-loop graphs	26
11 Noriko Yui, Modularity of certain Calabi-Yau threefolds over \mathbb{Q} (Wednesday, July 20)	27
12 Johannes Walcher, Rationality of MUMs and 2-functions (Thursday, July 21)	29
12.1 Rational 2-functions are abelian	29
12.2 MUM	29
13 Matt Kerr, K_2 and quantum curves (Thursday, July 21)	30
13.1 A miraculous coincidence	30
13.2 Beilinson’s Formula	31
13.3 Families of elliptic curves	31
13.4 Quantum Curves	32
14 Chloe Xiaohua Ai, From Feynman Amplitudes to multiple L-values (Thursday, July 21)	33
15 Lawrence Barrott, The Doran-Harder-Thompson conjecture via the Gross-Siebert program? (Thursday, July 21)	34
15.1 DHT Conjecture	34
16 Amanda Francis, Hanging gardens and divisors from Adinkras (Thursday, July 21)	36
17 Elana Kalashnikov, Unwinding the Gelfand—Cetlin toric degeneration on the mirror (Thursday, July 21)	37
17.1 Toric un-degenerating	41

1 | Preface

These are some (extremely sketchy) notes I've taken while attending the [last part of the program on K-theory, algebraic cycles and motivic homotopy theory at the Isaac Newton Institute \(INI\) in Cambridge](#). I don't expect these will be even remotely useful to anyone else, but perhaps this will have some value to those who just want some vague idea of what topics were discussed in talks. If you find yourself perplexed by anything here, the fault is almost surely entirely my own, since I was likely perplexed myself when I wrote it in the first place. Moreover, if anything appears incorrect, that too is surely my fault, either due to my own misunderstandings, typesetting errors, or just not hearing things correctly. The moral of the story: abandon all hope, all ye who continue reading!

2 | Talk 1: Tyler Kelly, Open Saito-Givental Invariants and Mirror Symmetry (Tuesday, July 19)

Remark 2.0.1: Realizations of cycles and motives play a role in the physics of amplitudes in string theory and the study of Feynman integrals. Today's focus: mirror symmetry, with a focus on certain integrals over cycles. Goal of mirror symmetry: link

- Enumerative data for a symplectic manifold, algebraic variety, or complex function, or
- Periods of a so-called mirror

The ur-example is a quintic threefold.

Remark 2.0.2: How this example works: count degree d rational curves in $X \subset \mathbb{P}^4$ a quintic threefold. The mirror is a crepant resolution of

$$Z(\sum x_i^5 - 5\psi \prod x_i) \subseteq \mathbb{P}^4 / (C_5)^3,$$

a 1-parameter deformation of the quintic. Compute as some integral $\int_{\gamma} \Omega_{\psi}$.

Question 2.0.3

How to construct the mirror? And how to distill the enumerative information?

Remark 2.0.4: Today's goal: answer these questions for Landau-Ginzberg (LG) models (X, G, W) where X is a quasi-affine variety $G \curvearrowright X$ is a finite group, $W = ?$. These give mirrors to Fanos, and noncommutative symplectic deformations of complete intersections in toric varieties.

Remark 2.0.5: Example LG model: $(\mathbb{C}^n, \prod \mu_{r_i}, \sum x_i^{r_i}) \rightleftharpoons (\mathbb{C}^n, 1, \sum x_i^{r_i})$. Answers to these questions for LG models: the mirror is built via a combinatorial construction, which annoyingly

isn't geometric. For enumerative data: oscillatory integrals associated to the RHS using Saito-Givental theory the encapsulate FJRW invariants. Issue: deformation must be built perturbatively term-by-term.

2.1 Saito-Givental Theory in dimension 1

Remark 2.1.1: Start with $W : \mathbb{C}^n \rightarrow \mathbb{C}$ given by $\sum x_i^{r_i}$. Build a twisted de Rham complex where $\delta = d + h^{-1}dW \vee (-)$ to get $0 \rightarrow \Omega^0 \xrightarrow{\delta} \dots$, then take hypercohomology. For e.g. $W = x^r$, a basis can be computed explicitly for H^1 . One can cook up a perfect pairing between a certain relative homology $H_n(X, \mathfrak{R}(W/H) \ll 0, \mathbb{C})$ and this hypercohomology, roughly

$$(\psi, \omega) \mapsto \int_{\xi} e^{\frac{W}{h}} \omega$$

where $\mathfrak{R}(W/h) \ll 0$ so that the exponential term is small. Now take deformations to distill enumerative info from this periods. First approximation: take the versal deformation

$$W^s = \sum x_i^{r_i} + \sum s_{a_1, \dots, a_n} \prod x_i^{a_i}.$$

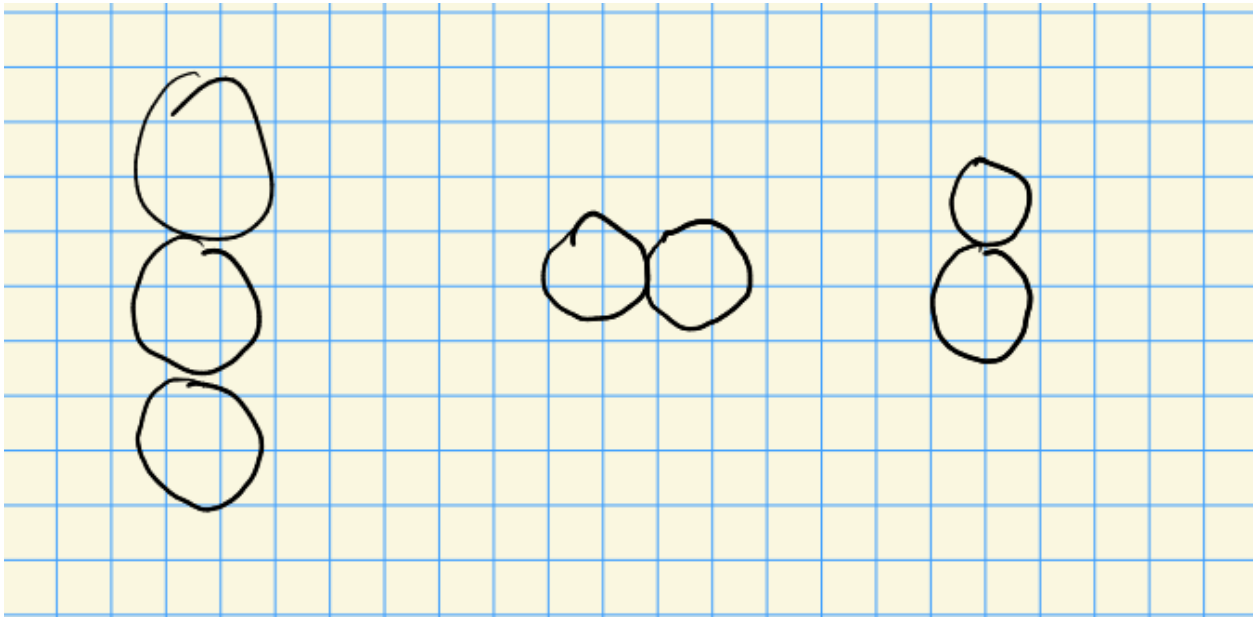
Integrate to get a power series whose coefficients are $\psi_k \in \mathbb{C}[[s_{a_1, \dots, a_n}]]$. Compute this using a dual basis and integration by parts.

Can try $W = x^4 + s_2 s^2 + s_1 x + s_0$. Problem: lack of linearity! But this can be fixed to obtain flat coordinates and a primitive form ω .

Remark 2.1.2: Theorem: there is a deformed potential W^t which has primitive form $\omega = dx$ and flat coordinates t_0, \dots, t_{r-2} for the Frobenius manifold construction via Saito-Givental theory for the LG model $W = x^r$. This is a generating function for open enumerative invariants.

Remark 2.1.3: Strategy: start with X , where we want to understand its enumerative geometry. Build the mirror \tilde{X} using some type of *open* enumerative geometry, along with a deformation. Look at integrals of forms over pieces of \tilde{X} , and that should recover the enumerative counts.

Remark 2.1.4: Build a moduli of orbifolds \mathcal{M} by taking configurations of marked points on \mathbb{P}^1 , where points off of \mathbb{R} come in pairs of complex conjugates. The local picture at marked points: $[\mathbb{C}/\mu_r]$. Compactify by considering degenerations of configurations – internal nodes, boundary nodes, contracted boundaries:



The compactified moduli space is a real manifold with corners.

Remark 2.1.5: Take the universal line bundle $\mathcal{L} \rightarrow \mathcal{M}$ and define the Witten bundle $R^1(\pi_*\mathcal{L})^*$. One needs to specify a canonical multisection s_{can} to control what happens on the boundary in order to integrate over \mathcal{M} . Theorem: in dimension one, the open r -spin invariants exist and don't depend on choice of this multisection, enjoy topological recursion, and there are closed form solutions of certain invariants. Parts of this theorem fail in dimension 2, although analogous statements can be made.

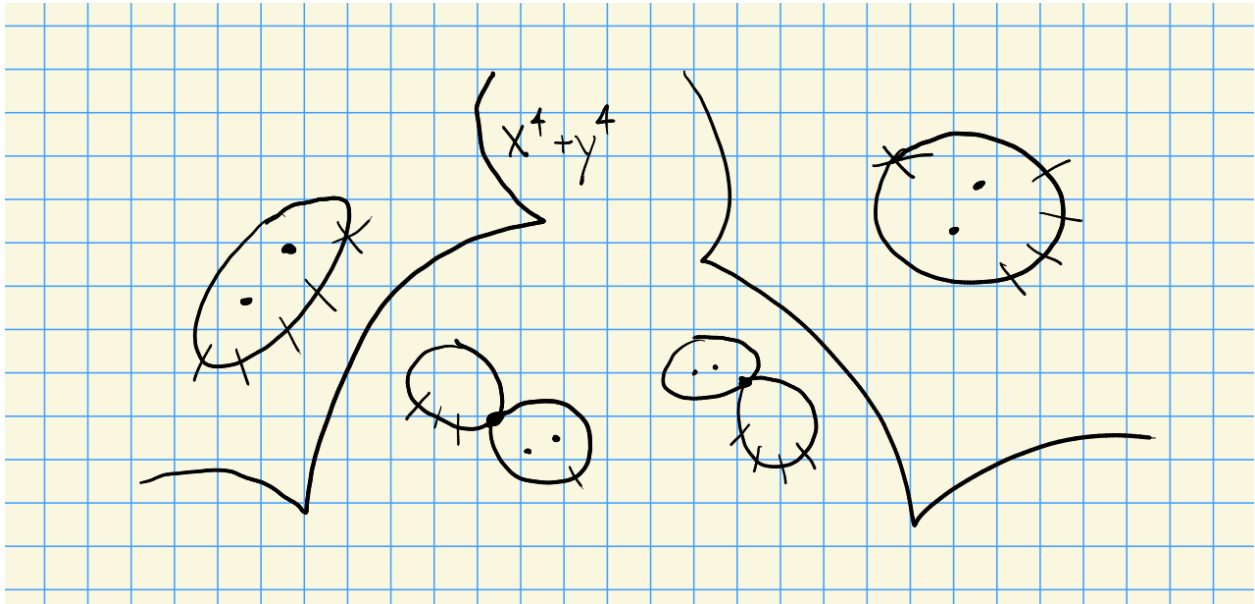
Theorem 2.1.6 (Gross-K-Tessler).

Open FJRW invariants exist for $(\mathbb{C}^2, \mu_r \times \mu_s, x^r + y^2)$ and do depend on the choice of s_{can} .

2.2 Dimension 2

Remark 2.2.1: Proceed as before, but run into issues with extra terms involving Gamma functions in h^0 , as opposed to terms attached to h^{-1} previously. A solution: insert a hypergeometric function to cancel it, as Li-Li-Saito-Shen do.

What's the picture?



There are some cases with mild singularities (ADE and elliptic type), outside of these things become unmanageable. Key foothold in the proof: no wall-crossing to deal with, which is advantageous. Proof uses topological recursion: compute something in two different ways, set them equal, and play the two formulae off of each other.

3

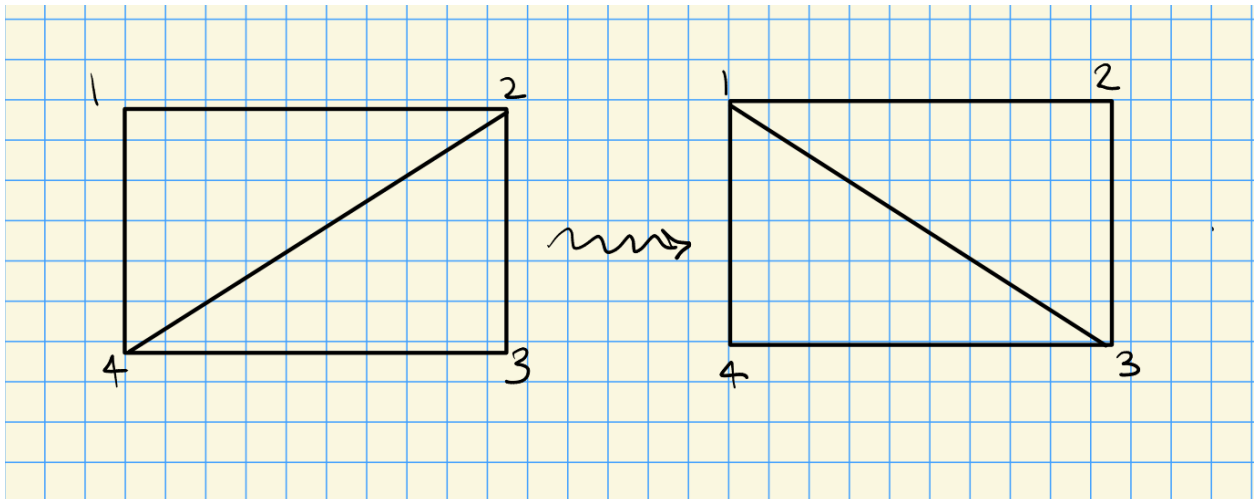
Talk 2: Man Wai Cheung, Cluster dualities and superpotentials for Grassmannian (Tuesday, July 19)

Remark 3.0.1: Goal: Marsh-Rietsch potential write down a potential for Grassmannians. Gross-Hacking-Keel-Kontsevich do as well, we'll construct a map P going from the former to the latter. Makes use of cluster structure on Gr , cluster varieties, and a GHKK construction.

Remark 3.0.2: Cluster algebras: start with a seed $s = (\{A_1, \dots, A_n\}, B = (b_{ij}))$ where A_i are variables and $B = (b_{ij})$ is a skew-symmetrizable matrix. Start with an initial seed, allow "mutations" taking $s \rightarrow s'$ where the new A_i are functions of the old, and take all such possible iterations to generate the algebra.

Example 3.0.3(?): Example: $s = \left(\{A_1, A_2\}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \right)$. Mutate at 1 to get $A_1 A'_1 = 1 + A_2$; write $A_3 := A'_1$ to get $A_3 = \frac{1 + A_2}{A_1}$. The next seed is $(\{A_3, A_2\}, B')$.

Remark 3.0.4: Can express cluster structures in terms of triangulations of $(n + 3)$ -gons:



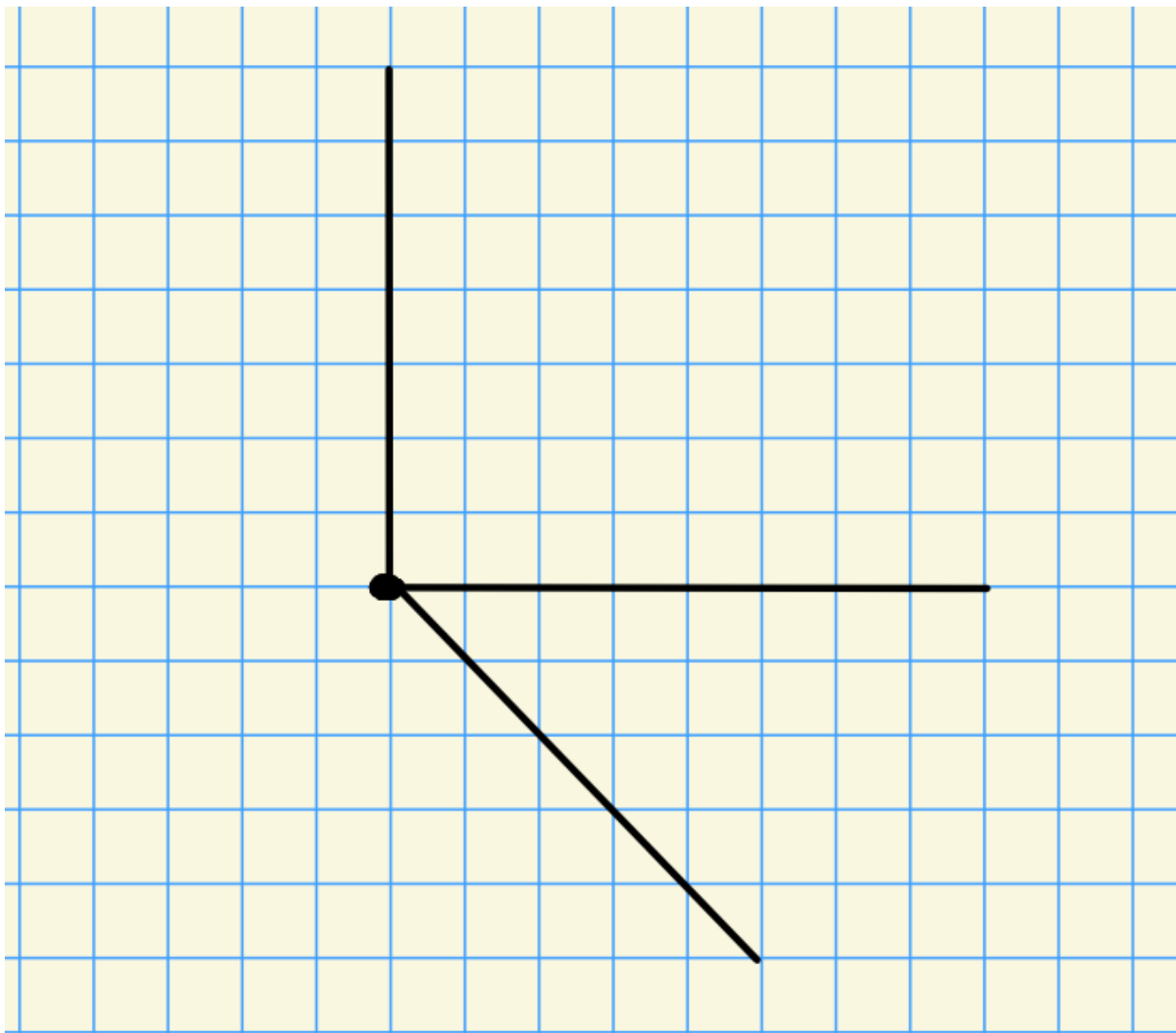
This flip corresponds to the mutation relation $A_{24}A_{13} = A_{12}A_{34} + A_{14}A_{23}$. Since we can not flip the boundary, some variables are frozen – namely those corresponding to adjacent nodes, so $A_{12}, A_{23}, A_{34}, A_{41}$.

Remark 3.0.5: The homogeneous coordinate ring of $\mathbb{C}[\text{Gr}_k(n)]$ carries a cluster structure. On the geometry side, cluster varieties are log Calabi-Yau whose rings of regular functions carry cluster structures. For seeds, variables correspond to algebraic tori $(\mathbb{C}^\times)^n$, and mutations are birational maps between tori and defines a gluing between them.

Remark 3.0.6: Formally defining a cluster variety: start with a lattice N and skew-symmetric form $\{-, -\}$ on N , and integers d_i , and a seed given by a basis for N . Take a dual basis M and scale to M° by taking a basis $\frac{1}{d_i}e_i^*$. Take the corresponding tori A, X and glue by mutations to get the corresponding cluster variety. The form gives a cluster ensemble lattice map $p^* : N \rightarrow M^\circ$ where $n \mapsto \{n, -\}$. Since these commute with mutation, and produce a map of varieties $A \rightarrow X$.

Remark 3.0.7: There is a Langlands dual given by sending $B \mapsto B^{-t}$. A conjecture (Bardwellcot-Evans-C-Hong-Lin) of mirror symmetry: $A \rightleftharpoons X^L$ and $X \rightleftharpoons A^L$. For today, the exchange matrices are skew-symmetric, so the expectation is that $A \rightleftharpoons X$ are mirrors.

Remark 3.0.8: Cluster varieties can be described by scattering diagrams: defining $N_{\mathbb{R}}, M_{\mathbb{R}}$, this is a collection of walls with finiteness and consistency conditions. A wall is a pair (d, f_d) where $d \subseteq M_{\mathbb{R}}$ is the support of walls, a convex rational polyhedral cone of codimension 1.



Remark 3.0.9: Crossing walls: $p_\gamma : z^m \mapsto z^m f_d^{\pm \langle n_0, m \rangle}$ where n_0 is a primitive normal to a wall and γ is a path between chambers. A motivating example for theta functions: $(\mathbb{C}^\times)^2$ satisfies $H^0((\mathbb{C}^\times)^2; \mathcal{O}) = \bigoplus_{m_1, m_2 \in \mathbb{Z}} \mathbb{C} z_1^{m_1} z_2^{m_2}$. To each nonzero point in M° assign a theta function θ_m , defined

from a collection of broken lines with initial slope m and endpoint Q in the positive chamber. Identify points in the scattering diagram with tropical points of Langlands dual cluster varieties.

Remark 3.0.10: Cluster structures for Gr: have a distinguished anticanonical divisor $D_{ac} = \sum_{i=1}^n D_i$

where $D_k = D_{[i+1, i+k]} = \{p_{[i+1, i+k]} = 0\}$. Let $\text{Gr}_k(\mathbb{C}^n)^\circ = \text{Gr}_k(\mathbb{C}^n)/D$, the positroid... something. Compactify using boundary divisors $D_k = D_{[i+1, i+(n-k)]}$. Each irreducible component yields a term in the potential of the mirror (something about LG models). Can somehow get valuations on the coordinate ring and write the potential directly.

Remark 3.0.11: Marsh-Rietsch define an isomorphism from the Jacobi ring of $(\text{Gr}_k(\mathbb{C}^n)^\circ, W)$ to the equivariant quantum cohomology ring of $\text{Gr}_{n-k}(\mathbb{C}^n)$. Rietsch-Williams: there is a Newton-

Okounkov body on one side $\Delta(D)$, identifies points in this as tropical points on functions extending to the boundary of... a potential Γ ? Use theta reciprocity to get $\Delta(D) \cong \Gamma$ via p^* .

4

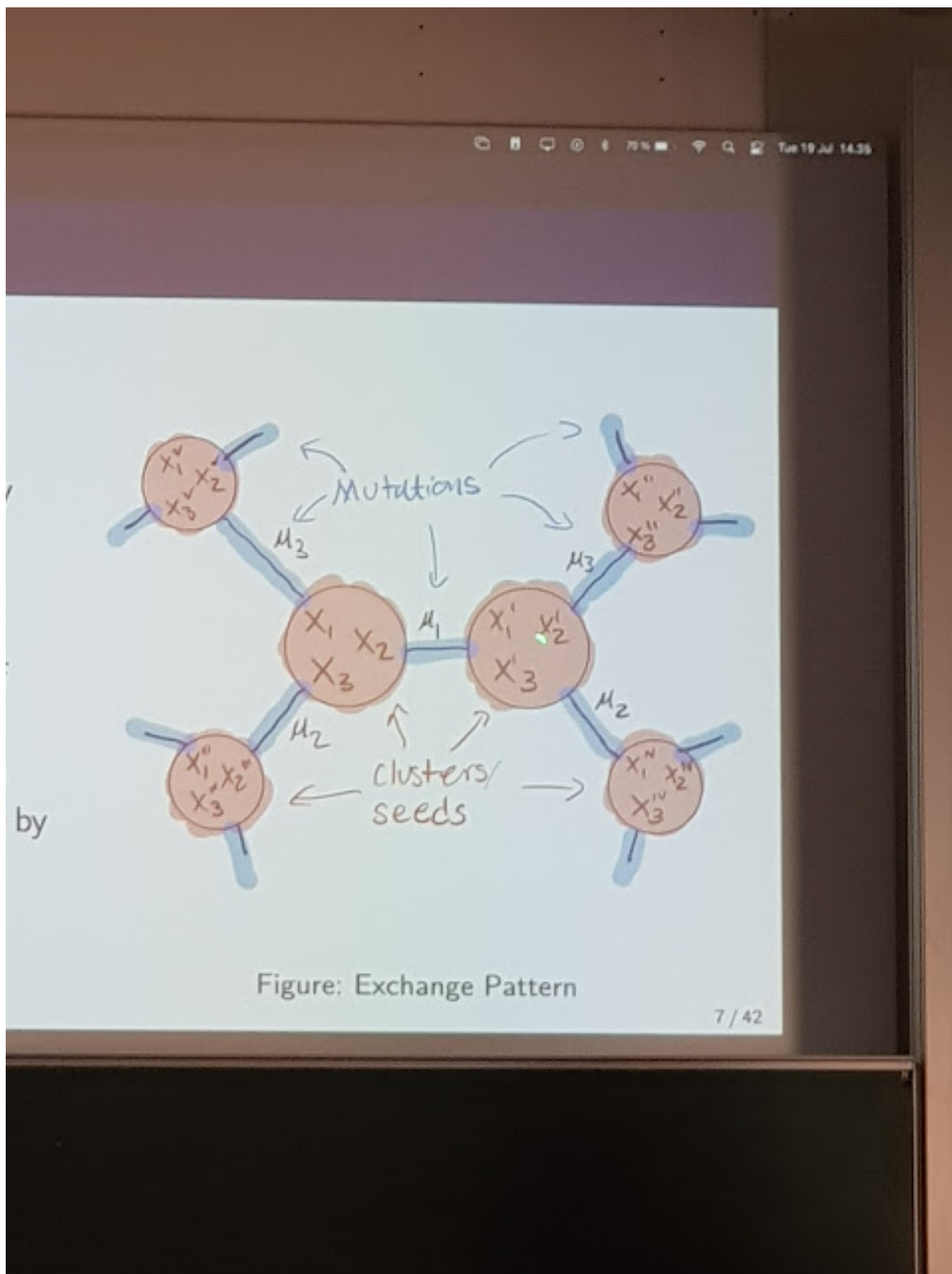
Talk 3: Dani Kaufman, Cluster varieties, amplitude symbols, and mutation invariants (Tuesday, July 19)

4.1 Cluster Varieties

Remark 4.1.1: Recall: cluster varieties are varieties with a toric atlas generated by some extra combinatorial data. Encodes unexpected coordinate functions, and automorphisms of the cluster structure encode exotic automorphisms of the variety. Cluster structure for the Grassmannian: uses Plucker coordinates. New coordinate functions appear as part of a *symbol alphabet* for scattering amplitudes for $N = 4$ SYM (super Yang-Mills) theory.

Remark 4.1.2: Idea: the same function can be obtained from several different constructions, classifying mutation invariants shows that these must produce the same functions. Examples of such special functions: traces of monodromy, cluster character, Diophantine equations of Markov numbers, theta functions, Skein modules elements of closed loops, etc.

Remark 4.1.3: Cluster varieties: varieties built out of seed tori glued along birational morphisms called mutations.



Geometric type cluster varieties can be tracked with quivers – a quiver is used as the initial seed and generated a cluster variety after taking mutations. Cluster ensembles associated with a quiver Q : a pair of cluster varieties (A, X) and a map $p : A \rightarrow X$ defined on coordinates on the same

seed by $p^*(x_i) = \prod_j a_j^{e_{ij}}$. Introduced by Fock, Goncharov toward an algebraic theory of higher Teichmüller spaces. Example: $\text{Gr}_p(p+q)$ and $\text{Conf}_n(\mathbb{P}^{k-1})$ are an (A, X) pair. Interestingly, the immutable seeds for $\text{Gr}_3(7)$ look like an E_6 diagram! There are “phase transitions” for values of p, q where there is a jump from finitely many clusters to infinitely many.

4.2 Amplitudes

Remark 4.2.1: Scattering amplitudes in the planar limit $N = 4$ SYM live in $\text{Conf} + n(\mathbb{P}^3)$, and the amplitudes for up to two loops can be written in terms of polylogarithms. The arguments of these polylogarithms come from the X cluster variety structure on this configuration space, and constitute the *symbol* of the amplitude which encodes its singularity and branch cut structure. Goal: understand what happens as cluster ensembles transition from finite type to infinite type.

4.3 Cluster modular groups

Remark 4.3.1: Idea: automorphism group Γ of mutation structures, so similar to a mapping class group. Can define an action of this on the rational function fields of these varieties. Mutation invariant functions: those invariant under some subgroup of Γ .

Remark 4.3.2: Example: take the Markov quiver, which is associated to the punctured torus and the affine root system of type $A_1^{(1,1)}$ and $\Gamma = \text{PSL}_2(\mathbb{Z})$. Similarly a 4-punctured sphere is of type elliptic $D_4^{(1,1)}$. See Somos sequences.

4.4 Surface cluster ensembles

Remark 4.4.1: Look at S associated to a marked surface and take $G = \text{PSL}_2(\mathbb{R})$, then A corresponds to Penner’s decorated Teichmüller spaces and X with Teichmüller space with Fock coordinates parameterizing hyperbolic somethings with an “opening”.

Remark 4.4.2: Main theorem of their thesis: classification of invariants for the action of Dehn twists on surface cluster ensembles. The invariant ring $\mathbb{R}(X)^{\langle \gamma \rangle}$ is generated by traces of monodromy operators of excised closed curves and invariant X coordinates for an excising triangulation of δ a simple closed curve, where S is a marked surface and $\gamma \in \Gamma_S$ is a Dehn twist about δ . For A , it’s generated by curves that don’t intersect δ .

Remark 4.4.3: There’s some way to obtain a quiver from a triangulation of a surface. More generally, each cluster modular group has an element that looks like a Dehn twist (a *cluster Dehn twist*), even when the cluster doesn’t come from a surface at all.

Question 4.4.4

Some questions:

- What can we say about cluster invariants from higher Teichmüller spaces? Are these related to higher laminations, tensor webs?
- Are the square root symbols related to polylogarithm relations?
- The collection of invariants for a Dehn twist behave like cluster variables for a limit cluster ensemble. How is this limit related to compactifications of Teichmüller space?

5

Talk 4: Goncalo Tabuada, Noncommutative generalized Riemann hypothesis (Tuesday, July 19)

5.1 GRH

Remark 5.1.1: Recall $\zeta(s) = \sum_{n \geq 1} \frac{1}{n^s}$, which admits an Euler product expansion $\prod_{p \text{ prime}} \frac{1}{1 - p^{-s}}$, which converges for $\Re(s) > 1$ and has a simple pole at $s = 1$.

Conjecture 5.1.2 (Riemann, 1859).

$$\zeta(s) = 0 \implies \Re(s) = \frac{1}{2}.$$

Remark 5.1.3: Formulating GRH: let

- K be a global field (e.g. $K = \mathbb{Q}$), so for $\text{ch}(K) = 0$ a finite extension of \mathbb{Q} and for $\text{ch}(K) > 0$ an extension of $\mathbb{F}_q(t)$.
- X be a smooth proper K -scheme.
- $0 \leq w \leq 2 \dim(X)$
- Σ_K the infinite set of nonarchimedean places of K
- For $v \in \Sigma_K$,
 - K_V the completion of K at v (e.g. \mathbb{Q}_p)
 - \mathcal{O}_v the valuation ring of K_V (e.g. \mathbb{Z}_p)
 - κ_v the residue field of \mathcal{O}_v (e.g. $\mathbb{Z}/p\mathbb{Z}$)
 - p_v the characteristic of κ_v (e.g. p)
 - N_v the cardinality of κ_v (e.g. p)

There is a SES

$$1 \rightarrow I_v \rightarrow G_{K_v} \rightarrow G_{\kappa_v} \cong \widehat{\mathbb{Z}} \rightarrow 1$$

where I_v is the inertia group. Let the Frobenius be π_v , defined as the inverse of $\lambda \mapsto \lambda^{N_v} \in \widehat{\mathbb{Z}}$. Choose primes ℓ_v and field embeddings $\iota_v : \mathbb{Q}_{\ell_v} \rightarrow \mathbb{C}$.

Definition 5.1.4 (Serre)

$$L_w(X; s) = \prod_{v \in \Sigma_k} L_{w,v}(X, s),$$

$$L_{w,v}(X; s) := \frac{1}{\det \left(\text{id} - N_v^{-s}(\pi_v \otimes \mathbb{C}) \mid H_{\text{ét}}^w(X \times_{K_v} \overline{K}_v; \mathbb{Q}_{\ell_v})^{I_v} \otimes \mathbb{C} \right)}.$$

Theorem 5.1.5 (Serre).

$L_w(X; s)$ is a meromorphic function on the half-plane $\Re(s) > \frac{w}{2} + 1$. Some conditions:

- $L_w(X; s)$ admits a meromorphic continuation to \mathbb{C}
- $L_w(X; s)$ has no poles in the critical strip $\frac{w}{2} < \Re(s) < \frac{w}{2} + 1$.

Conjecture 5.1.6 (GRH).

$$L_w(X; s) = 0 \text{ and } \frac{w}{2} < \Re(s) < \frac{w}{2} + 1 \implies \Re(s) = \frac{w + 1}{2}.$$

Example 5.1.7 (?): For $K = \mathbb{Q}$, $X = \text{Spec } \mathbb{Q}$, $w = 0$ then $L_0(X; s) = \zeta(s)$.

For K a number field and $X = \text{Spec } K$, $L_0(X; s) = \sum_{I \trianglelefteq \mathcal{O}_K} \frac{1}{N(I)^s}$ is the Dedekind-zeta function.

Remark 5.1.8: Status of GRH:

- For $\text{ch } k = 0$, no cases are known. The case of a point $\text{Spec } \mathbb{Q}$ is already classical RH!
- $\text{ch } k > 0$ follows from Grothendieck-Deligne.

5.2 Noncommutative geometry

Remark 5.2.1: Recall that a DG-category is a category enriched over complexes of K -modules.

Example 5.2.2(?): Examples: DG-algebras and $\text{Perf}_{\text{dg}}(X)$ for $X \in \text{Sch}/K$. By Beilinson, $X = \mathbb{P}^1$, $\text{Perf}_{\text{dg}}(\mathbb{P}^1)$ is Morita-equivalent to $\begin{bmatrix} K & K \oplus K \\ 0 & K \end{bmatrix}$.

Definition 5.2.3 (?)

Let \mathbf{A} be a DG-category, and say \mathbf{A} is a **geometric NC scheme** if there exists a smooth proper $X \in \text{Sch}/K$ such that there is an admissible embedding $\mathbf{A} \hookrightarrow \text{Perf}_{\text{dg}}(X)$.

Example 5.2.4(?): A a finite dimensional k -algebra with finite global dimension where A/K is K -separable defines a geometric NC scheme. For the quiver $Q := (\cdot \rightarrow \cdot)$, $KQ = \begin{bmatrix} K & K \\ 0 & K \end{bmatrix}$ is not Morita-equivalent to $\text{Perf}_{\text{dg}}(Y)$ for any smooth proper Y . So there exist NC schemes which are not themselves schemes.

Remark 5.2.5: Can GRH be formulated in the NC setting? Attempt to define NC ℓ -adic cohomology: let

- K be a field
- $\ell \neq \text{ch}(K)$ a prime
- A a geometric NC scheme

Define the étale K -theory of A as

$$\mathbf{K}^{\text{ét}}(A)_{\widehat{\ell}} := \text{hocolim}_{n \geq 0} L_{\text{KU}} \mathbf{K}(A; \mathbb{Z}/\ell^n).$$

Define

$$\begin{cases} H_{\text{even}, \ell}(A) & \pi_0 \mathbf{K}^{\text{ét}}((A \otimes_K \overline{K})_{\widehat{\ell}}) \otimes_{\mathbb{Z}} \mathbb{Z}[\frac{1}{\ell}] \\ H_{\text{odd}, \ell}(A) & \pi_1 \mathbf{K}^{\text{ét}}((A \otimes_K \overline{K})_{\widehat{\ell}}) \otimes_{\mathbb{Z}} \mathbb{Z}[\frac{1}{\ell}]. \end{cases}$$

These are \mathbb{Q}_{ℓ} -linear G_K -modules, and can be used to define NC L -functions.

Definition 5.2.6 (Noncommutative L -functions)

Let K be a global field, A a NC geometric K -scheme. Define

$$L_{\text{even}}(A, s) := \prod_{v \in \Sigma_K} L_{\text{even}, v}(A; s)$$

$$L_{\text{even}, v}(A; s) = \frac{1}{\det \left(\text{id} - N_v^{-s} (\pi_v \otimes \mathbb{C}) \mid H_{\text{even}, \ell_v}(A \otimes_K K_v)_{\mathbb{V}}^I \otimes \mathbb{C} \right)},$$

and $L_{\text{odd}, v}$ similarly.

Theorem 5.2.7(?).

$L_{\text{even}}(A; s)$ is a meromorphic function on $\Re(s) > 1/2$, and odd for $\Re(s) > 3/2$.

Conjecture 5.2.8 (NC generalized RH).

The even zeros in the critical strip are on $\Re(s) = \frac{1}{2}$, and the odd ones on $\Re(s) = 1$.

Theorem 5.2.9 (Tob?).

Let X be a smooth proper K -scheme. Then $R_{\text{even}}(\text{Perf}_{\text{dg}}(X)) \iff \{R_w(X)\}_{w \text{ even}}$, and similarly for odd, where R denotes the various forms of GRH.

5.3 Applications to commutative geometry

Theorem 5.3.1 (?).

Let K be a global field with $\text{ch}(K) = 0$ and let X, Y be smooth proper K -schemes and assume condition (C2). Suppose $\text{Perf}(X) \cong \text{Perf}(Y)$ is a Fourier-Mukai equivalence, then $\{R_w(X)\}_w \iff \{R_w(Y)\}_w$. So this version of GRH is invariant under derived equivalence.

Theorem 5.3.2 (?).

Suppose X is homologically projectively dual (HPD) to Y , with $X \rightarrow \mathbb{P}(V)$ and $Y \rightarrow \mathbb{P}(V^\vee)$. For $L \subseteq V^\vee$, write $X_L := X \times_{\mathbb{P}(V)} \mathbb{P}(L^\perp)$ and $Y_L = Y \times_{\mathbb{P}(V^\vee)} \mathbb{P}(L)$.

Remark 5.3.3: See

- Veronese-Clifford duality
- Grassmannian-Pfaffian duality
- Spinor duality?
- Determinantal duality?

Theorem 5.3.4 (?).

Let $V_1 \cdots V_n$ be simple A/J -modules and consider $\text{End}(V_i)$ which are division K -algebras. Then $K \hookrightarrow K_i \hookrightarrow \text{End}(V_i)$ where the K_i are finite separable extensions, and

$$\{R_0(\text{Spec } K_i)\}_{1 \leq i \leq m} \implies R_{\text{even}}(A).$$

Remark 5.3.5: This doesn't exactly help prove GRH, but does help establish properties of various conjectures. See paper of Thomason for paper on étale K-theory.

Some analogs: $H_{\text{dR}} \mapsto HP$, $H_{\text{crys}} \mapsto TP$ (topological periodic cyclic homology) which comes equipped with a Frobenius and can be used to define zeta functions.

6 | Herbert Gangl, The Aomoto polylogarithm via iterated integrals (Wednesday, July 20)

Remark 6.0.1: The plan:

- Iterated integrals and formal versions, leading to the “algebraic fingerprint” (i.e. symbols).
- Aomoto polylogarithms
- A new choice of coordinates on configuration spaces
- Relating Aomoto polylogarithms to iterated integrals via the symbol

6.1 Classical Polylogarithms

Remark 6.1.1: Recall

$$\mathrm{Li}_n(z) = \sum_{k \geq 1} \frac{z^k}{k^n}.$$

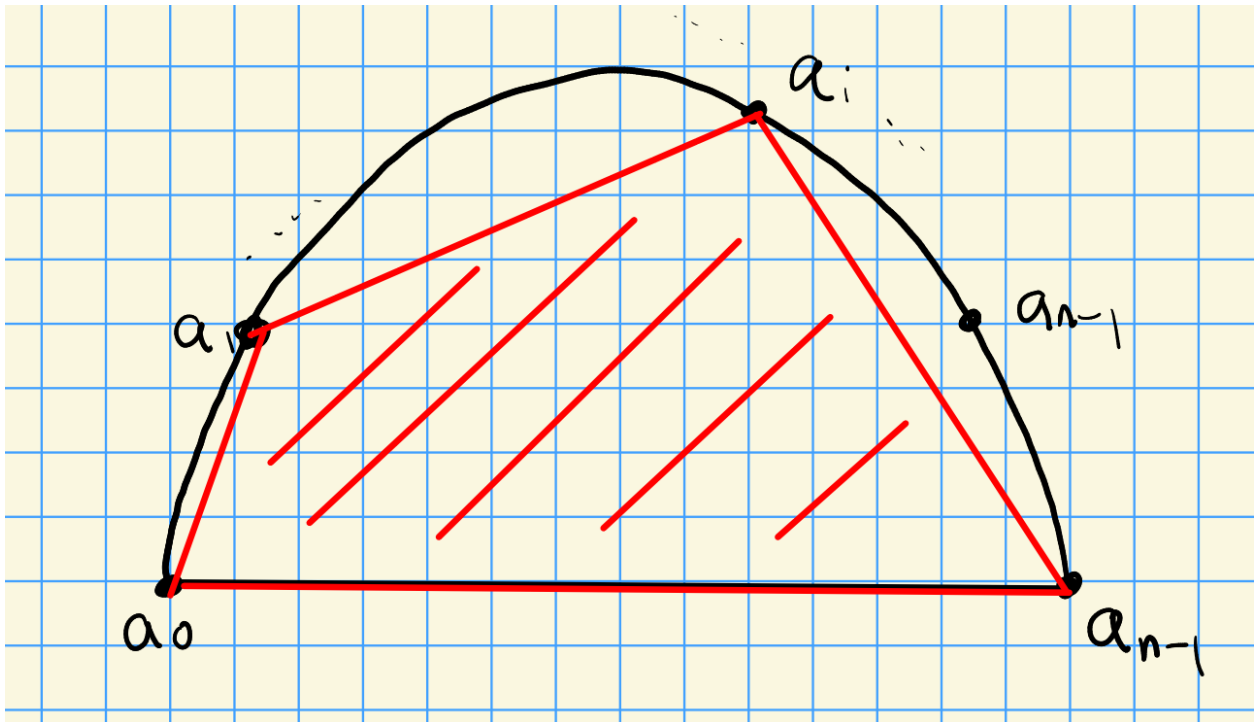
Here n is a *weight*, and the *depth* is 1 (the number of variables). More generally, for example, $\mathrm{Li}_{m,n}(x, y)$ has weight $m + n$ and depth 2. This can be written as an iterated integral:

$$\Pi(a_0; a_1, \dots, a_n : a_{n+1}) = \int_{\Delta_\gamma} \frac{dt_1}{t_1 - a_1} \wedge \dots \wedge \frac{dt_n}{t_n - a_n}$$

where $\gamma : [0, 1] \rightarrow \mathbb{C} \setminus \{a_1, \dots, a_n\}$ with $\gamma(0) = a_0, \gamma(1) = a_{n+1}$ and Δ_γ is a certain simplex. Some properties:

- Form an algebra under the shuffle product
- Path composition
- Path reversal
- An associated differential equation

Remark 6.1.2: One can encode this formally as an algebra over a field F with generators $I(a_0, \dots, a_{n+1})$ and relations coming from the above properties. A new feature: a “semicircle coproduct” due to Goncharov, whose terms are indexed by polygons involving any subset of the a_i when arranged along a semicircle:



In general, the symbol for $I(\dots)$ will take values in $(F^\times)^{\otimes n}$.

6.2 Aomoto Polylogarithms

Remark 6.2.1: Idea: attach a period to any pair of simplices (L, M) in $\mathbb{P}^n(\mathbb{C})$ determined by coordinates $(\ell_0, \dots, \ell_n; m_0, \dots, m_n)$.

- To L , attach ω_L a differential form, $d\text{Log}\left(\frac{z_1}{z_0}\right) \wedge \dots \wedge d\text{Log}\left(\frac{z_n}{z_0}\right)$.
- To M attach an integral cycle.

This gives an assignment $(L, M) \mapsto \int_{\Delta_M} \omega_L$, which generalizes the classical polylogarithm. Some properties:

- Degeneracies go to zero (?)
- Skew-symmetry: $(\sigma L, M) = (L, \sigma M) = \text{sign}(\sigma)(L, M)$ for any $\sigma \in S_n$.
- Invariance for $g \in \text{PGL}_{n+1}(\mathbb{C})$ in the sense that $(gL, gM) = (L, M)$.
- Additivity, a type of scissors congruence property: $\sum_{i=0}^{n+1} (-1)^i (\ell_0, \dots, \widehat{\ell}_i, \dots, \ell_{n+1}; M) = 0$.

BGSV ('89) formalized this in an F -algebra with generators (L, M) and relations coming from these new properties, and found a good candidate for the category of mixed Tate motives over a field. An issue is that the coproduct is only defined on generic such pairs.

Example 6.2.2(?): Consider $n = 1$ and (L, M) with $(\ell_0, \ell_1, m_0, m_1)$, then

$$A_1(L, M) = \int_{m_0}^{m_1} d\text{Log} \left(\frac{z_1 - \ell_1}{z_0 - \ell_0} \right).$$

One can extract a cross-ratio from this which lives in F^\times .

Remark 6.2.3: A surprise to BGSV: a certain expression can be written as a combination of Steinberg elements $a \otimes (1 - a)$ which is related to motivic cohomology and the Bloch-Suslin complex. Goncharov also writes $A_3(L, M)$ in terms of polylogarithms, and attaches a symbol to general $A_n(L, M)$ in terms of Plucker coordinates.

Theorem 6.2.4 (G-Charlton-Radchenko).

The symbol of $A_n(\mathbf{v})$ can be written as a sum of iterated integrals.

7 | Adriana Salerno and Ursula Whitcher, Diagonal pencils and Hasse-Witt invariants (Wednesday, July 20)

Remark 7.0.1: Calabi-Yau threefolds generalize elliptic curves, and K3s are somewhere in-between. Idea: use mirror symmetry to prove arithmetic properties of higher dimensional varieties, generalizing things we know about elliptic curves.

Remark 7.0.2: Consider the Legendre family $X_\psi : y^2 = x(x-1)(x-\psi)$ satisfies Picard-Fuchs, and over \mathbb{F}_p the trace of Frobenius is $a_p = 1 + p - \sharp X_\psi(\mathbb{F}_p)$. Igusa noticed that the Picard-Fuchs equations for the holomorphic form is hypergeometric, and a_p satisfies a truncated hypergeometric formula. Some progress on arithmetic properties of the Fermat quintic.

Remark 7.0.3: What is the mirror of smooth quintics in \mathbb{P}^4 ? Start with the Fermat quintic pencil $\sum x_i^5 - \psi \prod x_i = 0$, which admits a C_5^3 action. Taking the quotient and resolving singularities yields the mirror Y . For $\psi \in \mathbb{Z}$, the point counts agree: $\sharp X(\mathbb{F}_q) \cong \sharp Y(\mathbb{F}_q) \pmod q$.

Remark 7.0.4: Generalized hypergeometric functions:

$${}_A F_V(\alpha, \beta | z) = \sum_k \frac{\alpha_{1,k} \cdots \alpha_{A,k}}{\beta_{1,k} \cdots \beta_{B,k}} z^k.$$

Remark 7.0.5: Daqing-Wang: for the Fermat pencil X_ψ and its Greene-Plesser mirror Y_ψ , the unit roots of their zeta functions coincide, yielding the congruence of point counts. Kadir claims something similar works for generalize Fermat pencils of CYs in a Gorenstein Fano weighted projective space and its Greene-Plesser mirror. Recent work by (lots of people) uses BHK mirror symmetry to generalize to invertible pencils in \mathbb{P}^n .

Remark 7.0.6: Batyrev's construction: mirror families of CYs using reflexive polytopes. Look at polar polytopes, lattice polytopes Δ which contains 0 in its interior. The facet equation can be normalized to $\sum a_i x_i = -1$, and there is an associated **polar polytope** Δ° associated to the coefficients (a_i) . The polytope Δ is **reflexive** if Δ° is again a lattice polytope. Note that there is no classification of reflexive polytopes in dimensions $d \geq 5$.

Remark 7.0.7: Building hypersurfaces: for each lattice point choose a parameter α and define $p_\alpha = \sum_{M \cap \Delta} \alpha_m \prod x^m$. Can use this to somehow cook up mirror pairs of CYs? Using that CYs can be realized as hypersurfaces in toric varieties.

Remark 7.0.8: Kernel pairs: for Δ, Γ combinatorially equivalent polytopes, put vertices as columns of matrix and see if they have equal kernels. These are resolutions of quotients of a common toric variety, and the corresponding vertex pencils satisfy the same Picard-Fuchs equation.

Remark 7.0.9: An alternative to counting points for small primes: for X a smooth variety, Frobenius induces a p -linear operator $H^n(X; \mathcal{O}_X) \circlearrowleft$, and the matrix for this map in some basis is the **Hasse-Witt matrix** HW_p . Katz: for CYs, this determines the point counts mod p ? A theorem of Huang-Lian-Yau-Yu shows that HW_p for X a Calabi-Yau satisfies a truncated relation and shows that the period integral has a certain power series expansion. For smooth CYs, we have a good idea of what HW_p is.

Remark 7.0.10: Idea: for mirror kernel pairs, can sometimes replace a very complicated equation with a much simpler equation for its mirror, e.g. if one wants to do point counts or study the Picard-Fuchs equation. Can classify K3 surfaces by their HW_p .

Remark 7.0.11: Picard-Fuchs for rank 19 lattice-polarized K3s can be written as the symmetric square of a 2nd order linear Fuchsian differential equation. For hypergeometric functions, can use Clausen's formula.

Remark 7.0.12: Let $\Gamma \subseteq \mathrm{PSL}_2(\mathbb{R})$, and define a modular curve as $\overline{\Gamma \backslash \mathbb{H}}$. See Atkin-Lehner map. Group I in the classification is related to Monstrous Moonshine, and the classification shows it's associated to $\Gamma_0(3)^+$ where plus means retaining all Atkin-Lehner involutions.

8 Daniel Kaplan, Quiver varieties and symplectic resolutions of singularities (Wednesday, July 20)

Remark 8.0.1: Three main sections:

- Spaces you (yes you!) care about can be realized as quiver varieties,
- These are well-understood with respect to symplectic resolutions,
- (joint work with Travis Schedler) Even more spaces are "locally" quiver varieties.

Example 8.0.2(?): Let $C_2 \curvearrowright \mathbb{C}^2$ by negation, then $C^2 \curvearrowright \mathbb{C}[C^2] = \mathbb{C}[x, y]$ by $f(x, y) \mapsto f(-x, -y)$. The invariants are $R := \mathbb{C}[x, y]^{C_2} = \mathbb{C}[x^2, y^2, xy] = \mathbb{C}[y, v, w] / \langle uv - w^2 \rangle$ which is a type A_1 singularity at the origin. Take a blowup $\text{Bl}_0(X) \rightarrow X := \text{Spec } R$.

Example 8.0.3(?): Let's do the same thing but differently: consider the nilpotent cone $\text{Nil}(\mathfrak{sl}_2) = \left\{ m = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid \text{tr}(m) = \det(m) = 0 \right\} = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid a^2 - bc = 0 \right\}$, which is the same equation as before. Take pairs $(m, \ell \in \ker m)$, then the projection map $(m, \ell) \mapsto m$ is the cotangent bundle $\mathbf{T}^\vee \mathbb{P}^1 \rightarrow \text{Nil}(\mathfrak{sl}_2)$. Now perhaps it's clear how to generalize, e.g. by taking other lie algebras.

8.1 Quivers

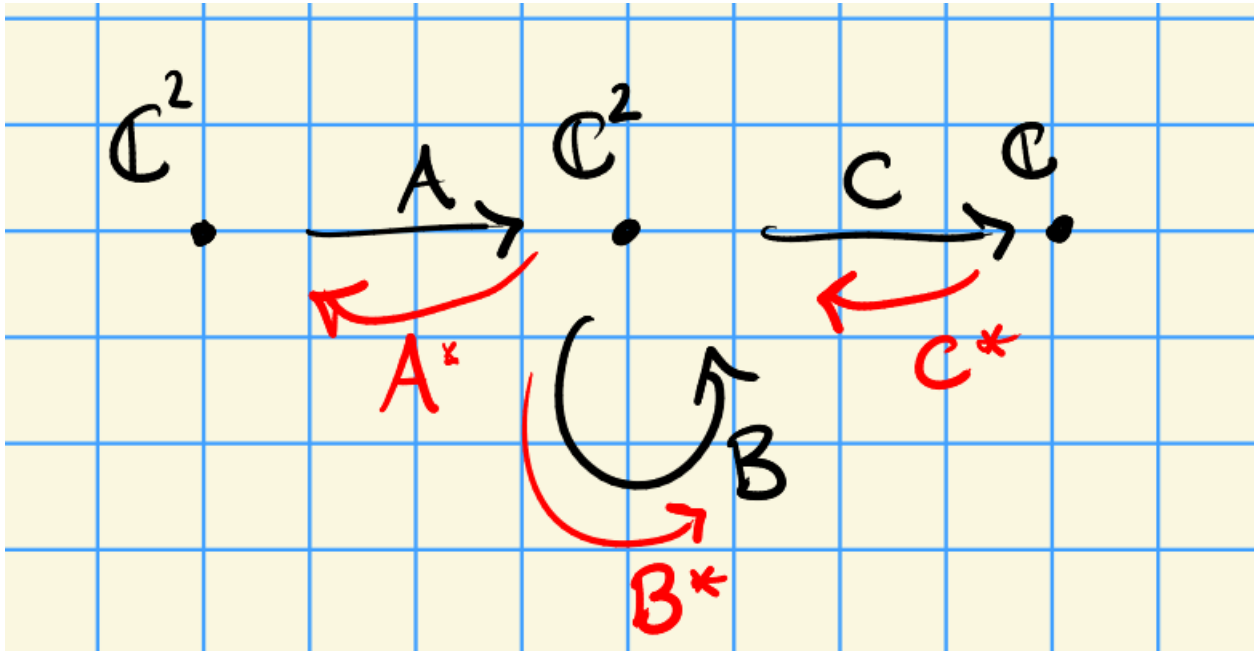
Remark 8.1.1: A quiver is the data $Q = (Q_0, Q_1, s, t : Q_1 \rightarrow Q_0)$ where Q_0 is a vertex set and Q_1 is an arrow set, and everything is over a field $k = \mathbb{C}$ in our case. Fix $d \in \mathbb{N}^{Q_0}$, a tuple $d = (d_i)_{i \in Q_0}$, then

$$\text{Rep}(Q, d) = \bigoplus_{q \in Q_1} \text{Hom}_{\mathbb{C}\text{-Mod}}(\mathbb{C}^{d_{s(q)}}, \mathbb{C}^{d_{t(q)}}).$$

There is an action $G_d := \prod_{i \in Q_0} \text{GL}_{d_i}(\mathbb{C})$ by conjugation, which we sometimes quotient by.

Remark 8.1.2: The G_d orbits are too big! Let A, B, C be matrices and consider the quiver $\mathbb{C}^2 \xrightarrow{A} \mathbb{C}^2 \circlearrowleft_B \xrightarrow{C} \mathbb{C}$. It's hard to find G_d -invariant functions, and in fact $\mathbb{C}[\text{Rep}(Q, d)]^{G_d} = \mathbb{C}[\{\text{tr}(\alpha), \alpha \text{ a cycle}\}]$. Note that B is a cycle in this case.

Remark 8.1.3: There is a process of doubling a quiver: add inverse arrows, and duplicate every loop:



Remark 8.1.4: Note

$$\text{Rep}(\bar{Q}, d) \rightarrow \text{Rep}(Q, d) \oplus \text{Rep}(Q^*, d) = \text{Rep}(Q, d) \oplus \text{Rep}(Q, d)^\vee \rightarrow \mathbf{T}^\vee \text{Rep}(Q, d),$$

which yields a cotangent bundle and puts us in the realm of symplectic geometry. The moment map is

$$\begin{aligned} \mu_d : \text{Rep}(\bar{Q}, d) &\rightarrow \mathfrak{g}_d^\vee \leftarrow \mathfrak{g}_d \\ \text{tr}(X(-)) &\leftarrow X. \end{aligned}$$

Write $\nu = \sum_{q \in Q_1} [q, q^*]$ where the commutator is $aa^* - a^*a$, then $p \in \mu_d^{-1}(0)$ if p is a module for the path algebra $\mathbb{C}\bar{Q}/\langle \nu \rangle = \Pi(Q)$.

Remark 8.1.5: Let $\theta \in \mathbb{Z}^{Q_0} = \text{Hom}(\prod_{\text{AlgGrp}} \text{GL}_{d_i}(\mathbb{C}), \mathbb{C}^\times)$ where $\theta \cdot \dim(\rho) = 0$. Say $\rho' \subseteq \rho$ is destabilizing if $\dim(\rho') \cdot \theta > 0$, and ρ is θ -semistable if it does not contain any destabilizing ρ' . One can then define

$$\mathcal{M}(Q, d, \theta) = \text{Proj} \bigoplus \mathbb{C}[\mu_d^{-1}(0)]^{k\theta}.$$

This is an important object: the extra θ parameter can be sent to zero, and generically will yield a smooth space.

8.2 Local to global obstructions

Theorem 8.2.1 (Bellamy-Schedler).

There is a decomposition into symmetric spaces:

$$\mathcal{M}(Q, d, \theta) = \prod_{i=1}^k \text{Sym}^{n_i} \mathcal{M}(Q, d_i, \theta)$$

where $d = \sum n_i d_i$ and $d_i \cdot \theta = 0$ for all i . Moreover, this has a symplectic resolution iff each factor does iff $\gcd(d_i) = 1$ or $(\gcd(d_i), p \left(\frac{d}{\gcd(d_i)} \right)) = (2, 2)$.

Theorem 8.2.2 (K-Schedler).

Let $\mathcal{M}(A, d, \theta)$ be the moduli space of θ -semisimple A -modules of dimension d , where e.g. $A = k\bar{Q}/\langle R \rangle$. There is an isomorphism of formal Poisson varieties after completing:

$$\widehat{\mathcal{M}}(\Lambda(Q), d, \theta)_m \xrightarrow{\sim} \widehat{\mathcal{M}}(Q', d', \theta)_0,$$

where e.g. $Q = Q'$ in type \tilde{A}_n .

Remark 8.2.3: Local to global obstructions: since X is Poisson, it has a stratification into symplectic leaves, $X = \coprod_i S_i$. There is an action of $\pi_1(S_i, x_i)$ on the set of local resolutions $\{\tilde{U}_{s_i} \xrightarrow{\pi} U_{s_i}\} / W$ where W is a Nakajima-Weyl group. This action can be nontrivial, and there is a compatibility with leaf closures: if $\bar{S}_i \supseteq S_j$ then $\pi_{\bar{S}_i}|_{S_j} = \pi_{S_j}$.

Example 8.2.4 (?): In the situation $C_2 \curvearrowright (\mathbb{C}^\times)^2$ there are four fixed points $(\pm 1, \pm 1)$ yielding simple singularities with local resolutions. The theorem implies there is a global symplectic resolution.

9 Jonathan Rosenberg, Twisted derived equivalences and string theory dualities with B-field (Wednesday, July 20)

Remark 9.0.1: Type II string theory: spacetime is $X = \mathbb{R}^{10-2n} \times M$ where M is compact Kähler with a Ricci-flat metric and trivial canonical (the Calabi-Yau condition), not necessarily simply connected. For us: M is smooth projective, X is quasi-projective. The physics is controlled by D-branes (D for Dirichlet), either complex submanifolds in IIB or isotropic submanifolds for the symplectic structure in IIA. More generally, IIA branes are coherent sheaves, and IIA branes are classified by the Fukaya category. In type IIB, there is interest in $D^b\text{Coh}(X)$, since complexes can capture joining branes, separating, collapsing into other configurations, etc. There is a coarser topological classification given by topological K-theory (Minasian-Moore, Witten). The D-branes

give classes in $KU^0(M)$ in IIB and $KU^{-1}(M)$ for IIA.

Remark 9.0.2: Today: orientifold string theories. Classically, strings moving in time yield maps $\Sigma \rightarrow X$ where Σ is the string worldsheet and X is spacetime. This is a σ -model. Orientifold theory: ask for strings to be equivariant with respect to involutions ι and worldsheet parity operators – if $\iota = \text{id}_X$, this recovers type I string theory (forgetting string orientation). In this theory, D-brane charges live in real K-theory $KR(X, \iota)$, which reduces to $KO(X)$ when $\iota = \text{id}_X$. 2015: joint work classifies orientifold string theories associated to (anti)holomorphic involutions on elliptic curves: there are 10. One case corresponds to Karoubi and Donovan’s twisted K-theory.

Remark 9.0.3: Can view an elliptic curve with antiholomorphic involution as a smooth genus one curve, possibly with no real points. Karoubi and Weibel (2003) show that KR of the complex curve is related to K^{Alg} of the real variety. Suggests that duality of orientifold string theories may be reflected in a equivalence of derived categories for *real* (rather than complex) curves.

Remark 9.0.4: Derived equivalence: prototypical example is $D(X) \cong D(Y)$ for X, Y smooth varieties over k via Mukai duality. Formulated in terms of the Poincaré sheaf on $X \times \widehat{X}$ for X an abelian variety and \widehat{X} its dual. Integration over the fiber is really derived pushforward, so define the transform as

$$\varphi(\mathcal{F}) = \mathbb{R}\pi_{2,*}(\mathcal{L} \otimes \pi_1^* \mathcal{F}).$$

AVs are simple examples of CYs?

Remark 9.0.5: Elliptic curves are self-dual, so it works here, which suggests it may also work fiberwise for elliptic fibrations. Căldăraru considers elliptic fibrations $f : X \rightarrow Z$ with no section. The role of \widehat{X} is the relative Jacobian $\widehat{f} : J \rightarrow Z$, the relative module space of semistable sheaves of rank 1 and degree zero on the fibers of f . Then \widehat{f} is an elliptic fibration with section, but there’s no universal sheaf to play the role of the Poincaré sheaf \mathcal{L} . Two reasons for this:

- J may have singularities. Not so serious, can resolve and replace J by a small resolution
- May exist locally but not globally.

It turns out that \mathcal{L} exists as a twisted sheaf: the transition functions may only agree up to a 2-cycle, defining a class α in the Brauer group $H_{\text{ét}}^2(J; \mathbb{G}_m)$. A similar recipe goes through: pull up to $X \times_Z J$, tensor by the α^{-1} -twisted sheaf \mathcal{L} , and push down to X .

Theorem 9.0.6 (JR 2017).

View a complex elliptic curve \overline{E} defined over $\overline{\mathbb{R}} = \mathbb{C}$ with a free antiholomorphic involution, regarded as $C(\mathbb{C})$ with its $\text{Gal}(\mathbb{C}/\mathbb{R})$ action for C a smooth projective curve with $g = 1$ defined over \mathbb{R} and $C(\mathbb{R}) = \emptyset$. Think of $C \rightarrow \text{Spec } \mathbb{R}$ as an elliptic fibration without a section, and the relative Jacobian as $E \rightarrow \text{Spec } \mathbb{R}$. There is a Mukai equivalence $D(C) \xrightarrow{\sim} D(E, \alpha)$ where $\alpha \in \text{Br}(E)/\text{Br}(\mathbb{R})$ is the unique nonzero class.

Remark 9.0.7: This is an AG version of T -duality of orientifold theories, and α corresponds to the B-field.

9.1 Generalizations

Remark 9.1.1: Generalizing from genus 1 curves to K3s: let E_1, E_2 be elliptic curves over \mathbb{C} with real j -invariant, so they admit antiholomorphic involutions. Cook up a real structure on $X = (E_1 \times E_2)/\mu_2$ and hence on its blowup \widehat{X} which is a Kummer K3. Projection gives an elliptic fibration without a section. The relative Jacobian gives a dual K3 fibred over \mathbb{P}^1 with a canonical section and the same derived equivalence as in the theorem. Can interpret as duality of orientifold type IIB string theories on K3s.

Remark 9.1.2: Generalizing to genus 1 curves over general fields: let $\text{ch } k = 0$ and C/k be a smooth connected projective curve with $g = 1$ and let E be its Jacobian. Then J is elliptic over k and C is an E -torsor, so defines a class in $\delta(C) \in H_{\text{ét}}^1(\text{Spec } k, E(\bar{k}))$. Can how a d_2 vanishes in the Leray spectral sequence

$$E_2^{p,q} H_{\text{ét}}^p(\text{Spec } k; \mathbb{R}^q \pi_* \mathbb{G}_m) \Rightarrow H_{\text{ét}}^{p+q}(E; \mathbb{G}_m).$$

Use Tsen's theorem to get vanishing. Then $\text{Br}(E) \cong \text{Br}(E, \text{Spec } k) \oplus \text{Br}(k)$ where the first term is the *relative Brauer group*.

10

Andrew Harder, Hyperelliptic curves and planar 2-loop Feynman graphs (Wednesday, July 20)

10.1 Introductory

Remark 10.1.1: Feynman graphs Γ : vertices, edges, and half-edges where (for us) all vertices has a unique adjacent half-edge. Can attach a Feynman integral: to any internal edge, attach $q_e = \sum \delta_{e,\ell} k_\ell + \sum \delta_{e,v} p_v$ where $\delta \in \{0, \pm 1\}$ and $I_\Gamma = c \int_{\mathbb{R}^?} \frac{dk}{\prod_e (q_e^2 - m_e^2)}$ for $m_e \in \mathbb{C}$. Can reduce dimension to write $G = \sum x_e (q_e^2 - m_e^2)$, look at discriminant locus for a family of quadrics? Yields a locus $UF = 0$ for two polynomials U and F . Somehow get a parametric form

$$I_\Gamma = c \int_\sigma \frac{U^?}{F^?} \Omega_0$$

where σ is a cycle given by $\{[x_1 : \dots : x_E] \mid x_i \in \mathbb{R}_{\geq 0}\}$; this is an integral on the complement of $Z(UF) = 0$.

Theorem 10.1.2(?)

After blowing up $\mathbb{P}^{|E|-1}$, the integral I_Γ is well-defined and a relative period of the pair $(P \setminus Z(U^b F^a), B)$ where P is a toric variety and B its toric boundary, and $a, b \in \{0, 1\}$

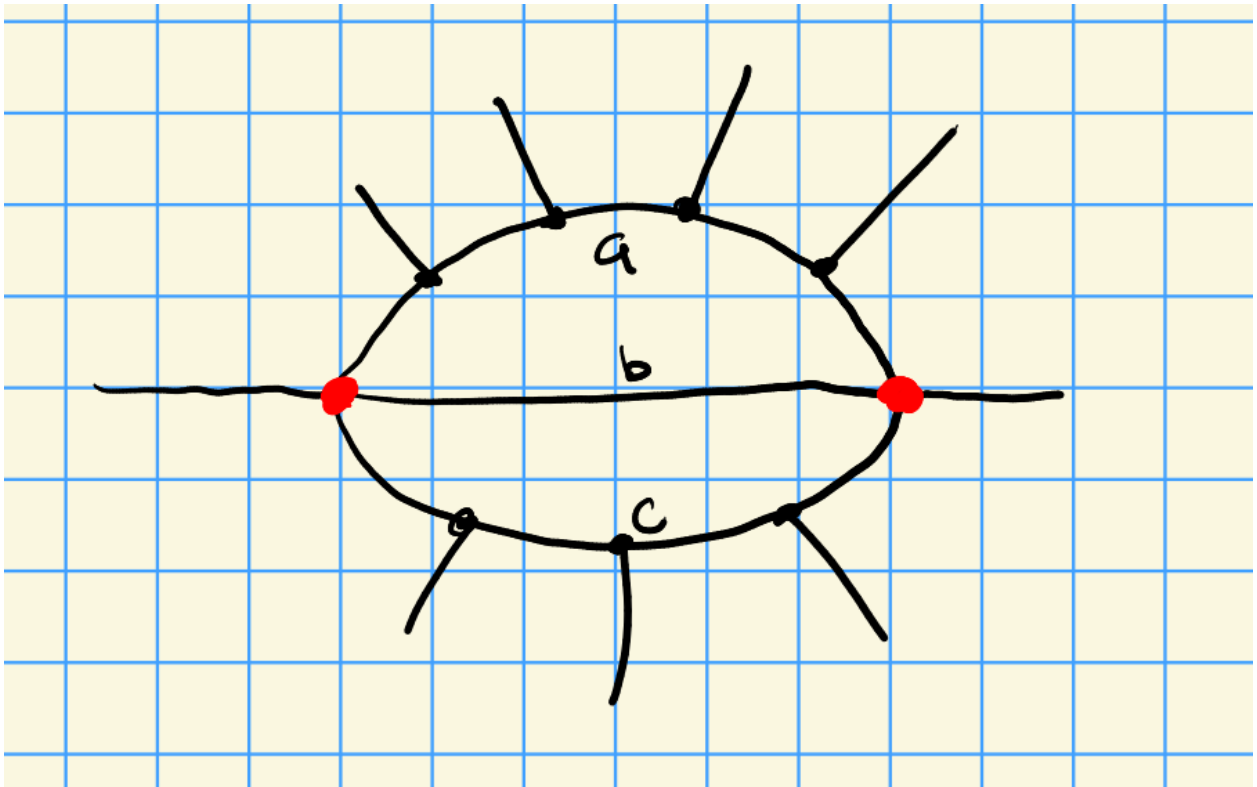
depending on the dimension D . Thus the geometry of $Z(U)$ and $Z(F)$ determine I_Γ .

Remark 10.1.3: The U and F are defined combinatorially from the Feynman graph. Let $\Gamma_\epsilon = (V, E)$ and let S_1 be the set of all spanning trees of Γ_ϵ . Let $x^T = \prod_{e \notin T} x_e$ for T any spanning tree, where we have coordinates $\{x_e\}_{e \in E}$. This somehow defines U , and F is defined by looking at spanning 2-trees (subgraphs with 2 connected components, no loops, containing all vertices). It turns out F is homogeneous, so defines a projective variety.

Example 10.1.4(?): The theta graph yields a smooth elliptic curve in \mathbb{P}^2 . More generally, connected the two vertices with more edges yields Calabi-Yau $(n - 1)$ -folds.

10.2 Extensions to 2-loop graphs

Remark 10.2.1: 2-loop graphs: two trivalent nodes connected by chains of edges:



What does the mixed Hodge structure on cohomology look like? This is an important step toward understanding periods, and the answer depends heavily on D . Bloch works out kite and double box graphs.

Theorem 10.2.2 (DHNV).

For $(a, 1, 1)$ hypersurfaces, identifies when certain homology (or some of its weight-filtered pieces) are Hodge-Tate under conditions on a and D .

Remark 10.2.3: Note that a Tate HS has $h^{i,j} = 0$ for $i \neq j$, and mixed Tate is an iterated extension of such a Hodge structure. This means it's as simple as it could reasonably be. Idea of proof: blow up a linear subspace, remove some singular hyperplanes.

Interestingly, for $D = 4$ only the cases $(3, 1, 1)$ and $(1, 1, 1)$ yields a non-Hodge-Tate mixed Hodge structure. These correspond to multiple polylogarithms.

Remark 10.2.4: WIP: can say more in $(a, 1, c)$ case when $a, c > 1$. Note that the $(a, 1, c)$ case can't yield K3s.

See Prym varieties?

11

Noriko Yui, Modularity of certain Calabi-Yau threefolds over \mathbb{Q} (Wednesday, July 20)

Remark 11.0.1: Definition: smooth projective variety X over \mathbb{C} , $\dim_{\mathbb{C}} X = 3$, and

- $H^i(X; \mathcal{O}_X) = 0$ for $i = 1, 2$, and
- $K_X \cong \mathcal{O}_X$.

By conjugation, $h^{i,j} = h^{j,i}$, and by Serre duality $h^{i,j} = h^{3-i, 3-j}$. Note $h^{1,1}, h^{2,2} > 0$ since Calabi-Yau threefolds are Kähler. The outer edge of the Hodge diamond contains only zeros and ones and has 7 rows, up to 4 columns. A typical example: quintic threefolds $\sum x_i^5 = 0$, or elliptic curves are dimension 1 CYs and K3s are dimension 2 CYs. More generally, CYs can be constructed as intersection threefolds in weighted projective spaces. Other examples:

- Double octics $w^2 = f_8(x, y, z, w)$
- $E \times S/\iota$ where E is an elliptic curve, S is a K3, ι is a non-symplectic involution
- Toric CYs

Remark 11.0.2: Let X/\mathbb{Q} be Calabi-Yau iff $X \otimes_{\mathbb{Q}} \mathbb{C}$ is Calabi-Yau in the usual sense. Suppose it has defining equations in $\mathbb{Z}[\frac{1}{m}]$ for some m , pick a *good prime* p coprime to m so that X has good reduction: X_p is smooth over $\overline{\mathbb{F}}_p$. Define the zeta function $Z_p(X, T) \in \mathbb{Q}[[T]]$ as usual, then the Weil conjectures hold for X . Since $P_1 = P_5 = 1$,

$$Z_p(X, T) = \frac{P^3(T)}{(1-T)P^2(T)P^2(pT)(1-p^3T)}.$$

There is a compatible system of ℓ -adic Galois representations on étale cohomology. Define an L -function $L_i(X, s) = \prod_{p \neq \ell \text{ good}} P^i(p^{-s})^{-1} \times (\dots)$, where the last factor accounts for when $\ell = p$.

Can compute $\sharp X(\mathbb{F}_p)$ using the Lefschetz trace formula for $\text{Frob}_p \curvearrowright H_{\text{ét}}(\bar{X}; \mathbb{Q}_\ell)$. Here $\sharp X(\mathbb{F}_p) = 1 + p^3 + (1+p)t_2(p) - t_3(p)$ where t_2, t_3 are bounded above by $h^{1,1}, \beta_?$ by RH.

Remark 11.0.3: The modularity question: are there global functions or automorphic forms that determine $L(X, s)$? This would be the Langlands philosophy.

Say X is rigid if $h^{2,1} = 0$ so $\beta_3 = 2$ and the Hodge diamond only has two unknowns, $h^{1,1}, h^{2,2}$. These resemble elliptic curves in terms of the characteristic polynomial.

Theorem 11.0.4(?).

Every rigid Calabi-Yau threefold is modular: there is some modular form f of weight 4 on some $\Gamma_0(N)$ such that $L(f, s) = L(X, s)$.

Remark 11.0.5: There are 14 one-parameter families of CYs defined over \mathbb{Q} of hypergeometric type, and at conifold points each family as a rigid motive and corresponds to a special cusp form of weight 4 on some $\Gamma_0(N)$ where only $2, 3, 5 \mid N$. The next simplest case is when $h^{2,1} = h^{1,2} = 1$ and not zero, so $\beta_3 = 4$. Constructing such CYs: Nygaard and van Geemen take a certain complete intersection whose singular locus has 16 ordinary double points and 4 plane conics intersecting in a square; take its resolution. Van Geemen and Werner/Consani and Scholten construct another using the affine equation of a skew pentagon, yielding a quintic Calabi-Yau threefold defined over \mathbb{Q} with bad primes 2, 3, 5.

Theorem 11.0.6(?).

For X the Calabi-Yau coming from a complete intersection as above,

$$L(X, s) = L(\chi^3, s) \cdot L(\chi_{(-1)}, s),$$

where χ is a Hecke character for $\mathbb{Q}(i)$?

Remark 11.0.7: Note that something similar happens for elliptic curves with CM. Idea of proof: split H^3 into two rank 2 motives to decompose the L function.

Remark 11.0.8: Since X is covered by some E^3 where $E: y^2 = 1 + x^4$ which has CM by $\mathbb{Q}(i)$, so $L(X, s) \mid L(E^3, s)$ and we can use this to define when a Calabi-Yau has CM in terms of the cover. General expectation: if X has CM for an imaginary quadratic number field K , then L splits into factors involving Hecke characters of K . To avoid Hecke characters, define *real* multiplication instead: if X is Calabi-Yau with $h^{p,q} = 1$ for $p+q=3$ and $\beta_3 = 4$, set $K = \mathbb{Q}(\sqrt{d})$ and say X has RM with K if

- There is an algebraic correspondence $\Psi: X \rightarrow X$ defined over K .
- There is an automorphism of H^3 preserving the Hodge structure,
- There is an automorphism of H^3 preserving Galois representations.

Remark 11.0.9: See Hilbert modular forms, Adrianov L -functions.

12 | Johannes Walcher, Rationality of MUMs and 2-functions (Thursday, July 21)

12.1 Rational 2-functions are abelian

Remark 12.1.1: Motivations: mirror symmetry links enumerative geometry (A model) to variations of Hodge structure (B model). On the B side, arithmetic properties naturally arise. Do they have an interpretation on the A side?

Classical mirror symmetry involves maximal unipotent monodromy (MUM) in the moduli space of CYs. Johannes entered this area by looking at extensions of algebraic cycles.

Remark 12.1.2: 2-functions: for $s \in \mathbb{Z}_{>0}$, and s -function is a power series $W \in z\mathbb{Q}[[z]]$ of the form $\sum_d n_d \text{Li}_s(z^d)$. Motivating examples: $g = 0$ prepotential expanded around MUM in flat coordinates. Equates generating functions for GW invariants and that for BPS invariants, $\sum a_d q^d = \sum n_d \text{Li}_3(q^d)$.

A new feature: framing, where $Y \in 1 + z\mathbb{Z}[[z]]$ defined an automorphism of $z\mathbb{Z}[[z]]$. See Lagrange inversion for finding coefficients of power series. Theorem: if W is a rational 3-function, then all of its framings are 3-functions. Relates to some classical theorems, e.g. the Wolstenholme theorem and the Jacobsthal-Kazandzidis congruence.

Remark 12.1.3: Over a number field, the s -functions form a finite rank free module. Muller shows that there can't be a natural choice of basis for K an arbitrary number field – if $W \in zK[[z]]$ is rational 2-function, so $\exp(\delta W) \in K(z)$, then K/\mathbb{Q} must be an abelian extension.

12.2 MUM

Remark 12.2.1: For $\pi : Y \rightarrow B$ a smooth family of Calabi-Yau threefolds of mirror quintic type, there is a \mathbb{Z} VHS (pure, weight 3). By analog with p -adic Hodge theory, Kontsevich et al have a theorem that lets you conclude integrality from rationality in some special cases? Look at monodromy of Picard-Fuchs equations. See Bogner's "strange operator".

Remark 12.2.2: Early 2000s: Almkvist-Zudilin look at Calabi-Yau differential operators, which are 4th order operators with MUM points, duality, integral fundamental periods, etc so that it could come from the integral cohomology of a CY.

See Beauville-Apery families of elliptic curves, Hadamard product.

Remark 12.2.3: Idea: take a family $Y \rightarrow B$ and $C \hookrightarrow Y$ some 2-cycles (curves) which is also a nice family over B , this gives an extension of VHSs. Somehow classified by intermediate Jacobian? By Griffith's transversality, the extension is fully determined by a piece in a certain Ext^1 group. Extraction of integers appears to depend on finding a good basis for 2-functions. Speculation: analytic 2-functions with coefficients in a number field arise as solution to inhomogeneous linear ODEs with algebraic coefficients, and unravelling the enumerative interpretation (e.g. in open GW/BPS theory) will involve natural such bases. Maybe K is the trace field of a hyperbolic 3-manifold?

13 | Matt Kerr, K_2 and quantum curves (Thursday, July 21)

Holy moly, Matt has a lot of equations and writes super fast! I was not able to record even half of what was written.

13.1 A miraculous coincidence

Remark 13.1.1: Suppose $f \in \text{Mero}(\mathbb{C}, \mathbb{C})$ with no poles in the horizontal strip $|\Im(z)| < 2\pi$, then $e^{\pm 2\pi i \frac{\partial}{\partial z}}$ is a shift operator $f(z) \mapsto f(z + 2\pi i)$. Example: $e^{2\pi i \frac{\partial}{\partial z}} z^k = \sum_{\ell} \left(\frac{(2\pi i)^\ell}{\ell!} \right) \left(\frac{\partial}{\partial z} \right)^\ell z^k = (z + 2\pi i)^k$. Consider

$$\hat{\varphi} := e^r + e^{-r} + e^{2\pi \frac{\partial}{\partial r}} + e^{-2\pi i \frac{\partial}{\partial r}} \curvearrowright L^2(\mathbb{R}).$$

This is some kind of quantum Hamiltonian for a Fermi gas? Note that this is unbounded, but its inverse is bounded and self-adjoint with spectrum that can be enumerated. This suggests a curve of the form $x + x^{-1} + y + y^{-1} = \lambda$, is there any AG or Hodge theory arising in this spectral problem?

Remark 13.1.2: Consider elliptic curves, pullbacks of the Legendre family $y^2 = x(x-1)(x-16t^2)$. This has two periods $A(t), B(t)$ given by integrating over the two torus cycles α, β , which are complicated but annihilated by the Picard-Fuchs operator $L = D_t^2 - 16t^2(D_t + 1)^2$. Solve $L(-) = c$ a constant using variation of parameters to get

$$C(t) \approx B(t) \int A(t) \frac{dt}{t} - A(t) \int B(t) \frac{dt}{t}.$$

Can choose constant terms such that the monodromy of C lies in $\mathbb{Z}(2) \langle A(t), B(t) \rangle$ where $\mathbb{Z}(2) = 2\pi i \mathbb{Z}$. Set $\nu(t) = \frac{1}{\pi^2} \log^2(-t) + \frac{7}{6} + O(t \log(t))$ and define t_n by $\nu(t_n) = n + 2$. Then $-\log(-t_n) = \pi \sqrt{n + \frac{5}{6}}$, which yields a sequence matching the exponents in the eigenvalues of $\hat{\varphi}$. Something is going on!

Remark 13.1.3: Local mirror symmetry conjecture linking inhomogeneous Picard-Fuchs equations of Hori-Vafa models and enumerative geometry of local CYs: the B model mirror of $K_{\mathbb{P}^1 \times \mathbb{P}^1}$ is the VMHS of H^3 of $Y_a = V(x + x^{-1} + y + y^{-1} + a + uv) \subseteq (\mathbb{C}^\times)^2 \times \mathbb{C}^2$ with coordinates $[x, y]$ and $[u, v]$. See Frobenius duality for differential operators? The Geneva conjectures connected difference operators and quantum curves to enumerative geometry of CYs.

13.2 Beilinson's Formula

Remark 13.2.1: Like X/\mathbb{C} smooth projective, and note

$$\mathcal{K}_2(\mathbb{C}(X)) = \frac{\mathbb{C}(X)^\times \wedge \mathbb{C}(X)^\times}{\langle F \wedge (1 - F) \rangle} \xrightarrow{T} \bigoplus_{p \in ?} \mathbb{C}^\times$$

where T is the *tame symbol*. We want to compute the regulator

$$\ker(T) = H_M^2(X, \mathbb{Z}(2)) \xrightarrow{R_X} H_D^2(X, \mathbb{Z}(2)) \cong H^1(X^{\text{an}}, \mathbb{C}/\mathbb{Z}(2)).$$

Beilinson shows that if $\{f, g\} \in \ker(T)$ and $\gamma \in \pi_1(X \setminus \{x_0\})$ then

$$\langle R_X \{f, g\}, [\gamma] \rangle \equiv \int_\gamma \log(f) \frac{dg}{g} - \log g(x_0) \int_\gamma \frac{df}{f}.$$

Remark 13.2.2: Consider $X \xrightarrow{\pi} S$ a family of smooth curves and $z \in H_m^2(X, \mathbb{Z}(2))$ we can consider their fibers $z_2 \in H_M^2(X_s, \mathbb{Z}(2))$. This maps to

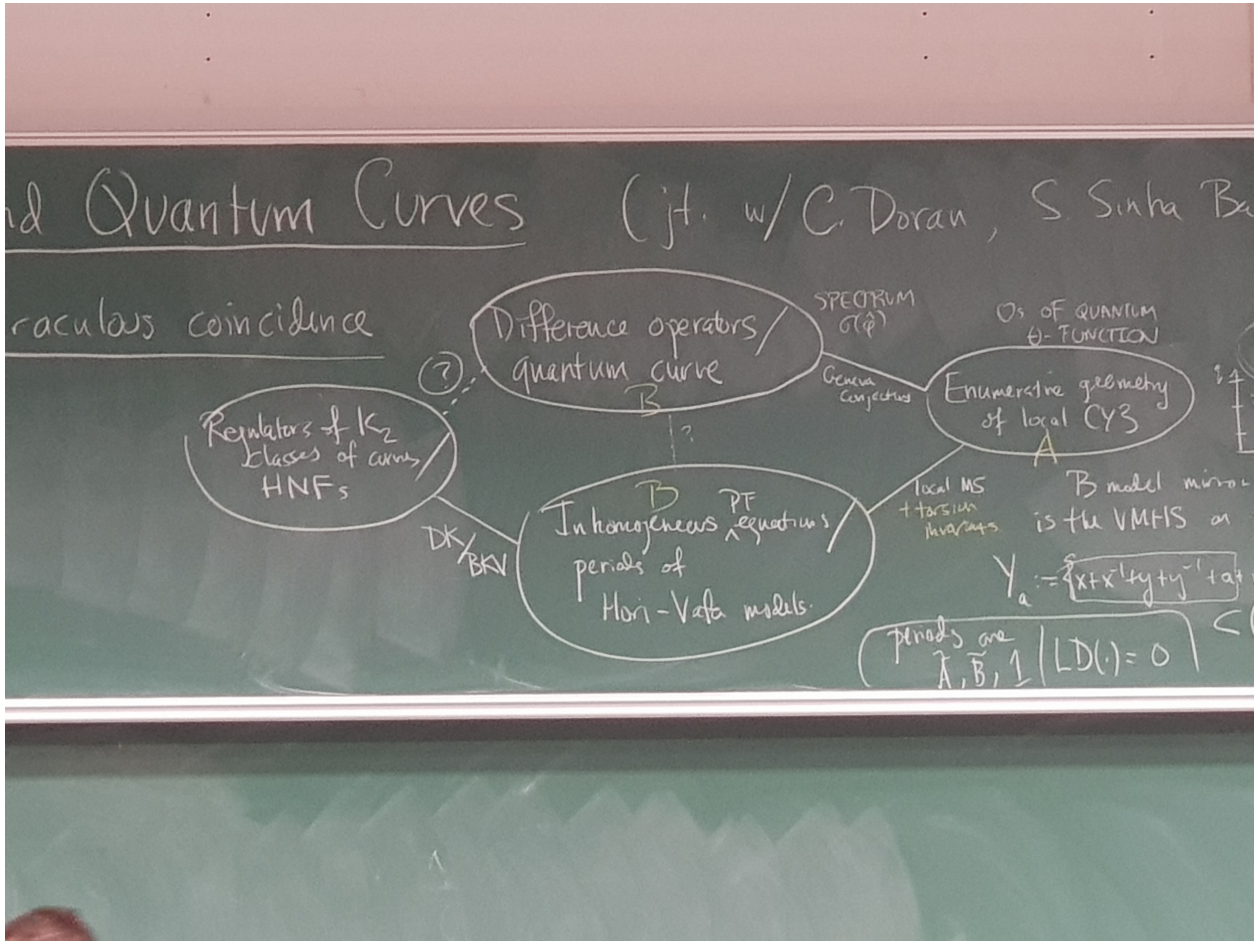
$$R(S) = H^1(X_s, \mathbb{C}/\mathbb{Z}(2)) \cong \text{Ext}_{\text{MHS}}^1(\mathbb{Z}(2), H^1(X_s, \mathbb{Z})).$$

By Griffiths transversality, $\nabla R = F^1 H_{X/S}^1 \otimes \omega_S^1$. If z lifts to $\{F, G\}$ (?) then $\nabla R = \frac{dF}{F} \wedge \frac{dG}{G}$.

13.3 Families of elliptic curves

Remark 13.3.1: Let $\Delta \subseteq \mathbb{R} \times [-1, 1]$ be a reflexive polytope, e.g. the polytope for \mathbb{P}^n , let Δ° be its polar dual. Somehow construct an elliptic surface \mathcal{E} out of this! The symbol $\{-x, -y\} \in \mathcal{K}_2(\mathbb{C}(\mathcal{E}))$ lifting to $z \in H_M^2(\Sigma \setminus \mathcal{E}_\infty, \mathbb{Z}(2))$ and take its regulator in $H^1(E_\gamma, \mathbb{C}/\mathbb{Z}(2))$.

Idea: regulators of \mathcal{K}_2 of curves relate to Hori-Vafa models, so we've reduced to relating regulators to quantum curves:



13.4 Quantum Curves

Remark 13.4.1: Theorems:

- For all $a \in V$, there is a $\psi_a \in L^2(\mathbb{R}) \setminus \{0\}$ such that $\widehat{\varphi}\psi_a = a\psi_a$.
- Asymptotically, 100% of $\sigma(\widehat{\varphi})$ belong to V .

The first primarily uses Beilinson’s formula, the second uses two-variable Fourier transforms. These establish a link between the spectrum $\sigma(\widehat{\varphi})$ (quantum curve side) to the \mathbb{Z} -locus of truncated HNFs (regulator side).

Remark 13.4.2: See Grothendieck period conjecture.

14

Chloe Xiaohua Ai, From Feynman Amplitudes to multiple L-values (Thursday, July 21)

Question 14.0.1

Recall that the multiple zeta function is defined as

$$\zeta(\mathbf{s}) = \sum_{n_1 > n_2 > \dots > 0} n_1^{-s_1} n_2^{-s_2} \dots n_k^{-s_k}.$$

The classical zeta generalizes to Dedekind zeta functions $\zeta_K(S) = \sum_{0 \neq n \subset \mathcal{O}_K} N_{K/\mathbb{Q}}(n)^{-s}$, is there an analog for the multiple zeta function?

Remark 14.0.2: The Hecke formula: takes automorphic periods to L -functions by integrating a well-chosen Eisenstein series with respect to a normalized Haar measure. Can this idea be extended to multiple variables?

Remark 14.0.3: Goncharov's theory on Hodge correlators: complex numbers given by integrals assigned to complex curves or varieties. Can construct integrals all arising from a single object: Green functions/currents. Many L -functions can be interpreted as Hodge correlators: Rankin-Selberg integral, polylogarithms, multiple Eisenstein-Kronecker series. Hodge correlators are periods of motivic correlators. The Green functions in Goncharov's construction work over \mathbb{C} , what about number fields?

Remark 14.0.4: Plectic conjecture: in the presence of real multiplication by a totally real number field, motives have canonical functorial additional structure. See Hilbert modular varieties, Hilbert modular forms. Goal: define higher plectic Green functions, and define multivariate zeta functions by integrating them. These new multivariate functions will depend on some combinatorial data: a graph Γ . If Γ is a special tree, then the generalized multiple zeta values $Z_{I,v}(\Gamma, \partial\Gamma)$ can be expressed as finite \mathbb{Z} -linear combinations of classical multiple zeta values. In the totally real case, Z be expanded as sum in values of \mathcal{L}_m , a generalized m -logarithm which is a nontrivial iterated integral.

15 | Lawrence Barrott, The Doran-Harder-Thompson conjecture via the Gross-Siebert program? (Thursday, July 21)

Remark 15.0.1: Goal: understand proportion of period integral via enumerative invariants on the mirror. For mirror symmetry, start with a family of CYs $X \xrightarrow{f} \mathbb{A}^1$ with X smooth and $f^{-1}(0) = \bigcup D_i$ an SNC divisor. Locally this is the map $f^* : t \rightarrow \prod_{i=1}^n x_i$ for t a coordinate on \mathbb{A}^1 , and we ask that X is étale locally isomorphism to $\text{Spec } k[x_i, y_i]$ for y_i some free variables. Want to encode fibers of this family via period integrals; use the dual intersection complex Σ_X , the subset of the cone $\bigoplus \mathbb{R}_{\geq 0} D_i$ where $\mathbf{a} \in \Sigma_X$ if the D_i with $a_i > 0$ all intersect with nonempty intersection.

Example 15.0.2(?): Take a quartic K3 degenerating to a toric boundary. Resolve 24 singularities via blowups to get a cone over a standard simplex. Sum coordinates to $\sum D_i$ to get a map $\Sigma_X \rightarrow \mathbb{R}_{\geq 0}$, then $\Sigma_X(Z)$ gives a collection of integer points.

Theorem 15.0.3 (Gross-Siebert).

There is a ring structure on formal monomials $k[\Sigma_X(Z)]$ extending the one induced by “conewise addition”.

Remark 15.0.4: This is a graded ring and the mirror is $\widehat{X} = \text{Proj } k[\Sigma_X(Z)]$ where $z^p z^q = \sum N_r z^r$ where N_r is a count of punctured (open) GW invariants. This also works for pairs (Y, D) of a variety and an anticanonical divisor $D \in |-K_Y|$. Having a family with a map to \mathbb{A}^1 yields a Landau-Ginzburg model. In this case, $\text{Proj } \Sigma_{(Y,D)}(Z)$ has a canonical map to \mathbb{A}^1 , and superpotential $\widehat{Y} \xrightarrow{W} \mathbb{A}^1$.

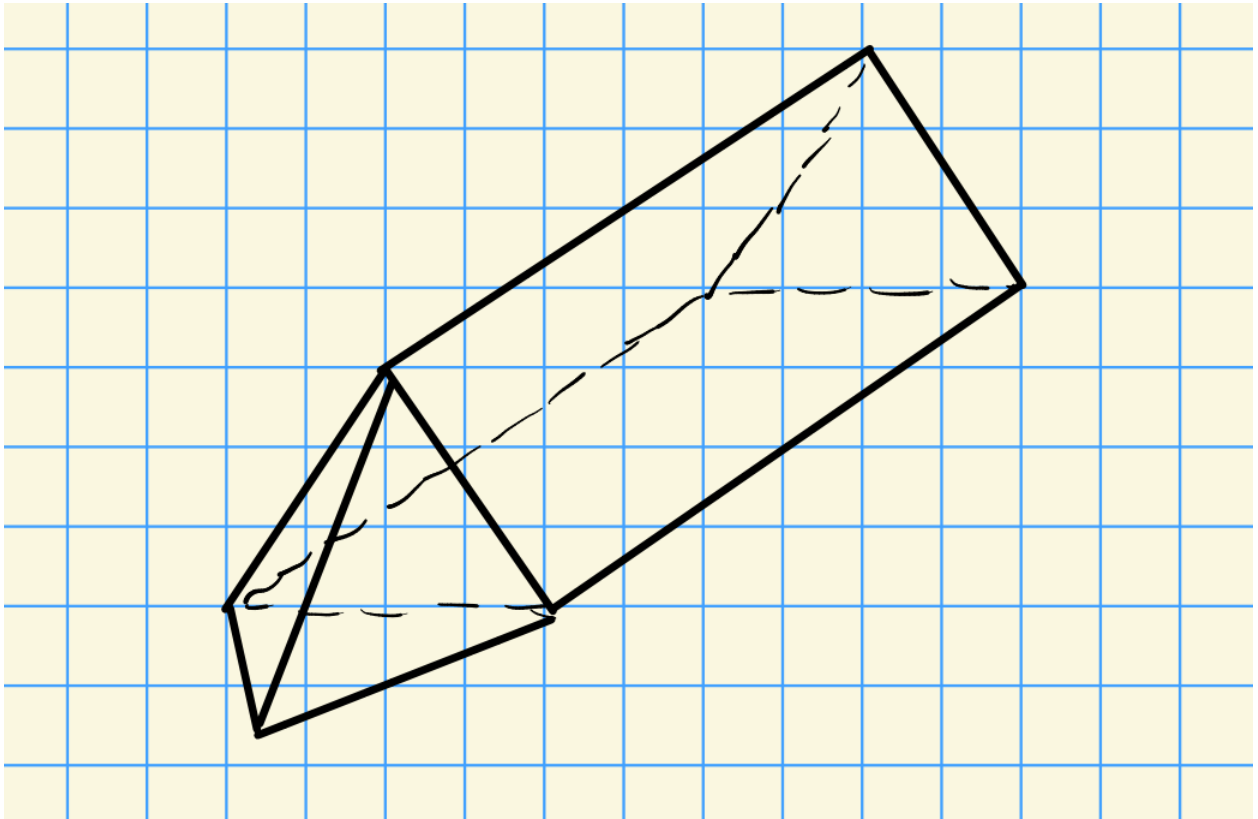
Remark 15.0.5: Tyurin degenerations: a family $X \xrightarrow{f} \mathbb{A}^1$ CYs with X smooth and $f^{-1}(0) = Y_1 \cup_D Y_2$ glued along an anticanonical divisor $D \in |-K_X|$.

15.1 DHT Conjecture

Remark 15.1.1: Let $A \xrightarrow{f} \mathbb{A}^2$ be a family of CYs and suppose

- $f^{-1}(t, 0) = Y_1 \cup_D Y_2$
- $f^{-1}(0, t)$ is an SNC with a maximality condition

Here locally $t_1 = \prod x_i, t_2 = \prod y_i$, and X is étale-isomorphic to $\text{Spec } k[x_i, y_i, z_i]$ where the z_i are free variables. For some reason, Σ_X looks like a Toblerone, and we now get *two* maps to \mathbb{A}^1 .



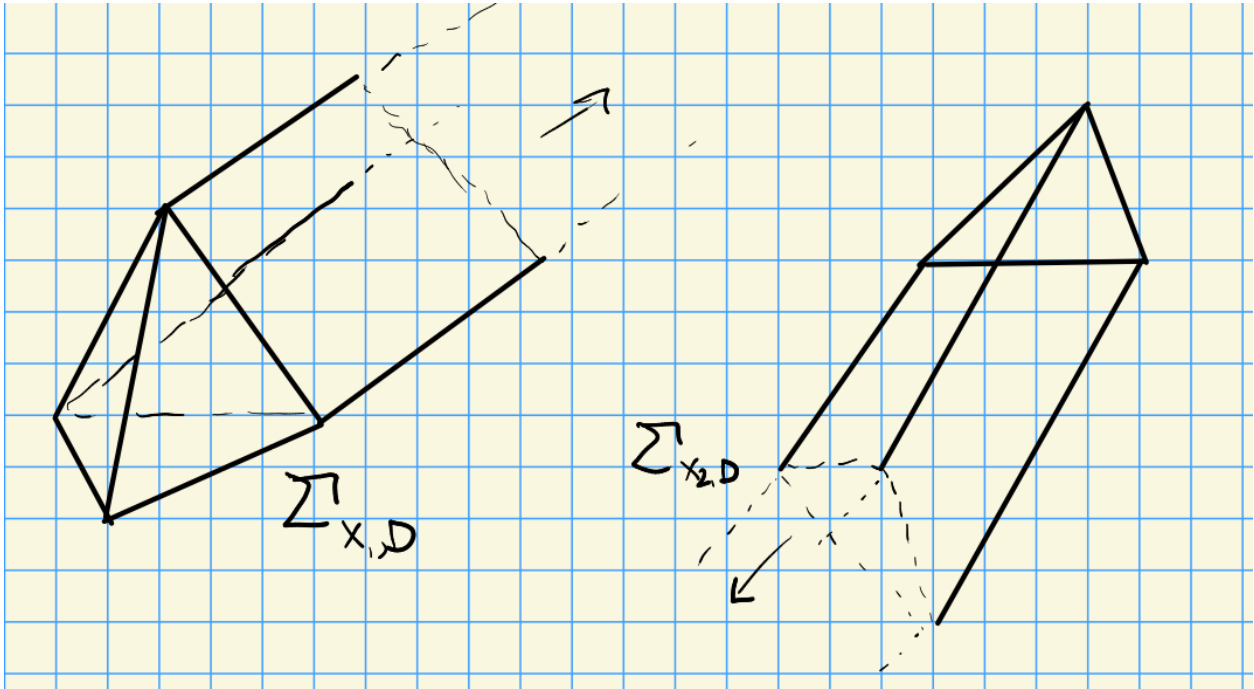
Remark 15.1.2: The DHT conjecture: \widehat{X} is a fibred Calabi-Yau and $\widehat{X} \rightarrow \mathbb{P}^1$ with fibers mirror to D . Topologically, the two maps to \mathbb{A}^1 can be glued to get the map to \mathbb{P}^1 , and here $\widehat{Y}_1 \xrightarrow{W_1} \mathbb{A}^1, \widehat{Y}_2 \xrightarrow{W_2} \mathbb{A}^1$ can be glued to get the fibration. Doran-Kostiuk-You construct examples as complete intersections on toric varieties of the form $\Pi \times \mathbb{P}^1$ and related the periods of the fibres via Hadamard products of relative I functions. Moral: can glue periods for $\widehat{Y}_1, \widehat{Y}_2$ to get periods for all of X .

Remark 15.1.3: The GS program gives a scattering diagram (pairs of walls and functions) on the Toblerone which gives gluing instructions for the mirror family. Scattering diagrams have canonical differential forms, i.e. canonical sections of $\omega_{\widehat{X}}$, but $\omega_{\widehat{X}}|_{\widehat{D}} \neq \omega_{\widehat{D}}$ – there are correction terms coming from other parts of the scattering diagram. Ruddat-Seibert give a way to compute period integrals associated to 1-dimensional balanced subsets of Σ_X and form an n -cycle β and integrating the canonical differential from from the scattering diagram, yielding $\int_{\beta} \omega_{\widehat{X}}$. Since β is contained in a fiber, one gets a relation

$$\int_{\beta} \omega_{\widehat{D}}(1 + \sum N_k) = \int_{\beta} \omega_{\widehat{X}}|_{\widehat{D}},$$

and the N_k count curves with $k + 1$ marked points lying on the fiber, all tangent to Y_1, Y_2 (and possible negative or punctured).

Remark 15.1.4: An ideal theorem: any purple curve (DZG: forgot to mark which was purple!) can be formed by gluing components on $\Sigma_{X_1, D}, \Sigma_{X_2, D}$:



16 | Amanda Francis, Hanging gardens and divisors from Adinkras (Thursday, July 21)

Remark 16.0.1: Physics motivations: standard model has 5 bosons and 12 fermions, and supersymmetry switches and matches these in pairs. Form an algebra from supersymmetry operators Q_i transforming bosons to fermions, where $Q_i\varphi = \pm(iH)^s\psi$, where H is a Hamiltonian. Visualize these with Adinkras (etymology: Ghanaian art): a bipartite graph (bosons/fermions), Q_i operators represented by colored edges,

Remark 16.0.2: Chromotopology: bipartite, N -regular with a consistent edge coloring (each vertex has N incident edges, one of each color), and other conditions.

Remark 16.0.3:

Remark 16.0.4: Somehow get an algebraic (Riemann) surface X from this. Match up with dessin d'enfants?

If a height function h on a triangular mesh sends adjacent vertices to distinct values, this is a discrete Morse function. So we can triangulate these graphs by adding new vertices and new fractional heights to get Morse functions. Build divisors from critical points of h by defining multiplicities m_v ; this yields a Morse divisor $D_h = \sum_{v \in M} -m_v[v]$.

Remark 16.0.5: The Jacobian of X carries interesting data. Neat fact: the Belyi map $X \rightarrow \mathbb{P}^1$ factors through a beach ball B_N , and the map $X \rightarrow B_N$ is less ramified. See Carvacho-Hidalgo-

Quispe: $N = 5$ hypercube curve is the generalized Fermat curve, a complete intersection $X \hookrightarrow \mathbb{P}^4$.
 Remainder of talk: discussion of counting height functions on N -hypercubes. The number blows up very quickly!

17 | Elana Kalashnikov, Unwinding the Gelfand—Cetlin toric degeneration on the mirror (Thursday, July 21)

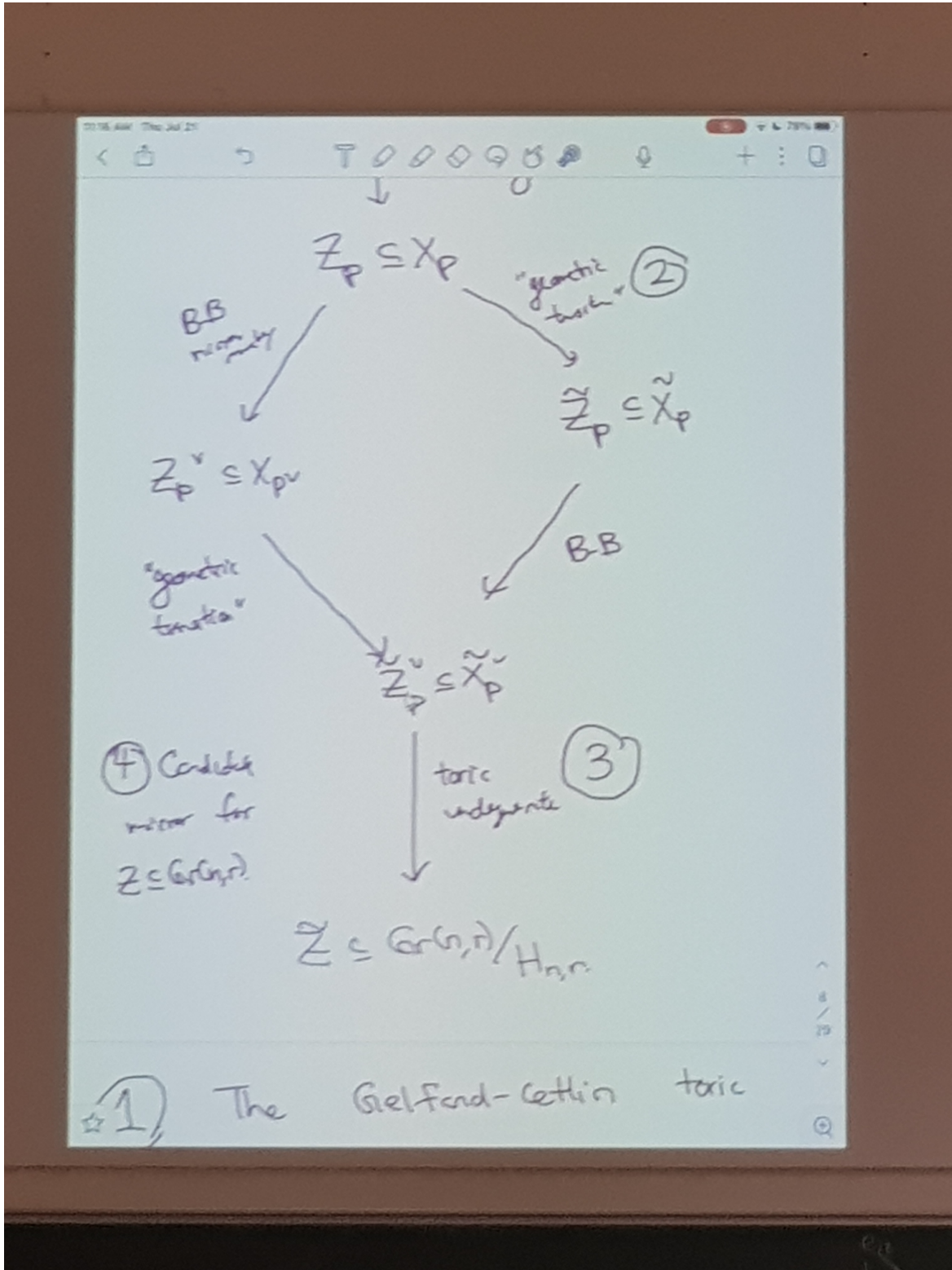
Remark 17.0.1: Batyrev mirror symmetry: anticanonical CYs have mirrors which are Fanos in toric varieties, where $Z \subseteq X_p$ and $Z^\vee \subseteq X_{p^\vee}$ for p a polytope and p^\vee its dual polytope. How to extend from mirror hypersurfaces in a Calabi-Yau to mirrors in a Grassmannian? Gelfand-Cetlin degenerate $\text{Gr}_n(m) \rightsquigarrow X_p$ a toric variety and proceed as before (?), how do we leave the toric world? It turns out that there is a Grassmannian on the mirror, and a natural generalization of the Greene-Plesser quintic mirror.

Remark 17.0.2: Generalizing Greene-Plesser mirrors: mirror pairs of the form $Z \subset \mathbb{P}^{n-1} \rightleftharpoons Z^\vee \subseteq \mathbb{P}^{n-1}/H$ for H a finite group constructed as follows. Write $\mathbb{P}^{n-1} = \mathbb{C}^n // \mathbb{C}^\times$ as a GIT quotient. Think of this as $\text{Mat}_{1 \times n}(\mathbb{C}) // \text{GL}_1(\mathbb{C})$. Define $\tilde{H}_n = \{(\zeta_1, \dots, \zeta_n) \mid \zeta_i^n = 1, \prod \zeta_i = 1\}$, then $\tilde{H}_n \curvearrowright \text{Mat}_{1 \times n}(\mathbb{C})$, and finally $H_n = \tilde{H}_n / \tilde{H}_n \cap \text{GL}_1(\mathbb{C})$. Replacing $\text{GL}_1(\mathbb{C})$ with $\text{GL}_r(\mathbb{C})$ yields $\text{Gr}_n(r) = \text{Mat}_{r \times n}(\mathbb{C}) // \text{GL}_r(\mathbb{C})$, and one defines $H_{n,r}$.

Theorem 17.0.3(?)

The CYs $Z \subseteq \text{Gr}_n(r)$ and $\tilde{Z} \in \text{Gr}_n(r)/H_{n,r}$ are related.

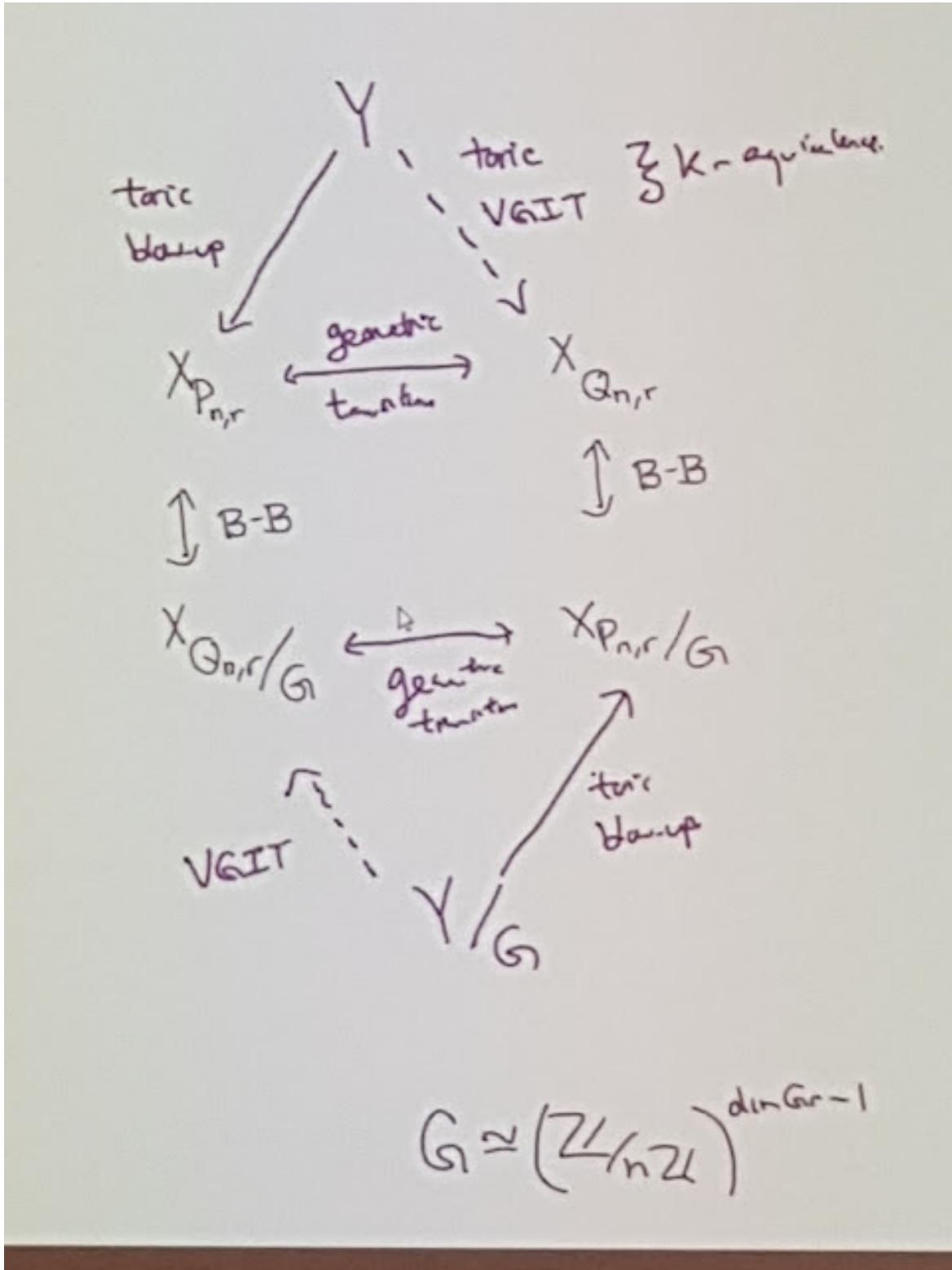
Remark 17.0.4: Take a toric degeneration $Z \subseteq \text{Gr}_n(r) \rightsquigarrow Z_p \subseteq X_p$ a Fano toric, apply BB mirror symmetry to get $Z_p^\vee \subseteq X_{p^\vee}$, take a “geometric transition” (blow up, change the stability conditions, variation of GIT) to get $\tilde{Z}_p \subseteq \tilde{X}_p$, and smooth to get \tilde{Z} .



Remark 17.0.5: Degenerate toric variety to one associated to a certain reflexive polytope coming from a quiver moduli spaces $\mathcal{M}_\theta(LQ, (1, 1, \dots, 1))$ which is a Fano where θ is a stability condition. Here LQ is a ladder quiver. There are correspondences between arrows in LQ and coordinates

on $X_{P_{n,r}}$, paths $0 \rightarrow 1$ and partitions $\lambda \vdash r \times (n - r)$, subsets of $[n]$ of size r , and sections of a very ample line bundle $\mathcal{O}(D)$ where $S_\lambda = \prod_{a \in \lambda} x_a$ yielding $X_{P_{n,r}} \hookrightarrow \mathbb{P}^{\binom{n}{r}-1}$ with Plucker coordinates p_λ . What equations cut out $X_{P_{n,r}}$ in this projective space? Can define operations $\sigma \wedge \lambda, \sigma \vee \lambda$ for two “incomparable” grid paths (pairs of paths which cross) through an $r \times (n - r)$ grid by taking maximal or minimal parts of each, and these generate the correct equations. This yields the toric degeneration described as step 1 in the image above.

Remark 17.0.6: Step 2: use a new version of polytope duality. Let $P \subseteq N_{\mathbb{R}}$ be a reflexive polytope, let Q denote the **primitive dual** of P . For fixed P, Q , BB mirror symmetry looks like $X_P \rightleftharpoons X_Q/G$ and $X_Q \rightleftharpoons X_P/G$, where $G = M/\overline{M} \cong \overline{M}^\vee/N$. See Gelfand-Cetlin polytope, toric variation of GIT. What geometric transition means in this context:



17.1 Toric un-degenerating

Remark 17.1.1: We get $H_{n,r}$ from mirror symmetry and G from the Gelfand-Cetlin polytope, and it turns out that $H_{n,r} \leq G$ is the maximal subgroup for which $\text{Gr}_n(r)$ is an equivariant subvariety. Constructing the toric degeneration as an equivariant family: let $Z \subseteq X := \mathbb{P}^{\binom{n}{r}-1} \times \mathbb{C}^k$ be the family cut out by Plucker relations with extra parameters tracked by the \mathbb{C}^k part. Extend the action of G to \mathbb{C}^k so that Z is G -equivariant. Then $Z/G \subseteq X/G \rightarrow \mathbb{C}^k/G$ is a family with special fiber $X_{P_{n,r}}/G$ and general fiber $\text{Gr}_n(r)/H_{n,r}$, so $X_{P_{n,r}}/G$ smooths to the Grassmannian analogy of Greene-Plesser. This more or less comes from the orbit-stabilizer theorem and carefully tracking the stabilizer.

Theorem 17.1.2(?)

The family of hypersurfaces $\sum_{\lambda} p_{\lambda}^n + \psi \prod_{\lambda} p_{\lambda} = 0$ in $\text{Bl } \text{Gr}_n(r)/H_{n,r}$ is a Calabi-Yau compactification of a Laurent polynomial mirror for $\text{Gr}_n(r)$. This is a candidate mirror to $Z \subseteq \text{Gr}_n(r)$.