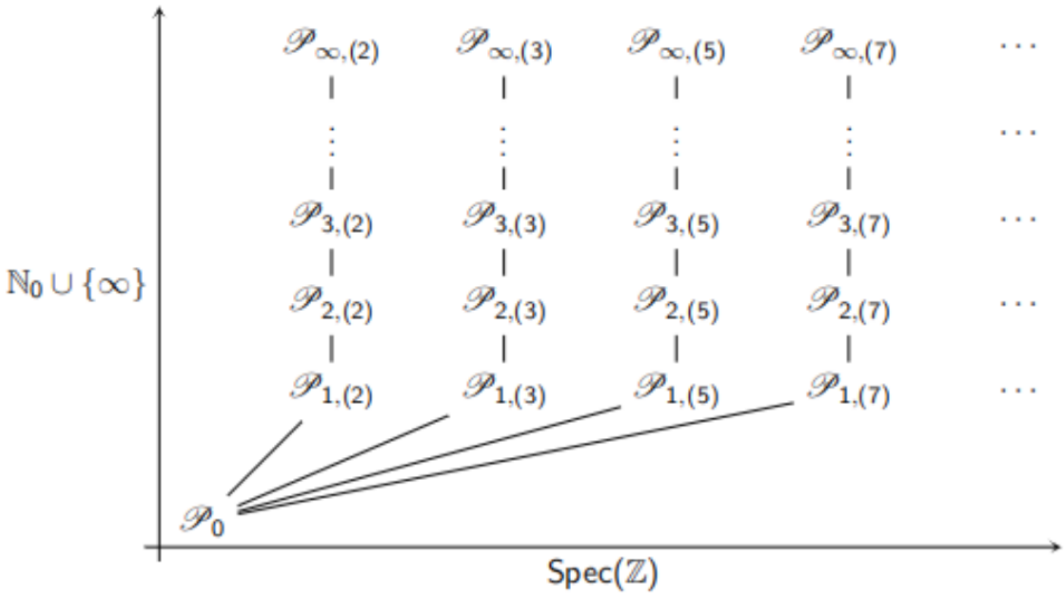

Notes: These are notes live-tex'd from a graduate course in Cohomology in Representation Theory taught by Dan Nakano at the University of Georgia in Spring 2022. As such, any errors or inaccuracies are almost certainly my own.

Cohomology in Representation Theory

Lectures by Dan Nakano. University of Georgia, Spring 2022

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1 Introduction and Background (Tuesday, January 11)

Remark 1.0.1: References: [Jac09].

Remark 1.0.2: Idea: study representation by studying associated geometric objects, and use homological methods to bridge the two. The representation theory side will mostly be rings/modules, and the geometric side will involve algebraic geometry and commutative algebra. Throughout the course, all rings will be unital and all actions on the left.

Example 1.0.3 ***(of categories of modules):*** Recall the definition of a left R-module. Some examples:

- $k \in \mathsf{Field} \implies \mathsf{k\text{-}Mod} = \mathsf{Vect}_k$
- $R = \mathbb{Z} \implies \mathbb{Z}\text{-}\mathsf{Mod} = \mathsf{AbGrp}$.
- $A \in \mathsf{Alg}_{/k}$, which is a ring $(A, +, \cdot)$ where $(A, +, .)$ (using scalar multiplication) is a vector space.
  - E.g. $\mathrm{Mat}(n \times n, \mathbb{C})$.
  - E.g. for G a finite group, the group algebra kG for $k \in \mathsf{Field}$.
  - E.g. $U(\mathfrak{g})$ for $\mathfrak{g} \in \mathsf{LieAlg}$ or a super algebra.

Remark 1.0.4: Connecting this to representation theory: for $A \in \mathsf{Alg}_{/k}$ and $M \in \mathsf{A\text{-}Mod}$, a representation of A is a morphism of algebras $A \xrightarrow{\rho} \mathfrak{gl}_n(k)$, the algebra of all $n \times n$ matrices (not necessarily invertible). Note that for groups, one instead asks for maps $kG \to \mathrm{GL}_n$, the invertible matrices. There is a correspondence between $\mathsf{A\text{-}Mod} \rightleftharpoons \mathsf{Rep}(A)$: given M, one can define the action as

$$\begin{aligned} \rho : A &\to \operatorname{End}_k(M) \\ \rho(a)(m) &= a.m. \end{aligned}$$

Remark 1.0.5: Recall the definitions of:

- Morphisms of R-modules: $f(r.m_1 + m_2) = r.f(m_1) + f(m_2)$
- Submodules: $N \leq M \iff r.n \in N$ and N is closed under $+$.
- Quotient modules: $M/N = \{m + N\}$.
- The fundamental homomorphism theorem: for $M \xrightarrow{f} N$, there is an induced $\psi : M/\ker f \to N$ where $M/\ker f \cong \operatorname{im} f$.

$$
\begin{tikzcd}
M \arrow[r, "f"] \arrow[d, "\eta"] & N & f(m) \\
M/\ker f \arrow[ur, dashed, "\exists\psi"] & & \\
m + \ker f \arrow[uurr, mapsto] & &
\end{tikzcd}
$$

Link to Diagram

- The fundamental SES

  $$0 \to \ker f \xhookrightarrow{g} M \xrightarrow{f} \operatorname{im} f \to 0,$$

  where one generally needs $\operatorname{im} g = \ker f$ for exactness.

  – More generally, need monomorphisms, epimorphisms.

Example 1.0.6 ***(?):*** Some examples:

- $f : \mathbb{Z} \to \mathbb{Z}$ where $f(m) := 4m$ yields $0 \to \mathbb{Z} \xrightarrow{f} \mathbb{Z} \to \mathbb{Z}/4 \to 0$ in $\mathbb{Z}$-Mod.
- In $\mathbb{C}$-Mod, one can take $0 \to \mathbb{C} \xrightarrow{\Delta: x \mapsto (x,x)} \mathbb{C}^{\times 2} \to \mathbb{C} \to 0$.

Remark 1.0.7: Direct sums, products, and indecomposables. Let I be an index set and $\{M_k\}_{k\in I}$ R-modules to define the **direct product** $\prod_{k\in I} M_k := \left\{ (m_k)_{k\in I} \mathrel{\Big|} m_k \in M_k \right\}$, the set of all ordered sequences of elements from the M_k, with addition defined pointwise. For the **direct sum** $\bigoplus_{k\in I} M_k$ to be those sequences with only finitely many nonzero components. For internal direct sums, if $M = M_1 + M_2$ then $M \cong M_1 \oplus M_2$ iff $M \cap M_2 = 0$. An **irreducible representation** is a simple R-module, and an **indecomposable representation** is an indecomposable R-module. An R-module is **simple** iff its only submodules are $0, M$, and **indecomposable** iff $M \ncong M_1 \oplus M_2$ for any $M_i \ncong M$. Note that simple $\implies$ indecomposable.

Note: is it possible for $M \cong M \oplus M$?

Example 1.0.8 ***(?):*** Some examples:

- Simple objects in k-Mod are isomorphic to k, and indecomposables are also isomorphic to k if we restrict to finite dimensional modules.

- Simple objects in $\mathbb{Z}$-Mod are cyclic groups of prime order, C_p. Indecomposables are $\mathbb{Z}, C_{p^k}$, using the classification theorem to rule out composites.

- For $A \in \mathsf{Alg}^{\mathrm{fd}}_{/k}$, the simple objects in $\mathsf{A\text{-}Mod}$ are hard to determine in general. The same goes for indecomposables, and is undecidable in many cases (equivalent to the word problem in finite groups).

*See **finite**, **tame**, and **wild** representation types.*

Remark 1.0.9: Toward homological algebra: free and projective modules. An R-module M is **free** iff $M \cong \bigoplus_{i \in I} R_i$ for some indexing set where $R_i \cong R$ as a left R-module. Equivalently, M has a linearly independent spanning set, or there exists an X and a unique φ such that the following diagram commutes:

$$\begin{tikzcd}
M \arrow[dr, dashed, "\exists!\varphi\in\mathsf{R\text{-}Mod}"] & \\
X \arrow[u, hook, "\iota\in\mathsf{R\text{-}Mod}"] \arrow[r, "\mathsf{Set}"'] & N
\end{tikzcd}$$

Link to Diagram

Every $M \in \mathsf{R\text{-}Mod}$ is the image of a free R-module: let $X := \{m_i\}_{i\in I}$ generate M, so $X \hookrightarrow M$ by inclusion. Define $X \to \bigoplus \bigoplus_{i\in I} R_i$ sending $m_i \to (0, \cdots, 1, \cdots, 0)$ with a 1 in the ith position, then since X is a generating set this will lift to a surjection $\bigoplus_i R_i \to M$. We can use this to define a free resolution:

$$\begin{tikzcd}
 & \ker\delta_1 \arrow[dr, hook] & & & & & \\
\cdots & & \exists F_1 \arrow[rr, dashed, "\exists\delta_1"] \arrow[dr, two heads] & & F_0 \arrow[r, "\delta_0"] & M \arrow[r] & 0 \\
 & & & \ker\delta_0 \arrow[ur, hook] \arrow[dr] & & & \\
 & & 0 \arrow[ur] & & 0 & &
\end{tikzcd}$$

Link to Diagram

Remark 1.0.10: Let $A \in \mathsf{Alg}^{\mathrm{fd}}_{/k}$ and $F \cong \bigoplus A$ be free, and suppose $e \in A$ is idempotent, so $e^2 = e$ – these are useful because they can split algebras up. There is a *Pierce decomposition* of 1 given by $1 = e + (1 - e)$. Noting that $1 - e$ is also idempotent, there is a decomposition $A \cong Ae \oplus A(1-e)$. Since Ae is direct summand of A which is free, this yields a way to construct projective modules.

2 | Thursday, January 13

Remark 2.0.1: Last time:

- R-modules and their morphisms
- Free resolutions $F \twoheadrightarrow R$.

Today: projective modules and their resolutions.

See Krull-Schmidt theorem.

Remark 2.0.2: Recall the definition of projective modules P and injective modules I:

$$\begin{tikzcd}
 & & & & P \arrow[d] \arrow[dl, dashed, "\exists"'] & \\
\forall\xi : & 0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[dl, dashed, "\exists"] & C \arrow[r] & 0 \\
 & & I & & &
\end{tikzcd}$$

Link to Diagram

Exercise 2.0.3 (?)
Show that free implies projective using the universal properties, and conclude that every R-module has a projective cover.

Remark 2.0.4: Forming projective resolutions: take the minimal $P_0 \xrightarrow{\delta_0} M \to 0$ such that $\Omega^1 := \ker \delta_0$ has no projective summands. Continue in such a minimal way:

$$\begin{tikzcd}
 & 0 \arrow[d] & & 0 \arrow[d] & & & \\
 & \Omega^2 \arrow[dr, hook] & & \Omega^1 \arrow[dr, hook] & & & \\
\cdots \arrow[rr, dashed] & & P_1 \arrow[ur, two heads] \arrow[rr, dashed, "\exists"] & & P_0 \arrow[r, two heads] & M \arrow[r] & 0
\end{tikzcd}$$

Link to Diagram

Remark 2.0.5: For modules M over an algebra A, if $\dim_k(M)$ is finite, then each P_i can be chosen to be finite dimensional. Otherwise, define a **complexity** or **rate of growth** $sc_A(M) \geq 0$ such that $\dim P_n \leq Cn^{s-1}$ for some constant C. A theorem we'll prove is that s is finite when $A = kG$ for every finite dimensional G-module. When $A = kG$, this is a numerical invariant but has a nice geometric interpretation in terms of support varieties $V_A(M)$, an affine algebraic variety where $\dim V_A(M) = c_A(M)$.

Exercise 2.0.6 (?)
Recall the definition of a SES $\xi : 0 \to A \xrightarrow{d_1} B \xrightarrow{d_2} C \to 0$ and show that TFAE:

- ξ splits
- ξ admits a right section $s_r : C \to B$
- ξ admits a left section $s_\ell B \to A$

Hint: for the right section, show that s_r is injective. Get that $\operatorname{im} f + \operatorname{im} h \subseteq M_2$, use exactness to write $\operatorname{im} d_1 = \ker d_2$ and show that $\ker d_2 \cap \operatorname{im} s_r = \emptyset$.

Warning 2.0.7
It's not necessarily true that if $B \cong A \oplus C$ that ξ splits: consider

$$0 \longrightarrow C_2 \longrightarrow C_4 \longrightarrow C_2 \longrightarrow 0$$

Link to Diagram

Exercise 2.0.8 (?)
Show that for $P \in \mathsf{R\text{-}Mod}$, TFAE:

- P is projective.
- Every SES $\xi : 0 \to A \to B \to P \to 0$ splits.
- There exists a free module F such that $F = P \oplus K$.

Exercise 2.0.9 (?)
Show that $\bigoplus_{i \in I} P_i$ is projective iff each P_i is projective.

Example 2.0.10 *(?):*

- If $R = k \in \mathsf{Field}$, then every $M \in \mathsf{k\text{-}Mod}$ is free and thus projective since $M \cong \bigoplus_{i \in I} k$ with k free in $\mathsf{k\text{-}Mod}$.
- If $R = \mathbb{Z}$, let $P \in \mathbb{Z}\text{-}\mathsf{Mod}$ be projective and F free and consider $0 \to K \to F \to P \to 0$. Since $F \cong P \oplus K$, P is a submodule of F, making P free since $\mathbb{Z}$ is a PID. So projective implies free.

- Not every $M \in \mathbb{Z}\text{-Mod}$ is projective: take $C_6 \in \mathbb{Z}\text{-Mod}$, then $C_6 \cong C_2 \oplus C_3$ so C_2, C_3 are projective in $\mathsf{C}_6\text{-Mod}$ but not free here.

Exercise 2.0.11 (?)
Let $Q \in \mathsf{R}\text{-Mod}$ and show TFAE:

- Q is injective
- Every SES $\xi : 0 \to Q \to B \to C \to 0$ splits.

Exercise 2.0.12 (?)
Show that $\prod_{i \in I} Q_i$ is injective iff each Q_i is injective. Note that one needs to use direct products instead of direct sums here.

Theorem 2.0.13 *(?).*
The category $\mathsf{R}\text{-Mod}$ has enough injectives, i.e. for every $M \in \mathsf{R}\text{-Mod}$ there is an injective Q and a SES $0 \to M \hookrightarrow Q$.

Proof (Sketch).
See Hungerford or Weibel. Prove it first for $\mathsf{C} = \mathsf{Z}\text{-Mod}$. The idea now is to apply

$$F(-) := \underset{\mathbb{Z}}{\mathrm{Hom}}(R, -) : (\mathbb{Z}, \mathbb{Z})\text{-biMod} \to (R, \mathbb{Z})\text{-biMod},$$

the left-exact contravariant hom. Using that $R \in (R, R)\text{-biMod} \hookrightarrow (\mathbb{Z}, R)\text{-biMod}$, one can use the right action R on itself to define a left action on $\underset{\mathbb{Z}}{\mathrm{Hom}}(R, M)$. Then check that

- f is left exact
- f sends injectives to injectives.
- If $R \in \mathbb{Z}\text{-Mod}$ has an R-module structure, then $F(R)$ is again an R-module.

■

Exercise 2.0.14 (?)
Show that for $M \in \mathsf{R}\text{-Mod}$ that $\underset{\mathbb{Z}}{\mathrm{Hom}}(R, M) \cong M$.

Hint: try $f \mapsto f(1)$.

Remark 2.0.15: Next week:

- Tensor products
- Categories
- Tensor and Hom

3 | Tensor Products (Tuesday, January 18)

Remark 3.0.1: Setup: $R \in \mathsf{Ring}$, $M_R \in \mathsf{Mod\text{-}R}$, and ${}_R N \in \mathsf{R\text{-}Mod}$. Note that R is not necessarily commutative. The goal is to define $M \otimes_R N$ as an abelian group.

Definition 3.0.2 (The Tensor Product)
The **balanced product** of M and N is a $P \in \mathsf{AbGrp}$ with a map $f : M \times N \to P$ such that

- $f(x + x', y) = f(x, y) + f(x', y)$
- $f(x, y + y') = f(x, y) + f(x, y')$
- $f(ax, y) = f(x, ay)$.

The **tensor product** $(M \otimes_R N, \otimes)$ of M and N is the initial balanced product, i.e. if P is a balanced product with $M \times N \xrightarrow{f} P$ then there is a unique map $\psi : M \otimes_R N \to P$:

$$\begin{tikzcd}
& M \otimes_R N \arrow[dr, dashed, "\exists ! \psi"] & \\
M \times N \arrow[ur, "\otimes"] \arrow[rr, "f"'] & & P
\end{tikzcd}$$

Link to Diagram

Uniqueness follows from the standard argument on universal properties:

$$\begin{tikzcd}
& (M \otimes N)_1 \\
M \times N \arrow[ur, "\otimes_1"] \arrow[r, "\otimes_2"] \arrow[dr, "\otimes_1"'] & (M \otimes N)_2 \arrow[u, dashed, bend left, "\exists \psi_{21}"] \\
& (M \otimes N)_1 \arrow[u, dashed, bend left, "\exists \psi_{12}"] \arrow[uu, bend right=60, "\mathrm{id}"']
\end{tikzcd}$$

Link to Diagram

Existence: let $\mathsf{Free}(-) : \mathsf{Set} \to \mathsf{AbGrp}$ and $F \coloneqq \mathsf{Free}(M \times N)$, then set $M \otimes_R N \coloneqq F/G$ where G is generated by

- $(x + x', y) - ((x, y) + (x', y))$
- $(x, y + y') - ((x, y) + (x, y'))$
- $(ax, y) - (x, ay)$.

Then define the map as

$$\begin{aligned}
\otimes : M \times N &\to F \\
(x, y) &\mapsto x \otimes y \coloneqq (x, y) + G.
\end{aligned}$$

Why it satisfies the universal property: use the universal property of free groups to get a map to F and check that the following diagram commutes:

$$\begin{tikzcd}
M \times N \arrow[r, "\otimes"'] \arrow[dr] & F \arrow[r, "(-)/G"'] \arrow[d, dashed, "\exists"'] & M \otimes_R N := F/G \arrow[dl, "\exists\psi"] \\
& P &
\end{tikzcd}$$

Link to Diagram

Morphisms: for $f : M \to M'$ and $g : N \to N'$, form

$$\begin{aligned} f \otimes g : M \otimes N &\to M' \otimes N' \\ x \otimes y &\mapsto f(x) \otimes g(y). \end{aligned}$$

Warning 3.0.3
Note every $z \in M \otimes_R N$ is a simple tensor of the form $z = x \otimes y$!

Example 3.0.4*(?)*:

- For $R = k \in \mathsf{Field}$, $M \otimes_k N \in (k,k)\text{-}\mathsf{biMod}$. If $M = \langle m_i \rangle$ and $N = \langle n_j \rangle$, then $M \otimes_k n = \langle m_i \otimes n_j \rangle$ and $\dim_k M \otimes_k N = \dim_k M \cdot \dim_k N$.

- For $A \in \mathsf{AbGrp}$, $A \otimes_{\mathbb{Z}} \mathbb{Z} \cong A$ since $x \otimes y = xy \otimes 1$.

- $M := C_p \otimes_{\mathbb{Z}} \mathbb{Q} = 0$. It suffices to check on simple tensors:

$$\begin{aligned} x \otimes y &= x \otimes \frac{p}{p} y \\ &= x \otimes p \left(\frac{1}{p} \right) y \\ &= px \otimes \left(\frac{1}{p} \right) y \\ &= 0 \otimes \frac{1}{p} y \\ &= 0. \end{aligned}$$

- More generally, if $A \in \mathsf{AbGrp}$ is torsion then $A \otimes_{\mathbb{Z}} \mathbb{Q} = 0$.

Definition 3.0.5 (Categories)
A category C is a class of objects $A \in \mathsf{C}$ and for any pair (A, B), a set of morphism $\underset{\mathsf{C}}{\mathrm{Hom}}(A, B)$ such that

1. $(A, B) \neq (C, D) \implies \mathrm{Hom}(A, B)$ and $\mathrm{Hom}(C, D)$ are disjoint.
2. Associativity of composition: $(h \circ g) \circ f = h \circ (g \circ f)$
3. Identities: $\exists!\, \mathrm{id}_A \in \underset{\mathsf{C}}{\mathrm{Hom}}(A, A)$ for all $A \in \mathsf{C}$.

A **subcategory** $\mathsf{D} \leq \mathsf{C}$ is a subclass of objects and morphisms, and is **full** if $\underset{\mathsf{D}}{\mathrm{Hom}}(A, B) =$

$\operatorname{Hom}_{\mathsf{C}}(A, B)$ for all objects in D.

Example 3.0.6 ***(?):*** Examples of categories:

- $\mathsf{C} = \mathsf{Set}$,
- $\mathsf{C} = \mathsf{Grp}$,
- $\mathsf{C} = \mathsf{R\text{-}Mod}$,
- $\mathsf{C} = \mathsf{Top}$ with continuous maps.

Example 3.0.7 ***(?):*** Examples of fullness:

- $\mathsf{Grp} \leq \mathsf{Set}$ is not a full subcategory, since not all set morphisms are group morphisms.
- $\mathsf{AbGrp} \leq \mathsf{Grp}$ is a full subcategory.

Remark 3.0.8: Recall the definition of covariant and contravariant functors, which requires that $F(\mathrm{id}_A) = \mathrm{id}_{F(A)}$.

4 | Thursday, January 20

Remark 4.0.1: RIP Brian Parshall and Fred Cohen… 😔

Remark 4.0.2: Recall the definition of a covariant functor. Some examples:

- $F(R) = U(R) = R^\times = \mathbb{G}_m(R)$, the group of units of R.
- The forgetful functor $\mathsf{Grp} \to \mathsf{Set}$.
- $\operatorname{Hom}_{\mathbb{Z}}(R, -)$ for $R \in (\mathbb{Z}, R)\text{-}\mathsf{biMod}$ is a functor $\mathbb{Z}\text{-}\mathsf{Mod} \to \mathsf{R\text{-}Mod}$.

Exercise 4.0.3 (?)
Formulate $\operatorname{Hom}_{\mathbb{Z}}(-, -)$ in terms of functors between bimodule categories. How does this "use up an action" in the way $- \otimes_{\mathbb{Z}} -$ does?

Remark 4.0.4: Recall that contravariant functors reverse arrows. Functors with the same variance can be composed.

Definition 4.0.5 (Full and Faithful Functors)
Let $F : \mathsf{C} \to \mathsf{D}$ and consider the set map

$$\begin{aligned} F_{AB} : \operatorname{Hom}(A, B) &\to \operatorname{Hom}(FA, FB) \\ f &\mapsto F(f). \end{aligned}$$

We say F is **full** if F_{AB} is injective for all $A, B \in \mathsf{C}$, and **faithful** if F_{AB} is surjective for all A, B.

Definition 4.0.6 (Natural Transformations)
A morphism of functors $\eta : F \to G$ for $F, G : \mathsf{C} \to \mathsf{D}$ is a **natural transformation**: a family of maps $\eta_A \in \operatorname{Hom}_{\mathsf{D}}(FA, GA)$ satisfying the following naturality condition:

$$\begin{tikzcd}
A \arrow[dd, "f \in \mathsf{C}"'] & & FA \arrow[r, "\eta_A"] \arrow[dd, "F(f)"] & GA \arrow[dd, "G(f)"] & \\
& \Longrightarrow & & & \in \mathsf{D} \\
B & & FB \arrow[r, "\eta_B"'] & GB &
\end{tikzcd}$$

Link to Diagram

If η_A is an isomorphism for all $A \in \mathsf{C}$, then η is a **natural isomorphism**.

Exercise 4.0.7 (?)
For $\mathsf{C}, \mathsf{D} = \mathsf{Vect}^{\mathrm{fd}}_{/k}$ finite-dimensional vector spaces, take $F = \mathrm{id}$ and $G(-) = (-)^{\vee\vee}$. Note that $\operatorname{Hom}(FV, GV) \cong \operatorname{Hom}(V, V^{\vee\vee}) \cong \operatorname{Hom}(V, V)$, so set η_V to be the image of id_V under this chain of isomorphisms. Show that $\{\eta_V\}_{V \in \mathsf{C}}$ assemble to a natural transformation $F \to G$.

Definition 4.0.8 (Isomorphisms and Equivalences of categories)
Two categories C, D are **isomorphic** if there are functors F, G with $F \circ G = \mathrm{id}_{\mathsf{D}}, G \circ F = \mathrm{id}_{\mathsf{C}}$ *equal* to the identities. They are **equivalent** if $F \circ G, G \circ F$ are instead *naturally isomorphic* to the identity.

Example 4.0.9 ***(?):*** Some examples:

- $\mathsf{C} = \mathsf{AbGrp}$ and $\mathsf{D} = \mathbb{Z}\text{-}\mathsf{Mod}$ by taking $G : \mathsf{D} \to \mathsf{C}$ the forgetful functor, and for F, using the same underlying set and defining the $\mathbb{Z}$-module structure by $n \cdot m \coloneqq m + m + \cdots + m$.

- $\mathsf{C} = \mathsf{R}\text{-}\mathsf{Mod}$ and $\mathsf{D} = \mathrm{Mat}_{\mathsf{n}\times\mathsf{n}}(\mathsf{R})\text{-}\mathsf{Mod}$. For $\mathsf{k}\text{-}\mathsf{Mod}$, the simple objects are k, but for $\mathrm{Mat}_{\mathsf{n}\times\mathsf{n}}(\mathsf{R})\text{-}\mathsf{Mod}$, the simple objects are k^n, so these categories are not isomorphic. However, it turns out that they are equivalent.

Producing inverse functors can be difficult, so we have the following:

Proposition 4.0.10 ***(A useful criterion for equivalence of categories).***
Let $F : \mathsf{C} \to \mathsf{D}$, then there exists an inverse inducing an *equivalence* iff

- F is fully faithful,
- Surjectivity on objects: for every $A' \in \mathsf{D}$, there exists an $A \in \mathsf{C}$ such that $F(A) \cong A'$.

Proof (?).
$\implies$: Suppose F, G induce an equivalence $\mathsf{C} \simeq \mathsf{D}$, so $F \circ G \simeq \mathrm{id}_{\mathsf{D}}$ and $G \circ F \simeq \mathrm{id}_{\mathsf{C}}$. To show $f \to F(f)$ is injective, check that

$$\begin{aligned} & F(f) = F(g) \\ \implies & GF(f) = GF(g) \\ & \mathrm{id}(f) = \mathrm{id}(g) \\ \implies & f = g. \end{aligned}$$

■

Exercise 4.0.11 (?)
Show surjectivity.
A hint:
Let $A' \in \mathsf{D}$ with $FG \simeq \mathrm{id}_{\mathsf{D}}$ and $\eta_{A'} \in \underset{\mathsf{D}}{\mathrm{Hom}}(A', FGA')$ is an iso. Set $A := GA' \in \mathsf{C}$ and use that

$$\underset{\mathsf{D}}{\mathrm{Hom}}(A', FGA') := \mathrm{Hom}(A', FA),$$

So if there is an isomorphism in $\mathrm{Hom}(A', FA)$, there exists an isomorphism in $\mathrm{Hom}(FA, A')$ and thus $FA \cong A'$.

#todo Missed a bit here so this doesn't make sense as-is!

Proposition 4.0.12 *(?).*
Let $R \in \mathsf{Ring}$ and set $S := \mathrm{Mat}_{n \times n}(R)$, then $\mathsf{R\text{-}Mod} \simeq \mathsf{S\text{-}Mod}$.

5 | Tuesday, January 25

Remark 5.0.1: Recall isomorphisms $\mathsf{C} \cong \mathsf{D}$ of categories, so $F \circ G = \mathrm{id}$, vs equivalences of categories $\mathsf{C} \simeq \mathsf{D}$ so $F \circ G \cong \mathrm{id}$.

Theorem 5.0.2 *(?).*
For $F : \mathsf{C} \to \mathsf{D}$ and $G : \mathsf{D} \to \mathsf{C}$ and write $\psi_F : \underset{\mathsf{C}}{\mathrm{Hom}}(A, B) \to \underset{\mathsf{D}}{\mathrm{Hom}}(F(A), F(B))$. This pair induces an equivalence iff

1. F is faithful, i.e. ψ_F is injective,
2. F is full, i.e. ψ_F is surjective,
3. For any $D \in \mathsf{D}$, there exists a $C \in \mathsf{C}$ with $F(C) \cong D$.

Proposition 5.0.3 *(?).*
Let $R \in \mathsf{Ring}$ and $S = \mathrm{Mat}_{n\times n}(R)$, then $\mathsf{R\text{-}Mod} \simeq \mathsf{S\text{-}Mod}$.

Proof (?).
Define a functor $F : \mathsf{R\text{-}Mod} \to \mathsf{S\text{-}Mod}$ by $F(M) := \prod_{k\le n} M$, regarding this as a column vector and letting S act by matrix multiplication. On morphisms, define $F(f)(\mathbf{x}) = [f(x_1), \cdots, f(x_n)]$ for $\mathbf{x} \in \prod M$. Then $F(\mathrm{id}) = \mathrm{id}$, and (exercise) $F(f)$ is a morphism of S-modules and composes correctly:

$$F(g\circ f)(\mathbf{x}) = [gf(x_1), \cdots, gf(x_n)] = F(g)[f(x_1), \cdots, f(x_n)] = (F(g)\circ F(f))\,\mathbf{x}.$$

So this defines a functor.

Claim: F is fully faithful.

- Faithfulness: if $F(f_1) = F(f_2)$, then $f_1(x_j) = f_2(x_j)$ for all j, making $f_1 = f_2$.
- Fullness: let $g \in \underset{S}{\mathrm{Hom}}(M^n, N^n)$ for $M, N \in \mathsf{R\text{-}Mod}$ and e_{ij} be the elementary matrix with a 1 only in the i,j position. Check that $e_{11}M^n = \left\{[x, 0, \cdots] \;\middle|\; x \in M\right\}$, $e_{11}N^n = \left\{[y, 0, \cdots] \;\middle|\; y \in N\right\}$, and $\mathrm{diag}(x)$ be a matrix with only copies of x on the diagonal. Then $g(e_{11}M^n) \subseteq e_{11}g(M^n) \subseteq e_{11}N^n$ and $g[x, 0, \cdots] = [y, 0, \cdots]$. Define $f : M \to N$ by $f(x) = y$, then on one hand,

  $$g(\mathrm{diag}(a)[x, 0, \cdots]) = g[ax, 0, \cdots] = [f(ax), 0, \cdots],$$

  but since g is a morphism of S-modules, this also equals $\mathrm{diag}(a)\cdot g[x, 0, \cdots] = [ay, 0, \cdots]$. Then $f(ax) = ay = af(x)$, so f is a morphism of R-modules.

  Note that $e_{j1}\mathbf{x} = [0, \cdots, x, \cdots 0]$ with x in the jth position. Check that $g(e_{j1}\mathbf{x}) = g[0, \cdots, x, \cdots, 0]$. The LHS is

  $$e_{j1}g(\mathbf{x}) = e_{j1}[f(x), 0, \cdots] = [0, \cdots, f(x), \cdots, 0]$$

  with $f(x)$ in the jth position. Hence $g(\mathbf{x}) = [f(x_1), \cdots, f(x_n)]$, making F full.

See also Jacobson Basic Algebra Part II *p.31.*

■

Exercise 5.0.4 (Tensors commute with direct sums)
Show that

$$\left(\bigoplus_{\alpha\in I} M_\alpha\right) \otimes_R N \cong \bigoplus_{\alpha\in I} \left(M_\alpha \otimes_R N\right),$$

and similarly for $M \otimes (\oplus N_\alpha)$.

Remark 5.0.5: Define functors $F, G\mathsf{R}\text{-}\mathsf{Mod} \to \mathbb{Z}\text{-}\mathsf{Mod}$ by $F(-) := M \otimes_R (-)$ and $G(-) := (-) \otimes_R N$ on objects, and on morphisms $f : N \to N'$, set $F(f) := \operatorname{id} \otimes f$ and similarly for G. Recall the definition of exactness, left-exactness, and right-exactness.

Example 5.0.6 ***(Tensoring may not be left exact):*** Consider

$$\xi : 0 \to p\mathbb{Z} \xrightarrow{f} \mathbb{Z} \xrightarrow{g} \mathbb{Z}/p\mathbb{Z} \to 0$$

and apply $(-) \otimes_{\mathbb{Z}} \mathbb{Z}/p\mathbb{Z}$. Use that $p\mathbb{Z} \cong \mathbb{Z}$ in $\mathbb{Z}\text{-}\mathsf{Mod}$ to get

$$F(\xi) : C_p \xrightarrow{f \otimes \mathrm{id}} C_p \xrightarrow{g \otimes \mathrm{id}} C_p,$$

and

$$(f \otimes \mathrm{id})(px \otimes y) = px \otimes y = x \otimes py = 0,$$

using that f is the inclusion.

Exercise 5.0.7 (?)
Show that $M \otimes_R (-)$ and $(-) \otimes_R N$ are right exact for any $M, N \in \mathsf{R}\text{-}\mathsf{Mod}$.

Solution:
Let $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ which maps to $M \otimes A \xrightarrow{\operatorname{id} \otimes f} M \otimes B \xrightarrow{\operatorname{id} \otimes g} C$.

- Show $\operatorname{id} \otimes g$ is surjective: write $m \in M \otimes C$ as $m = \sum x_i \otimes y_j$, pull back the y_j via g to get z_j with $g(z_j) = y_j$. Then

$$(\operatorname{id} \otimes g)(\sum x_i \otimes z_J) = \sum x_i \otimes g(z_j) = \sum x_i \otimes y_j.$$

- Exactness, $\operatorname{im}(\operatorname{id} \otimes f) = \ker(\operatorname{id} \otimes g)$: Use that $gf = 0$ by exactness of the original sequence, and $(\operatorname{id} \otimes g) \circ (\operatorname{id} \otimes f) = \operatorname{id} \otimes (g \circ f) = 0$, so $\operatorname{im}(\operatorname{id} \otimes f) \subseteq \ker(\operatorname{id} \otimes g)$.
  - For the reverse containment, use that $\operatorname{id} \otimes g : M \otimes B \to M \otimes C$ and define a map

$$\begin{aligned} \Gamma : \frac{M \otimes B}{\operatorname{im}(\operatorname{id} \otimes f)} &\to M \otimes C \\ m \otimes n + \operatorname{im}(\operatorname{id} \otimes f) &\mapsto m \otimes g(n). \end{aligned}$$

    Then φ is an isomorphism iff $\operatorname{im}(\operatorname{id} \otimes f) = \ker(\operatorname{id} \otimes g)$. Define

$$\begin{aligned} \Psi : M \times C &\to \frac{M \otimes B}{\operatorname{im}(\operatorname{id} \otimes f)} \\ (x, y) &\mapsto x \otimes z + \operatorname{im}(\operatorname{id} \otimes f), \end{aligned}$$

    where $g(z) = y$, so z is a lift of y.

  Why is this well-defined? Check $g(z_1) = y = g(z_2)$ implies $z_1 - z_2 \in \ker g = \operatorname{im} f$, so write $f(y) = z_1 - z_2$ for some y. Then $x \otimes z_1 + \operatorname{im} f = x \otimes z_2 + \operatorname{im} f$.

Why does this factor through the tensor product? Check that Ψ is a balanced product, this yields $\overline{\Psi} : M \otimes C \to \dfrac{M \otimes B}{\mathrm{im}(\mathrm{id} \otimes f)}$. Now check that $\overline{\Psi}, \Gamma$ are mutually inverse:

$$\Gamma\Psi(x \otimes y) = \Gamma(x \otimes z + \mathrm{im}(\mathrm{id} \otimes f)) = x \otimes g(z) = x \otimes y$$
$$\Psi\Gamma(x \otimes z + \mathrm{im}(\mathrm{id} \otimes f)) = (x \otimes g(z)) = x \otimes z + \mathrm{im}\, f.$$

Question 5.0.8
When is $M \otimes_R (-)$ exact?

6 | Thursday, January 27

Remark 6.0.1: Recall that $M \in \mathsf{R\text{-}Mod}$ is flat iff for every N, N' and $f \in \underset{\mathsf{R\text{-}Mod}}{\mathrm{Hom}}(N, N')$, the induced map

$$\mathrm{id}_M \otimes f : M \otimes_R N \to M \otimes_R N'$$

is a monomorphism. Equivalently, $M \otimes_R (-)$ is left exact and thus exact.

Proposition 6.0.2 *(?).*
$M := \bigoplus_{\alpha \in I} M_\alpha$ is flat iff M_α is flat for all $\alpha \in I$.

Proof (?).

$$M \otimes_R (-) := (\bigoplus M_\alpha) \otimes_R (-) \cong \bigoplus (M_\alpha \otimes_R (-)).$$

■

Exercise 6.0.3 (?)
Show that projective $\implies$ flat.

Exercise 6.0.4 (?)
Prove that the hom functors $\underset{\mathsf{R\text{-}Mod}}{\mathrm{Hom}}(M, -), \underset{\mathsf{R\text{-}Mod}}{\mathrm{Hom}}(-, M) : \mathsf{R\text{-}Mod} \to \mathbb{Z}\text{-}\mathsf{Mod}$ are left exact.

Exercise 6.0.5 (?)
Show that

- P is projective iff $\underset{\mathsf{R\text{-}Mod}}{\mathrm{Hom}}(P, -)$ is exact

- I is projective iff $\underset{\textsf{R-Mod}}{\mathrm{Hom}}(-, I)$ is exact

Remark 6.0.6: An object $Z \in \mathsf{C}$ is a zero object iff $\underset{\mathsf{C}}{\mathrm{Hom}}(A, Z), \underset{\mathsf{C}}{\mathrm{Hom}}(Z, A)$ are singletons for all $A \in \mathsf{C}$. Write this as $0_A \in \underset{\mathsf{C}}{\mathrm{Hom}}(A, Z)$. If C has a zero object, define the zero morphism as $0_{AB} \coloneqq 0_B \circ 0_A \in \underset{\mathsf{C}}{\mathrm{Hom}}(A, B)$.

7 | Tuesday, February 01

Definition 7.0.1 (Additive categories)
A category C is **additive** iff

- C has zero object
- There exists a binary operation $+ : \mathrm{Hom}(A, B)^{\times 2} \to \mathrm{Hom}(A, B)$ for all $A, B \in \mathsf{C}$ making $\mathrm{Hom}(A, B)$ an abelian group.
- Distributivity with respect to composition: $(g_1 + g_2)f = g_1 f + g_2 f$
- For any collection $\{A_1, \cdots, A_n\}$, there exists an object A, projections $p_j : A \to A_j$ with sections $i_k : A_k \to A$ with $p_j i_j = \mathrm{id}_A$, $p_j i_k = 0$ for $j \neq k$, and $\sum i_j p_j = \mathrm{id}_A$.

Definition 7.0.2 (Monomorphisms and epimorphisms)
A morphism: $k : K \to A$ is **monic** iff whenever $g_1, g_2 : L \to K$, $kg_1 = kg_2 \implies g_1 = g_2$:

$$\begin{tikzcd} L \arrow[r, shift left, "g_1"] \arrow[r, shift right, "g_2"'] & K \arrow[r, "k"] & A \end{tikzcd}$$

Link to Diagram

Define k to be **epic** by reversing the arrows.

Definition 7.0.3 (Kernel)
Assume C has a zero object. Then for $f : A \to B$, the *morphism* $k : K \to A$ is the **kernel** of f iff

- k is monic
- $fk = 0$
- For any $g : G \to A$ with $fg = 0$, there exists a g' with $g = kg'$.

Example 7.0.4 ***(?):*** For $f \in \textsf{R-Mod}(A, B)$, take $k : \ker f \hookrightarrow A$. If $g \in \mathsf{C}(G, A)$ with $f(g(x)) = 0$ for all $x \in G$, then $\mathrm{im}\, g \subseteq \ker f$ and we can factor g as $G \xrightarrow{g'} \ker f \overset{k}{\hookrightarrow} A$.

Definition 7.0.5 (Cokernel)
For $f : A \to B$, a morphism $c : B \to C$ is a **cokernel of f** iff

- c is epic,
- $cf = 0$
- For any $h \in \mathsf{C}(B, H)$ with $hf = 0$, there is a lift $h' : C \to G$ with $h = h'c$.

Example 7.0.6 ***(?):*** For $\mathsf{C} = \mathsf{R\text{-}Mod}$ and $f \in \mathsf{R\text{-}Mod}(A, B)$, set $c : B \to B/\operatorname{im} f$.

Exercise 7.0.7 (?)
Show that kernels are unique. Sketch:

- Set $k : K \to A$, $k' : K' \to A$.
- Factor $k = k'u_1$ and $k' = ku_2$.
- Then $k \operatorname{id} = k(u_2u_1) \implies \operatorname{id} = u_2u_1$, similarly $u_1u_2 = \operatorname{id}$.

Definition 7.0.8 (Abelian categories)
C is **abelian** iff C is additive and

- A5: Every morphism admits kernels and cokernels.
- A6: Every monic is the kernel of its cokernel, and every epic is the cokernel of its kernel.
- A7: Every morphism f factors as $f = me$ with m monic and e epic.

Example 7.0.9 ***(?):*** For $f \in \mathsf{R\text{-}Mod}(A, B)$,

- A5: Take $k : \ker f \hookrightarrow A$ and $c : B \twoheadrightarrow B/\operatorname{im} f$
- A6: For $m : A \hookrightarrow B$ monic, consider the composition $A \hookrightarrow B \xrightarrow{\operatorname{coker} m} B/A$ and check $A \cong \ker(\operatorname{coker} m)$.
- A7: Use the 1st isomorphism theorem:

$$
\begin{tikzcd}
A \arrow[rrr, "f"] \arrow[dr, "i"'] & & & B \\
& A/\ker f \arrow[r, "\text{1st iso}"'] & \operatorname{im} f \arrow[ur, hook, "m"'] &
\end{tikzcd}
$$

Link to Diagram

Remark 7.0.10: Some notes:

- Recall the definition the category of chain complexes $\mathsf{Ch}(\mathsf{C})$ over an abelian category: $d_i d_{i+1} = 0$, so $\operatorname{im} d_i \subseteq \ker d_{i+1}$.

- Every exact sequence is an acyclic complex.
- $\mathsf{C} \hookrightarrow \mathsf{Ch}(\mathsf{C})$ by $M \mapsto \cdots \to 0 \to M \to 0 \to \cdots$. Note that this isn't an acyclic complex.
- Morphisms between complexes: chain maps, just levelwise maps forming commutative squares, i.e. maps commuting with the differentials.
- $\mathsf{Ch}(\mathsf{C})$ is additive: given $\alpha_\bullet, \beta_\bullet \in \mathsf{ChC}((A,d),(B,\delta))$, check that $(\alpha_{i-1}+\beta_{i-1})d_i = \delta_i(\alpha_i+\beta_i)$.
- There are direct sums: $(A \oplus B)_i := A_i \oplus B_i$ with $d := d_A + d_B$.
- Define cycles as $Z_i := \ker\left(C_i \xrightarrow{d_i} C_{i-1}\right)$ for $C_\bullet \in \mathsf{Ch}(\mathsf{C})$, and boundaries $B_i := \operatorname{im}\left(C_{i+1} \xrightarrow{d_{i+1}} C_i\right) \subseteq \ker d_i$.
- Define $H_i(C_\bullet) := Z_i / B_i$.
- Show that chain morphisms induce morphisms on homology:
  - Let $\alpha \in \mathsf{Ch}(\mathsf{C})(C, C')$, then $\alpha_i(Z_i) \subseteq Z_i'$.
  - Check $d_2(a_i(Z_i)) = a_{i-1}d_i(Z_i) = 0$.
  - Factor $Z_i \xrightarrow{\alpha_i} Z_i' \twoheadrightarrow Z_i'/B_i'$.
  - Show that $x \in B_i$ maps lands in B_i' to get well-defined map on H_i.
  - Use $\alpha(B_i) \subseteq Z_i'$, so pull back $x \in B_i$ to $y \in C_{i+1}$.
  - Check $d_{i+1}(y) = x$, so $\alpha(d_{i+1}(y)) = \alpha(x)$.
  - The LHS is $d_{i+1}'(\alpha_{i+1}(y))$, so $\alpha_i(x) in \operatorname{im} d_{i+1}' = B_{i+1}'$
- Chain homotopies: for $\alpha, \beta \in \mathsf{Ch}(\mathsf{C})(C, C')$, write $\alpha \simeq \beta$ iff there exists $\left\{ s_i : C_i \to C_{i+1}' \right\}$ with $\alpha_i - \beta_i = d_{i+1}' s_i + s_{i-1} d_i$.

$$
\begin{tikzcd}
\cdots \arrow[r] & C_{i+1} \arrow[r, "d_{i+1}"] \arrow[d] & C_i \arrow[r, "d_{i+1}"] \arrow[dl, "s_i"'] \arrow[d, "\alpha_i - \beta_i"] & C_{i-1} \arrow[r] \arrow[dl, "s_{i-1}"'] \arrow[d] & \cdots \\
\cdots \arrow[r] & C_{i+1}' \arrow[r, "d_{i+1}'"] & C_i' \arrow[r, "d_i'"] & C_{i-1}' \arrow[r] & \cdots
\end{tikzcd}
$$

Link to Diagram

8 | Thursday, February 03

8.1 Projective Resolutions and Chain Maps

Remark 8.1.1: Also check that $\simeq$ is an equivalence relation, i.e. it is symmetric, transitive, and reflexive. For transitivity: given

$$
\begin{aligned}
\alpha_i - \beta_i &= d_{i+1}' s_i + s_{i-1} d_i \\
\beta_i - \gamma_i &= d_{i+1}' t_i + t_{i-1} d_i,
\end{aligned}
$$

one can write

$$\alpha_i - \gamma_i = d'_{i+1}(s_i + t_i) + (s_{i-1} + t_{i-1})d_i.$$

Theorem 8.1.2 *(?).*
Let $\alpha, \beta \in \mathsf{ChC}(A, B)$ with induced maps $\widehat{\alpha}, \widehat{\beta} \in \mathsf{ChC}(H^*A, H^*B)$ on homology. If $\alpha \simeq \beta$, then $\widehat{\alpha} = \widehat{\beta}$.

Proof (?).
A computation:

$$\begin{aligned}
\widehat{\alpha}_1(\ z_1 + B_i) &= \alpha_1\,(z_i) + B'_i \\
&= \beta_i\,(z_i) + \delta'_{i+1} s_1\,(z_i) + s''_{i-1}\delta_i\,(z_i) + B'_i \\
&= \beta_i\,(z_i) + B'_i \\
&= \widehat{\beta}_i\,(z_i + B_i)
\end{aligned}$$

■

Remark 8.1.3: Roadmap:

- Homological algebra
- Commutative rings
- Support theory
- Tensor triangular geometry

Definition 8.1.4 (?)
Let $M \in \mathsf{R\text{-}Mod}$. A **projective complex** for M is a chain complex $(C_i, d_i)_{i\in\mathbb{Z}}$, indexed homologically:

$$\cdots \to C_2 \xrightarrow{d_2} C_1 \xrightarrow{d_1} C_0 \xrightarrow{d_0 := \varepsilon} 0.$$

In particular, $d^2 = 0$, but this complex need not be exact. A **projective resolution** of M is an *exact* projective complex in the following sense:

- $H_{k\geq 1}(C_\bullet) = 0$
- $H_0(C_\bullet) = C_0/d(C_1) = C_0/\ker\varepsilon \cong M$.

Example 8.1.5 *(?):* Some projective resolutions:

- For $M \in \mathsf{R\text{-}Mod}$, projective resolutions exist since we can find covers by free modules:

$$\begin{tikzcd}
\cdots \arrow[r] & F_2 \arrow[r] & F_1 \arrow[r, "d_1"] \arrow[d] & F_0 \arrow[r, two heads, "\varepsilon"] \arrow[d] & M \arrow[r] & 0 \\
& & \ker d_1 \arrow[ur] & \ker \varepsilon \arrow[ur] & &
\end{tikzcd}$$

Link to Diagram

- For $M \in \mathsf{Z\text{-}Mod}$, every module has a 2-stage resolution:

$$\begin{tikzcd}
0 \arrow[r] & \ker \varepsilon \cong \mathbb{Z}^{\oplus m} \arrow[r] & \mathbb{Z}^{\oplus n} \arrow[r, two heads, "\varepsilon"] & M \arrow[r] & 0
\end{tikzcd}$$

Link to Diagram

Theorem 8.1.6 *(?).*
For $\mu \in \mathsf{C}(M, M')$ and $C \coloneqq (C_\bullet, d) \twoheadrightarrow M, C' \coloneqq (C_\bullet{}', d') \twoheadrightarrow M'$, there is an induced chain map $\alpha \in \mathsf{ChC}(C, C')$. Moreover, any other chain map β is chain homotopic to α.

Note that C can in fact be any projective complex over M, not necessarily a resolution.

Proof (?).
Using that C_0 is projective, there is a lift of the following form:

$$\begin{tikzcd}
C_0 \arrow[r, two heads, "\varepsilon"] \arrow[d, dashed, "\exists \alpha_0"'] & M \arrow[d, "\mu"] \\
C_0' \arrow[r, two heads, "\varepsilon"'] & M'
\end{tikzcd}$$

Link to Diagram

Now inductively, we want to construct the following lift:

$$\begin{tikzcd}
C_n \arrow[rr, "d_n"] \arrow[d, dashed, "\exists"'] & & C_{n-1} \arrow[r, "d_{n-1}"] \arrow[d, "\alpha_{n-1}"] & C_{n-2} \arrow[d, "\alpha_{n-2}"] \\
C_n' \arrow[rr, "d_n'"'] \arrow[dr, two heads] & & C_{n-1}' \arrow[r, "d_{n-1}'"'] & C_{n-2}' \\
& \operatorname{im} d_n' = \ker d_{n-1}' \arrow[ur, hook] & &
\end{tikzcd}$$

Link to Diagram

STS $\operatorname{im} \alpha_{n-1} d_n \subseteq \ker d_{n-1}'$, which follows from

$$d_{n-1}' \alpha_{n-1} d_n(x) = \alpha_{n-1} d_{n-1} d_n(x).$$

So there is a map $C_n \to \operatorname{im} d'_n$, and using projectivity produces the desired lift by the same argument as in the case case:

$$\begin{tikzcd}
C_n \arrow[rr, "d_n"] \arrow[d, dashed, "\exists \text{ by projectivity}"'] \arrow[dr, dashed, "\exists"] & & C_{n-1} \arrow[r, "d_{n-1}"] \arrow[d, "\alpha_{n-1}"] & C_{n-2} \arrow[d, "\alpha_{n-2}"] \\
C'_n \arrow[rr, "d'_n"] \arrow[dr, two heads] & & C'_{n-1} \arrow[r, "d'_{n-1}"'] & C'_{n-2} \\
& \operatorname{im} d'_n = \ker d'_{n-1} \arrow[ur, hook] & &
\end{tikzcd}$$

Link to Diagram

To see that any two such maps are chain homotopic, set $\gamma := \alpha - \beta$, then

$$\varepsilon'(\gamma_0) = \varepsilon'(\alpha_i - \beta_i) = \mu\varepsilon - \mu\varepsilon = 0,$$

and

$$\begin{aligned}
d'_n(\gamma_n) - d'_n(\alpha_n - \beta_n) \\
&= d'_n\alpha_n - d'_n\beta_n \\
&= \alpha_{n-1}d_n - \beta_{n-1}d_n \\
&= \gamma_{n-1}d_n,
\end{aligned}$$

so γ yields a well-defined chain map.
We'll now construct the chain homotopy inductively. There is a lift s_0 of the following form:

$$\begin{tikzcd}
& C_0 \arrow[dl, dashed, "\exists s_0"'] \arrow[d, "\gamma_0"] & & \\
C'_1 \arrow[r] \arrow[d, two heads, "d'_1"'] & C'_0 \arrow[r, two heads, "\varepsilon'"] & M' \arrow[r] & 0 \\
\operatorname{im} d'_1 \arrow[ur, hook] & & &
\end{tikzcd}$$

Link to Diagram

This follows because $\operatorname{im} d'_1 = \ker \varepsilon'$ and $\varepsilon'\gamma_0 = 0$ by the previous calculation.
Assuming all $s_{i \le n-1}$ are constructed, set $\gamma_i = d'_{i+1}s_i + s_{i-1}d_i$. Setting $\gamma_n - s_{n-1}d_n : C_n \to C'_n$, then

$$\begin{aligned}
d'_n(\gamma_n - s_{n-1}d_n) &= d'_n\gamma_n - d'_n s_{n-1}d_n \\
&= \gamma_{n-1}d_n - d'_n s_{n-1}d_n \\
&= (\gamma_{n-1} - d'_n s_{n-1})d_n \\
&= s_{n-2}d_{n-1}d_n \\
&= 0,
\end{aligned}$$

using $d^2 = 0$. Now there is a lift s_n of the following form:

$$
\begin{tikzcd}
& & C_n \arrow[lld, dashed, "s_n"'] \arrow[d, "\gamma_n - s_{n-1}d_N"] & \\
C'_{n+1} \arrow[rr] \arrow[rd, "d_{n+1}"'] & & C'_n \arrow[r, "d'_n"] & C_{n-1} \\
& \operatorname{im} d_{n+1} = \ker d'_n \arrow[ru] & &
\end{tikzcd}
$$

Link to Diagram

Thus follows from the fact that $\operatorname{im} \gamma_n - s_{n-1}d_n \subseteq \ker d'_n$ and projectivity of C_n.

■

Remark 8.1.7: Dually one can construct injective resolutions $0 \to M \overset{\eta}{\hookrightarrow} D_\bullet$

8.2 Derived Functors

Remark 8.2.1: Setup: $F : \mathsf{R\text{-}Mod} \to \mathbb{Z}\text{-}\mathsf{Mod}$ is an additive covariant functor, e.g. $(-) \otimes_R N$ or $M \otimes_R (-)$, and $C_\bullet \overset{\varepsilon}{\twoheadrightarrow} M$ a complex over M. We define the left-derived functors as $(L_n F)(M) \coloneqq H_n(F(C_\bullet))$.

9 | Tuesday, February 08

Remark 9.0.1: Defining derived functors: for F an additive functor and $M \in \mathsf{R\text{-}Mod}$, take a projective resolution and apply F:

$$\cdots \to C_2 \xrightarrow{d_2} C_1 \xrightarrow{d_1} C_0 \xrightarrow{\varepsilon = d_0} M \to 0 \rightsquigarrow F(C_2) \xrightarrow{Fd_2} F(C_1) \xrightarrow{Fd_1} \cdots,$$

so $C_\bullet \rightrightarrows F$.

Define the left-derived functor

$$\mathbb{L}FM \coloneqq H_n F C_\bullet.$$

Remark 9.0.2: Any $\mu \in \mathsf{R\text{-}Mod}(M, M')$ induces a chain map $\widehat{\alpha} \in \mathsf{ChR\text{-}Mod}(H_* F C_\bullet, H_* F C_\bullet')$, where α is any lift of μ to their resolutions.

$$
\begin{tikzcd}
C_\bullet \arrow[r, Rightarrow, "\varepsilon"] \arrow[d, "\alpha"'] & M \arrow[d, "\mu"] \\
C_\bullet' \arrow[r, Rightarrow, "\varepsilon'"'] & M'
\end{tikzcd}
$$

Link to Diagram

Exercise 9.0.3 (?)
Show that any two lifts α, α' induce the same map on homology.

Remark 9.0.4: Similarly, $\mathbb{L}F(M)$ does not depend on the choice of resolution:

$$
\begin{tikzcd}
C_\bullet \arrow[r, "\varepsilon"] \arrow[d, "\alpha"] \arrow[dd, bend right, "\mathrm{id}_{C_\bullet}"'] & M \arrow[d, "\mathrm{id}_M"] \\
C_\bullet' \arrow[r, "\varepsilon"] \arrow[d, "\beta"] & M \arrow[d, "\mathrm{id}_M"] \\
C_\bullet \arrow[r, "\varepsilon"] & M
\end{tikzcd}
\quad \rightsquigarrow \quad
\begin{tikzcd}
FC_\bullet \arrow[r] \arrow[d, "F(\alpha)"] \arrow[dd, bend right, dashed, "\therefore \mathrm{id}_{FC_\bullet}"'] & F(M) \arrow[d] \\
FC_\bullet' \arrow[r] \arrow[d, "F(\beta)"] & F(M) \arrow[d] \\
FC_\bullet \arrow[r] & F(M)
\end{tikzcd}
$$

Link to Diagram

Definition 9.0.5 (Projective resolution of a SES)
For $0 \to M' \to M \to M'' \to 0$ in C, a **projective resolution** is a collection of chain maps forming projective resolutions of each of the constituent modules:

$$
\begin{tikzcd}
0 \arrow[r] & C_\bullet' \arrow[r] \arrow[d, Rightarrow] & C_\bullet \arrow[r] \arrow[d, Rightarrow] & C_\bullet'' \arrow[r] \arrow[d, Rightarrow] & 0 \\
0 \arrow[r] & M' \arrow[r] & M \arrow[r] & M'' \arrow[r] & 0
\end{tikzcd}
$$

Link to Diagram

Exercise 9.0.6 (?)
Show that such resolutions exist. This involves constructing $\varepsilon : C_0 \to M$:

$$
\begin{tikzcd}
0 \arrow[r] & C_0' \arrow[r, hook, "\iota_0"] \arrow[d, "\varepsilon"'] & C \cong C_0' \oplus C_0'' \arrow[r, two heads, "p_0"] \arrow[d, dashed, "\therefore \exists \varepsilon"'] & C_0'' \arrow[r] \arrow[dl, dashed, "\exists \varepsilon^*"'] \arrow[d, two heads, "\varepsilon''"'] & 0 \\
0 \arrow[r] & M' \arrow[r, hook, "\gamma"] & M \arrow[r, two heads, "\sigma"] & M'' \arrow[r] & 0
\end{tikzcd}
$$

Link to Diagram

The claim is that $\varepsilon(x, x'') \coloneqq \gamma\varepsilon'(x') + \varepsilon^*(x'')$ works. To prove surjectivity, use the following:

Proposition 9.0.7 ***(Short Five Lemma).***
Given a commutative diagram of the following form

$$
\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "p"] \arrow[d, "f"'] & B \arrow[r, "q"] \arrow[d, "g"'] & C \arrow[r] \arrow[d, "h"'] & 0 \\
0 \arrow[r] & A' \arrow[r, "s"] & B' \arrow[r, "t"] & C' \arrow[r] & 0
\end{tikzcd}
$$

Link to Diagram

If g, h are mono (resp. epi, resp. iso) then f is mono (resp. epi, resp. iso).

Proof (of surjectivity, alternative by diagram chase).

- Let $x \in M$
- Set $y = \sigma(x)$
- Find $z \in C_0$ such that $\varepsilon'' p_0(z) = y$.
- Consider $\varepsilon(z) - x$ and apply σ:

$$
\begin{aligned}
\sigma(\varepsilon(z) - x) &= \sigma\varepsilon(x) - \sigma(x) \\
&= \varepsilon'' p_0(x) - \sigma(x) \\
&= y - y \\
&= 0.
\end{aligned}
$$

- So $\varepsilon(z) - x \in \ker \sigma = \operatorname{im} \gamma$
- Pull back to $w \in C_0'$ such that $\gamma\varepsilon'(w) = \varepsilon(z) - x$
- Check $\varepsilon i_0(w) = \gamma\varepsilon'(w) = \varepsilon(z) - x$, so $\varepsilon(i_0(w) - z) = -x$.

■

Proof (of existence).
The setup:

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \ker \varepsilon' \arrow[r, hook, "f"] \arrow[d, hook] & \ker \varepsilon \arrow[r, "g"] \arrow[d, hook] & \ker \varepsilon'' \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & C_0' \arrow[r, hook, "\iota_0"] \arrow[d, "\varepsilon'"] & C_0 \arrow[r, two heads, "p_0"] \arrow[d, dashed, "\therefore\exists\varepsilon"'] & C_0'' \arrow[r] \arrow[d, two heads, "\varepsilon''"] \arrow[dl, dashed, "\exists\varepsilon^*"'] & 0 \\
0 \arrow[r] & M' \arrow[r, hook, "\gamma"] & M \arrow[r, two heads, "\sigma"] & M'' \arrow[r] & 0
\end{tikzcd}
$$

Link to Diagram

This is exact and commutative by a diagram chase:

- $f = i \circ \downarrow_{\ker \varepsilon'}$ shows $g(\ker \varepsilon) \subseteq \ker \varepsilon''$
- $g = p \circ \downarrow_{\ker \varepsilon}$ shows $f(\ker \varepsilon') \subseteq \ker \varepsilon$.

To show exactness along the top line:

- f is injective, since it's the restriction of an injective map.
- g is surjective:
  - Let $x \in \ker \varepsilon''$, so $\varepsilon''(x) = 0$.
  - $\exists y \in C_0$ with $p_0(y) = x$ by surjectivity of p_0.
  - Check $\varepsilon''(p_0(y)) = \varepsilon(x) = 0$ in M'', so $\sigma\varepsilon(y) = 0$
  - Thus $\varepsilon(y) \in \ker \sigma = \operatorname{im} \gamma$
  - By surjectivity there exists $w \in C_0'$ such that $\gamma(\varepsilon'(w)) = \varepsilon(y)$.
  - Use commutativity to verify

$$
\begin{aligned}
\varepsilon(i_0(w) - y) &= \varepsilon(i_0(w)) - \varepsilon(y) \\
&= \gamma\varepsilon'(w) - \varepsilon(y) \\
&= \varepsilon(y) - \varepsilon(y) \\
&= 0.
\end{aligned}
$$

  - Then

$$
\begin{aligned}
g(i_0(w) - y) &= p_0(i_0(w)) - g(y) \\
&= -g(y) \\
&= -p_0(y) \\
&= -x.
\end{aligned}
$$

- Exactness at the middle, i.e. $\operatorname{im} f = \ker g$:

- $\operatorname{im} f \subseteq \ker g$ by exactness of the second row, so it STS $\ker g \subseteq \operatorname{im} f$.
- Let $y \in \ker g$, then by commutativity $y \in \ker p_0 = \operatorname{im} i_0$. Note that $y \in \ker \varepsilon$ by definition.
- Write $y = i_0(x)$ for some $x \in C_0'$
- Note $\gamma\varepsilon'(x) = \varepsilon i_0(x) = \varepsilon(y) = 0$ since $y \in \ker \varepsilon$.
- Since γ' is mono, $\varepsilon'(x) = 0$, so $y = i_0(x) = f(x)$.

■

Proposition 9.0.8 *(?).*
For $F : \mathsf{R\text{-}Mod} \to \mathbb{Z}\text{-}\mathsf{Mod}$ additive and a SES

$$\xi : 0 \to M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0,$$

note that there are morphisms

$$\mathbb{L}FM'' \to \mathbb{L}FM \to \mathbb{L}FM'.$$

There is a connecting morphism

$$\Delta : \mathbb{L}FM'' \to \Sigma^{-1}\mathbb{L}FM',$$

which in components looks like

$$\begin{tikzcd}
0 & \mathbb{L}_0F(M'') \arrow[l] & \mathbb{L}_0F(M) \arrow[l] & \mathbb{L}_0F(M') \arrow[l] \\
 & \mathbb{L}_1F(M'') \arrow[rru] & \mathbb{L}_1F(M) \arrow[l] & \mathbb{L}_1F(M') \arrow[l] \\
 & \mathbb{L}_2F(M'') \arrow[rru] & \cdots \arrow[l] &
\end{tikzcd}$$

Link to Diagram

10 Thursday, February 10

Missed! Please send me notes. :)

11 | Tuesday, February 15

12 | Tuesday, February 22

12.1 Prime Ideals

Remark 12.1.1: Plan: commutative ring theory, aiming toward tensor triangular geometry.

Remark 12.1.2:

- Recall the definition of prime ideals.
- Show $\mathfrak{p} \in \operatorname{Spec} R \iff R/\mathfrak{p}$ is an integral domain.
- Recall $\mathfrak{m} \in \operatorname{mSpec} R \iff R/\mathfrak{m}$ is a field.
- Recall the definition of a monoid
- Note that $R \setminus \mathfrak{p} \ni 1$ and $R \setminus \mathfrak{p}$ is a submonoid of $(R, \cdot)$.
- Examples of primes:
  - $\langle p \rangle \in \operatorname{Spec} R$ and if $p \neq 0$ then $\langle p \rangle \in \operatorname{mSpec} R$.
  - $R = k[x]$ is a PID and $\langle f \rangle \in \operatorname{Spec} R \iff f$ is irreducible.
- Recall the set of nilpotent elements and the nilradical $\sqrt{0_R}$.
  - Show $\sqrt{0_R} \in \operatorname{Id}(R)$.
  - Show that $R_{\text{red}} \coloneqq R/\sqrt{0_R}$ is reduced (no nonzero nilpotents).

Lemma 12.1.3 ***(Prime Avoidance).***
Let $A, I_j \in \operatorname{Id}(R)$ where at most two of the I_j are not prime and $A \subseteq \bigcup_j I_j$. Then $A \subseteq I_j$ for some j.

Proof (of lemma).
The case $n = 1$ is clear. For $n > 1$, if $A \subseteq \tilde{I}_k \coloneqq I_1 \cup I_2 \cup \cdots \widehat{I_k} \cup \cdots \cup I_n$ then the result holds by the IH. So suppose $A \not\subseteq \tilde{I}_k$ and pick some $a_k \notin \tilde{I}_k$. Since $A \subseteq \bigcup I_j$, we must have $a_k \in I_k$.
Case 1: $n = 2$. If $a_1 + a_2 \in A$ with $a_1 \in I_1 \setminus I_2$ and $a_2 \in I_2 \setminus I_1$, then $a_1 + a_2 \notin I_1 \cup I_2$ – otherwise $a_1 + a_2 \in I_1 \implies a_2 \in I_1$, and similarly if $a_1 + a_2 \in I_2$. So $A \subseteq I_1 \cup I_2$.
Case 2: $n > 2$. At least one I_j is prime, without loss of generality I_1. However, $a_1 + a_2 a_3 \cdots a_n \in A \setminus \bigcup_{j \geq 1} I_j$. Since $a_j \in I_j$, we have $a_2 \cdots a_n \in I_j$, contradicting $a_1 \notin I_j$ for $j \neq 1$. ■

Proposition 12.1.4 ***(?).***
Let $S \leq (R, \cdot)$ be a submonoid and $P \in \operatorname{Id}(R)$ proper with $P \cap S = \emptyset$ and P is maximal with

respect to this property, so if $P' \supseteq P$ and $P' \cap S = \emptyset$ then $P' = P$. Then $P \in \operatorname{Spec} R$ is prime.

Proof (?).
By contrapositive, we'll show $a, b \notin P \implies ab \notin P$. If $a, b \notin P$, then $P \subsetneq aR + P, bR + P$ is a proper subset. By maximality, $(aR + P) \cap S \neq \emptyset$ and $(bR + P) \cap S \neq \emptyset$. Pick $s_1, s_2 \in S$ with $s_1 = x_1 a + p_1, s_2 = x_2 b + p_2$. Then $s_1 s_2 \in S$ and thus

$$s_1 s_2 = x_1 x_2 ab + x_1 a p_2 + x_2 b p_1 + p_1 p_2 \in x_1 x_2 ab + P + P + P,$$

hence $ab \notin P$ – otherwise $S \cap P \neq \emptyset$. ↯

■

Proposition 12.1.5 ***(?).***
Let $S \leq R$ be a monoid and let $I \in \operatorname{Id}(R)$ with $I \cap S = \emptyset$. Then there exists some $p \in \operatorname{Spec} R$ such that

- $I \subseteq p$
- $p \cap S = \emptyset$

Proof (?).
Set $B = \left\{ I' \supseteq I \;\middle|\; I' \cap S = \emptyset \right\}$, then $B \neq \emptyset$. Apply Zorn's lemma to get a maximal element p, which is prime by the previous proposition.

■

Theorem 12.1.6 ***(Krull).***

$$\sqrt{0_R} = \cap_{p \in \operatorname{Spec} R} p.$$

Exercise 12.1.7 (?)
Prove this!

12.2 Localization

Remark 12.2.1: Recall the definition of $\mathbb{Q}$ as $\mathbb{Z}[\frac{1}{S}]$ where $S = \mathbb{Z} \setminus \{0\}$ using the arithmetic of fractions. More generally, for D an integral domain, there is a field of fractions F with $D \hookrightarrow F$ satisfying a universal property and thus uniqueness. Recall the definition of localization and the universal property: if $\eta : R \to R'$ with $\eta(S) \subseteq (R')^\times$ then $\exists \tilde{\eta} : R\left[s^{-1}\right] \to R'$.

Remark 12.2.2: Next time:

- Existence of $R\left[S^{-1}\right]$
- Localization for R-Mod.
- Localization using tensor products.

13 | Tuesday, March 01

Remark 13.0.1: Recall the definition of the localization of an $R \in \mathsf{CRing}^{\text{unital}}$ at a submonoid $S \leq (M, \cdot)$, written $R\left[S^{-1}\right]$. Similarly for $M \in \mathsf{R\text{-}Mod}$, one can form $M\left[S^{-1}\right]$, and $(-)\left[S^{-1}\right]$ is a functor where the induced map on $M \xrightarrow{f} N$ is $f_S(m/s) \coloneqq f(m)/s$.

Proposition 13.0.2 *(?).*
For $I \in \operatorname{Id}(R)$, let $j(I) \coloneqq \left\{a \in R \;\middle|\; a/s \in I \text{ for some } s \in S\right\}$ which is again an ideal in R. Then

1. $j(I)_S = I$,
2. $I_S = R_S \iff I$ contains an element of S.

Proof (of 2).
$\impliedby$: $I_S \subseteq R_S$ is clear. Let $x/t \in R_S$ and $s \in I \cap S$, then $\dfrac{sx}{st} = \dfrac{x}{t} \in I_S$.
$\implies$: Write $1 = i/s$ to produce $t \in s$ with $t(s-i) = 0$. Then $z = ts \in S$ and $z = it \in I$ so $z \in I \cap S$.

■

Proposition 13.0.3 *(?).*
Let $P \in \operatorname{Spec} R$ with $S \cap p = \emptyset$, then $j(P_S) = P$.

Proof (?).
$\supseteq$: Clear.
$\subseteq$: Let $a \in j(P_S)$, so $a/s = p/t$ for $s, t \in S, p \in P$ and $\exists u \in S$ such that $u(at - sp) = 0 \in P$, so $uat - usp \in P$ where $usp \in P$. Thus $uat \in P \implies a(ut) \in P \implies a \in P$, since $ut \in S$ and $ut \notin P$.

■

Proposition 13.0.4 *(?).*
There is an order-preserving correspondence

$$\left\{p \in \operatorname{Spec} R \;\middle|\; p \cap S = \emptyset\right\} \rightleftharpoons \operatorname{Spec} R\left[S^{-1}\right]$$
$$P \mapsto P\left[S^{-1}\right]$$
$$j(P') \mapsfrom P'.$$

Proof (?).
We need to show

1. $P[S^{-1}] \in \operatorname{Spec} R[S^{-1}]$ is actually prime.
2. If $P' \in \operatorname{Spec} R[S^{-1}]$ then $j(P') \in \operatorname{Spec} R$ with $j(P') \cap S = \emptyset$.

For one:

$$\begin{aligned}\frac{x}{t}, \frac{y}{t} \in P_S &\implies \frac{xy}{st} \in P_S \\ &\implies xy \in j(P_S) = P \\ &\implies x \in P \text{ or } y \in P \\ &\implies x/s \in P \text{ or } y/s \in P.\end{aligned}$$

For two:

$$\begin{aligned}xy \in j(P') &\implies \frac{xy}{s} \in P' \\ &\implies \frac{x}{1}\frac{y}{s} \in P' \\ &\implies \frac{x}{1} \in P' \text{ or } \frac{y}{s} \in P' \\ &\implies x \in P' \text{ or } y \in P'\end{aligned}$$

.

If $x \in j(P') \cap S$ then $\frac{x}{t} \in P'$ so $\frac{t}{x}\frac{x}{t} \in P'$. ↯
One can then check that these two maps compose to the identity. ■

Exercise 13.0.5 (?)
Show that if $p \in \operatorname{Spec} R$ then $R_p \in \mathsf{LocRing}$ is local. Use that the image of p in R_p is $P_p = R_p \setminus R_p^\times$, making it maximal and unique.

Exercise 13.0.6 (?)
Show that

1. $M = 0 \iff M_S = 0$ for all S,
2. $M = 0 \iff M_p = 0 \,\forall p \in \operatorname{mSpec} R$,
3. $M = 0 \iff M_p = 0 \,\forall p \in \operatorname{Spec} R$, noting that this is a stronger condition than maximal.

For (2), use that $\operatorname{Ann}_R(x)$ is a proper ideal and thus contained in a maximal, and show by contradiction that $x/1 \neq 0 \in M_p$.

Exercise 13.0.7 (?)
Show that if $f \in \mathsf{R\text{-}Mod}(M, N)$ then

- f injective (resp. surjective) $\implies$ f_S injective (resp. surjective)
- If f_p is injective for all $p \in \operatorname{Spec} R$, then f is injective (resp. surjective)
- If M is flat then M_S is flat
- If M_p is flat for all p then M is flat.

Remark 13.0.8: Recall that for $A \subseteq R$, $V(A) := \left\{ p \in \operatorname{Spec} R \;\middle|\; p \supseteq A \right\}$. Letting $I(A)$ be the ideal generated by A, then check that $V(I(A)) = V(A)$ and $V(I) = V(\sqrt{I})$.

Exercise 13.0.9 (?)
Check that defining closed sets as $\left\{ V(A) \;\middle|\; A \subseteq R \right\}$ forms the basis for a topology on $\operatorname{Spec} R$, and $V(p) \cap V(q) = V(pq)$.

Remark 13.0.10: Next time: generic points, idempotents, irreducible sets.

14 | Tuesday, March 15

See https://www.math.ucla.edu/~balmer/Pubfile/TTG.pdf

Remark 14.0.1: Recall that $V(B) := \left\{ p \in \operatorname{Spec} R \;\middle|\; p \supseteq B \right\}$ are the closed sets for the Zariski topology, and $V(B) = V(\langle B \rangle)$. Write $I(A) = \bigcap_{p \in A} p$ for the vanishing ideal of A, and note $V(I(A)) = \operatorname{cl}_{\operatorname{Spec} R} A$. Recall $\sqrt{J} = \bigcap_{p \supseteq J} = \left\{ x \in R \;\middle|\; \exists n \text{ such that } x^n \in J \right\}$, so $\sqrt{0}$ is the nilradical, i.e. all nilpotent elements. An ideal J is radical iff $\sqrt{J} = J$.

Theorem 14.0.2 *(?)*.
For $X = \operatorname{Spec} R$, $I(V(J)) = \sqrt{J}$, and there is a bijection between closed subsets of X and radical ideals in R.

Proof (?).

$$I(V(J)) = \bigcap_{p \in V(J)} p = \bigcap_{p \supseteq J} p = \sqrt{J},$$

and

$$J \xrightarrow{V} V(J) \xrightarrow{I} I(V(J)) = \sqrt{J} = J.$$

■

Remark 14.0.3: Recall that X is **reducible** iff $X = X_1 \cup X_2$ with X_i nonempty proper and closed.

Theorem 14.0.4 *(?).*
For $R \in \mathsf{CRing}$, a closed subset $A \subseteq X$ is irreducible iff $I(A)$ is a prime ideal.

Proof (?).
$\implies$: Suppose A is irreducible, let $fg \in I(A) = \bigcap_{p \in A} p$. Then $fg \in p \implies f \in [$ without loss of generality for all $p \in A$, and $A = (A \cap V(f)) \cup (A \cap V(g))$ so $A \subseteq V(f)$ or $A \subseteq V(g)$. Thus $f \in \sqrt{\langle f \rangle} = I(V(f)) \subseteq I(A)$ (similarly for g).
$\impliedby$: Suppose $I(A)$ is a prime ideal and $A = A_1 \cup A_2$ with A_j closed, so $I(A) \subseteq I(A_j)$. Then

$$I(A) = I(A_1 \cup A_2) = I(A_1) \cap I(A_2).$$

If $I(A_j) \subsetneq I(A)$ are proper containments, then one reaches a contradiction: if $x \in I(A_1)$ and $y \in I(A_2)$, use that $xy \in I(A)$ to conclude $x \in I(A)$ or $y \in I(A)$.

■

Theorem 14.0.5 *(?).*
Let $X \in \mathsf{Top}$; TFAE:

1. X is irreducible.
2. Any two open nonempty sets intersect.
3. Any nonempty open is dense in X.

Proposition 14.0.6 *(?).*

1. Any irreducible subset of X is entirely contained in a single irreducible component.
2. Any space is a union of its irreducible components.

Remark 14.0.7:
- A space is Noetherian iff any descending chain of closed sets stabilizes, and if R is a Noetherian ring then $X = \operatorname{Spec} R$ is a Noetherian space. Note that the converse may not hold in general!
- A Noetherian space has a unique decomposition into irreducibles.
- Any irreducible component is the closure of a point.
- Any nonempty irreducible closed subset $A \subseteq \operatorname{Spec} R$ contains a unique generic point $p = I(A)$.

Remark 14.0.8: Coming up:

- Group cohomology, the Hopf algebra structure on kG
- Cohomology using minimal resolutions
- $R = H^0(G; k) = \operatorname{Ext}^0_{kG}(k, k)$ which is a Noetherian ring
- Use minimal resolutions to define $c_{kG}(M)$, the rate of growth of a minimal projective resolution of M (1977)

- Support varieties: $R := \mathrm{Ext}^i_{kG}(k,k) \curvearrowright \tilde{M} := \mathrm{Ext}^0_{kG}(M,M)$, let $J = \mathrm{Ann}_R(\tilde{M})$ and $V_G(M) = \mathrm{Spec}(R/J)$.
- An equality of numerical invariants: $c_{kG}(M) = \dim V_G(M)$.
- Paul Balmer's tensor triangular geometry.

15 | Tuesday, March 22

15.1 Hilbert-Serre

Remark 15.1.1: Setup: $V \in \mathsf{gr}_{\mathbb{Z}}\mathsf{k}\text{-}\mathsf{Mod}$ a graded vector space, so $V = \bigoplus_{r \geq 0} V_r$ with $\dim_k V_r < \infty$.

Define the **Poincare series**

$$p(V,t) = \sum_{r \geq 0} \dim V_r t^r.$$

Theorem 15.1.2 ***(Hilbert-Serre).***
Let $R \in \mathsf{gr}_{\mathbb{Z}}\mathsf{CRing}$ be of finite type over A_0 for $A \in k\text{-}\mathsf{Alg}$ and suppose R is finitely generated over A_0 by homogeneous elements of degrees $k_1, \cdots, k_s$. Supposing $V \in \mathsf{A}\text{-}\mathsf{Mod}^{\mathrm{fg}}$,

$$p(V,t) = \frac{f(t)}{\prod_{1 \leq j \leq s} 1 - t^{k_j}}, \qquad f(t) \in \mathbb{Z}[t].$$

Proposition 15.1.3 ***(?).***
Suppose that

$$p(V,t) = \frac{f(t)}{\prod_{1 \leq j \leq s} 1 - t^{k_j}} = \sum_{r \geq 0} a_r t^r, \qquad f(t) \in \mathbb{Z}[t], a_r \in \mathbb{Z}_{\geq 0}.$$

Let γ be the order of the pole of $p(t)$ at $t = 1$. Then

1. There exists $K > 0$ such that $a_n \leq K n^{\gamma - 1}$ for $n \geq 0$
2. There does *not* exist $k > 0$ such that $a_n \leq k n^{\gamma - 2}$.

Definition 15.1.4 (?)
Let V be a graded vector space of finite type over k. The **rate of growth** $\gamma(V)$ of V is the smallest γ such that $\dim V_n \leq C n^{\gamma - 1}$ for all $n \geq 0$ for some constant C.

Remark 15.1.5: Compare this to the complexity $C_G(M) = \gamma(P_0)$ where $P^0 \rightrightarrows M$ is a minimal projective resolution.

15.2 Finite Generation of Cohomology

Remark 15.2.1: Fix $G \in \mathsf{FinGrp}$. Recall that $H^\bullet(G;k)\mathrm{Ext}^\bullet{}_G(k,k)$ has an algebra structure given by concatenation of LESs:

$$\begin{tikzcd}
\xi_M : & 0 \arrow[r] & k \arrow[r] & M_1 \arrow[r] & \cdots \arrow[r] & M_n \arrow[r] & k \arrow[r] \arrow[dllll, dashed, "\xi_M \cdot \xi_N"'] & 0 & \in \mathrm{Ext}^n_G(k,k) \\
\xi_N : & 0 \arrow[r] & k \arrow[r] & N_1 \arrow[r] & \cdots \arrow[r] & N_m \arrow[r] & k \arrow[r] & 0 & \in \mathrm{Ext}^m_G(k,k)
\end{tikzcd}$$

Link to Diagram

Recall that $\mathrm{Ext}^n_G(k,k) = \operatorname*{Hom}_{kG}(P_n, k)$, providing the additive structure. Moreover, $\mathrm{Ext}_{kG}(M,M)$ is a ring, and if $N \in \mathsf{kG\text{-}Mod}$, then $\mathrm{Ext}_{kG}\, N, M \in \mathrm{Ext}_{\mathsf{kG}}(\mathsf{M},\mathsf{M})\text{-}\mathsf{Mod}$. Similarly $\mathrm{Ext}^0_{kG}(N,M) \in \mathrm{Ext}^\bullet(\mathsf{k},\mathsf{k})\text{-}\mathsf{Mod}$ by tensoring LESs.

Remark 15.2.2: There is a coproduct

$$\begin{aligned} kG &\xrightarrow{\Delta} kG \otimes_k kG \\ g &\mapsto g \otimes g. \end{aligned}$$

There is a cup product:

$$\begin{tikzcd}
\bigoplus_{s+t=m} \mathrm{Ext}^s_{kG}(k,N) \otimes_k \mathrm{Ext}^t_{kG}(k,M) \arrow[r, leftrightarrow, "\cong"] \arrow[dr, "(a,b)\mapsto a\smile b"'] & \mathrm{Ext}^m_{kG^{\otimes^2_k}}(k \otimes_k N, k \otimes_k M) \arrow[d] \\
& \mathrm{Ext}^m_{kG}(N,M)
\end{tikzcd}$$

Link to Diagram

It is a theorem that this coincides with the Yoneda product.

Theorem 15.2.3 *(?).*

- $H^0(G,k)$ is a graded commutative ring, so $xy = (-1)^{|x||y|}yx$
- The even part $H^{\bullet\text{even}}(G;k)$ is a (usual) commutative ring.

Theorem 15.2.4 ***(Evans-Venkov, 61).***

- $H^0(G;k)$ is a finitely generated in $\mathsf{Alg}_{/k}$
- If $M \in \mathsf{kG\text{-}Mod}$ then $\mathrm{Ext}^0_{kG}(k, M) \in \mathsf{H^\bullet(G;k)\text{-}Mod}$.

Remark 15.2.5: Quillen described $\operatorname{mSpec} H^\bullet(G,k)^{\mathrm{red}}$ in the 70s. Idea: look at $E \hookrightarrow G$ the elementary abelian subgroups, so $E \cong {C_p}^{\times^m}$ where $p = \operatorname{ch} k$, and consider $V_G(k) = \bigcup\limits_{E \leq G} V_E(k)/\sim$ the union of all elementary abelian subgroups, where $V_G(k) := \operatorname{mSpec} H^\bullet(G;k)^{\mathrm{red}}$. Note that in characteristic zero, this is semisimple and only $H^0 = k$ survives.

Example 15.2.6 ***(?):***

- For $A = C_p$ with $\operatorname{ch} k = p > 0$, then

$$R := H^0(C_p;k) \cong \begin{cases} k[x,y]/\left\langle y^2 \right\rangle, |x| = 2, |y| = 1 & p \geq 3 \\ k[x], |x| = 1 & p = 2. \end{cases}, \qquad \operatorname{mSpec} R \cong \mathbb{A}^1_{/k}.$$

- Dan's favorite: $A = u(\mathfrak{g})$ for $\mathfrak{g} = \mathfrak{sl}_2$ with $\operatorname{ch} k = p \geq 3$ for u the *small enveloping algebra*. Friedlander-Parshall show $\operatorname{mSpec} R = k[\mathcal{N}]$ for $\mathcal{N} := \left\{ M \begin{bmatrix} a & b \\ c & -a \end{bmatrix} \;\middle|\; M \text{ is nilpotent} \right\}$. This can be presented as

$$k[\mathcal{N}] \cong k[x,y,z]/\left\langle z^2 + xy \right\rangle, |x|, |y|, |z| = 2,$$

and we'll see how finite generation is used in this setting.

16 | Tuesday, March 29

Remark 16.0.1: Setup: for $G \in \mathsf{FinGrp}, k \in \mathsf{Field}$ with $\operatorname{ch} k = p \mid \sharp G$. For $M \in \mathsf{kG\text{-}Mod}$, we associate $V_G(M) \subseteq \operatorname{mSpec}(R)$ for $R := H^0(G;k)$. There is a ring morphism $\Phi_M : R \to \mathrm{Ext}^0_{kG}(M, M)$, we set $I_G(M) = \left\{ x \in R \;\middle|\; \Phi_M(x) = 0 \right\}$ and define the support variety as $V_G(M) = \operatorname{mSpec}(R/I_G(M))$.

Example 16.0.2 ***(?):*** Let $G = {C_p}^{\times^n}$, then

- $H^2(G;k) = k[x_1, \cdots, x_n]$ for $\operatorname{ch} k = p \geq 3$.
- $\operatorname{mSpec} R = \mathbb{A}^n \supseteq V_E(M)$

16.1 Rank Varieties

Definition 16.1.1 (Rank varieties)
For $kG = k[z_1, \cdots, z_n]/\langle z_1^p, \cdots, z_n^p \rangle$, let $x_{\mathbf{a}} := \sum a_i z_i$ for $a_i \in k$. Define the **rank variety**

$$V_E^r(M) = \left\{ \mathbf{a} \;\middle|\; \operatorname{Res}_{\langle x_{\mathbf{a}} \rangle}^{kG} \text{ is not free} \right\} \cup \{0\}.$$

Theorem 16.1.2 *(Carlson).*

$$V_E(M) \cong V_E^r(M).$$

Remark 16.1.3: Note that $\operatorname{Ext}^0(M, M) \curvearrowright \operatorname{Ext}^0(M', M)$ by splicing, so we can define $I_G(M', M) := \operatorname{Ann}_R \operatorname{Ext}_{kG}^1(M', M)$ and the **relative support** variety $V_G(M', M) = \operatorname{mSpec}(R/I_G(M', M))$. This recovers the previous notion by $V_G(M, M) = V_G(M)$.

Remark 16.1.4: Since $I_G(M', M) \supseteq I_G(M) + I_G(M')$,

$$V_G(M', M) \subseteq V_G(M) \cap V_G(M'),$$

which relates relative support varieties to the usual support varieties.

Remark 16.1.5: If $0 \to A \to B \to C \to 0$ is a SES, there is a LES in Ext_{kG} and by considering annihilators we have

$$I_G(A, M) \cdot I_G(B, M) \subseteq I_G(C, M) \implies V_G(C, M) \subseteq V_G(A, M) \cup V_G(C, M).$$

Proposition 16.1.6 *(?).*
Let $M \in$ kG-Mod, then

$$V_G(M) \subseteq \bigcup_{S \leq M \text{ simple}} V_G(S, M).$$

Proof (?).
Take the SES $0 \to S_1 \to M \to M/S_1 \to 0$, then $V_G(M) = V_G(M, M) \subseteq V_G(S_1, M) \cup V_G(M/S_1, M)$. Continuing this way yields a union of $V(T, M)$ over all composition factors T. Conversely, by the intersection formula above, this union is contained in $V_G(M)$, so these are all equal. ∎

Theorem 16.1.7 *(?)*.
Let $M \in$ kG-Mod, then

1. $c_G(M) = \dim V_G(M)$
2. $V_G(M) = \{0\}$ (as a conical varieties) iff M is projective.

Proof (?).
Note (2) follows from (1), since complexity zero modules are precisely projectives. Consider $\Phi_M : R \to \mathrm{Ext}^0_{kG}(M, M)$, which induces $R/I_G(M) \hookrightarrow \mathrm{Ext}^0_{kG}(M, M)$ which is finitely generated over $R/I_G(M)$. A computation shows

$$\begin{aligned} c_G(M) &= \gamma(\mathrm{Ext}^0_{kG}(M, M)) \\ &= \gamma(R/I_G(M)) \\ &= \mathrm{krulldim}(R/I_G(M)) \\ &= \dim V_G(M). \end{aligned}$$

■

Remark 16.1.8: Consider a LES $0 \to M \to M_1 \to \cdots \to M_n \to M \to 0 \in \mathrm{Ext}^n_{kG}(M, M)$. Apply $\Omega^n(-)$, which arises from projective covers $P^\bullet \rightrightarrows M$ and truncating to get $0 \to \Omega^n \to P^{n-1} \to \cdots \to P_0 \to M \to 0$. Similarly define Ω^{-n} in terms of injective resolutions. There is an isomorphism $\mathrm{Ext}^n_{kG}(M, M) \cong \mathrm{Ext}^n_{kG}(\Omega^s M, \Omega^s M)$ which is compatible with the R action. Thus $V_G(M) \cong V_G(\Omega^s M)$ for any s. Since kG is a Hopf algebra, dualizing yields $\mathrm{Ext}^n_{kG}(M, M) \cong \mathrm{Ext}^n_{kG}(M^\vee, M^\vee)$ and thus $V_G(M) \cong V_G(M^\vee)$.

16.2 Properties of support varieties

Proposition 16.2.1 *(?)*.

$$V_G(M_1 \bigoplus M_2) \cong V_G(M_1) \cup V_G(M_2).$$

Proof (?).
Distribute:

$$\mathrm{Ext}^0_{kG}(M_1 \oplus M_2, M_1 \oplus M_2) \cong \mathrm{Ext}^0_{kG}(M_1, M_1) \oplus \mathrm{Ext}^0_{kG}(M_1, M_2) \oplus \mathrm{Ext}^0_{kG}(M_2, M_1) \oplus \mathrm{Ext}^0_{kG}(M_3, M_2).$$

Now $I_G(M_1 \bigoplus M_2) \subseteq I_G(M_1) \oplus I_G(M_2)$, so $V_G(M_1) \cup V_G(M_2) \subseteq V_G(M_1 \oplus M_2)$. Applying the 2 out of 3 property, $V_G(M_1 \oplus M_2) \subseteq V_G(M_1) \cup V_G(M_2)$ since there is a SES $0 \to M_1 \to M_1 \oplus M_2 \to M_2 \to 0$.

■

Theorem 16.2.2 *(Tensor product property).*
Let $M, N \in \mathsf{kG\text{-}Mod}$, then

$$V_G(M \otimes_k N) = V_G(M) \cap V_G(N).$$

Remark 16.2.3: Conjectured by Carlson, proved by Arvrunin-Scott (82). Prove for elementary abelians, piece together using the Quillen stratification.

Theorem 16.2.4 *(Carlson).*
Let $X = \operatorname{mSpec} R$, which is a conical variety, and let $W \subseteq X$ be a closed conical subvariety (e.g. a line through the origin). Then there exists an $M \in \mathsf{kG\text{-}Mod}$ such that $V_G(M) = W$.

Remark 16.2.5: Take $\zeta : \Omega^n k \to k$, so $\zeta \in R/I_G(M)$, and define certain L_ζ modules and set $Z(\zeta) \coloneqq V_G(L_\zeta)$.

Theorem 16.2.6 *(Carlson).*
Let $M \in \mathsf{kG\text{-}Mod}$ be indecomposable. Then the projectivization $\operatorname{Proj} V_G(M)$ is connected.

16.3 Supports using primes

Remark 16.3.1: As before, set $R = H^{\text{even}}(G; k), X = \operatorname{Spec} R$, and now define

$$V_G(M) = \left\{ p \in X \;\middle|\; \operatorname{Ext}^0_{kG}(M, M)_p \neq 0 \right\}.$$

All of the theorems mentioned today go through with this new definition.

Exercise 16.3.2 (?)
Let $I_G(M) = \operatorname{Ann}_R \operatorname{Ext}^0_{kG}(M, M) \trianglelefteq R$, and show

$$V_G(M) = \left\{ p \in X \;\middle|\; p \supseteq I_G(M) \right\} = V(I_G(M))$$

is a closed set.

Remark 16.3.3: Let $\mathfrak{g} \in \mathsf{LieAlg}_{/k}$ with $\operatorname{ch} k = p > 0$, e.g. $\mathfrak{g} = \mathfrak{gl}_n(k)$. Then there is a pth power operation $x^{[p]} = x \cdot x \cdots x$. The pair $(\mathfrak{g}, [p])$ forms a restricted Lie algebra. Consider the enveloping algebra $U(\mathfrak{g})$, and define

$$u(\mathfrak{g}) \coloneqq U(\mathfrak{g}) / \left\langle x^p - x^{\otimes_k^p} \;\middle|\; x \in \mathfrak{g} \right\rangle,$$

which is a finite-dimensional Hopf algebra:

- The counit is $\varepsilon(g) = 0$ for $g \in \mathfrak{g}$

- The antipode is $\theta(g) = -g$
- The comultiplication is $\Delta(g) = g \otimes 1 + 1 \otimes g$.

The dimension is given by $\dim u(\mathfrak{g}) = p^{\dim \mathfrak{g}}$.

17 | Tuesday, April 05

17.1 Lie Theory

Remark 17.1.1: Setup: $k = \bar{k}$, $\operatorname{ch} k = p > 0$, $\mathfrak{g}$ a restricted Lie algebra (e.g. $\mathfrak{g} = \mathsf{Lie}(G)$ for $G \in \mathsf{AffAlgGrp}_{/k}$). Write $A^{[p]} = AA \cdots A$ and set $A = u(\mathfrak{g}) = U(\mathfrak{g})/J$ where $J = \left\langle x^{\otimes_k^p} - x^{[p]} \right\rangle$ which is an ideal generated by central elements. Note that A is a finite-dimensional Hopf algebra.

Proved last time: $H^0(A; k) \in \mathsf{Alg}_{/k}{}^{\text{fg}}$, using a spectral sequence argument. From the spectral sequence, there is a finite morphism

$$\Phi : S(\mathfrak{g}^+)^{(1)} \to H^0(A; k),$$

making $H^0(A; k)$ an integral extension of $\operatorname{im} \Phi$. This induces a map

$$\Phi : \operatorname{mSpec} H^0(A; k) \hookrightarrow \mathfrak{g}.$$

Theorem 17.1.2 ***(Jantzen).***

$$\operatorname{mSpec} H^0(A; k) \cong \mathcal{N}_p := \left\{ x \in \mathfrak{g} \;\middle|\; x^{[p]} \right\}.$$

Example 17.1.3 ***(?):*** For $\mathfrak{g} = \mathfrak{gl}_n$, $\mathcal{N}_p \leq \mathcal{N}$ is a subvariety of the nilpotent cone. Moreover $\mathcal{N}_p$ is stable under $G = \mathrm{GL}_n$, and there are only finitely many orbits. There is a decomposition into finitely many irreducible orbit closures

$$\mathcal{N}_p = \bigcup_i \overline{G x_i}.$$

This corresponds to Jordan decompositions with blocks of size at most p.

Remark 17.1.4: Using spectral sequences one can show that if $M, N \in \mathsf{A\text{-}Mod}$ then $\operatorname{Ext}_A^0(M, N)$ is finitely-generated as a module over $R := H^0(A; k)$. So one can define support varieties $V_{\mathfrak{g}}(M) = \operatorname{mSpec} R/J_M$ where $I_M = \operatorname{Ann}_R \operatorname{Ext}_A^0(M, M)$. Some facts:

- $V_{\mathfrak{g}}(M) \subseteq \mathcal{N}_p \subseteq \mathfrak{g}$
- If M is a G-module in addition to being a $\mathfrak{g}$-module, then $V_G(M)$ is a G-stable closed subvariety of $\mathcal{N}_p$.

Theorem 17.1.5 ***(Friedlander-Parshall (Inventiones 86)).***
Given $M \in \mathsf{u}(\mathfrak{g})\text{-}\mathsf{Mod}$,

$$V_{\mathfrak{g}}(M) \cong \left\{ x \in \mathfrak{g} \;\middle|\; x^{[p]} = 0, M \downarrow_{U(\langle x \rangle)} \text{ is not free over } u(\langle x \rangle) \leq u(\mathfrak{g}) \right\} \cup \{0\},$$

which is similar to the rank variety for finite groups, concretely realize the support variety.

Remark 17.1.6: Here $\langle x \rangle = kx$ is a 1-dimensional Lie algebra, and if $x^{[p]} = 0$ then $u(\langle x \rangle) = k[x]/\langle x^p \rangle$ is a PID. We know how to classify modules over a PID: there are only finitely many indecomposable such modules.

17.2 Reductive algebraic groups

Example 17.2.1 ***(?):*** For type $A_n \sim \mathrm{GL}_{n+1}$, $\alpha_0 = \tilde{\alpha}_n = \sum_{1 \leq i \leq n} \alpha_i$ and $h = n + 1$. For G_2, $\tilde{\alpha}_n = 3\alpha_1 + 2\alpha_2$ and $h = 6$.

Fact 17.2.2
If $p \geq h$ then $\mathcal{N}_p(\mathfrak{g}) = \mathcal{N}$.

Definition 17.2.3 (Good and bad primes)
A prime is *bad* if it divides any coefficient of the highest weight. By type:

Type	Bad primes
A_n	None
B_n	2
C_n	2
D_n	2
E_6	2,3
E_7	2,3
E_8	2,3,5
F_4	2,3
G_2	2,3

Theorem 17.2.4 ***(Carlson-Lin-Nakano-Parshall (good primes), UGA VIGRE (bad primes)).***
$\mathcal{N}_p = \overline{\mathcal{O}}$ is an orbit closure, where $\mathcal{O}$ is a G-orbit in $\mathcal{N}$. Hence $\mathcal{N}_p(\mathfrak{g})$ is an irreducible variety.

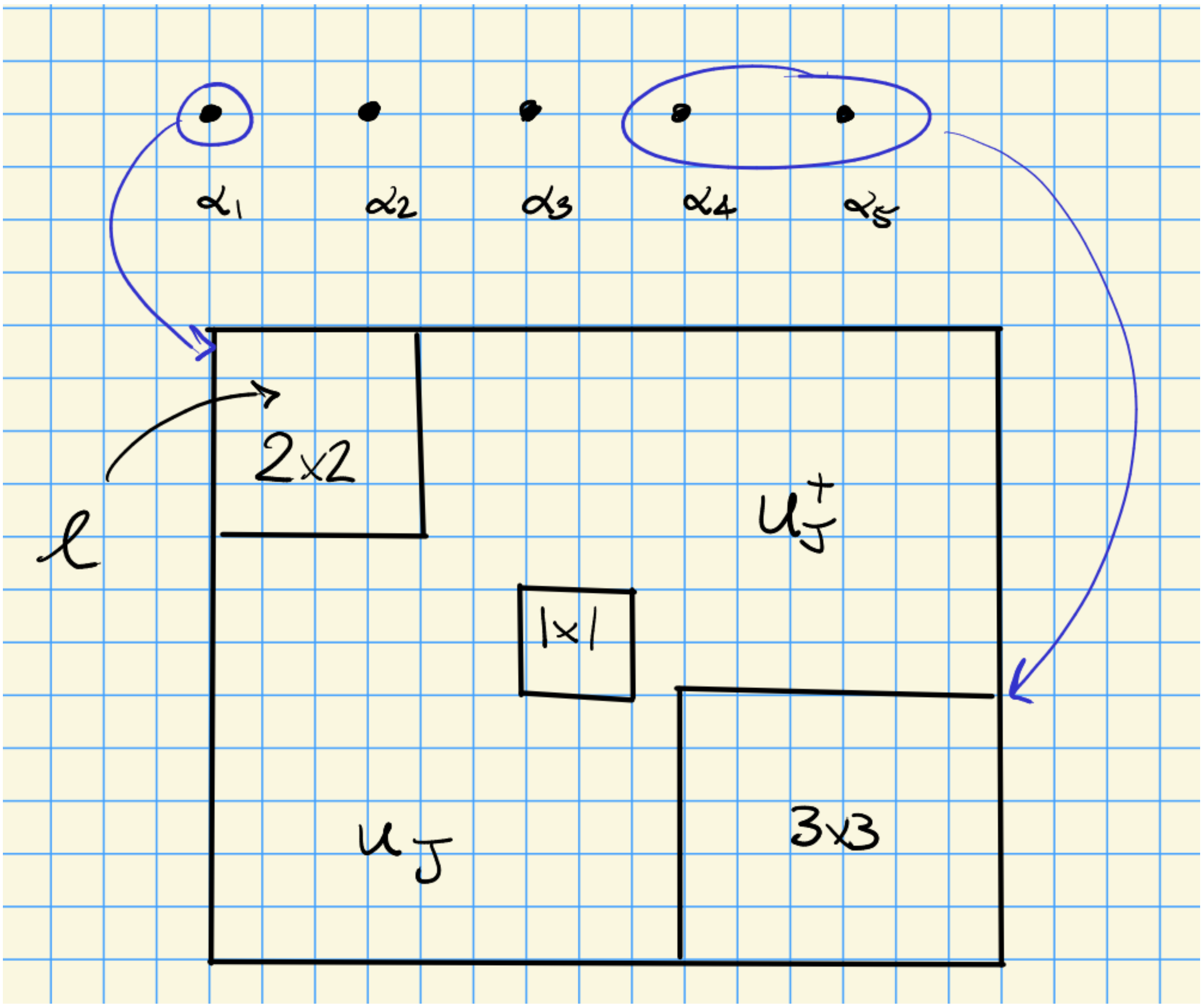
Remark 17.2.5: Let $X = X(T)$ be the weight lattice and let $\lambda \in X$, then

$$\Phi_\lambda := \left\{ \alpha \in \Phi \;\middle|\; \langle \lambda + \rho,\ \alpha^\vee \rangle \in p\mathbb{Z} \right\}.$$

Under the action of the affine Weyl group, this is empty when λ is on a wall (non-regular) and otherwise contains some roots for regular weights. When p is a good prime, there exists a $w \in W$ with $w(\Phi_\lambda) = \Phi_J$ for $J \subseteq \Delta$ a subsystem of simple roots. In this case, there is a **Levi decomposition**

$$\mathfrak{g} = u_J \oplus \ell_J \oplus u_J^+.$$

Remark 17.2.6: On Levis: consider type $A_5 \sim \mathrm{GL}_6$ with simple roots α_i.



Remark 17.2.7: Consider induced/costandard modules $H^0(\lambda) = \mathrm{Ind}_B^G \lambda = \nabla(\lambda)$, which are nonzero only when $\lambda \in X_+$ is a dominant weight. Their characters are given by Weyl's character formula, and their duals are essentially *Weyl modules* which admit Weyl filtrations. What are their support varieties?

Theorem 17.2.8 ***(Nakano-Parshall-Vella, 2008).***
Let $\lambda \in X_+$ and let p be a good prime, and let $w \in W$ such that $w(\Phi_\lambda) = \Phi_J$ for $J \subseteq \Delta$. Then

$$V_{\mathfrak{g}} H^0(\lambda) = G \cdot u_J = \overline{\mathcal{O}}$$

is the closure of a "Richardson orbit".

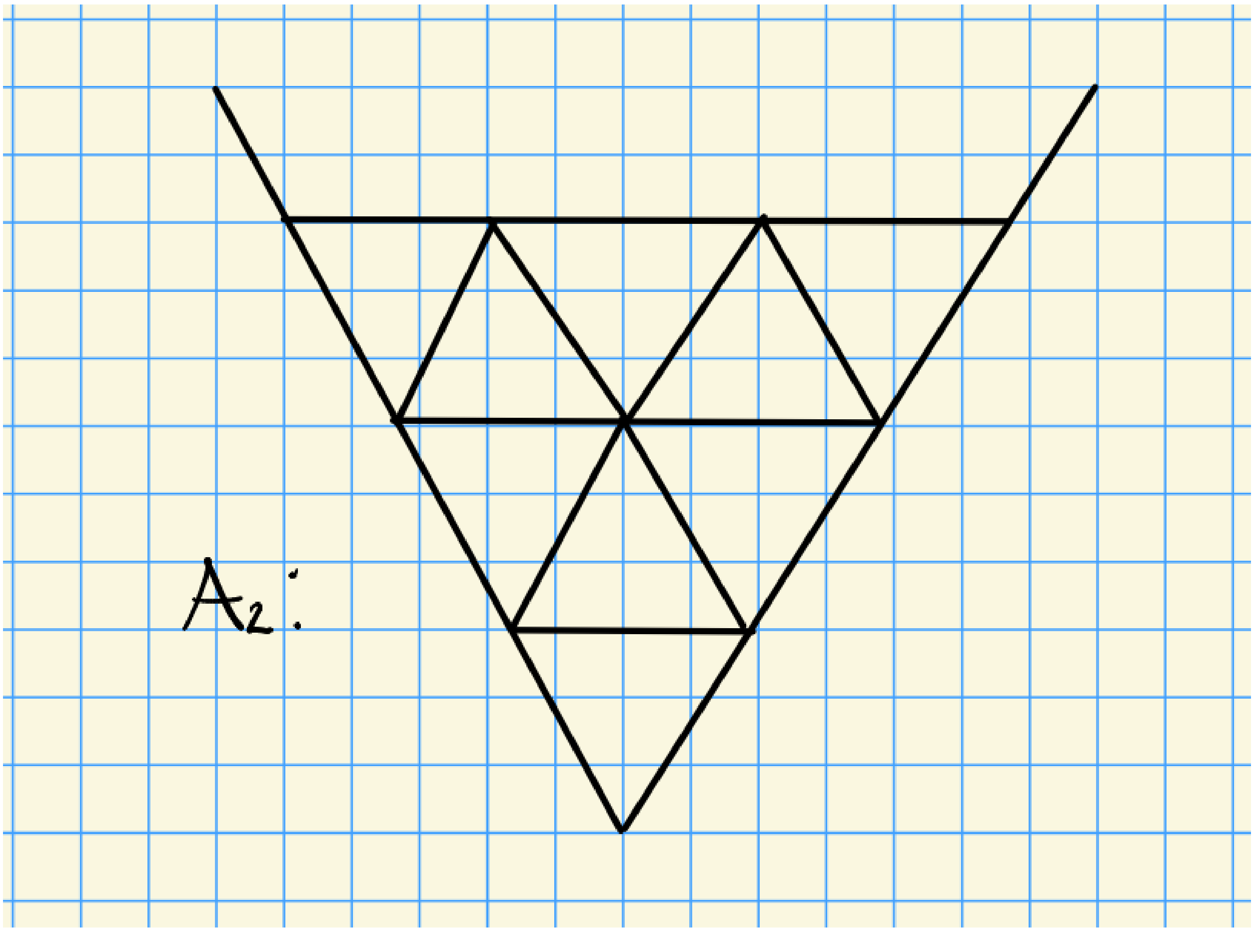
Remark 17.2.9:

- This theorem was conjectured by Jantzen in 87, proved for type A.
- For bad primes, $H^0(\lambda)$ is computed in one of seven VIGRE papers (2007). These still yield orbit closures that are irreducible, but need not be Richardson orbits.

Natural progression: what about tilting modules (good filtrations with costandard sections and good + Weyl filtrations)? We're aiming for the Humphreys conjecture.

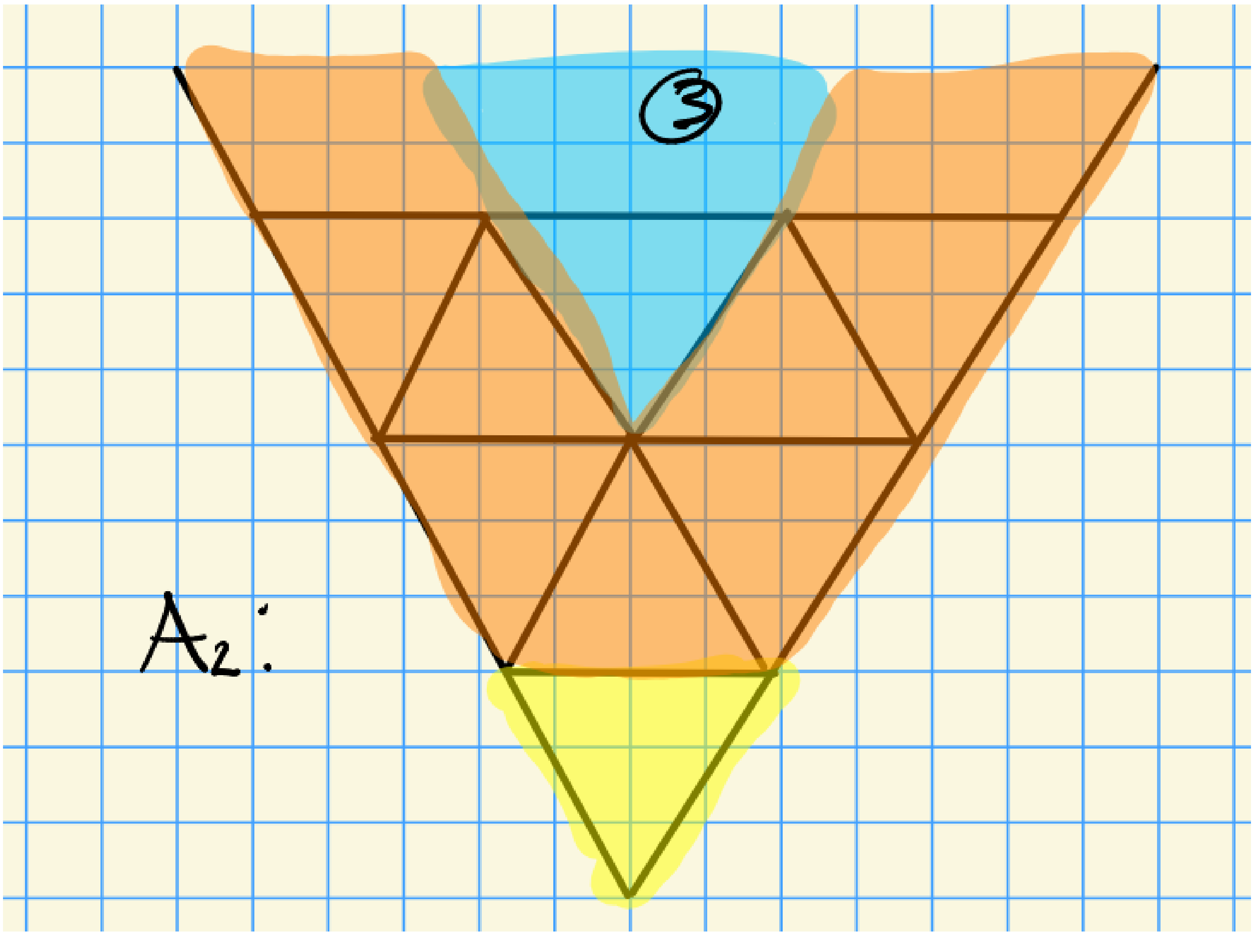
Remark 17.2.10: Let $T(\lambda)$ be a tilting module for $\lambda \in X_+$. A conjecture of Humphreys: $V_{\mathfrak{g}} T(\lambda)$ arises from considering 2-sided cells of the affine Weyl group, which biject with nilpotent orbits.

Example 17.2.11 ***(?):*** In type A_2:



There are three nilpotent orbits corresponding to Jordan blocks of type $X\alpha_1 : (1,0)$ and $X_{\text{reg}} : (1,1)$ in $\mathfrak{gl}_3$. Three cases:

- $V_{\mathfrak{g}}T(\lambda) = \mathcal{N} = \overline{GX_{\text{reg}}}$
- $V_{\mathfrak{g}}T(\lambda) = \overline{GX_{\alpha_1}}$
- $V_{\mathfrak{g}}T(\lambda) = \{0\}$



Remark 17.2.12: The computation of $V_G T(\lambda)$ is still open. Some recent work:

- $p = 2, A_n$: done by B. Cooper,
- $p > n+1, A_n$ by W. Hardesty,
- $p \gg 1$, Achar, Hardesty, Riche.

Remark 17.2.13: What about simple G-modules? Recall $L(\lambda) = \operatorname{Soc}_G \nabla(\lambda) \subseteq \nabla(\lambda)$ – computing $V_G L(\lambda)$ is open.

Theorem 17.2.14 ***(Drupieski-N-Parshall).***

Let $p > h$ and $w(\Phi_\lambda) = \Phi_J$, then

$$V_{u_q(\mathfrak{g})} L(\lambda) = G u_J,$$

i.e. the support varieties in the quantum case are known. This uses that the Lusztig character formula is know for $u_q(\mathfrak{g})$.

18 | Tuesday, April 12

18.1 Tensor triangular geometry

Remark 18.1.1: Last time: tensor categories and triangulated categories. Idea due to Balmer: treat categories like rings.

Definition 18.1.2 (Tensor triangulated categories)
A **tensor triangulated category** (TTC) is a triple $(K, \otimes, 1)$ where

- K is a triangulated category
- $(K, \otimes)$ is a symmetric monoidal category
- 1 is a unit, so $X \otimes 1 \xrightarrow{\sim} X \xrightarrow{\sim} 1 \otimes X$ for all X in K.

Remark 18.1.3: We'll have notions of ideals, thick ideals, and prime ideals in K. Define $\operatorname{Spc} K$ to be the set of prime ideals with the following topology: for a collection $C \subseteq \operatorname{Spec} K$, define $Z(C) = \left\{ p \in \operatorname{Spc} K \;\middle|\; C \cap p = \emptyset \right\}$. Note that there is a universal categorical construction of $\operatorname{Spc} K$ which we won't discuss here.

Remark 18.1.4: TTC philosophy: let K be a compactly generated TTC with a generating set K^c. Note that K can include "infinitely generated" objects, while K^c should thought of as "finite-dimensional" objects. Problems:

- What is the homeomorphism type of $\operatorname{Spc} K^c$?
- What are the thick ideals in K^c?

Although not all objects can be classified, there is a classification of thick tensor ideals. Idea: use the algebraic topology philosophy of passing to infinitely generated objects to simplify classification.

Remark 18.1.5: We'll need a candidate space $X \cong_{\mathsf{Top}} \operatorname{Spc}(K^c)$, e.g. a Zariski space: Noetherian, and every irreducible contains a generic point. We'll also need an assignment $V : K^c \rightsquigarrow X_{\mathrm{cl}}$ (the closed sets of X) satisfying certain properties, which is called a *support datum.* For I a thick tensor ideal, define

$$\Gamma(I) := \bigcup_{M \in I} V(M) \in X_{\mathrm{sp}},$$

a union of closed sets which is called *specialization closed*. Conversely, for W a specialized closed set, define a thick tensor ideal

$$\Theta(W) := \left\{ M \in K^c \;\middle|\; V(M) \subseteq W \right\}.$$

One can check that a tensor product property holds: if $M \in K^c$ and $N \in \Theta(W)$, check $V(M \otimes N) = V(M) \cap V(N) \subseteq W$. Under suitable conditions, a deep result is that $\Gamma \circ \Theta = \mathrm{id}$ and $\Theta \circ \Gamma = \mathrm{id}$. This yields a bijection

$$\begin{aligned} \{\text{Thick tensor ideals of } K^c\} &\rightleftharpoons \{\text{Specialization closed sets of } X\} \\ I &\mapsto \Gamma(I) \\ \Theta(W) &\leftarrow\!\shortmid W \end{aligned}$$

Remark 18.1.6: Define

$$\begin{aligned} f: X &\to \operatorname{Spc} K^c \\ x &\mapsto P_x := \left\{ M \in K^c \;\middle|\; x \not\in V(M) \right\}. \end{aligned}$$

This is a prime ideal: if $M \otimes N \in P_x$, then $x \notin V(M \otimes N) = V(M) \cap V(N)$, so $M \in P_x$ or $N \in P_x$.

18.2 Zariski spaces

Definition 18.2.1 (Zariski spaces)
A space $X \in \mathsf{Top}$ is a **Zariski space** iff

1. X is a Noetherian space, and
2. Every irreducible closed set has a unique generic point.

Note that since X is Noetherian, it admits a decomposition into irreducible components $X = \bigcup_{1 \leq i \leq t} W_i$.

Example 18.2.2 *(?)*: The basic examples:

- For R a unital Noetherian commutative ring, $X = \operatorname{Spec} R$ is Zariski.
- For R a graded unital Noetherian ring, taking homogeneous prime ideals $\operatorname{Proj} R$.

- For $G \in \mathsf{AffAlgGrp}$ with $G \curvearrowright R$ a graded ring by automorphisms (permuting the graded pieces), the stack $X := \underset{G}{\operatorname{Proj}}(R)$ (which is not Proj of the fixed points) is the set of G-invariant homogeneous prime ideals. There's a map $\rho : \operatorname{Proj} R \to \underset{G}{\operatorname{Proj}} R$ where $P \mapsto \cap_{g \in G} gP$ which gives $\underset{G}{\operatorname{Proj}} R$ the quotient topology: $W \in \underset{G}{\operatorname{Proj}} R$ is closed iff $\rho \in R$ is close din $\operatorname{Proj} R$. This topologizes orbit closures.

Remark 18.2.3: Notation:

- $\mathcal{X} = 2^X$ for the powerset of X,
- $\mathcal{X}_{\mathrm{cl}}$ the closed sets,
- $\mathcal{X}_{\mathrm{irr}}$ the irreducible closed sets,
- $\mathcal{X}_{\mathrm{sp}}$ the specialization-closed sets.

18.3 Support data

Remark 18.3.1: Recall

- $M = \mathsf{kG\text{-}Mod}$
- $R = H^{\mathrm{even}}(G; k)$
- $V_G(M) = \left\{ p \in \operatorname{Proj} R \;\middle|\; \operatorname{Ext}_{kG}(M, M)_p \neq 0 \right\}$.

Note that $V_G(P) = \emptyset$ for any projective and $V_G(k) = \emptyset$. In general, we'll similarly want $V_G(0) = \emptyset$ and $V_G(1) = X$.

Definition 18.3.2 (Support data)
A **support datum** is an assignment $V : K \to \mathcal{X}$ such that

1. $V(0) = \emptyset$ and $V(1) = X$.
2. $V\left(\bigoplus_{i \in I} M_i = \bigcup_{i \in I} V(M_i) \right)$
3. $V(\Sigma M) = V(M)$ (similar to Ω)
4. For any distinguished triangle $M \to N \to Q \to \Sigma M, V(N) \subseteq V(M) \cup V(Q)$.
5. $V(M \otimes N) = V(M) \cap V(N)$.

We'll need two more properties for the Balmer classification:

6. Faithfulness: $V(M) = \emptyset \iff M \cong 0$.
7. Realization: for any $W \in \mathcal{X}_{\mathrm{cl}}$ there exists a compact $M \in K^c$ with $V(M) = W$.

Remark 18.3.3: Note that (6) holds for group cohomology, and (7) is Carlson's realization theorem.

Lemma 18.3.4 *(?)***.**
Let K be a TTC which is closed under set-indexed coproducts and let $V : K \to \mathcal{X}$ be a support datum. Let C be a collection of objects in K and suppose $W \subseteq X$ with $V(M) \subseteq W$ for all $M \in C$. Then $V(M) \subseteq W$ for all M in $\mathsf{Loc}(C)$.

Proof (?).
Note that $\mathsf{Loc}(C)$ is closed under

- Applying Σ or Σ^{-1},
- 2-out-of-3: if two objects in a distinguished triangle are in $\mathsf{Loc}(C)$, the third is in $\mathsf{Loc}(C)$,
- Taking direct summands,
- Taking set-indexed coproducts.

These follow directly from the properties of support data and properties of $\mathsf{Loc}(C)$. ■

18.4 Extension of support data

Remark 18.4.1: Let X be a Zariski space and let $K \supseteq K^c$ be a compactly generated TTC. Let $V : K^c \to \mathcal{X}_{\mathrm{cl}}$ be a support data on compact objects, we then seek an *extension*: a support datum $\mathcal{V}$ on K forming a commutative diagram:

$$\begin{tikzcd}
K \arrow[r, "\mathcal{V}"] & \mathcal{X} \\
K^c \arrow[u, hook] \arrow[r, "V"] & \mathcal{X}_{\mathrm{cl}} \arrow[u, hook]
\end{tikzcd}$$

Link to Diagram

Definition 18.4.2 (?)
Let K be a compactly generated TTC and $V : K^c \to \mathcal{X}_{\mathrm{cl}}$ be a support datum. Then $\mathcal{V} : K \to \mathcal{X}$ **extends** V iff

- $\mathcal{V}$ satisfies properties (1) – (5) above,
- $V(M) = \mathcal{V}(M)$for any $M \in K^c$.
- If V is faithful then $\mathcal{V}$ is faithful.

Remark 18.4.3: We'll need Hopkins' theorem to analyze such extensions.

19 | Tuesday, April 19

19.1 Hopkins' Theorem

Remark 19.1.1: Let K be a compactly generated tensor triangulated category with K^c a subcategory of compact objects. Goal: classify $\operatorname{Spc}\mathsf{K}^c$. A candidate for its homeomorphism type: we'll build a Zariski space X and a homeomorphism $\operatorname{Spc}\mathsf{K}^c \to X$. We'll use support data $\mathbf{V} : \mathsf{K}^c \to \mathcal{X}_{\mathrm{cl}}$ which satisfies the faithfulness and realization properties. We'll extend this to $\mathcal{V} : \mathsf{K} \to \mathcal{X}$. So we need

- A Zariski space X,
- Support data $\mathbf{V}$,
- An extension $\mathcal{V}$.

19.2 Localization functors

Remark 19.2.1: Let $\mathsf{C} \leq \mathsf{K}$ be a thick subcategory for $\mathsf{K} \in \mathsf{triangCat}$. A mysterious sequence:

$$\Gamma_c(M) \to M \to L_c(M).$$

Suppose $W \in \mathcal{X}_{\mathrm{irr}}$ is nonempty and let $Z = \left\{x \in X \;\middle|\; w \not\subseteq \mathrm{cl}_X\left\{x\right\}\right\}$. Define a functor $\nabla_W = \Gamma_{I_W} L_{I_Z}$ and $\mathcal{V}(M) \coloneqq \left\{x \in X \;\middle|\; \nabla_{\{x\}}(M) = 0\right\}$.

Theorem 19.2.2 ***(Hopkins-Neeman).***
Let K be a compactly generated tensor triangulated category, X a Zariski space, and $\mathcal{X}_{\mathrm{cl}}$ the closed sets. Given a compact object $M \in \mathsf{K}^c$, let $\langle M \rangle_{\mathsf{K}^c}$ be the thick tensor ideal in K^c generated by M. Let $\mathbf{V} : \mathsf{K}^c \to \mathcal{X}_{\mathrm{cl}}$ be support data satisfying the faithfulness condition and suppose $\mathcal{V} : \mathsf{K} \to \mathcal{X}$ is an extension. Set $W = \mathbf{V}(M)$ and $I_W = \left\{N \in \mathsf{K}^c \;\middle|\; V(N) \subseteq W\right\}$. Then

$$I_W = \langle M \rangle_{\mathsf{K}^c},$$

i.e. this is generated by a single object.

Proof (?).
Let $I \coloneqq I_W$ and $I' \coloneqq \langle M \rangle_{\mathsf{K}^c}$.
$I' \subseteq I$: If $N \in I'$, then N is obtained by taking direct sums, direct summands, distinguished triangles, shifts, etc. These all preserve support containment, so $\mathbf{V}(N) \subseteq W$ and $N \in I = I_W$.

$I \subseteq I'$: Let $N \in \mathsf{K}^{\mathsf{c}}$. Apply the functorial triangle $\Gamma_{I'} \to \mathrm{id} \to L_{I'}$ to $\Gamma_I(N)$ to obtain

$$\Gamma_{I'}\Gamma_I N \to \Gamma_I(N) \to L_{I'}\Gamma_I N.$$

From above, $I' \subseteq I$ so the first term is in $\mathsf{Loc}(I)$. Since the second term is as well, the 2-out-of-3 property guarantees that the third term satisfies $L_{I'}\Gamma_I N \in \mathsf{Loc}(I)$. By the lemma, $V(L_{I'}\Gamma_I N) \subseteq W$. There are no nonzero maps $I' \to VL_{I'}\Gamma_I N$, therefore for $S \in \mathsf{K}^{\mathsf{c}}$, noting that $S \otimes M \in I'$,

$$0 = \underset{\mathsf{K}}{\mathrm{Hom}}(S \otimes M, L_{I'}\Gamma_I M) = \underset{\mathsf{K}}{\mathrm{Hom}}(S, M^\vee \otimes L_{I'}\Gamma_I N),$$

and since S is an arbitrary compact object, this forces $M^\vee \otimes L_{I'}\Gamma_I N = 0$. By faithfulness, and the tensor product property,

$$\begin{aligned}
\emptyset &= \mathcal{V}(M^\vee \otimes L_{I'}\Gamma_I N) \\
&= \mathcal{V}(M^\vee) \cap \mathcal{V}(L_{I'}\Gamma_I N) \\
&= \mathbf{V}(M) \cap \mathcal{V}(L_{I'}\Gamma_I N) \\
&= W \cap \mathcal{V}(L_{I'}\Gamma_I N) \\
&= \mathcal{V}(L_{I'}\Gamma_I N),
\end{aligned}$$

so by faithfulness (again) $L_{I'}\Gamma_I N = 0$. Thus by the localization triangle, $\Gamma_{I'}\Gamma_I N \cong \Gamma_I N$. Now specialize to $N \in I$; the localization triangle yields

$$\Gamma_I N \to N \xrightarrow{0} L_I(N) \implies \Gamma_I N \cong N.$$

Now replacing I with I' yields $\Gamma_{I'} N \cong N$ since $L_{I'} N \cong L_{I'}\Gamma_I N \cong 0$ by the previous part. Thus $N \in \mathsf{Loc}(I')$ by applying a result of Neeman, implying $N \in I'$ and $I \subseteq I'$.

■

Remark 19.2.3: Many different takes on classification of thick tensor ideals:

- Benson, Carlson, Rickard at UGA in the late 90s, for finite group representations (now extended).
- Benson, Iyengar, Krause: axiomatic approach and description of supports.
- Dell'Ambrogio
- Boe, Kujawa, Nakano

Theorem 19.2.4 *(?).*
Let

- K be a compactly generated tensor triangulated category,
- X be a Zariski space,
- $\mathbf{V} : \mathsf{K}^{\mathsf{c}} \to \mathcal{X}_{\mathrm{cl}}$ be a support datum satisfying both the faithfulness *and* realization properties,
- $\mathcal{V} : \mathsf{K} \to C$ be an extension of $\mathbf{V}$.

Let $\mathrm{Id}(\mathsf{K}^{\mathsf{c}})$ be the set of thick tensor ideals in K^{c}, then there is a bijection

$$\begin{aligned}\mathrm{Id}(\mathsf{K}^{\mathsf{c}}) &\rightleftharpoons \mathcal{X}_{\mathrm{sp}} \\ I &\mapsto \Gamma(I) \coloneqq \bigcup_{M \in I} \mathbf{V}(I) \\ \Theta(W) = I_W \coloneqq \left\{ N \in \mathsf{K}^{\mathsf{c}} \;\middle|\; \mathbf{V}(N) \subseteq W \right\} &\leftarrow W.\end{aligned}$$

Exercise 19.2.5 (?)
Show that $I_W \in \mathrm{Id}(\mathsf{K}^{\mathsf{c}})$ is in fact a thick tensor ideal.

Proof (?).
$\Gamma \circ \Theta = \mathrm{id}$: Check that

$$\Gamma \Theta W = \Gamma(I_W) = \bigcup_{M \in I_W} \mathbf{V}(M) \subseteq W.$$

For the reverse inclusion, let $W = \bigcup_{j \in W} W_j$ where $W_j \in \mathcal{X}_{\mathrm{cl}}$. By the realization property, there exist $N_j \in \mathsf{K}^{\mathsf{c}}$ such that $\mathbf{V}(N_j) = W_j$, so $N_j \in I_W$. Now $W \subseteq \bigcup_{M \in I_W} \mathbf{V}(M)$, so $W = \bigcup_{M \in I_W} \mathbf{V}(M)$.

$\Theta \circ \Gamma = \mathrm{id}$: For $I \in \mathrm{Id}(\mathsf{K}^{\mathsf{c}})$, set $W \coloneqq \Gamma(I) = \bigcup_{M \in I} \mathbf{V}(M)$, then

$$\Theta \Gamma I = \Theta(W) = I_W \supseteq I.$$

For the reverse inclusion $I_W \subseteq I$: let $N \in I_W$. Since X is a Zariski space, X is Noetherian and there is an irreducible component decomposition $V(N) = \bigcup_i W_i$ with each W_i irreducible with a unique generic point, so $W_i = \mathrm{cl}_{W_i}\{x_i\}$. Since each $W_i \subseteq W$, each $x_i \in W = \bigcup \mathbf{V}(M)$, so there exist M_i with $x_i \in \mathbf{V}(M_i)$. Since supports are closed, $W_i = \mathrm{cl}_{W_i}\{x_i\} \subseteq \mathbf{V}(M_i)$. Setting $M \coloneqq \bigoplus_i M_i \in I$ yields $V(N) \subseteq \bigcup V(M_i) = V(M) \subseteq W$.

Claim:

$$N \in \langle M \rangle_{\mathsf{K}^{\mathsf{c}}}.$$

Proving the claim will complete the proof, since I is a thick ideal containing M, so $\langle M \rangle_{\mathsf{K}^{\mathsf{c}}} \subseteq I$ and $N \in I$.

Proof (of claim).
By Hopkins' theorem, $\langle M \rangle_{\mathsf{K}^{\mathsf{c}}} = I_Z$ where $Z = \mathbf{V}(M)$. Since $V(N) \subseteq V(M) = Z$, we have $N \in I_Z = \langle M \rangle_{\mathsf{K}^{\mathsf{c}}}$.

■

■

Remark 19.2.6: Next time:

- Showing $\operatorname{Spc}\mathsf{K}^{\mathsf{c}} \underset{\mathsf{Top}}{\cong} X$
- Examples: $\mathsf{kG}\text{-}\mathsf{stMod}$, $\mathsf{u}(\mathfrak{g})\text{-}\mathsf{stMod}$, and $\mathbb{D}\mathsf{R}\text{-}\mathsf{Mod}$.

20 | Thursday, April 21

20.1 Classification theorem

Theorem 20.1.1 *(?).*
Let K be a compactly generated tensor-triangulated category and let X be a Zariski space. Suppose that

1. $\mathbf{V} : \mathsf{K}^c \to \mathcal{X}_{\mathrm{cl}}$ is a support datum,
2. $\mathbf{V}$ satisfies the faithfulness property,
3. $\mathcal{V} : \mathsf{K} \to \mathcal{X}$ extends $\mathbf{V}$.

Then there exists a bijective correspondence

$$\operatorname{Id}(\mathsf{K}^c) \underset{\Theta}{\overset{\Gamma}{\rightleftarrows}} \mathcal{X}_{\mathrm{sp}}$$

where $\Gamma(I) := \bigcup_{M \in I} \mathbf{V}(M)$ and $\Theta(W) := \left\{ N \in \mathsf{K}^c \,\middle|\, \mathbf{V}(N) \subseteq W \right\}$.

Remark 20.1.2: This relies on Hopkins' theorem.

20.2 Balmer spectrum

Theorem 20.2.1 *(?).*
Let K and X be as in the previous theorem, satisfying the same assumptions. Then there exists a homeomorphism $f : X \to \operatorname{Spc}\mathsf{K}^c$.

Proof (?).
Since $\mathbf{V} : \mathsf{K}^c \to \mathcal{X}_{\mathrm{cl}}$ is a support datum, Balmer shows there exists a continuous map

$$\begin{aligned} f : X &\to \operatorname{Spc}\mathsf{K}^c \\ x &\mapsto P_x := \left\{ M \,\middle|\, x \notin \mathbf{V}(M) \right\}. \end{aligned}$$

Note that P_x is a prime ideal:

$$\begin{aligned} M \otimes N \in P_x &\implies x \notin \mathbf{V}(M \otimes N) \\ &\implies x \notin \mathbf{V}(M) \cap \mathbf{V}(N) \\ &\implies x \notin \mathbf{V}(M) \text{ or } x \notin \mathbf{V}(N) \\ &\implies M \in P_x \text{ or } N \in P_x. \end{aligned}$$

Applying the classification theorem, this yields a bijection.

■

Remark 20.2.2: Examples of classification:

For $G \in \mathsf{FinGrp}, \operatorname{ch} k = p \mid \sharp G$, take $\mathsf{K} = \mathsf{kG\text{-}stMod}$, $R = H^{\text{even}}(G; k)$, and $X = \operatorname{Proj} R = \operatorname{Proj}(\operatorname{Spec} R)$. Checking that this satisfies the 4 properties in the theorem:

1. For $M \in \mathsf{K}^c$, we take $\mathbf{V}(M) = \left\{ p \in X \,\middle|\, \operatorname{Ext}^{\bullet}{}_{kG}(M, M)\,[p^{-1}] \neq 0 \right\}$. This yields a support datum.

2. The tensor product property holds because $\mathbf{V}_E(M) = \mathbf{V}_E^r(M)$ (the rank variety), and we showed that $\mathbf{V}$ satisfies faithfulness and (Carlson) realization properties.

3. We can use localization functors to define $\mathcal{V} : \mathsf{K} \to \mathcal{X}$ which satisfies the same support data properties. For this to be an extension, one should check that

- $\mathbf{V}(M) = \mathcal{V}(M)$ for every compact $M \in \mathsf{K}^c$.
- $\mathbf{V}(M \otimes N) = \mathcal{V}(M) \cap \mathcal{V}(N)$ for all $M, N \in \mathsf{K}$
- If $\mathcal{V}(M)$ is empty then $M = 0$.

Remark 20.2.3: To prove these properties, Benson-Carlson-Rickard start with E elementary abelian, so $E = \langle x_1, \cdots, x_n \rangle \cong {C_p}^{\times^n}$ with $o(x_i) = p$ for all i. Set $y_i = x_i - 1 \in kE$, so $y_i^p = 0$, and define cyclic subgroups $\alpha = [\alpha_1, \cdots, \alpha_n] \in L^n$ where L/k is a field of large transcendence degree. Define $y_\alpha \coloneqq \sum_{1 \leq i \leq n} \alpha_i y_i$ and define a rank variety

$$\mathcal{V}_E^r(M) = \left\{ \alpha \in L^n \,\middle|\, L \otimes_k M \downarrow_{\langle y_\alpha \rangle} \text{ is not free } \right\} \cup \{0\}\,.$$

Theorem 20.2.4 *(?).*
Let E be as above and suppose $\operatorname{trdeg}(L/k) \geq n$. Then if $M \in \mathsf{K}$, $\mathcal{V}_E(M) \cong \mathcal{V}_E^r(M)$, and the three properties for (3) above hold for E.

Theorem 20.2.5 *(?).*
Let $A = kG$ for G a finite group scheme, and let $R = H^{\text{even}}(G; k)$ and $X = \operatorname{Proj}(R)$. Then

- There is a bijective correspondence

$$\mathsf{kG\text{-}stMod} \underset{\Theta}{\overset{\Gamma}{\rightleftarrows}} \mathcal{X}_{\mathrm{sp}}.$$

- $\mathrm{Spc}(\mathsf{kG\text{-}stMod}) \underset{\mathsf{Top}}{\cong} X$.

Remark 20.2.6: Some remarks:

- This theorem is an indication of why cohomology is central in understanding the tensor structure of representation categories. If $G \in \mathsf{FinGrpSch}_{/k}$ then the coordinate ring $k[G]$ is a commutative Hopf algebra, so $A = kG = k[G]^\vee$ is a finite dimensional cocommutative Hopf algebra. So there is an equivalence of categories between $\mathsf{Rep}G$ and $\mathsf{Rep}A$ for A such a Hopf algebra. By a result of Friedlander-Suslin, R is finitely generated.

- The realization of $\mathbf{V}$ and $\mathcal{V}$ for a general group scheme involve so-called *π-points* developed be Friedlander-Pevtsovz and the construction of explicit rank varieties.

Remark 20.2.7: A special case: let $\mathfrak{g} = \mathsf{Lie}G$ for $G \in \mathsf{AlgGrp}_{/k}$ reductive and k positive characteristic. Let $A = u(\mathfrak{g})$, which is a finite-dimensional cocommutative Hopf algebra. If $p > h$ for h the Coxeter number,

$$\mathcal{N}_p = \left\{ x \in \mathfrak{g} \;\middle|\; x^{[p]} = 0 \right\} = \mathcal{N}, \text{ the nilpotent cone,}$$

$R = H^{\mathrm{even}}(u(\mathfrak{g}); k) = k[\mathcal{N}]$, and $X = \mathrm{Proj}(k[\mathcal{N}])$, then applying the theorem,

- There is a correspondence

$$\mathsf{u}(\mathfrak{g})\text{-}\mathsf{stMod} \rightleftarrows \mathcal{X}_{\mathrm{sp}}.$$

- There is a homeomorphism

$$\mathrm{Spc}\left(\mathsf{u}(\mathfrak{g})\text{-}\mathsf{stMod}\right) \underset{\mathsf{Top}}{\cong} \mathrm{Proj}(k[\mathcal{N}]).$$

Theorem 20.2.8 ***(Arkhipov-Bezrukavikov-Ginzburg).***
Let $\tilde{\mathcal{N}} \to \mathcal{N}$ be the Springer resolution. There is an equivalence of derived categories

$$\mathbb{D}^b \mathsf{u}_\zeta(\mathfrak{g})_0\text{-}\mathsf{Mod} \xrightarrow{\sim} \mathbb{D}^b \mathsf{Coh}^{G \times \mathbb{C}^\times} k[\tilde{\mathcal{N}}] \xrightarrow{\sim} \mathbb{D}^b \mathsf{Perv}(\Omega \mathrm{Gr}).$$

where $\mathsf{Perv}(-)$ is the category of perverse sheaves and $\Omega\mathrm{Gr}$ is the loop Grassmannian.

Remark 20.2.9: For M a $u_\zeta(\mathfrak{g})$-module and $R = H^{\mathrm{even}}(u_\zeta(\mathfrak{g}); M) = \mathbb{C}[\mathcal{N}] \cong \mathbb{C}[\tilde{\mathcal{N}}]$. There is an action of R on $H^\bullet(u_\zeta(\mathfrak{g}); M)$. Next time: examples for Lie superalgebras and Thomason's reconstruction theorem for rings.

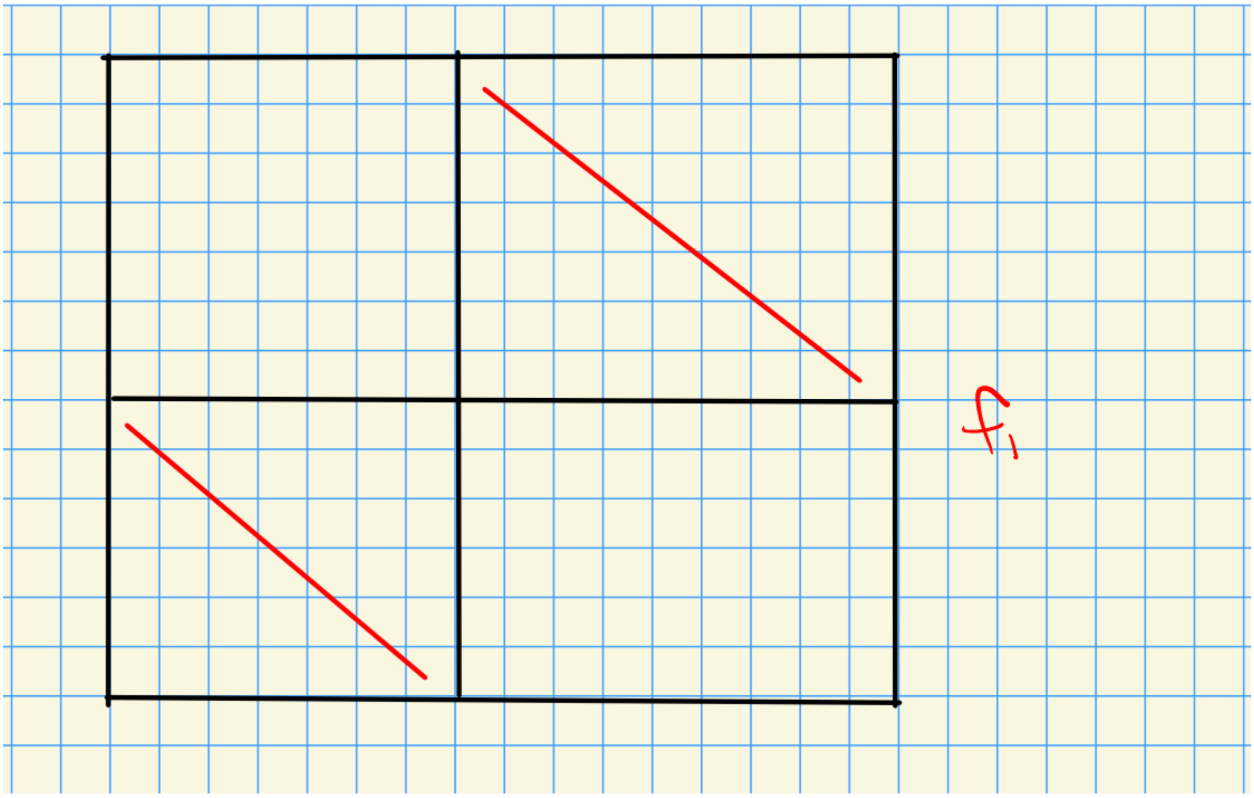
21 | Tuesday, April 26

See Boe-Kujawa-Nakano, Adv. Math 2017.

Remark 21.0.1: Setup: $\mathsf{K}^c \leq \mathsf{K} \in \mathsf{TTC}$, X a Zariski space, $V : \mathsf{K}^c \to \mathcal{X}_{\mathrm{cl}}$ with an extension $\mathcal{V} : \mathsf{K} \to \mathcal{X}$. Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be a Lie superalgebra with a C_2 grading over $k = \mathbb{C}$ where $\mathfrak{g}_0 \curvearrowright \mathfrak{g}_1$, e.g. $\mathfrak{gl}_{m,n} = \mathfrak{gl}_m \times \mathfrak{gl}_n$ with matrices $\begin{bmatrix} \mathfrak{g}_0 & \mathfrak{g}_1 \\ \mathfrak{g}_1 & \mathfrak{g}_0 \end{bmatrix}$ with the bracket action. Write $\mathsf{Lie}G_0 = \mathfrak{g}_0$, and note that G_0 is reductive. Let $\mathcal{F}(\mathfrak{g}, \mathfrak{g}_0)$ be the category of finite-dimensional $\mathfrak{g}$-supermodules which are completely reducible over $\mathfrak{g}_0$. Take $\mathsf{K}^c = \mathcal{F}(\mathfrak{g}, \mathfrak{g}_0)\text{-}\mathsf{stMod} \leq \mathsf{K} = \mathsf{C}(\mathfrak{g}, \mathfrak{g}_0)\text{-}\mathsf{stMod}$, where for C we drop the finite-dimensional condition.

Set $R = H^0(\mathfrak{g}_1, \mathfrak{g}_0; \mathbb{C}) = \operatorname{Ext}(\mathbb{C}, \mathbb{C}) \cong S({\mathfrak{g}_0}^\vee)^{G_0}$. By a theorem of Hilbert, $\operatorname{Ext}(M, M)$ is finitely generated over R. Write $V_{\mathfrak{g},\mathfrak{g}_0}(M) = \operatorname{spec} R/J_M$ – for Kac modules $K(\lambda) = U(\mathfrak{g}) \otimes_{U(P^0)} L_0(\lambda)$, $V = 0$ but not every $K(\lambda)$ is projective.

Remark 21.0.2: Idea: use detecting subalgebras. For $\mathfrak{g} = \mathfrak{gl}_{n,n}$, let f_1 be the "torus":



Then define $f_0 = [f_1, f_1]$.

Remark 21.0.3: Let $X = N \operatorname{Proj}(S^*({f_1}^\vee))$ where $S^*({f_1}^\vee) \cong \operatorname{Ext}_{f_1,f_0}(\mathbb{C}, \mathbb{C}) = R'$ and $N = N_{G_0}(f_1)$, which is a reductive algebraic group. Define a support datum by $\mathbf{V}(M) = \left\{ p \in X \;\middle|\; \operatorname{Ext}_{f,f_0}(M, M)_p = 0 \right\}$.

The goal is to construct $\mathcal{V} : K \to \mathcal{X}$ using localization functors – one needs to show the tensor product formula, and the faithfulness and realization properties, which follows from Dede's lemma. It turns out that $f_1 \cong \mathfrak{sl}(1,1)^{\times m}$ and it suffices to define the rank variety on f_1. Define

$$V_{f_1}^{\text{rank}}(M) = \left\{ \bar{x} = \tilde{K} \otimes_{\mathbb{Q}} f_1 \;\middle|\; K \otimes_{\mathbb{C}} M \downarrow \langle \bar{x} \rangle \text{ is not projective} \right\}$$

where $\tilde{K} \supseteq \mathbb{C}$ is an extension with $\operatorname{trdeg}_{\mathbb{C}} \tilde{K} \geq \dim f_1$. A theorem shows $\mathcal{V}(M) = V_{f_1}^{\text{rank}}(M)$ for $M \in K$. This yields a classification for $\mathfrak{gl}_{m,n}$ of thick tensor ideals in K^c in terms of $\mathcal{X}_{\text{sp}}$.

Remark 21.0.4: What is the classification of other Lie superalgebras? This is an open problem.

21.1 Noncommutative theory

Remark 21.1.1: How does one extend this theory to noncommutative TTCs? See Nakano-Vashaw-Yakomov, to appear in Amer J. Math.

Remark 21.1.2: Let K be a compactly generated monoidal triangulated category, not necessarily symmetric. One approaches this via noncommutative ring theory, where e.g. even the definition of prime ideals differs. We'll only consider 2-sided ideals.

Definition 21.1.3 ((Noncommutative) prime ideals)
A thick triangulated subcategory P is a **completely prime** ideal iff $M \otimes N \in P \implies M \in P$ or $N \in P$. The ideal P is **prime** iff $I \otimes J \subseteq P \implies I \subseteq P$ or $J \subseteq P$, where I, J are themselves ideals. Define $\operatorname{spc} K$ to be prime ideals and $\operatorname{CP} \operatorname{Spc} K$ to be completely prime ideals.

Example 21.1.4*(?)*: Let $A \in \mathsf{HopfAlg}_{/k}^{\text{fd}}$ where the coproduct $\Delta : A \to A^{\otimes_k^2}$ is not necessarily commutative, e.g. in the setting of quantum groups. Some remarks:

- Note that $M \otimes N \ncong N \otimes M$ in general.
- Here $\operatorname{spc} K^c$ is not known, but there is a conjectural answer.
- In general $\operatorname{spc} K^c \ncong \operatorname{Proj} R$ for $R = H(A; k)$.
- R is not known to be finitely-generated. Etingof-Ostrik conjecture this in the setting of finite tensor categories.
- The definition of prime ideals is due to Buan-Krause-Solberg in 2007.
- A weird example: there are nilpotents where $M \neq 0$ (is not projective) but $M^{\otimes_k^2} = 0$ (is projective).
- Being a prime ideal P is equivalent to $A \otimes C \otimes B \in P$ for all $C \implies A \in P$ or $B \in P$.

Definition 21.1.5 ((Noncommutative) support data)
Let K be a monoidal triangulated category, X a Zariski space, and $\mathcal{X} = 2^X$ the subsets of X. A map $\sigma : K \to \mathcal{X}$ is a **weak support datum** iff

- $\sigma(0) = \emptyset$ and $\sigma(\mathbb{1}) = X$
- $\sigma(A \otimes B) = \sigma(A) \cup \sigma(B)$

- $\sigma(\Sigma A) = \sigma(A)$
- If $A \to B \to C$ is exact then $\sigma(A) \subseteq \sigma(B) \cup \sigma(C)$.

Set $\Phi_\sigma(I) \coloneqq \bigcup_{M \in I} \sigma(I)$; Then σ is a **support datum** if additionally

- $\bigcup_{C \in K} \sigma(A \otimes C \otimes B) = \sigma(A) \cap \sigma(B)$ and
- $\Phi_\sigma(I \otimes J) = \Phi_\sigma(I) \cap \Phi_\sigma(J)$.

Remark 21.1.6: Next time:

- Classification theorems
- The NVY conjecture for finite-dimensional Hopf algebras.
- Tensor product theorems.
- Examples of applications.

ToDos

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Bibliography

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