FLOER HOMOTOPY THEORY

ABSTRACT. The goal of this note is to summarize and contextualize Abouzaid-Blumberg's proof of the Arnold conjecture using Floer homotopy types with coefficients in Morava K-theories.

Contents

1. An Introduction to Floer Homotopy

The main reference for this note is [AB21], however much of the content for this section comes from the earlier overview [Coh19].

1.1. **What is a homotopy type?**

Let us first start with a toy model of the type of question we are interested in:

Question 1.1.1. Given a bounded chain complex of $\mathbb{Z}$-modules

$$C_\bullet : \qquad \cdots \to C_{n-1} \xrightarrow{\partial_{n-1}} C_n \xrightarrow{\partial_n} C_{n+1} \to \cdots ,$$

does there exist a topological space X whose cellular chain complex ${C_\bullet}^{\mathrm{cell}}(X;\mathbb{Z})$ is quasi-isomorphic to $C_\bullet$?

Remark 1.1.2. For nice spaces X, we can immediately reduce this to a question of homotopy theory by noting that a homotopy equivalence of spaces $X \xrightarrow{f} X'$ induces (at least!) a quasi-isomorphism on their cellular chain complexes

$$f_* : {C_\bullet}^{\mathrm{cell}}(X) \xrightarrow{\sim} {C_\bullet}^{\mathrm{cell}}(X').$$

So we turn our attention to searching for a *homotopy type* $[X]$.

Remark 1.1.3. Permit me, if you will, the following notation:

- C will generally denote a (small) category, and lowercase letters $x, y \in \mathsf{C}$ will denote objects in this category. Cat will denote the infinity category of small categories, and we'll abuse notation by writing $x \in \mathsf{C}$ when x is an object of C.
- Top denotes the category of (completely arbitrary) topological spaces whose morphisms are continuous maps.
- $\mathsf{C}(x, y) := \mathrm{Hom}_{\mathsf{C}}(x, y)$ denotes the set of morphisms from x to y. So for example, $\mathsf{Top}(x, y)$ denotes the set of continuous maps from the space x to the space y.
- hoTop denotes the associated (naive) homotopy category whose objects are precisely those in Top, but whose morphisms are given by $\mathsf{hoTop}(x, y) = \mathsf{Top}(x, y)/\sim$ where $f \sim g$ if f is homotopic to g. Note that this is more properly thought of as $\mathsf{Top}[\frac{1}{W}]$, the category obtained as a certain categorical localization of Top at a class W of *weak equivalences*.
- $\mathsf{R\text{-}Mod}$ will denote the category of modules over a ring R, and $\mathsf{Ch(R\text{-}Mod)}$ will denote the category of chain complexes of R-modules. Its homotopy category will be written as $\mathsf{hoCh(R\text{-}Mod)}$, and the derived category as $\mathbf{D}\mathsf{Ch(R\text{-}Mod)}$.
- $\mathsf{SHC} = \mathsf{hoSp}$ is the homotopy category of spectra, sometimes called the *stable homotopy category*.

Definition 1.1.4 (Homotopy equivalence and weak equivalence)**.** Recall that two spaces $x, y \in \mathsf{Top}$ are **homotopy equivalent** precisely when there exist witnessing maps $f \in \mathsf{Top}(x, y)$ and $g \in \mathsf{Top}(y, x)$ whose compositions are homotopic to the identities on x and y respectively. Note that as a consequence, there is an induced isomorphism of graded $\mathbb{Z}$-modules

$$\pi(f) : \pi_*(X) \xrightarrow{\sim} \pi_*(Y).$$

The spaces x and y are **weakly equivalent** when there is a witnessing morphism $f \in \mathsf{Top}(x, y)$ which induces such an isomorphism on homotopy groups. The difference is slightly subtle: we do not require the existence of a homotopy inverse for such a morphism, and indeed such an inverse may not exist – there are weakly equivalent spaces that are *not* homotopy equivalent, which witnesses the fact that this is truly a weaker condition!

Remark 1.1.5. In the setup of Floer homotopy, we'll want to associate not a space or a chain complex to some fixed starting set of Floer data, but rather a *spectrum*, which can be provisionally regarded as a "generalized space".

The general state of affairs is that one is usually interested in only the homotopy groups of a space, or only the homology of a chain complex. We seek to enrich this information by remembering not only the results of such "homotopical extractions" (here taking homotopy groups or homology) but rather the object from whence it came. Note that this strictly enriches the situation – for example, with a space in hand, one can apply *other* homology theories to extract new and potentially interesting information. However, we have some flexibility in the situation – trading in a given space for another, homotopy-equivalent space recovers precisely the same homotopy groups. Similarly trading a chain complex for a homotopy equivalent complex recovers precisely the same homology, which leads us to consider spaces or chain complexes up to homotopy or *homotopy types*.

Remarkably, the notion of a *homotopy type* is used in quite a few papers and books and seems to have a clear intended meaning but no precise definition. I would love to be corrected on this, but in the meantime, let us proceed with the following provisional definition:

Definition 1.1.6 (Homotopy type (provisional))**.** A **(topological) homotopy type** is an object $x \in \mathsf{hoTop}$, which can be concretely thought as a topological space up homotopy equivalence. Note that one could instead only require *weak* equivalence here.

A **chain homotopy type** is an object in $x \in \mathsf{hoCh}(\mathsf{R\text{-}Mod})$, regarded as a chain complex up chain homotopy. Similarly, one could relax this slightly and only require $x \in \mathbf{D}\mathsf{Ch}(\mathsf{R\text{-}Mod})$, so that x is a complex up to quasi-isomorphism instead.

A **stable homotopy type** is an object $x \in \mathsf{SHC}$, regarded as a spectrum up to weak equivalence.

Remark 1.1.7. Since we'll need some infinity-categorical machinery eventually, it is perhaps worth introducing an equivalent definition of topological homotopy types: there is a guiding principle, the *homotopy hypothesis*, which posits that there is an equivalence of infinity categories

$$\Pi_* : \mathsf{hoTop} \xrightarrow{\sim} \underset{\infty}{\mathsf{Grpd}} := (\infty, 1)\text{-}\mathsf{Grpd}$$

where Π_* is a certain *fundamental groupoid* construction. This is a known theorem for source categories like Kan complexes, which are known to be closely related to hoTop via simplicial sets. Granted this hypothesis, we could equivalently define a homotopy type to just be an infinity-groupoid, although it is less clear (to me) what the corresponding definitions should be for chain homotopy types and stable homotopy types.

1.2. **The analogous question in Morse Theory.**

Remark 1.2.1. To help motivate what exactly a "Floer homotopy type" should be, let's first recall the simpler analogous situation in classical Morse theory. For M a smooth finite-dimensional Riemannian manifold and $f \in C^\infty(M, \mathbb{R})$ a smooth real-valued functional, we can extra the Morse chain complex:

$$C_\bullet(f) : \qquad \cdots \to C_{n-1}(f) \xrightarrow{\partial_{n-1}} C_n(f) \xrightarrow{\partial_n} C_{n+1}(f) \to \cdots .$$

Here the grading is by the indices n of critical points of f, and $n_p(f) := \mathbb{Z} \dagger \operatorname{Crit}_n(f)$ is the free $\mathbb{Z}$-module on all critical points of index n. The differential applied to a critical point a is given by the formula

$$\partial_n(a) := \sum_{b \in \operatorname{Crit}_{n-1}(f)} \sharp \mathcal{M}(a, b) \cdot b,$$

where $\mathcal{M}(a, b)$ is an appropriately normalized moduli space of gradient flow trajectories from a to b. Interestingly, Cohen notes that $\sharp \mathcal{M}(a, b) \in \mathbb{Z} \cong \pi_0 \mathrm{MSO}$, so that this count is somehow related to the homotopy groups of the Thom spectrum of the special orthogonal groups SO_n.

The following fact is most relevant to our goal:

Fact 1.2.2. If M is closed, then there is a quasi-isomorphism of chain complexes

$$C_\bullet(f) \simeq C_\bullet{}^{\mathrm{cell}}(\mathcal{E}_{M,f})$$

where $\mathcal{E}_{M,f}$ is a CW complex representation the homotopy type of M.

Remark 1.2.3. This seems to be a restatement (or perhaps a slight strengthening) of the standard theorems showing that f induces a handle decomposition of M, and in fact a deformation retract of M onto a CW skeleton, along with the fact that Morse homology recovers cellular (and thus singular) homology. We are thus lead to formulate the following analogous question in Floer theory:

Question 1.2.4. Let (M, H) be a fixed manifold and Hamiltonian. Is the Floer chain complex $\mathrm{CF}_\bullet(M, H)$ *realized* as $C_\bullet{}^{\mathrm{cell}}(\mathcal{E}_{M,H})$ for some CW complex $\mathcal{E}_{M,H}$?

Remark 1.2.5. In actuality, the precise questions is a bit more complicated than I've made it out to be here. More generally, one asks for this realization not in terms of CW complexes but rather in CW spectra, and Cohen proffers a definition of *realization* of a chain complex $C_\bullet$ that involves a filtered spectrum $X_1 \hookrightarrow X_2 \hookrightarrow \cdots \hookrightarrow X_n = X$ whose "associated graded" cofibers satisfy

$$X_i/X_{i-1} \coloneqq \mathrm{hocofib}(X_{i-1} \hookrightarrow X_i) \simeq \Sigma^k \bigvee \mathbb{S},$$

so that the "associated graded" is a suspension of wedges of spheres, where (roughly) $H^*(X_i/X_{i-1}) \cong C_i$.

Bibliography

[AB21] Mohammed Abouzaid and Andrew J. Blumberg. *Arnold Conjecture and Morava K-theory*. 2021. DOI: 10.48550/ARXIV.2103.01507. URL: https://arxiv.org/abs/2103.01507.

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