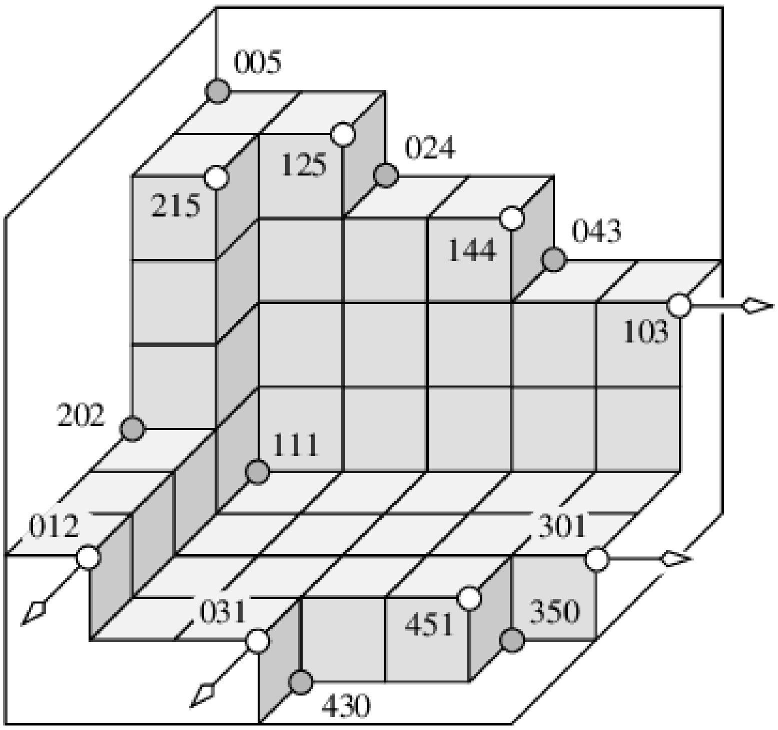

Notes: These are notes live-tex'd from a graduate course in Commutative Algebra taught by Daniel Litt at the University of Georgia in Spring 2022. As such, any errors or inaccuracies are almost certainly my own.

Commutative Algebra

Lectures by Daniel Litt. University of Georgia, Spring 2022

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1 | Thursday, January 06

Topics: Localization and completion, Nakayama's lemma, Dedekind domains, Hilbert's basis theorem, Hilbert's Nullstellensatz, Krull dimension, depth and Cohen-Macaulay rings, regular local rings.

Remark 1.0.1: References:

- Atiyah-MacDonald, *Commutative Algebra*. Be sure to check the erratum!
- Chambert-Loir, Mostly Commutative Algebra
- Miles Reid, *Undergraduate Algebraic Geometry*
- Altman-Kleiman, A Term of Commutative Algebra

Example 1.0.2 ***(?):*** Some examples of module morphisms:

- $\operatorname{Hom}_{\mathsf{Ring}}(\mathbb{Z}, S) = \{\text{pt}\}$, since $1 \to 1$ is necessary.
- $\operatorname{Hom}_{\mathsf{Ring}}(\mathbb{Z}[x], S) \cong S$. Why? Since $1 \to 1$ is forced, $x \mapsto s$ can be sent to any $s \in S$.
- $\operatorname{Hom}_{\mathsf{Ring}}\left(\frac{\mathbb{Z}[x,y]}{\langle y^2 - x^3 - 1\rangle}, S\right) = \left\{(a,b) \in S^{\times 2} \;\middle|\; a^2 - b^3 = 1\right\}$.
- $\operatorname{Hom}_{\mathsf{Ring}}(R/I, S) = \left\{f \in \operatorname{Hom}_{\mathsf{Ring}}(R, S) \;\middle|\; f(I) = 0\right\}$

Exercise 1.0.3 (?)
Show that $\operatorname{Id}(R/I) \rightleftharpoons \left\{J \in \operatorname{Id}(R) \;\middle|\; J \supseteq I\right\}$ using

$$\begin{aligned} \Pi : R &\to R/I \\ x &\mapsto [x] \\ \pi^{-1}(J) &\hookleftarrow J. \end{aligned}$$

Show that $\pi^{-1}(J)$ is in fact an ideal, construct a proposed inverse Π^{-1}, and show $\Pi \circ \Pi^{-1} = \Pi^{-1} \circ \Pi = \text{id}$.

Exercise 1.0.4 (?)
Show that R is a field iff R is a simple ring iff any ring morphism $R \to S$ is injective. For $3 \implies 1$, directly shows that every nonzero element is a unit.

Exercise 1.0.5 (?)
Chapter 1 of A&M:

- 1,8,10,13,15,16,19.

2 | Thursday, January 13

Remark 2.0.1: Last time: fields are simple rings.

Exercise 2.0.2 (?)
Let $k \in \mathsf{Field}$ and show that

- If $f \in R := k[x]$ is irreducible then $\langle f \rangle \in \operatorname{Spec} R$.
- $\langle xy \rangle \in \operatorname{Id}(k[x,y])$ is not prime and not maximal.
- There exist nonzero non-prime maximal ideals.
- Show that $I \in \operatorname{Spec} R \iff A/I$ is an integral domain.
- Show that $I \in \operatorname{mSpec} R \iff A/I \in \mathsf{Field}$.

Exercise 2.0.3 (?)
Prove that if $f : R \to S$ is a ring morphism then there is a well-defined map

$$\begin{aligned} f^* : \operatorname{Spec} S &\to \operatorname{Spec} R \\ \mathfrak{p} &\mapsto f^{-1}(\mathfrak{p}), \end{aligned}$$

i.e. if $\mathfrak{p}$ is prime in S then $f^{-1}(\mathfrak{p})$ is prime in R. Show that this doesn't generally hold with Spec replaced by mSpec.

Exercise 2.0.4 (?)
Show that defining $V(I) := \left\{ \mathfrak{p} \in \operatorname{Spec} R \;\middle|\; \mathfrak{p} \supseteq I \right\}$ as closed sets defines a topology on $\operatorname{Spec} R$. Show that $\operatorname{Spec} R$ is Hausdorff iff it's discrete, and V is functorial.

Exercise 2.0.5 (?)
Describe

- $\operatorname{Spec} k$ for $k \in \mathsf{Field}$
- $\operatorname{Spec} \mathbb{Z}$, show it is not Hausdorff, and $\langle 0 \rangle$ is a generic point, and the subspace topology on its closed points is cofinite.
- $\operatorname{Spec} R$ for R a local ring.
- $\operatorname{Spec} R$ for R a DVR
- $\operatorname{Spec} k[x,y]$
- $\operatorname{Spec} R$ for $R = \mathbb{Z}[i]$.

- $\operatorname{Spec} \mathcal{O}_K$, the ring of integers of a number field K.
- $\operatorname{Spec} R$ for R a Dedekind domain

Exercise 2.0.6 (?)
Show that every $I \in \operatorname{Id}(R)$ is contained in some $\mathfrak{m} \in \operatorname{mSpec} R$.

Exercise 2.0.7 (?)
Show that the following rings are local:

- For $p \in \mathbb{Z}$ prime, $R := \mathbb{Z}\left[S^{-1}\right]$ for $S := \left\{ \ell \in \mathbb{Z} \text{ prime} \;\middle|\; \ell \neq p \right\}$.
- $k \in \mathsf{Field}$
- $k[x]$ with $\mathfrak{m} = \langle x \rangle$
- $k[x, y]$. What is the maximal ideal?

Exercise 2.0.8 (?)
For $(R, \mathfrak{m})$ a local ring, show that $\mathfrak{m} = R \setminus (R^{\times})$.

Remark 2.0.9: Recall that

- $\sum I_j$ is the smallest ideal containing all of the I_j.
- $\bigcap I_j$ is again an ideal
- $IJ := \left\langle xy \;\middle|\; x \in I, y \in J \right\rangle$ is an ideal
- $IJ \subseteq I \cap J$.
- $\sqrt{0_R}$ is the set of nilpotent elements.

Theorem 2.0.10 *(?)*.
Show that $\sqrt{0_R}$ is the intersection of all prime ideals.

Slogan 2.0.11
Regarding elements $f \in R$ as functions on $\operatorname{Spec} R$, f nilpotent is like being zero at every point of $\operatorname{Spec} R$.

Exercise 2.0.12 (?)
Show that $x \in J(R) \iff 1 - xy \in R^{\times}$ for all $y \in R$.

3 | Tuesday, January 18

Remark 3.0.1: A reference for pictures: Mumford's red book. Note the typo in A&M problem 10: it is false for the zero ring.

Recall some definitions:

- R-modules, what are some examples?
- Submodules and quotient modules.
- The submodule generated by a subset.
- A morphism of modules and the R-module structure on $\underset{R}{\operatorname{Hom}}(M,N)$, $(rf)(x) := r \cdot f(x)$.
  - This makes $\mathsf{R\text{-}Mod}$ into a category enriched over itself.
- $\operatorname{im} f, \ker f, \operatorname{coker} f$.
- $\prod M_i, \bigoplus M_i$

Exercise 3.0.2 (Module structure on quotients)
Show that if $M \leq N \in \mathsf{R\text{-}Mod}$, then the following action makes M/N into an R-module:

$$r \cdot [x] := [rx],$$

i.e. that if $[x] = [y]$ then $r \cdot [x] = r \cdot [y]$.

Exercise 3.0.3 (Universal properties of kernels and cokernels)
Show the universal properties of kernels and cokernels: given $f : M \to N$ and $Q \in \mathsf{R\text{-}Mod}$,

$$\underset{R}{\operatorname{Hom}}(Q, \ker f) = \left\{ s \in \underset{R}{\operatorname{Hom}}(Q, M) \;\middle|\; f \circ s = 0 \right\}$$
$$\underset{R}{\operatorname{Hom}}(\operatorname{coker} f, Q) = \left\{ t \in \underset{R}{\operatorname{Hom}}(N, Q) \;\middle|\; t \circ f = 0 \right\}.$$

Exercise 3.0.4 (Direct sum and product coincide for finite index sets)
Show that if I is a finite indexing set, there is an isomorphism

$$\bigoplus_{i \in I} M_i \xrightarrow{\sim} \prod_{i \in I} M_i,$$

and that

$$\underset{R}{\operatorname{Hom}}\left(T, \prod M_i\right) \cong \prod_{i \in I} \underset{R}{\operatorname{Hom}}\left(T, M_i\right)$$
$$\underset{R}{\operatorname{Hom}}\left(\bigoplus M_i, T\right) \cong \bigoplus \underset{R}{\operatorname{Hom}}\left(M_i, T\right).$$

Show that by Yoneda, this satisfies the universal property.

Solution:
Sketch:

- Define projections $\pi_j : \prod_i M_i \to M_j$.
- Send $f \in \operatorname{Hom}(T, \prod_i M_i)$ to $\pi_j \circ f \in \prod_i \operatorname{Hom}(T, M_i)$
- For the other direction, given $(f_i) \in \prod_i \operatorname{Hom}(M_i, T)$, send (f_i) to $\sum f_i$.

Exercise 3.0.5 (Modules have products and coproducts)
Show that $\prod$ is a categorical product and $\bigoplus$ is a categorical coproduct. What are the product and coproduct in Top?

Definition 3.0.6 (Colon ideals and annihilators)
Given $N \leq M \in \mathsf{R\text{-}Mod}$, define the **colon ideal**

$$(N : M) := \left\{ a \in R \;\middle|\; aM \subseteq N \right\}$$

and the **annihilator** of M

$$\operatorname{Ann}(M) = (0 : M) = \left\{ a \in R \;\middle|\; aM = 0 \right\}.$$

Example 3.0.7 ***(Annihilators):*** Some annihilators:

- $C_n \in \mathbb{Z}\text{-}\mathsf{Mod}$, so $\operatorname{Ann}(C_n) = n\mathbb{Z} = \langle n \rangle$.
- Again in $\mathbb{Z}\text{-}\mathsf{Mod}$, $\operatorname{Ann}(C_n \oplus C_m) = n\mathbb{Z} \cap m\mathbb{Z} = \operatorname{lcm}(m, n)\mathbb{Z}$.

Remark 3.0.8: An R-module is free iff $R \cong \bigoplus_{i \in I} R$, where I can be infinite but (importantly) we need the direct sum instead of the direct product. Note that generally $\prod_{i \in I} R$ may not be free as an R-module.

A module is finitely generated if there exists a generating set $X := \{x_i\}_{i \leq n} \subseteq M$ such that any submodule containing X is all of M, or equivalently $x \in M \implies x = \sum r_i x_i$ for some $r_i \in R$.

Exercise 3.0.9 (Finitely generated iff surjective image of a free module)
Show that M is finitely-generated iff there is a surjective morphism $R^n \to M$ for some $n \in \mathbb{Z}_{\geq 0}$.

Solution:
Sketch:

- finitely-generated $\implies$ surjection:

- $\Longleftarrow$:
  - Take $e_i = \{0, \cdots, 1, \cdots, 0\} \in R^n$ and define $f(e_i) \coloneqq x_i$
  - Use the universal property of the direct sum.
- $\Longleftarrow$:
  - Show that the $f(e_i)$ generate M: by surjectivity, $m = f(x) = f(\sum r_i e_i) = \sum r_i f(e_i)$.

Remark 3.0.10: Thus every $M \in \mathsf{R\text{-}Mod}$ is the quotient of a free module: find a surjection $f : F \to M$, so $\operatorname{im} f = M$, then use that $\operatorname{im} f \cong / \ker f$. For example, one can take $F \coloneqq \bigoplus_{m \in M} R \to M$.

Recall

- The definition of exact sequences
- $0 \to A \xrightarrow{f} B$ is exact iff f is injective,
- $A \xrightarrow{f} B \to 0$ iff f is surjective,
- $0 \to A \xrightarrow{f} B \to 0$ iff f is an isomorphism,
- $0 \to A \to B \to C \to 0$ is called a short exact sequence.

Exercise 3.0.11 (Direct sums are exact)
Check that $(-) \oplus M$ is exact.

Exercise 3.0.12 (Left exactness of hom)
Show that $\operatorname{Hom}(N, -)$ and $\operatorname{Hom}(-, N)$ are both left-exact for any $N \in \mathsf{R\text{-}Mod}$, where $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is sent to $0 \to \operatorname{Hom}(N, A) \xrightarrow{f \circ -} \cdots$. Give an example of when right-exactness fails.

Hint: try $0 \to \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \to C_2 \to 0$ and apply $\operatorname{Hom}_{\mathbb{Z}}(-, C_2)$.

Remark 3.0.13: Recall that $\operatorname{Hom}_{\mathbb{Z}\text{-}\mathsf{Mod}}(\mathbb{Z}, -) \cong \operatorname{id}$ and $\operatorname{Hom}_{\mathbb{Z}\text{-}\mathsf{Mod}}(C_n, -) \cong (-)[n]$ picks out the n-torsion.

4 | Thursday, January 20

Remark 4.0.1: Last time: $\operatorname{Hom}(N, -)$ and $\operatorname{Hom}(-, N)$ are left exact. We can explicitly describe

$$\operatorname{Hom}_A(A/I, M) = \left\{ m \in M \;\middle|\; im = 0 \forall i \in I \right\},$$

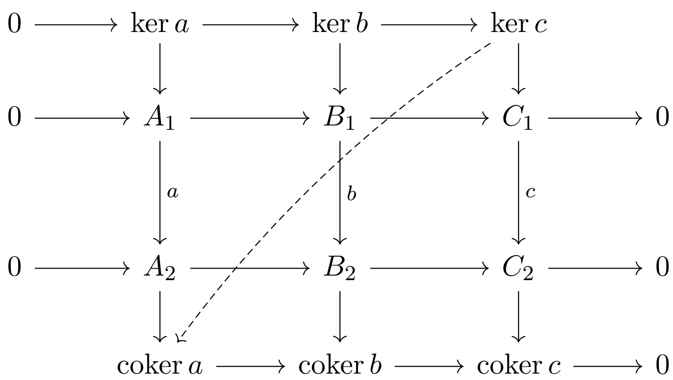
which is the I-torsion in M. Using that $0 \to I \to A \to A/I \to 0$ is short exact, we'll get a long exact sequence

$$0 \to \operatorname{Hom}(I, M) \to \operatorname{Hom}(A, M) \xrightarrow{f} \{I\text{-torsion in } M\} \to \operatorname{Ext}^1_A(I, M) \to \cdots,$$

where the Ext term measures failure of surjectivity of f.

Exercise 4.0.2 (?)
Show that $\operatorname{Hom}(N, -)$ and $\operatorname{Hom}(-, N)$ are left exact.

Remark 4.0.3: Recall the snake lemma:



Link to Diagram

The recipe for the connecting morphism:

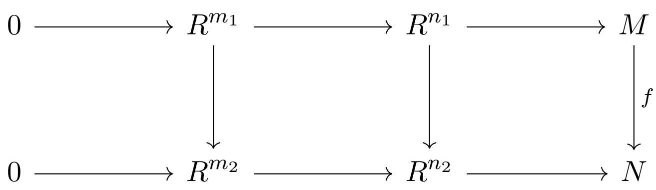
- Start with $x \in \ker(C_1 \to C_2)$, choose a preimage in B_1
- Push to B_2
- Use exactness to pull to A_2
- Project along $A_2 \to \operatorname{coker}(A_1 \to A_2)$

Definition 4.0.4 (Finite presentation)
An object $M \in$ R-Mod is of **finite presentation** iff there is an exact sequence

$$R^m \to R^n \to M \to 0,$$

i.e. there are finitely many generators and finitely many relations.

Remark 4.0.5: A nice application of the snake lemma: for finitely presented modules M, N, one can extend a morphism $M \to N$ to



Link to Diagram

Since $R \in \mathsf{R\text{-}Mod}$ is free, the extended maps can be represented by matrices, which is a significant simplification. In fact, f can be recovered uniquely by knowing the map on generators.

Exercise 4.0.6 (?)
Prove the snake lemma, and show exactness at all 6 places.

Remark 4.0.7: Recall the universal property of $M \otimes_R N$ in R-Mod in terms of bilinear morphisms.

Exercise 4.0.8 (?)
Prove uniqueness of any object satisfying a universal property using the Yoneda lemma.

Exercise 4.0.9 (?)
Prove using universal properties:

- $R \otimes_R R \cong R$, using universal properties. Why is this unique?
- $R \otimes_R M \cong M$.
- $(M_1 \oplus M_2) \otimes_R N \cong (M_1 \otimes_R N) \oplus (M_2 \otimes_R N)$.

5 | Tuesday, January 25

Exercise 5.0.1 (?)
Do A&M:

- 2.12, 24, 25
- 3.1, 4, 12

See website: https://www.daniellitt.com/commutative-algebra

Remark 5.0.2: Last time:

- Defined $M \otimes_R N$, need to show it exists.
- Showed $R \otimes_R M \cong M$
- Showed sums commute with tensor products

Exercise 5.0.3 (?)
Show that if $A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is exact and $A \otimes_R N, B \otimes_R N$ exist, then $C \otimes_R N$ also exists.

Hint: Use the universal property to produce a map $\psi : A \otimes N \to B \otimes N$ and set $C \otimes N := \operatorname{coker} \psi$.

Exercise 5.0.4 (?)
Prove that $M \otimes_R N$ exists by constructing it.

Solution:
Some hints: Construction 1: use $0 \to \ker f \to R^{\oplus^I} \xrightarrow{f} M \to 0$, and find $R^{\oplus^J} \to \ker f$ to assemble an exact sequence

$$R^{\oplus^J} \to R^{\oplus^I} \to M \to 0.$$

Now apply $(-)\otimes_R N$ to exhibit $M \otimes_R N$ as a cokernel $N^{\oplus^J} \to N^{\oplus^I} \to M \otimes_R N \to 0$. Separately, use the hands-on construction and prove it satisfies the universal property.

Exercise 5.0.5 (?)
Prove $C_2 \otimes_{\mathbb{Z}} C_3 \cong 0$ using construction 1 above.

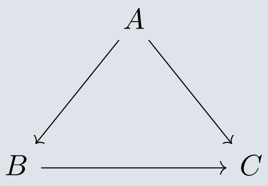
Solution:
Sketch:
Take $\mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \to C_2 \to 0$, apply $(-) \otimes C_3$, and check that $\operatorname{coker}(C_3 \xrightarrow{\times 2} C_3) = 0$ since multiplication by 2 is invertible. Alternatively, use bilinearity:

$$B(x,y) = 3B(x,y) - 2B(x,y) = B(x,3y) - B(2x,y) = B(x,0) - B(0,y) = 0.$$

Exercise 5.0.6 (?)
Show that

- $(-) \otimes_R N$ is right exact
- Tensoring is associative, distributive, commutative
- There is a canonical isomorphism $R \otimes_R M \to M$ induced by $r \otimes m \mapsto r.m$.
- Morphisms $f : A \to B \in \mathsf{CRing}$ induce functors $f^\sharp : \mathsf{A}\text{-}\mathsf{Mod} \to \mathsf{B}\text{-}\mathsf{Mod}$. Also show that $M \otimes_A B$ has a B-module structure given by $b_1(m \otimes b_2) := m \otimes (b_1 b_2)$.
- For $N \in \mathsf{B}\text{-}\mathsf{Mod}$, there is an isomorphism $\underset{A}{\operatorname{Hom}}(M, N) \xrightarrow{\sim} \underset{B}{\operatorname{Hom}}(M \otimes_A B, N)$.
    - Use $f \mapsto (m \otimes b \mapsto bf(m)) \in \operatorname{Bil}_B(M \times B, N)$, with inverse $Q \mapsto Q(-, 1)$.
- Show that $M \otimes_R R/I \cong M/IM$, using $R/I = \operatorname{coker}(I \hookrightarrow R)$ and applying $(-) \otimes M$. Also show that $I \otimes_R M \xrightarrow{\sim} IM$ canonically.
- Show that $k[t]^{\oplus^2} \otimes_{k[t]} \dfrac{k[t]}{\langle t^2 \rangle} \cong \left(\dfrac{k[t]}{\langle t^2 \rangle} \right)^{\oplus^2}$.
- Show that $R/I \otimes_R R/J \cong \dfrac{R/I}{I \cdot R/J} \cong \dfrac{R}{I+J}$.
- Tensoring need not be left exact.
- $C_p \notin \mathbb{Z}\text{-}\mathsf{Mod}^\flat$.
- $R \in \mathsf{R}\text{-}\mathsf{Mod}^\flat$.
- $\mathsf{R}\text{-}\mathsf{Mod}^\flat$ is closed under $\otimes_R$ and $\oplus$.
- $\mathbb{Q} \in \mathbb{Z}\text{-}\mathsf{Mod}^\flat$ but not in $\mathbb{Z}\text{-}\mathsf{Mod}^{\text{free}}$

Definition 5.0.7 (A-algebras)
$\mathsf{Alg}_{/A}$ is the coslice category $\mathsf{CRing}_{A/}$: objects are rings B equipped with ring morphisms $A \to B$, and morphisms are cones under A:



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Example 5.0.8 ***(?):*** Examples of algebras:

- $k[t] \in \mathsf{Alg}_{/k}$
- $R \in \mathsf{CRing} \implies R \in \mathsf{Alg}_{/\mathbb{Z}}$
- Every $B \in \mathsf{Alg}_{/A}$ is a quotient of some polynomial algebra $A[t_1, \cdots]$ on potentially infinitely many generators.

Definition 5.0.9 (Finiteness)
An object $B \in \mathsf{Alg}_{/A}$ is **finitely generated** if it is a quotient of some $A[t_1, \cdots, t_n]$. Equivalently, there exist $x_1, \cdots, x_n \in B$ such that any subring containing the x_i and the image of A is all of B.
B is **finite** if $B \in \mathsf{A\text{-}Mod}^{\mathrm{fg}}$.

Example 5.0.10 ***(?):*** Examples:

- $k[t] \in \mathsf{Alg}_{/k}$ is finitely generated but not finite.
- $k[t]/\left\langle t^2 \right\rangle \in \mathsf{Alg}_{/k}$ is finitely generated and finite.
- $(-) \otimes_A (-) \in \mathsf{Fun}(\mathsf{Alg}_{/A}{}^{\times^2}, \mathsf{Alg}_{/A})$, defined by $(b_1 \otimes c_1)(b_2 \otimes c_2) \coloneqq b_1 b_2 \otimes c_1 c_2$.
- $\underset{\mathsf{Alg}_{/A}}{\mathrm{Hom}}(B \otimes_A C, S) = \left\{ f : B \to S, g : C \to S \;\middle|\; f|_A = g|_A \right\}$.
- $k[t_1] \otimes_k k[t_2] \cong k[t_1, t_2]$ is not isomorphic to $k[t_1] \times k[t_2]$ via $(f(t_1), g(t_2)) \mapsto f(t_1) \cdot g(t_2)$, since e.g. $h(t_1, t_2) \coloneqq t_1 + t_2$ is not in the image of this map.

6 | Thursday, January 27

Remark 6.0.1: Nakayama: called a lemma, but arguably the most important statement in commutative algebra! Recall that $J(A) = \cap_{\mathfrak{m} \in \mathrm{mSpec}\, A} \mathfrak{m}$, and being zero mod every $\mathfrak{m} \in \mathrm{mSpec}\, A$ means being zero mod $J(A)$.

Theorem 6.0.2 ***(Nakayama's Lemma).***
Let $A \in \mathsf{CRing}$ with $I \subseteq J(I)$ and let $M \in \mathsf{A}\text{-}\mathsf{Mod}^{\mathrm{fg}}$. Then

$$M = IM \implies M = 0.$$

Example 6.0.3 ***(?):*** This reduces statements about local rings to statements about fields. Commonly used examples:

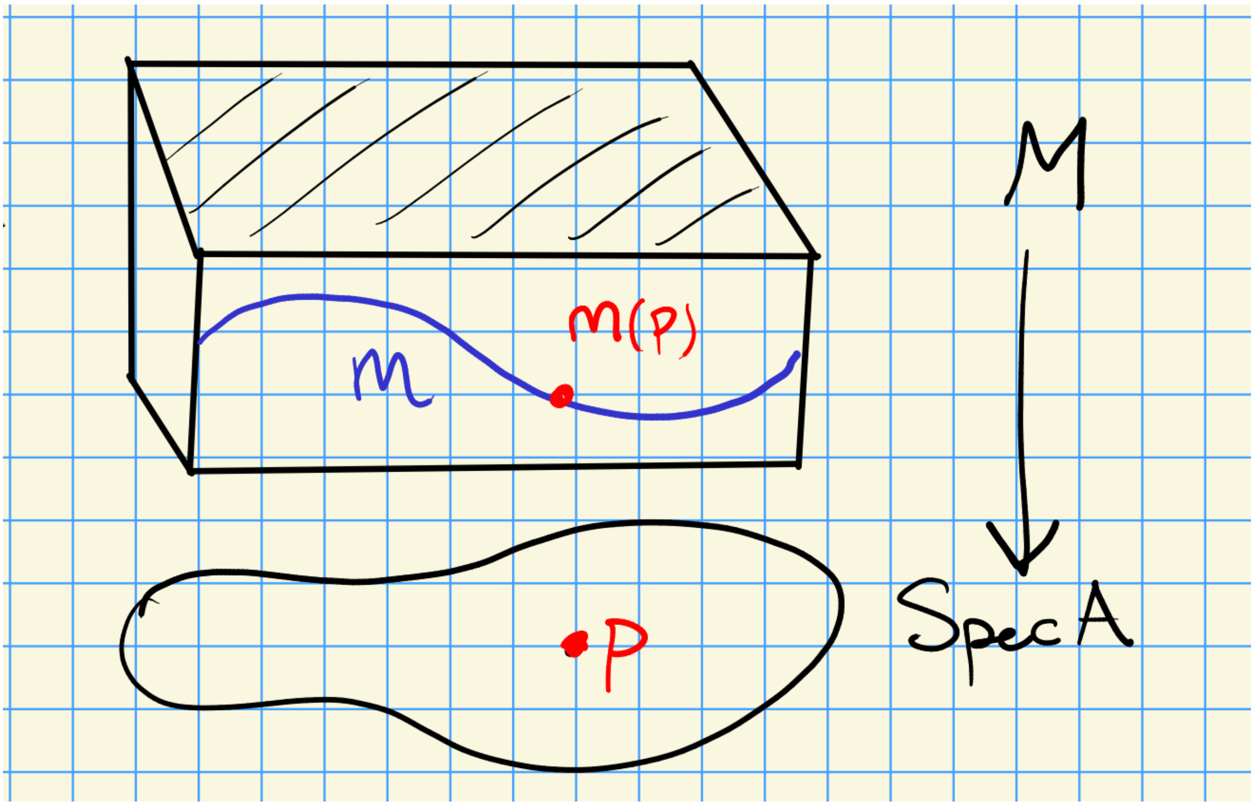
- $I = \sqrt{0_R}$.
- $A \in \mathsf{LocRing}$ with $I = \mathfrak{m}_A$ its maximal ideal.

Corollary 6.0.4 ***(?).***
If A is local and $M \in \mathsf{A}\text{-}\mathsf{Mod}^{\mathrm{fg}}$,

$$M/\mathfrak{m}_A M = 0 \iff M = 0.$$

Remark 6.0.5: M yields a sheaf (or vector bundle) on $\operatorname{Spec} A$, so think of $m \in M$ as a function on $\operatorname{Spec} A$ in the following way: if $m \in M$,

$$m : \mathfrak{p} \mapsto m \bmod \mathfrak{p} \in M/\mathfrak{p}M.$$



Proposition 6.0.6 ***(Equivalent formulation of Nakayama).***
If $n \in N \in \mathsf{A\text{-}Mod}^{\mathrm{fg}}$ and $n \equiv 0 \bmod \mathfrak{m}$ for all $\mathfrak{m} \in \operatorname{mSpec} A$, then $M = 0$.

Exercise 6.0.7 (?)
Show that if $A \in \mathsf{LocRing}$ and $M \in \mathsf{A\text{-}Mod}^{\mathrm{fg}}$ with $\{x_i\}_{i \leq n} \subseteq M$, then

$$\langle x_1, \cdots, x_n \rangle = M \iff \langle \overline{x_1}, \cdots, \overline{x_n} \rangle = M/\mathfrak{m}M, \qquad \overline{x_i} := x_i \bmod \mathfrak{m}.$$

Solution:
$\implies$: If $\bar{y} \in M/\mathfrak{m}M$, lift to $y \in M$ to write $y = \sum c_i x_i$ and thus $\bar{y} = \sum c_i \overline{x_i}$.
$\impliedby$: Present M by $A^n \xrightarrow{f} M \to C = \operatorname{coker} f \to 0$ where $f(e_i) = x_i$ – we want to show $C = 0$. Reduce mod $\mathfrak{m}$ by applying $(-) \otimes_A A/\mathfrak{m}$ to get

$$(A/\mathfrak{m})^n \xrightarrow{\bar{f}} M/\mathfrak{m}M \to C/\mathfrak{m}C \to 0.$$

This is surjective so $C/\mathfrak{m}C = 0$. By Nakayama, $C = 0$.

Example 6.0.8 ***(?):*** Let the local ring $R = k[t]$ with $\mathfrak{m}_R = \langle t \rangle$, and let $M = R^{\oplus^3}$. Consider

$$\left[1 + t^3 + t^5, t + t^2, t^3\right], \left[t^{22}, 1 + t^9, t^{10} + t^{10^{10}}\right], \left[1 + t^{10^{10^{10}}}, 1 + t^5 + t^9, 1 + t^7 + t^{11}\right],$$

which reduced $\bmod\, t$ yields

$$[1, 0, 0], [0, 1, 0], [1, 1, 1].$$

So the original elements generate M.

Remark 6.0.9: Next goal: proving Nakayama. We'll need a version of Cayley-Hamilton. Recall the definition of multiplicative subsets and localization, along with its universal property.

Example 6.0.10 ***(of multiplicative sets S):*** Examples:

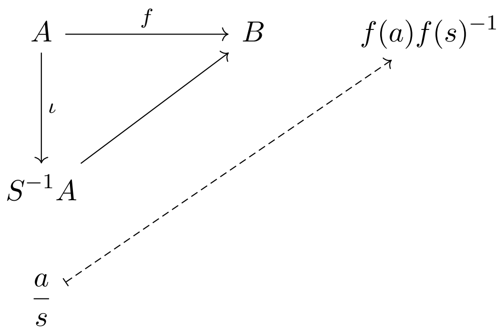
- For any element f, $S := \left\{1, f, f^2, \cdots\right\}$
- For A an integral domain, $S := A \setminus \{0\}$
- All nonzero zero divisors.

Exercise 6.0.11 (?)
Prove $S^{-1}A$ exists and is unique for $A \in \mathsf{CRing}$. Give an example where $s'a = sb$ is not sufficient.

Remark 6.0.12: Remarks on localization:

- $\dfrac{a}{s} = \dfrac{b}{s'} \iff s''s'a - s''sb$ for some $s'' \in S$ – this is needed because it will hold in $S^{-1}A$, since $s'' \in (S^{-1}A)^\times$ for any $s'' \in S$ by construction.

- $\frac{a}{s} = 0$ iff a is annihilated by an element of S.
- Producing the actual map for the universal property: if $f : A \to B$ sends S to invertible elements,



Link to Diagram

- $A_f := S^{-1}A$ for $S := \left\{1, f, f^2, \cdots\right\}$. For $A = \mathbb{Z}$ and $f = p$ a prime, $\mathbb{Z}_f = \mathbb{Z}\left[\frac{1}{p}\right] \subseteq \mathbb{Q}$ are fractions whose denominator is a power of p.
- For A an integral domain and $S = A \setminus \{0\}$ yields $S^{-1}A = \operatorname{ff}(A)$ the fraction field.
- For $\mathfrak{p} \in \operatorname{Spec} A$, $S := A \setminus \mathfrak{p}$, then $A_{\mathfrak{p}} := S^{-1}A$ is localization at a prime ideal
- $\mathbb{Z}_{\langle p \rangle} = \mathbb{Z}[\frac{1}{\ell}]_{\ell \neq p \text{ prime}}$, fractions in $\mathbb{Q}$ with denominators not divisible by p.

Exercise 6.0.13 (?)
Some exercises:

- Use the universal property to show $(\mathbb{Z}/15\mathbb{Z})_5 \cong \mathbb{Z}/3\mathbb{Z}$.
- Show that for $M \in \mathsf{A\text{-}Mod}$, $S^{-1}M$ exists and is unique.
- Show that $S^{-1}A \curvearrowright S^{-1}M$.
- Show that $S^{-1}M = M \otimes_A S^{-1}A$ using the universal property. Use

$$M = M \otimes_A A \xrightarrow{\operatorname{id} \otimes S^{-1}(-)} M \otimes_A S^{-1}A$$

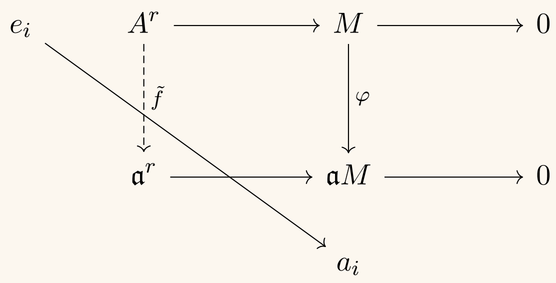
  where $m \mapsto m \otimes 1$ For $f : M \to N$ where s acts invertibly on N, produce a map $M \times S^{-1}A \to N$ where $(m, a/s) \mapsto as^{-1}f(m)$ where s^{-1} is the inverse of the action $s : N \to N$.
- Show that $(-) \otimes_A S^{-1}A$ is left exact and thus exact.
  - Injectivity: use that $\frac{m}{s} \mapsto 0 \iff s'f(m) = 0$ for some $s' \in S$.
- Show that $S^{-1}A \in \mathsf{A\text{-}Mod}^{\flat}$.

7 | Tuesday, February 01

Proposition 7.0.1 *(Cayley-Hamilton).*
For $A \in \mathsf{CRing}, M \in \mathsf{A\text{-}Mod}^{\mathrm{fg}}$, $\mathfrak{a} \in \mathrm{Id}(A)$, and $\varphi : M \to \mathfrak{a}M \subseteq M$, there exists $\{a_i\}_{i \le n} \subseteq \mathfrak{a}$ such that

$$\varphi^r + a_1\varphi^{r-1} + \cdots + a_r \,\mathrm{id} = 0.$$

Proof (Cayley-Hamilton: Reductions).
Reduce to showing this for $M = A^r$ a free module. Use the diagram:

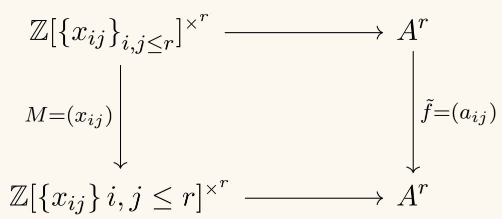


Link to Diagram

Lift by sending e_i to any element $a_i \in \mathfrak{a}$. Then: STS $\tilde f$ satisfies some polynomial, since φ will satisfy the same polynomial and map to zero by commutativity of the square above. $\tilde f : A^r \to A^r$ can be written as a matrix (a_{ij}). Now reduce to $\mathbb{C}$: consider the map

$$\mathbb{Z}[\{x_{ij}\}_{i,j \le r}] \to A$$
$$x_{ij} \mapsto a_{ij}.$$

Forming the matrix $M = (x_{ij})$ yields a commutative diagram:



Link to Diagram

Since $\mathbb{Z}[\{x_{ij}\}]$ is an integral domain, we can pass to the fraction field $\mathbb{Q}(\{x_{ij}\})$, where by linear algebra M has a characteristic polynomial in the x_{ij}. So for some homogeneous polynomials p_i in the x_{ij}, we have

$$\tilde f^r + p_1(x_{ij})\tilde f^{r-1} + \cdots + p_r(x_{ij}) = 0.$$

Now choose an arbitrary embedding $\mathbb{Q}(x_{ij}) \hookrightarrow \mathbb{C}$, and prove Cayley-Hamilton here using (e.g.) Jordan normal form.

■

Exercise 7.0.2 (Fun, everyone should have a proof!)
Write a careful proof of Cayley-Hamilton for arbitrary fields.

Remark 7.0.3: Useful strategy: reduce to the "universal matrix".

Proof (Cayley-Hamilton for arbitrary fields, sketch).
Recall that there exists an *adjugate* of any square matrix satisfying $M\operatorname{adj}(M) = \operatorname{adj}(M)M = \det(M)\operatorname{id}$. Apply this to $M := tI - \tilde{f} \in \operatorname{End}_{A[t]}(A(t)^{\times r})$. Write $\operatorname{adj}(tI - \tilde{f}) = \sum B_i t^i$ with $B_i \in \operatorname{Mat}_{n\times n}(A)$. We have

$$(tI - \tilde{f})\operatorname{adj}(tI - \tilde{f}) = \det(\tilde{f})I,$$

but we can't plug in $\tilde{f}$ here because we don't know if this lands in a *commutative* ring, so e.g. $tmt^r \neq mt^{r+1}$ doesn't necessarily hold.
Note that B_i commutes with $tI - \tilde{f} \iff B_i$ commutes with $\tilde{f}$, e.g. by equating coefficients. Write $R = Z(\tilde{f})$ for the centralizer, those matrices commuting with $\tilde{f}$. Then $(tI - \tilde{f}) \in R[t]$, which reduces us to the world of commutative rings. Then $\det(tI - \tilde{f})\Big|_{\tilde{f}} = (\tilde{f} - \tilde{f}) \cdot g = 0$. ■

Exercise 7.0.4 (?)
Show that if $M \in \mathsf{A\text{-}Mod}^{\mathrm{fg}}$ and $\mathfrak{a} \in \operatorname{Id}(A)$,

$$M = \mathfrak{a}M \implies \exists x \cong 1\mathfrak{a}\text{-}\mathsf{Mod} \text{ such that } xM = 0.$$

Hint: apply Cayley-Hamilton to id_M.

Theorem 7.0.5 ***(Nakayama).***
For $M \in \mathsf{A\text{-}Mod}^{\mathrm{fg}}, I \in \operatorname{Id}(A)$ with $I \in J(A)$,

$$M = IM \implies M = 0.$$

Remark 7.0.6: Apply the corollary to get $x \equiv 1 \bmod I$ with $xM = 0$. But then $x \equiv 1 \bmod J(A)$, so x is a unique and $xM = M$.

Alternatively, pick a minimal set of generators $\{x_i\}$ of M, so $m = \sum a_i x_i$ with $a_i \in I$ since $M = IM$. Since $1 - a_n \in J(A)$ and is a unit, so

$$(1 - a_n)x_n = \sum_{i \le n-1} a_i x_i \implies x_n = (1 - a_n)^{-1} \sum_{i \le n-1} a_i x_i.$$

$\lightning$

Remark 7.0.7: Notes:

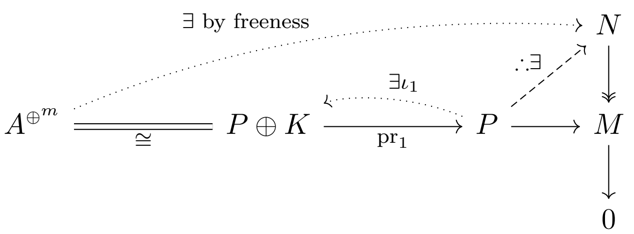
- Proved last time: $A \in \mathsf{LocRing}$, $M \in \mathsf{A\text{-}Mod}^{\mathrm{fg}}$, if $X := \{x_i\} \subseteq M$ with $\{\overline{x_i}\}$ generating $M/\mathfrak{m}M$, then X generates M.

- Suppose $f \in \mathsf{A\text{-}Mod}(M, N)$ with $\overline{f} : M/\mathfrak{m}_A M \to N/\mathfrak{m}_A N$ an isomorphism – f is not necessarily an isomorphism.
  - Counterexample: $k[[x]] \to k$ is not an isomorphism in $\mathsf{k\text{-}Mod}$ but reduces mod x to $k \xrightarrow{\text{id}} k$.
- Show that f need not be injective, but is always surjective.
  - For surjectivity: use $M \xrightarrow{f} N \to C \to 0$, use that $C/\mathfrak{m}C = 0$ to conclude $C = 0$ by Nakayama.
  - For injectivity: use $0 \to K \to M \xrightarrow{f} N \to 0$ and try to show $K = 0$. Not true, take $0 \to \langle x \rangle \to k[[x]] \to k \to 0$ and apply $(-) \otimes_{k[[x]]} k$ to get $k \xrightarrow{0} k \xrightarrow{\text{id}} k \to 0$.
- The special SES $0 \to M \to M \oplus N \to N \to 0$ has a left-section $s : M \oplus N \to M$; applying $(-) \otimes_A S$ or $\underset{A}{\operatorname{Hom}}(S, -)$ actually produces a SES, since this induces left sections on the resulting sequences.
- Prove that free modules are projective.
- Prove that divisible abelian groups are injective in $\mathbb{Z}\text{-}\mathsf{Mod}$.
  - Prove that $\mathbb{Q}$ and $\mathbb{Q}/\mathbb{Z}$ are injective.
- Show that a SES splits iff it admits a right section or a left section.
- Show that $0 \to I \to M \to P \to 0$ splits if either P is projective or I is injective.
- Show that $(-) \otimes_A S, \underset{A}{\operatorname{Hom}}(-, S), \underset{A}{\operatorname{Hom}}(S, -)$ are exact on SES's $0 \to A \to B \to P \to 0$ with P projective.

8 | Thursday, February 03

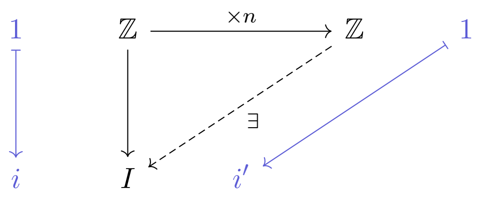
Remark 8.0.1: Exercises:

- Find a way to remember which hom is covariant vs contravariant.
- Show P is projective $\iff$ $\operatorname{Hom}(P, -)$ is exact, I is injective $\iff$ $\operatorname{Hom}(-, I)$ is exact (sends injections to surjections).
- Find a non-free projective module.
- Show that P is projective iff P is a direct summand of a free module. Use that $0 \to K \to A^{\oplus^I} \to P \to 0$ and lift $P \xrightarrow{\text{id}_P} P$ to get $A^{\oplus^I} = K \oplus P$. For the other direction:



Link to Diagram

- Show that in $\mathbb{Z}$-Mod, I is injective iff divisible.
  - For one direction, show $ni' = i$ using the following diagram:



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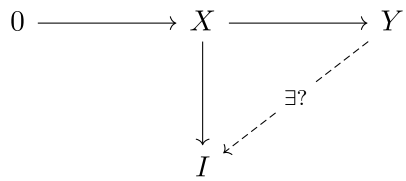
- For the other direction, produce a map:



Link to Diagram

- Use Zorn's lemma on pairs (Y, f) where $(Y, f) \leq (Y', g) \iff Y \subseteq Y'$ and $g|_Y = f$. Show every chain has an upper bound by setting $Y_\infty = \bigcup Y_i$ and $f_\infty = \cup f_i$, thus producing $Y_{\max}, f_{\max}$ Take a SES $0 \to \langle y \rangle \to \langle y, Y_{\max} \rangle \to \langle y, Y_{\max} \rangle / \langle y \rangle \to 0$ and map the last term to I.

- Recall that projective resolutions are complexes $P_\bullet = \cdots P_1 \to P_0 \to 0$ with $H_{i>0} P_\bullet = 0$ and $H_0 P_\bullet = M$, equivalently an exact complex $\cdots \to P_1 \to P_0 \to M \to 0$.

- Find a projective resolution of $C_2 \in \mathbb{Z}\text{-Mod}$ and $k[t]/\left\langle t^2 \right\rangle \in \text{k-Mod}$. Why must the latter necessarily be infinite length?

- Compute $\operatorname{Tor}_*^{\mathbb{Z}}(C_2, C_2) \in \mathbb{Z}\text{-Mod}$ and $\operatorname{Tor}_*^{\mathbb{Z}}(\mathbb{Z}^{\oplus^2}, C_2)$.

- Show $\operatorname{Tor}_*^R(k, k) = \bigoplus_{i \geq 0} k$ for $R = k[t]/\left\langle t^2 \right\rangle$? Use the resolution $P_i = k[t]/\left\langle t^2 \right\rangle$ with maps $(-) \times t$, using that $t \curvearrowright k$ by zero.

- Show that if $f \simeq g$, then $f - g$ induces the zero map in homology. Use $d_{i+1} s_i + s_{i-1} d_i : H_i(C) \to H_i(D)$, pick $\bar{x} \in C_i$ with $d_i \bar{x} = 0$ and check $(d_{i+1} s_i + s_{i-1} d_i)\bar{x} \in \operatorname{im} d_{i+1}$.

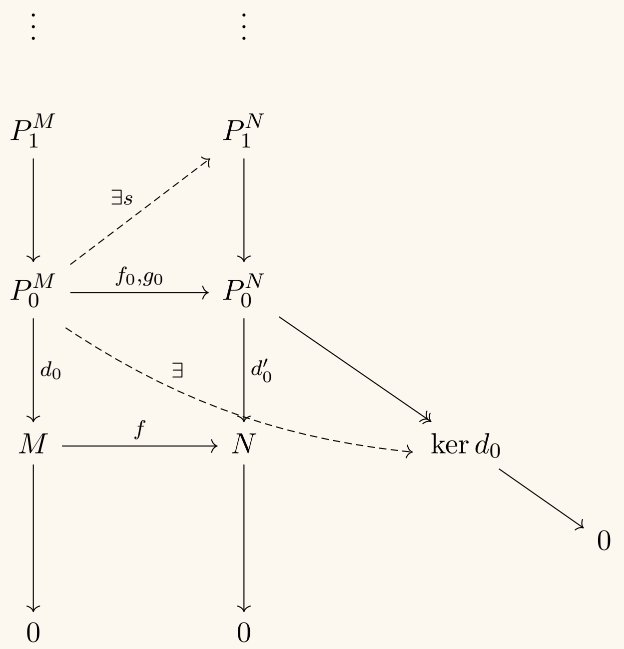
9 | Tuesday, February 08

Exercise 9.0.1 (?)
Show that chain-homotopic maps induce the same map in homology.

Corollary 9.0.2 *(?).*
$\operatorname{Tor}_\bullet^{\mathsf{R\text{-}Mod}}(M, N)$ for $M, N \in \mathsf{R\text{-}Mod}$ is well-defined, since the identity $M \xrightarrow{\mathrm{id}_M} M$ induces an isomorphism on $P_\bullet \otimes N \to P_\bullet' \otimes N$ for any two projective resolution $P_\bullet, P_\bullet' \rightrightarrows M$.

Proposition 9.0.3 *(?).*
If $f \in \mathsf{R\text{-}Mod}(M, N)$, then there is an induced morphism $\tilde{f} \in \mathsf{ChR\text{-}Mod}(P_\bullet{}^M, P_\bullet{}^N)$ between resolutions $P_\bullet{}^M \rightrightarrows M, P_\bullet{}^N \rightrightarrows N$, where $\tilde{f}$ is unique up to homotopy.

Proof (Hint).
For existence, use projectivity to lift through surjections onto kernels. For uniqueness, create s_0 such that $d_1 s_0 = d_0 - g_0$ and check that $\operatorname{im} f_0 - g_0 \subseteq \ker d_0$ after lifting through $P_1^N \to \ker d_0$:



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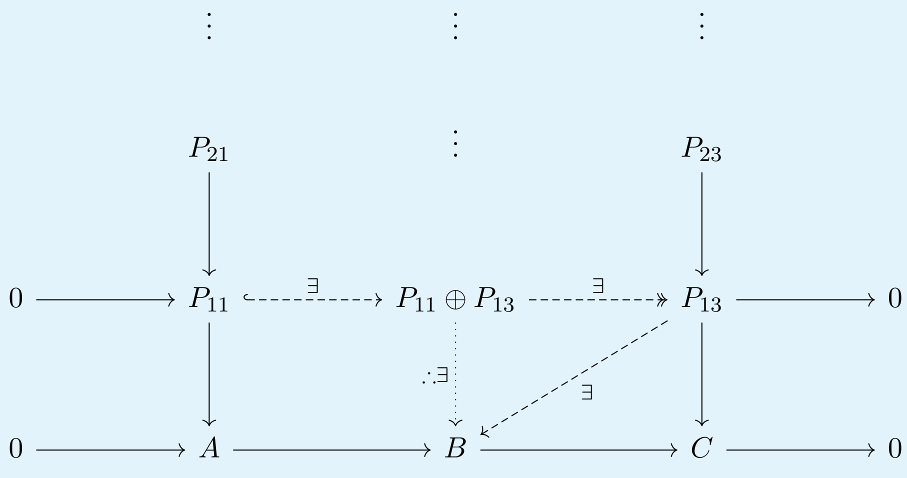
■

Exercise 9.0.4 (?)
Show that a SES of modules induces a SES of chain complexes between their projective

resolutions.
Hint: use the following diagram.



Link to Diagram

Exercise 9.0.5 (?)
Show that a SES of chain complexes induces a LES in their homology. There are about eight conditions one needs to check here (as in the snake lemma, of which this is a special case for two-term complexes).

Exercise 9.0.6 (?)
Show that given $0 \to M_1 \to M_2 \to M_3 \to 0$, taking projective resolutions and applying $(-) \otimes N$ yields a SES

$$0 \to P_\bullet{}^1 \otimes N \to P_\bullet{}^2 \otimes N \to P_\bullet{}^3 \otimes N \to 0,$$

so there is an induced LES in $\mathrm{Tor}_\bullet^{\mathsf{R\text{-}Mod}}$.

Exercise 9.0.7 (?)
Show

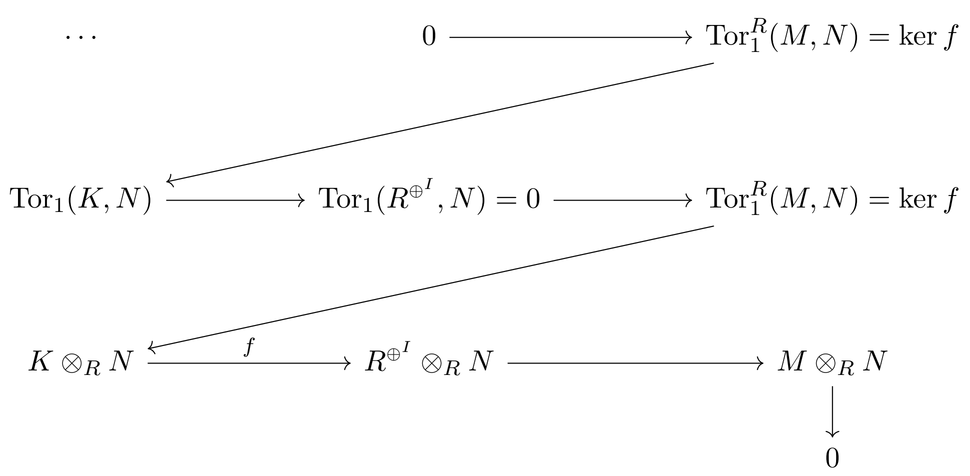
- $\mathrm{Tor}_0^{\mathsf{R\text{-}Mod}}(M, N) = M \otimes_R N$
- $\tau_{\geq 1}\mathrm{Tor}_\bullet{}^{\mathsf{R\text{-}Mod}}(M, N) = 0$ if either M or N is projective.
- $\mathrm{Tor}_\bullet{}^R(M, N)$ is uniquely determined by these properties.

Remark 9.0.8: Hint:

$$\begin{aligned}
\operatorname{Tor}_0^{\textsf{R-Mod}}(M,N) &= H_0(P_\bullet{}^M \otimes_R N) \\
&= \operatorname{coker}\left(P_1 \otimes N \to P_0 \otimes_R N\right) \\
&= \operatorname{coker}(P_1 \to P_0) \otimes_R N \\
&= M \otimes_R N.
\end{aligned}$$

For vanishing, use that projective implies flat and exact complexes have zero higher homology. Note that if M is projective, it is its own projective resolution.

For uniqueness, induct on i: write $0 \to K \to R^{\oplus^J} \to M \to 0$, use that free implies projective, and consider the LES:



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Now induct up using the isomorphisms in the LES.

Exercise 9.0.9 (?)
Show that

$$\operatorname{Tor}_\bullet^{\textsf{R-Mod}}(M,N) \cong H_\bullet(P_\bullet{}^M \otimes_R N) \cong H_\bullet(M \otimes_R P_\bullet{}^N) \cong \operatorname{Tor}_\bullet^{\textsf{R-Mod}}(N,M).$$

10 | Thursday, February 10

Exercise 10.0.1 (?)
Let $R = k[x,y]$ and $M = k \cong k[x,y]/\langle x,y \rangle$, and compute $\mathrm{Tor}_\bullet^{\mathsf{R\text{-}Mod}}(k,k)$.

Solution:
Hint: use the following resolution

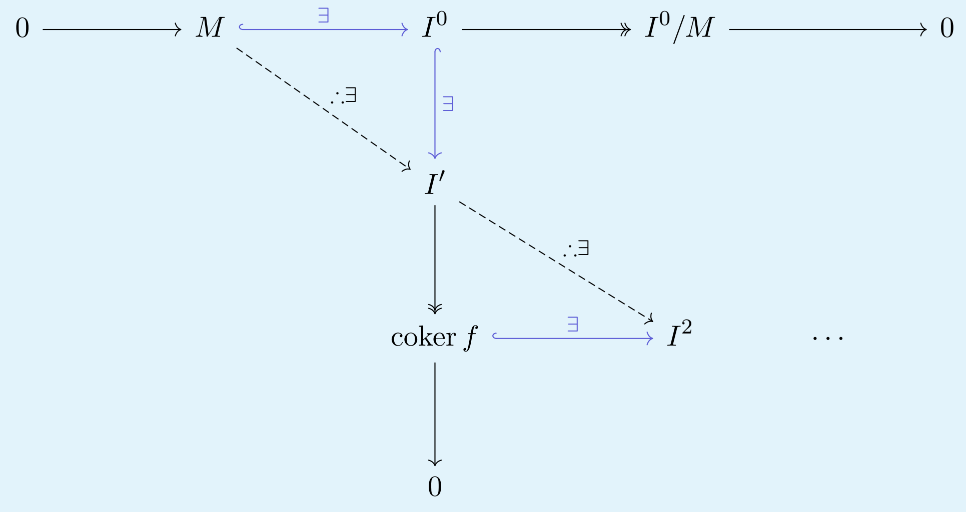
$$\begin{array}{ccccccccccc}
 & & 1 & \mapsto & (y,-x) & & 1 & \mapsto & 1 & & \\
0 & \to & k[x,y] & \to & k[x,y]^{\otimes_R^2} & \to & k[x,y] & \to & k & \to & 0 \\
 & & & & e_1 & \mapsto & x & & & & \\
 & & & & e_2 & \mapsto & y & & & &
\end{array}$$

Link to Diagram

Exercise 10.0.2 (?)
Compute $\mathrm{Tor}_\bullet^{\mathsf{R\text{-}Mod}}(k,k)$ for $R = k[x_1, \cdots, x_n]$ and $M = k = k[x_1, \cdots, x_n]/\langle x_1, \cdots, x_n \rangle$.

Remark 10.0.3: Tor measures failure of injectivity of tensoring against a module M, $\mathrm{Ext}^\bullet$ measures failure of surjectivity when mapping against M.

Exercise 10.0.4 (?)
Show that for $R \in \mathsf{CRing}$, every $M \in \mathsf{R\text{-}Mod}$ admits an injective resolution.
Hint: it suffices to show any M injects into an injective object. Use the following diagram:



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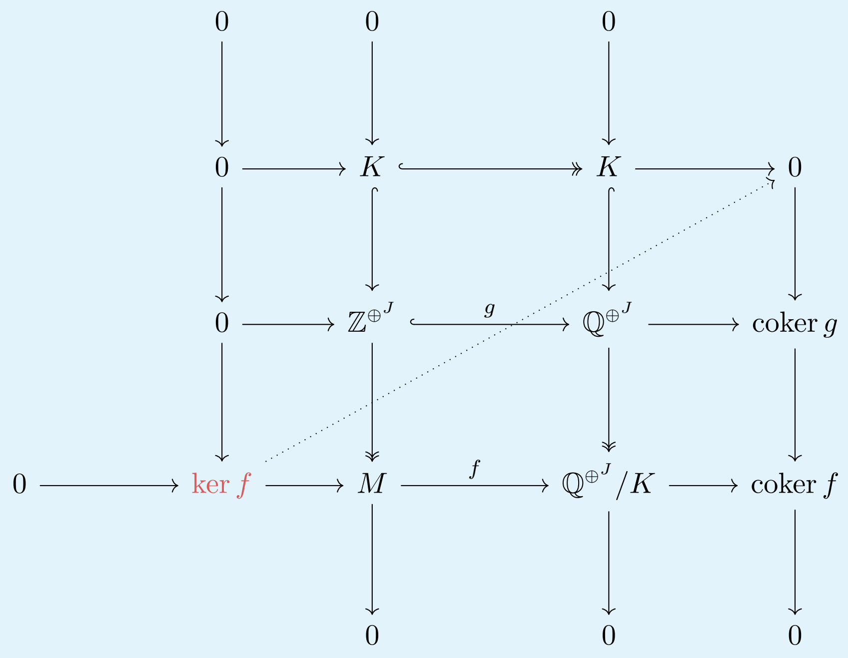
First show this for $R = \mathbb{Z}$ using that $\mathbb{Q}^{\oplus^J}/K$ is divisible and thus injective:



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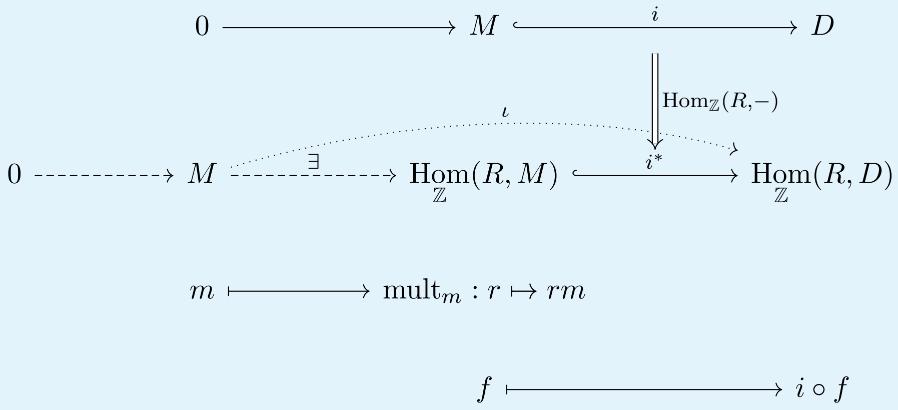
Now reduce to $R = \mathbb{Z}$ using that $M \hookrightarrow D$ for D a divisible abelian group, and show $M \hookrightarrow I := \underset{\mathbb{Z}\text{-Mod}}{\mathrm{Hom}}(R, D) \in \mathsf{R\text{-}Mod}$. Form the map as the composition:



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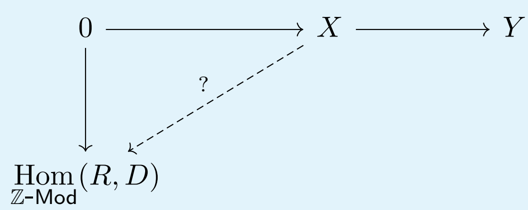
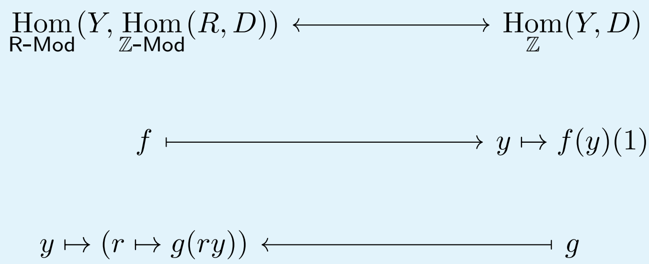
Then show that $I := \underset{\mathbb{Z}\text{-Mod}}{\text{Hom}}(R, D)$ is injective using the universal property:





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Exercise 10.0.5 (?)
Show $\mathbb{Z} \in \mathbb{Z}\text{-Mod}$ has an injective resolution.

Solution:

$$0 \to \mathbb{Z} \hookrightarrow \mathbb{Q} \twoheadrightarrow \mathbb{Q}/\mathbb{Z} \to 0.$$

Exercise 10.0.6 (?)
Show $\text{Ext}^{\bullet}{}_{\mathbb{Z}\text{-Mod}}(C_3, \mathbb{Z}) \cong C_3[1]$.

Hint: apply $\underset{\mathbb{Z}\text{-Mod}}{\text{Hom}}(C_3, -)$ *to the above resolution and use* $\underset{\mathbb{Z}\text{-Mod}}{\text{Hom}}(C_3, \mathbb{Q}/\mathbb{Z}) \cong C_3$

Exercise 10.0.7 (?)
Show that if $f \in \mathsf{R\text{-}Mod}(M, N)$ then there is an induced chain map $\tilde{f} \in \mathsf{ChR\text{-}Mod}(I^{\bullet}{}_M, I^{\bullet}{}_N)$ which is unique up to homotopy. Conclude that $\text{Ext}^{\bullet}{}_{\mathsf{R\text{-}Mod}}(M, N)$ is independent of injective resolution.

Hint: take $f = \text{id}_M$.

Exercise 10.0.8 (?)
Show that if $0 \to A \to B \to C \to 0$ is a SES in $\mathsf{R\text{-}Mod}$ with $I^{\bullet}{}_A \leftleftarrows A, I^{\bullet}{}_C \leftleftarrows B$, then there is a complex $I^{\bullet}{}_B \leftleftarrows B$ making $0 \to I^{\bullet}{}_A \to I^{\bullet}{}_B \to I^{\bullet}{}_C \to 0$ a SES in $\mathsf{ChR\text{-}Mod}$.

Exercise 10.0.9 (?)
Show that a SES in $\mathsf{R\text{-}Mod}$ induces a LES in $\text{Ext}^{\bullet}{}_{\mathsf{R\text{-}Mod}}$. Do this for both homs: start with $0 \to A \to B \to C \to 0$, and produce a LES for $\text{Ext}^{\bullet}{}_{\mathsf{R\text{-}Mod}}(M, -)$ and $\text{Ext}^{\bullet}{}_{\mathsf{R\text{-}Mod}}(-, M)$.

Hint: for the first case, apply $\underset{\mathsf{R\text{-}Mod}}{\text{Hom}}(M, -)$ *to the SES of chain complexes of injective resolutions, and use that if* I_1 *is injective then the SES splits. For the second, use that* $\text{Hom}(-, I)$ *is exact iff* I *is injective and take an injective resolution of* M.

Exercise 10.0.10 (?)
Show that $\text{Ext}^{\bullet}{}_{\mathsf{R\text{-}Mod}}(M, N)$ is uniquely characterized by

1. $\text{Ext}^0_{\mathsf{R\text{-}Mod}}(M, N) \cong \underset{\mathsf{R\text{-}Mod}}{\text{Hom}}(M, N)$
2. $\tau_{\geq 1}\text{Ext}^{\bullet}{}_{\mathsf{R\text{-}Mod}}(M, N) = 0$ if M is projective or N is injective.
3. The two LESs above exist.

Solution:
Hints:

- Resolve $I^{\bullet}{}_N \leftleftarrows N$, apply $\underset{\mathsf{R\text{-}Mod}}{\text{Hom}}(M, -)$, and identify $\text{Ext}^0_{\mathsf{R\text{-}Mod}}$ as a kernel.
- N is its own injective resolution when N is injective.
- $\underset{\mathsf{R\text{-}Mod}}{\text{Hom}}(M, -)$ is exact when M is projective.
- For uniqueness, use that if $0 \to N \to I \to C \to 0$ with I injective, then the middle terms in the LES vanish to get isomorphisms.

11 | Tuesday, February 15

Exercise 11.0.1 (?)
Check that $\operatorname{Ext}_R^i(M, -)$ is independent of injective resolutions, and $\operatorname{Ext}_R^i(-, N)$ is independent of projective resolutions.

Exercise 11.0.2
Check that $\operatorname{Ext}^\bullet$ is determined by

- $\operatorname{Ext}^0 = \operatorname{Hom}$
- $\operatorname{Ext}^{i>0}(P, I) = 0$ if P is projective or I is injective.
- It extends SESs to LESs

Exercise 11.0.3 (?)
Show

$$\operatorname{Ext}^\bullet{}_{k[t]/\langle t^2 \rangle}(k, k) = \bigoplus_{i \geq 0} k[i].$$

Use the projective resolution with entries $k[t]/\left\langle t^2 \right\rangle$ with differential $\partial = \cdot t$

Exercise 11.0.4 (?)
Compute $\operatorname{Ext}^\bullet{}_{k[x_1, \cdots, x_n]}(k, k)$ using the Koszul resolution $\bigwedge^\bullet k[x_1, \cdots, x_n] \rightrightarrows k$.

Remark 11.0.5: Defining Noetherian rings and modules:

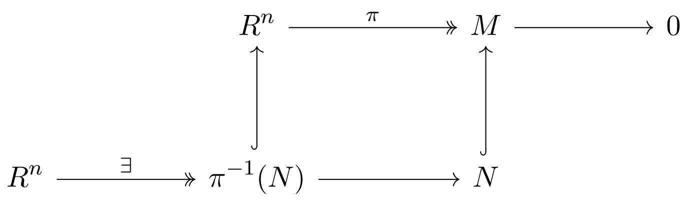
- $A \in \mathsf{CRing}$ is Noetherian iff every $\operatorname{Id}(A) \subseteq \mathsf{R\text{-}Mod}^{\mathrm{fg}}$.
- $M \in \mathsf{A\text{-}Mod}$ is Noetherian iff every $N \leq M$ satisfies $N \in \mathsf{A\text{-}Mod}^{\mathrm{fg}}$

Thank you Emmy Noether!!

Exercise 11.0.6 (?)
Show that TFAE:

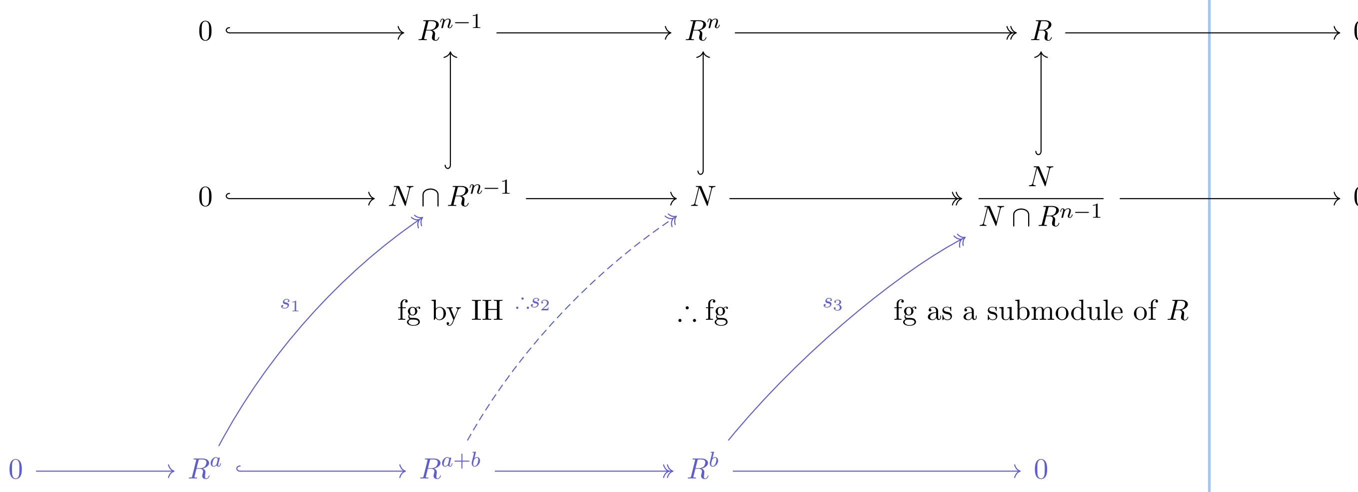
- R is Noetherian
- Every $M \in \mathsf{R\text{-}Mod}^{\mathrm{fg}}$ is Noetherian.

Solution:
For 1 $\implies$ 2, it STS that R^n is Noetherian. To reduce, use the diagram



Link to Diagram

To show R^n is Noetherian, use induction since we know R^1 is Noetherian. Use the following diagram, using the snake lemma on s_1, s_3 to show s_2 is surjective:



Link to Diagram

Exercise 11.0.7 (?)
Show TFAE:

- $M \in \mathsf{A\text{-}Mod}$ is Noetherian.
- The ACC for submodules.

Solution:
Hint: for $1 \implies 2$, set $M_\infty = \bigcup_i M_i \leq M \in \mathsf{A\text{-}Mod}$. Write $M_i = \langle x_1, \cdots, x_n \rangle$ and use that each $x_k \in M_{i_k}$ to choose $N \gg 1$ with $M_\infty = M_N$. For $2 \implies 1$, for M non-Noetherian find $S \leq M$ infinitely generated as $S = \langle x_1, \cdots \rangle$ and take the chain $\{S_k\}_{k \geq 0}$ where $S_k = \langle x_1, \cdots, x_k \rangle$.

Exercise 11.0.8 (?)
Show that the following are Noetherian:

- Fields
- PIDs

- R/I for R Noetherian and I arbitrary
- For A Noetherian, $A[x]$ (Hilbert's basis theorem).[a]
  - Similarly $A[[x]]$.
- Localizations of Noetherian rings

[a] A very important result! Marks the end of invariant theory historically.

Theorem 11.0.9 ***(?).***
If A a Noetherian local ring and $M \in \mathsf{A\text{-}Mod}^{\text{fg,proj}}$, then M is free.

Exercise 11.0.10 (?)
Prove this!

Solution:
Hints:

- Surjectivity:
  - Choose a basis $M/\mathfrak{m}M = \left\langle \{\bar{x}\}_{k \leq n} \right\rangle$ and lifts x_k to M.
  - Take a surjective map $A^n \to M$ where $e_i \mapsto x_i$.
  - Take the SES $A^n \xrightarrow{f} M \twoheadrightarrow \operatorname{coker} f \to 0$ and apply $(-) \otimes_A A/\mathfrak{m}$; use that $(A/\mathfrak{m})^n \xrightarrow{\sim} M/\mathfrak{m}M$ and apply Nakayama.
- Injectivity:
  - Write $0 \to K = \ker f \to A^n \xrightarrow{f} M \to 0$ and apply $(-) \otimes_A A/\mathfrak{m}$ to get

$$\cdots \to \operatorname{Tor}^0(M, A/\mathfrak{m}) \to K/\mathfrak{m}K \to (A/\mathfrak{m})^n \xrightarrow{\sim} M/\mathfrak{m}M \to 0,$$

then $K/\mathfrak{m}K = 0$ and apply Nakayama to conclude $K = 0$.

Remark 11.0.11: Try an example: $k[x] \in \mathsf{k\text{-}Mod}$, which is free after reduction mod $\mathfrak{m} = \langle x \rangle$ but not free before reduction.

Remark 11.0.12: Note that this works with projective replaced by flat.

Remark 11.0.13: Why care about Noetherian rings? Hilbert studied group actions G on modules over (say) polynomial rings and wanted to find the submodules of G-invariants. There was an industry of writing down generating sets in order to show existence and finiteness, and the basis theorem (which is partially effective) showed that this is no longer necessary – finite generating sets always exist.

Theorem 11.0.14 ***(Hilbert's basis theorem).***
If A is Noetherian then $A[x]$ is Noetherian.

Proof (?).
Fix $U \subseteq A[x]$ an ideal, so $I = \left\{ f \in I \;\middle|\; f = \sum a_{f,i} x^i \right\}$. Write $J = \langle a_{f,\deg f} \rangle$ be the ideal generated by all leading coefficients of elements in I. Since A is Noetherian, J is finitely-generated, so write $J = \langle a_1, \cdots, a_n \rangle$ and choose $\{f_1, \cdots, f_n\}$ so that a_i is the leading coefficient of f_i. Note that these exist since $J = A\{a_1, \cdots, a_n\}$ (i.e. these already form an ideal). Consider $L \coloneqq I \cap A[x]^{\deg \leq d}$: since A is Noetherian, $A[x]^{\deg \leq d} \in \mathsf{A\text{-}Mod}^{\mathrm{fg}}$, and L also forms a finitely generated A-module. Write the generators as $L = \{g_1, \cdots, g_m\}$.

Claim:

$$I = I_{fg} \coloneqq \langle f_1, \cdots, f_n, g_1, \cdots, g_m \rangle .$$

Proof (?).
If not, pick $f \in I \setminus I_{fg}$ of minimal degree. Then $\deg f > d$ by construction of L. Write $f = \sum b_i x^i$ where $b_{\deg f} = \sum c_i a_i$, and check $f - \sum c_i f_i x^{\deg f - \deg f_i} \in I$. ■

■

Remark 11.0.15: Idea: the f_k take care of low degree elements, the g_k knock things down in degree.

Corollary 11.0.16 *(?).*
Any quotient $k[x_1, \cdots, x_n]/I$ is Noetherian, as is $\mathbb{Z}[x_1, \cdots, x_n]/I$.

Example 11.0.17 ***(Non-Noetherian rings):*** Some examples:

- $k[x_1, \cdots,]$ is not Noetherian: take $I = \langle x_1, \cdots \rangle$.
- $k[t.t^{\frac{1}{2}}, t^{\frac{1}{3}}, t^{\frac{1}{4}}, \cdots]$.
- $\mathbb{Z}\left[\left\{ 2^{\frac{1}{n}} \right\}_{n \geq 0} \right]$. Note this is countable!

Remark 11.0.18: Next time: other finiteness conditions, integrality, the Nullstellensatz.

12 | Thursday, February 17

Exercise 12.0.1 (?)
Prove the correspondence theorem between $\mathrm{Id}(A)$ and $\mathrm{Id}(S^{-1}A)$.

$$\begin{array}{ccc} A & \xrightarrow{\iota} & S^{-1}A \\ \uparrow & & \uparrow \\ \iota^{-1}(I) & \longrightarrow & I \end{array}$$

Link to Diagram

Use that

- $\iota^{-1}I \in \mathrm{Id}(A)$
- $\iota(\iota^{-1}(I))$ generates I.
- $\frac{a}{s} = \frac{1}{s}\frac{a}{1}$ and $\frac{a}{1}$ is in the image of the lower map.

Exercise 12.0.2 (?)
Show that the localization of any Noetherian ring is again Noetherian.

Exercise 12.0.3 (?)
Say $M \in \mathsf{R\text{-}Mod}^{\mathrm{fp}}$ iff there is an exact sequence $R^a \to R^b \to M \to 0$. Show that if R is Noetherian, $\mathsf{R\text{-}Mod}^{\mathrm{fp}} = \mathsf{R\text{-}Mod}^{\mathrm{fg}}$.

Definition 12.0.4 (Integrality)
For $f \in \mathsf{CRing}(A, B)$ and $b \in B$ define

$$\begin{aligned} f_b : A[x] &\to B \\ x &\mapsto b \\ \sum a_i x^j &\mapsto \sum f(a_i) b^i. \end{aligned}$$

The element b is **integral over** A iff $\operatorname{im} f_b \in \mathsf{A\text{-}Mod}^{\mathrm{fg}}$. The ring B is **integral over** A iff every $b \in B$ is integral over A as above.

Exercise 12.0.5 (?)
Show that $\mathbb{Z}[\frac{1}{2}] \subseteq \mathbb{Q}$ is not integral.

Exercise 12.0.6 (?)
Show TFAE:

- b is integral over A
- b satisfies a monic polynomial over A, so $b^n + \sum_{i=1}^{n} a_i b^{n-i} = 0$.

Solution:
Hints:

- Write $\operatorname{im} f_b = I \coloneqq \left\langle 1, b, \cdots, b^{n-1} \right\rangle$, suppose $b^n \in I$, and show $b^{n+1} \in I$ by expanding as a sum.
- Set $M_i = \left\langle 1, b, \cdots, b^i \right\rangle$ and use that these stabilize to conclude $b^n \in M_{n-1}$ for some n.

⚠ **Warning 12.0.7**
B Noetherian over A does not imply B is Noetherian! Consider $\overline{\mathbb{Z}}$, the algebraic integers, which is integral over $\mathbb{Z}$. Note that $\left\langle 2^{\frac{1}{n}} \right\rangle$ is not finitely-generated.

13 | Toward the Nullstellensatz

Remark 13.0.1: Idea: there is a dictionary between $k[x_1, \cdots, x_n]$ and $\mathbb{A}^n_{/k}$ for $k = \bar{k}$:

- Points in $\mathbb{A}^n$ correspond to maps $k[x_1, \cdots, x_n] \to k$ given by $x_i \mapsto c_i \in k$.
- Rings R correspond to their prime spectra $\operatorname{Spec} R$.
- Kernels correspond to maximal ideals
- Ideals $I \in \operatorname{Id}(R)$ correspond to $V(I) = \left\{ \mathfrak{m} \;\middle|\; I \subseteq \mathfrak{m} \right\}$
- Ideals $I \in \operatorname{Id}(k[x_1, \cdots, x_n])$ correspond to $V(I) = \left\{ \mathbf{c} \in k^{\times^n} \;\middle|\; f(\mathbf{c}) = 0 \, \forall f \in I \right\}$.

Theorem 13.0.2 ***(Nullstellensatz V1).***
Any $\mathfrak{m} \in \operatorname{mSpec} k[x_1, \cdots, x_n]$ is given by $\ker \pi_p$ for some morphism

$$\begin{aligned} \pi_p : k[x_1, \cdots, x_n] &\to k \\ x_i &\mapsto c_i \end{aligned}$$

for some $p = [c_1, \cdots, c_n] \in k^{\times^n}$. Equivalently, $\mathfrak{m} = \langle x_1 - c_1, \cdots, x_n - c_n \rangle$.

Theorem 13.0.3 ***(Nullstellensatz).***
If $f \in k[x_1, \cdots, x_n]$ satisfies $f|_{V(I)} = 0$ for a fixed $I \in \operatorname{Id}(k[x_1, \cdots, x_n])$, then $f^n \in I$ for some $n > 0$. I.e. there is a bijective correspondence $I \rightleftharpoons V(I)$ for radical ideals $\sqrt{I} = I$.

Example 13.0.4 ***(?):*** Necessity of conditions:

- Why $k = \bar{k}$ is needed: take $\langle x^{-}2 \rangle \in \mathbb{Q}[x]$, this is a maximal ideal but $V(I) = 0$.

- Why the radical condition is needed: take $I = \left\langle x^{10} \right\rangle$ so $V(I) = \{0\} \subseteq k$, but $x|_{V(I)} = 0$.

Corollary 13.0.5 ***(?).***
For $k = \bar{k}$ and $R \in \mathsf{Alg}_{/k}{}^{\mathrm{fg}}$,

$$\operatorname{mSpec} R \rightleftharpoons \underset{\mathsf{Alg}_{/k}}{\operatorname{Hom}}(R, k).$$

Corollary 13.0.6 ***(?).***
$V(f) = k^{\times n} = V(0)$ iff $f \in \sqrt{0_R}$.

Corollary 13.0.7 ***(?).***
Let $R \in \mathsf{Alg}_{/k}{}^{\mathrm{fg}}$, so $R = k[x_1, \cdots, x_n]/I$ for some n, and let $V(I) \subseteq k^{\times n}$. Given $J \subseteq R$ radical, $V(J) \subseteq V(I)$, and $f \in R$ vanishes on $V(J)$ iff $f^n \in J$ for some $n > 0$.

Theorem 13.0.8 ***(Maximal idealansatz).***
For $k = \bar{k}$ and $A \in \mathsf{Alg}_{/k}{}^{\mathrm{fg}}$ with $\mathfrak{m} \in \operatorname{mSpec} A$,

$$A/\mathfrak{m} \cong k.$$

Theorem 13.0.9 ***(Maximal idealansatz 2).***
For k an arbitrary field and $A \in \mathsf{Alg}_{/k}{}^{\mathrm{fg}}$ with $\mathfrak{m} \in \operatorname{mSpec} A$,

$$A/\mathfrak{m} \text{ is a finite extension of } k.$$

Theorem 13.0.10 ***(EEKS / proto Zariski's lemma).***
If $k \subseteq F$ fields with $F \in \mathsf{Alg}_{/k}{}^{\mathrm{fg}}$, then $F_{/k}$ is a finite extension of fields.

Corollary 13.0.11 ***(?).***
If $R \in \mathsf{Alg}_{/\mathbb{Z}}{}^{\mathrm{fg}}$ and $\mathfrak{m} \in \operatorname{mSpec} R$, then $R/\mathfrak{m}$ is finite.

Proof (?).
Check $\mathfrak{m} \cap \mathbb{Z} = \mathfrak{p} \in \operatorname{Spec} \mathbb{Z}$ and $R/\mathfrak{m}$ is a finitely generated extension of $\mathbb{F}_p$ and hence finite. ■

Lemma 13.0.12 ***(Zariski).***
If $A \in \mathsf{CRing}^{\mathrm{Noeth}}$ and $A \subseteq B \subseteq C$ with

- $C \in A\text{-}\mathsf{Alg}^{\mathrm{fg}}$ and
- $C \in \mathsf{B\text{-}Mod}^{\mathrm{fg}}$,

then $B \in A\text{-}\mathsf{Alg}^{\mathrm{fg}}$.

Proof (?).
Sketch:

- Choose $x_1, \cdots, x_n$ generating C as an A-algebra
- Choose $y_1, \cdots, y_m$ generating C as a B-module.
- Write $x_i = \sum_j b_{ij} y_j$ with $b_{ij} \in B$
- Write $x_i x_j = \sum_k b_{ijk} y_k$ with $b_{ijk} \in B$
- Let $B_0 \subseteq B$ be the A-algebra generated by the b_{ij} and b_{ijk}.
- Observe that B_0 is Noetherian by the Hilbert basis theorem since it's finitely generated as an A-algebra.
- For any $c \in C$, write $c = \sum_i b_i y_i$ since the y_i generate C as a B_0 module.
- Idea: can rewrite in terms of lower degree monomials?
- Since each $b_i = \sum a_I^i x^I$, we have $c = \sum_i \sum_I a_I^i x^I y_i$, which is a polynomial in $\sum b_{ij} y_j$.
- Then $y_i \cdots y_{i_k} = \sum p_s y_s$ where the p_s are polynomials in the b_{ijk}?
- Since B_0 is Noetherian and $B \subseteq C$, B is finitely generated as a B_0 module and thus as a B_0 algebra.
- Since $A \subseteq B_0 \subseteq B$ and B_0 is finitely-generated as an A algebra and B is finitely-generated as a B_0 algebra, we have that B is finitely-generated as an A algebra.

■

14 | Thursday, February 24

14.1 Going up, going down

Theorem 14.1.1 ***(Going Up I).***
Let $A \subseteq B \in \mathsf{CRing}$ with B integral over A. Then for every $p \in \operatorname{Spec} A$ there is a $q \in \operatorname{Spec} B$ with $p = A \cap q$.

Remark 14.1.2: Idea: $\operatorname{Spec} B \to \operatorname{Spec} A$ has finite fibers.

Lemma 14.1.3 ***(?).***
If $A \hookrightarrow B$ is an integral extension, then $A \in \mathsf{Field} \iff B \in \mathsf{Field}$.

Proof (of lemma).
$\implies$: Last time.

$\Longleftarrow$: Given $x \in A$ we have $x^{-1} \in A$. Then if $f(x) = x^{-n} + \sum_{0 \leq k \leq n-1} a_k x^{-k}$, with $f(x) = 0$, then $x^{n-1} f(x) = 0 \implies x^{-1} = -\sum a_k x^k \in A$.

■

Proof (of Going Up).
Note $A\left[p^{-1}\right] \hookrightarrow B\left[p^{-1}\right]$ is still integral, and let $q' \in \operatorname{mSpec} B\left[p^{-1}\right] \neq \emptyset$. Write p for the maximal ideal of A, then (claim) $A\left[p^{-1}\right] \cap q' = p$. STS $A\left[p^{-1}\right] \cap q'$ is maximal, since these are local rings, and so it's ETS $A\left[p^{-1}\right] / (A\left[p^{-1}\right] \cap q') \in \mathsf{Field}$. Since this is integral iff $B\left[p^{-1}\right] / q'$ is integral, which it is, by the lemma this is a field too.

Exercise *(?)*
Show tat taking the preimage of q' in B works.

■

Theorem 14.1.5 ***(Going Up II: partial lifts of chains of primes extend).***
Given $A \hookrightarrow B$ integral and

$$\begin{aligned} p_1 \subseteq p_2 \subseteq \cdots \subseteq p_n \in \operatorname{Spec} A \\ q_1 \subseteq q_2 \subseteq \cdots \subseteq q_m \in \operatorname{Spec} B \end{aligned},$$

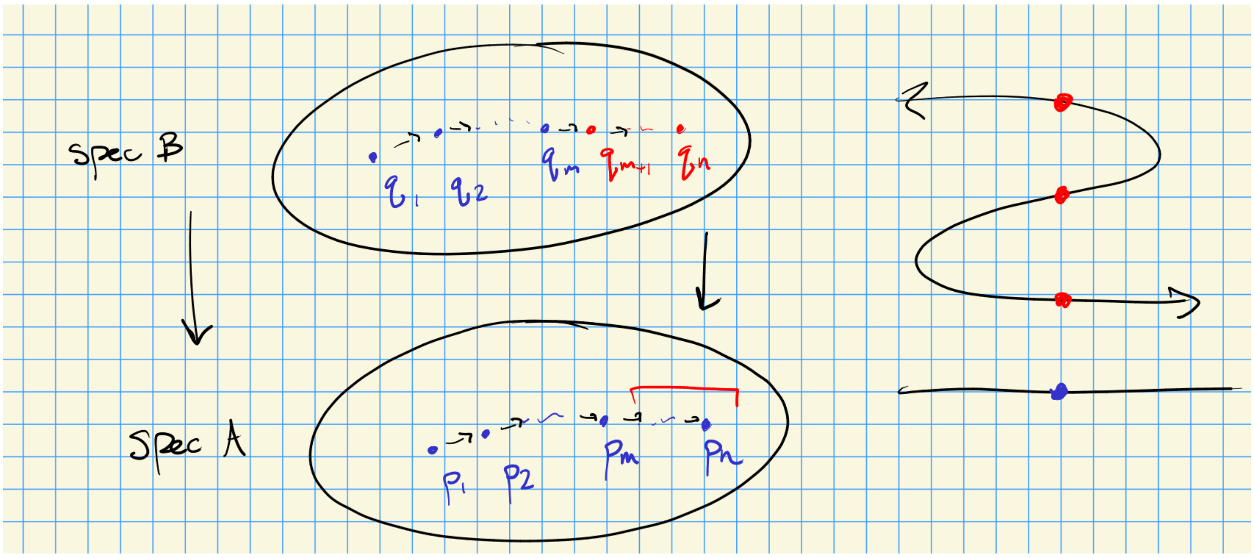
where $p_i = q_i \cap A$ and e.g. with $m \leq n$, then there exist $q_{m+1}, \cdots, q_n$ with $q_j \subseteq q_{j+1}$ and $q_i \cap A = p_i$.

Proof (?).
Note that it's enough to lift one stage and induct. So given $q_1 \subseteq \cdots q_n$, it's ETF $q_{m+1} \supseteq q_m$ with $q_{m+1} \cap A = p_{m+1}$. Strategy:

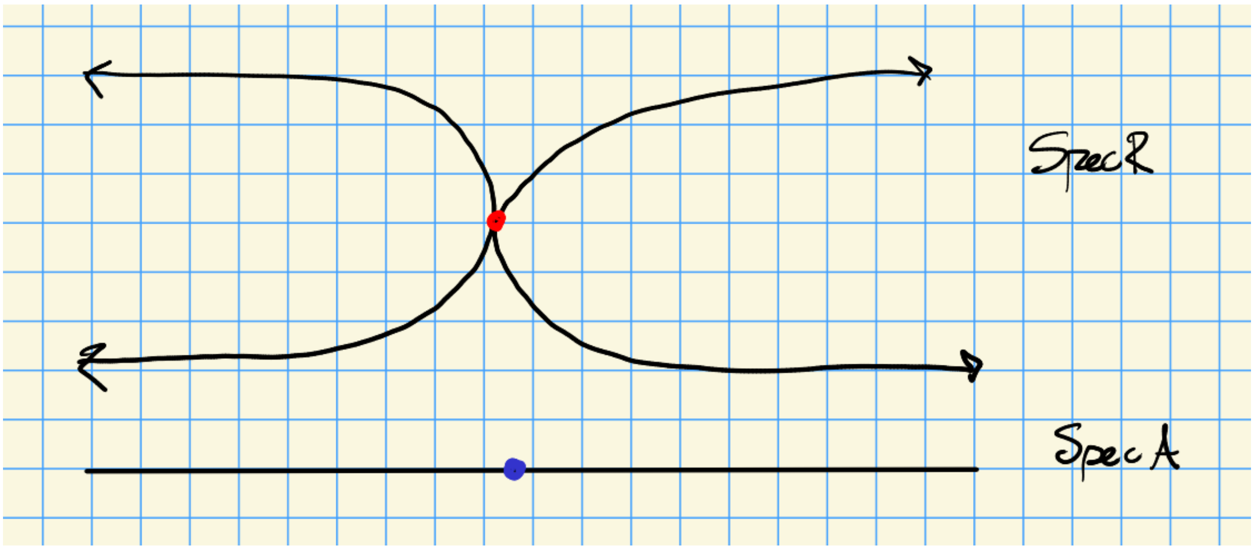
- Replace A with A/p_n and B with B/q_n.
- Find $\bar{q} \in B/q_m$ with $\bar{q} \cap A/p_n$ the image of p_{n+1}
- Use that $\bar{q}$ exists by Going Up I.

■

Remark 14.1.6: The geometry: $A \to B \rightsquigarrow \operatorname{Spec} B \xrightarrow{\pi} \operatorname{Spec} A$. Increasing chains p_i means $p_{i+1} \in \operatorname{cl}_{\operatorname{Spec} A} p_i$, and "going up" means sequences can be completed with points in closures in $\operatorname{Spec} B$ I.e. π is a closed map, i.e. closed under specialization (passing to a point in the closure). Idea: covering map, possibly with ramification or splitting.



Example 14.1.7 ***(?):*** Consider $k[x] \hookrightarrow k[x,y]/\left\langle y^2 - x\right\rangle$ over $\operatorname{ch} k = 0$ and $k = \bar{k}$. Take $p = \langle x - 2\rangle$ and $q = \left\langle y - \sqrt{2}\right\rangle$ or $\left\langle y + \sqrt{2}\right\rangle$ extend p.



Similarly, take $\mathbb{Z} \hookrightarrow \mathbb{Z}[i]$ and consider how primes lift (see inert, split, ramified).

> **Definition 14.1.8** (Integrally closed)
> For $A \subseteq B$, say A is **integrally closed in** B iff A contains every element of B which is integral over A.

Example 14.1.9 ***(?):*** $k[x] \subseteq k[x,y]$ is integrally closed, but $k[x] \subseteq k[x,y]/\left\langle y^2 - x\right\rangle$ is not.

Theorem 14.1.10 ***(Going Down).***
For $A \leq B \in \mathsf{IntDom}$ with A integrally closed in $\mathrm{ff}(A)$ and B integral over A. If

$$p_1 \supseteq p_2 \supseteq \cdots \supseteq p_n \in \operatorname{Spec} A$$
$$q_1 \supseteq q_2 \supseteq \cdots \supseteq q_m \in \operatorname{Spec} B$$

with $q_i \cap A = p_i$, then there exist $q_m \supseteq q_{m+1} \supseteq \cdots \supseteq q_{n+1} \supseteq q_n$ with $q_i \cap A = p_i$.

Proof (?).
See A&M, similar to proof of going up. ■

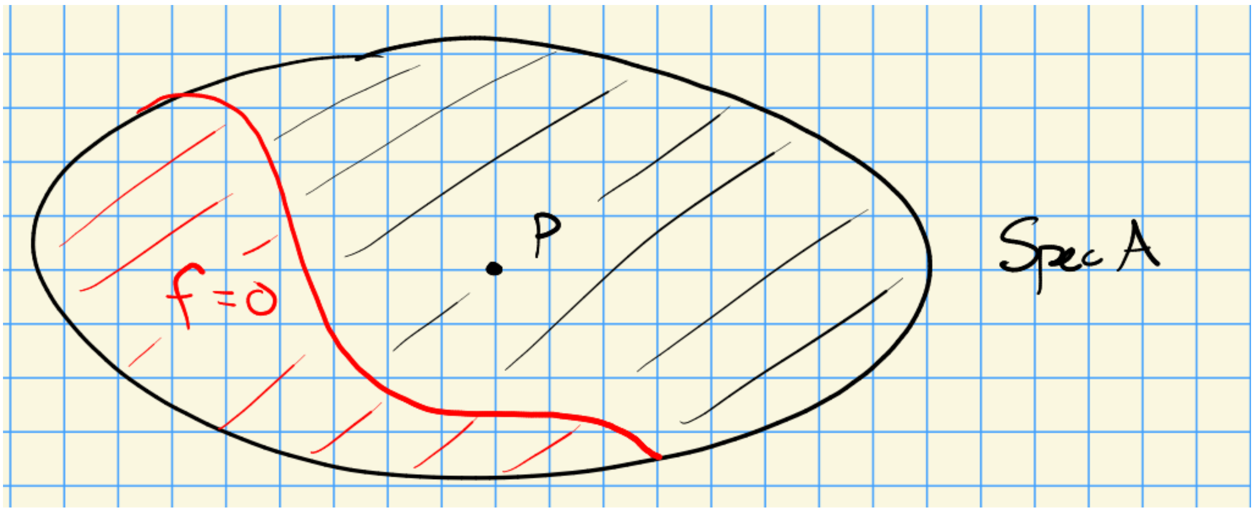
Remark 14.1.11: Idea: closed under *generization* (opposite of specialization, given x finding a point y with $x \in \mathrm{cl} y$), so the geometric map is almost open.

Example 14.1.12 ***(?):*** Being integrally closed corresponds to a variety being **normal** and is a smoothness condition.

Exercise 14.1.13 (Challenge)
Use $k[x,y]/\left\langle y^2 - x^3 \right\rangle \cong k[x^2, x^3] \hookrightarrow k[x,y]$ to construct a counterexample to the going down theorem when $A \hookrightarrow \mathrm{ff}(A)$ is *not* integrally closed.

14.2 Local Properties

Remark 14.2.1: On local properties: for $M \in \mathsf{R\text{-}Mod}^{\mathrm{fg,Noeth}}$, a HW problem shows if $p \in \operatorname{Spec} A$ and $M\left[p^{-1}\right] = 0$ then $M\left[f^{-1}\right] = 0$ for some f. This says that the property of being zero extends:



Definition 14.2.2 (Local)
A property Q is **local** iff given $M \in \mathsf{A}\text{-}\mathsf{Mod}$, TFAE:

- Q holds for M
- Q holds for $M\left[f_i^{-1}\right]$ for every $\{f_i\}$ with $\langle \{f_i\} \rangle = \langle 1 \rangle$.

Slogan 14.2.3
Local properties can be checked on an open cover of $\operatorname{Spec} A$, and $\mathsf{A}\text{-}\mathsf{Mod}$ corresponds to $\mathsf{QCoh}(\operatorname{Spec} A)$.

Remark 14.2.4: One can always take the set $\{f_i\}$ to be finite since if such a collection generates the unit ideal, there is some finite sum $\sum a_i f_i = 1$. One can also reformulate the second condition as follows: for each $p \in \operatorname{Spec} A$, there exists some $f_p \notin p$ such that every $M\left[f_p^{-1}\right]$ satisfies Q. $\implies$: Check that $\langle \{f_p\} \rangle = \langle 1 \rangle$; if not then there exists some $m \in \operatorname{mSpec} A$ with $\{f_p\} \subseteq m$ which is maximal and hence prime.

$\impliedby$: The claim is that given p there exist $f_i \notin p$. If not, $\{f_i\} \subseteq p$ and $1 \in p$.

Corollary 14.2.5 *(?).*
If $\langle \{f_i\} \rangle = 1$ then $\operatorname{Spec} A\left[f_i^{-1}\right] = \operatorname{Spec} A \setminus V(f_i) \subseteq \operatorname{Spec} A$ is an open cover of $\operatorname{Spec} A$. Thus $\operatorname{Spec} A$ is quasicompact for any ring A.

Remark 14.2.6: Some local properties:

- Being zero
- Being injective/surjective/bijective
- Being finitely generated (and projective)
- Being flat

15 | Tuesday, March 01

Remark 15.0.1: A property is *local* on A if it can be checked on an affine open cover of $\operatorname{Spec} A$. Note that e.g. $\mathbb{A}^2_{/k}$ is affine but $\operatorname{Spec} k[x, y]/\langle x, y \rangle \cong \mathbb{A}^2_{/k} \setminus \{0\}$ is not an affine open subset.

Exercise 15.0.2 (?)
Show that the property of being zero is local.

Hint: $M_{f_i} = 0 \implies$ for every $m \in M$ there is an n_i with $f_i^{n_i} m = 0$, write $\sum a_i f_i = 1$, and consider $(\sum a_i f_i)^N m$ for $N \gg 1$.

Exercise 15.0.3 (?)
Show that being injective/surjective/bijective is local.

Hint: localization is exact, so take the SES $0 \to \ker g \to M \xrightarrow{g} N \to \operatorname{coker} g \to 0$.

Exercise 15.0.4 (?)
Show that being finitely generated is local.

Hint: tensor a presentation. For the other direction, take generators $M_{f_i} = \left\langle \frac{x_{i1}}{a_{i1}}, \frac{x_{i2}}{a_{i2}}, \cdots, \frac{x_{in}}{a_{in}} \right\rangle$ *for* $\{f_1, \cdots, f_m\}$*, where without loss of generality* $a_{ij} = 1$*, and take* $\{x_{ij}\}$ *and check surjectivity locally.*

Exercise 15.0.5 (?)
Show that flatness is local, i.e. if $M \in \mathsf{A\text{-}Mod}^\flat$ then $M_f \in \mathsf{A_f\text{-}Mod}^\flat$.

Hint: to show M_f *is flat assuming* M *is flat, show that* $\operatorname{Tor}_i^{A_f}(A_f/a, M_f) = 0$ *by taking* $P^\bullet \twoheadrightarrow M$ *and* ${P^\bullet}_f \twoheadrightarrow M_f$*. Then compute*

$$\begin{aligned}
\operatorname{Tor}_1(M_f, A_f/a) &= H_1({P^\bullet}_f \otimes A_f/a) \\
&= H_1(P^\bullet \otimes A/a' \otimes A_f) \\
&= H_1(P^\bullet \otimes A/a') \otimes A_f \\
&= 0 \otimes A_f = 0,
\end{aligned}$$

using that localization is exact and thus commutes with taking homology. In the other direction, show that for $0 \to A \xrightarrow{f} B \to C \to 0$ *leads to* $A \otimes M \xrightarrow{\tilde f} B \otimes M$*, and injectivity can be checked locally.*

Remark 15.0.6: Define the **support** of M as $\operatorname{supp}(M) := \left\{ p \in \operatorname{Spec} A \;\middle|\; M_p \neq 0 \right\}$, thought of as points where "functions on A" defined by M do not vanish.

Exercise 15.0.7 (?)
Show that if $M \in \mathsf{A\text{-}Mod}^{\mathrm{fg}}$ then $\operatorname{supp} M = V(\operatorname{Ann}_A(M))$ where $\operatorname{Ann}_A(M) := \left\{ a \in A \;\middle|\; am = 0 \, \forall m \in M \right\}$.

Exercise 15.0.8 (?)
Show that for $M \in \mathsf{A\text{-}Mod}^{\mathrm{fg}}$ with A Noetherian, $M = 0 \iff \operatorname{supp} M = 0$.

Remark 15.0.9: Modules give sheaves over $\operatorname{Spec} A$, and the following theorem is a special case of faithfully flat descent:

Theorem 15.0.10 ***(Serre).***
If $M \in \mathsf{A\text{-}Mod}^{\text{fg}}$ and $\{f_i\}$ is a finite generating set, then the following sequence is exact:

$$0 \to M \to \bigoplus_{f_i} M_{f_i} \to \bigoplus M_{f_i f_i}$$
$$x \in M_{f_i} \mapsto \left[\cdots, \frac{x}{f_i f_j}, \cdots, -\frac{x}{f_j f_i}, \cdots \right].$$

with the positive sign in the ith component and the negative in the jth.

Exercise 15.0.11 (?)
Prove this: check injectivity locally, and use that localization commutes with direct sums. Note that essentially the same proof goes through for faithfully flat descent.

Theorem 15.0.12 ***(Classification of flat finitely-generated modules over a Noetherian ring).***
If $M \in \mathsf{A\text{-}Mod}^{\flat,\text{fg}}$ for A Noetherian, then M is locally free, i.e. there exist f_i generating the unit ideal with M_{f_i} free for all i.

Proof (?).
Philosophy: reduce to local case.

1. For A local: finitely-generated flat flat modules over a Noetherian *local* ring is free.
2. Hence M_p is free over A_p for all $p \in \operatorname{Spec} A$.
3. Spreading out: by the HW, there exist f_i^p not in p such that $M_{f_1^p \cdots f_\ell^p}$ is free and equals M_{f^p}.
4. The finite collection $\{P^*\}$ generated the unit ideal.

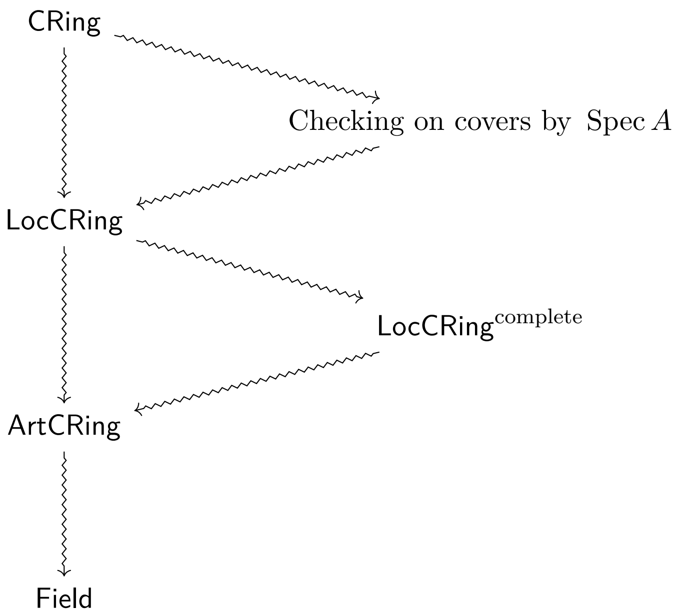
■

Remark 15.0.13: Next: Artinian, local, complete local rings, DVRs, etc – the building blocks of the theory of local rings!

16 | Thursday, March 03

Remark 16.0.1: Next topic: Artin rings. The general way we reduce the study of arbitrary rings:

16 Thursday, March 03



Link to Diagram

- Recall that Artin rings are defined by the DCC condition on ideals.
- The **length** of a module is the maximal length of a strictly increasing filtration.

Exercise 16.0.2 (?)
Show the following:

- $\mathbb{C}[t]/\langle t^n \rangle$ is Artin.
- C_{p^n} is Artin.
- $\mathbb{Z}$ is Noetherian but not Artin.
- Any finite product of fields is Artin.
- An Artin domain is a field.
- Prime implies maximal in any Artin ring.

Hint: quotient by the prime, and use that any element a satisfies $a^n = ba^{n+1}$ for some n and b to produce an inverse.

- $\operatorname{Spec} A = \{\text{pt}\}$ for a local Artin ring.
- Artin rings A have finite length in $A\text{-Mod}$.
- Quotients of Artin rings by their Jacobson radicals are products of fields.
- $\operatorname{Spec} A = \coprod_i \{\text{pt}_i\}$ is a disjoint union of points, and is Hausdorff.
  - The only Noetherian rings with Hausdorff spectra are Artin.

Theorem 16.0.3 *(?).*
Artin rings are Noetherian.

Proof (?).
Use that

- Any module has a maximal proper submodule.
- Choose $\mathfrak{a}_i \subseteq A$ simple, so $A/\mathfrak{a}_i$ is Artinian, to produce an increasing chain $0 \subseteq \mathfrak{a}_1 \subseteq \mathfrak{a}_2 \cdots$ where $\mathfrak{a}_i/\mathfrak{a}_{i+1}$ is simple for all i.
- Enumerate maximal ideals $\mathfrak{m}_i$ and produce a chain $\mathfrak{m}_1 \supseteq \mathfrak{m}_2 \supseteq \cdots$ and take colon ideals.
- Reduce to this case by showing every chain refines.

Steps:

- Make a descending filtration with semisimple associated graded, whose filtration is finite.
- Use Jordan-Holder, every such sequence has the same length.
- Refine an arbitrary filtration to one in which the quotients are simple.

■

Corollary 16.0.4 *(?).*
$\sharp\,\mathrm{mSpec}\, A < \infty$ for $A \in \mathsf{ArtCRing}$.

Proof (?).
Hints:

- If not, take a decreasing chain of $\mathfrak{m}_1 \supseteq \mathfrak{m}_1\mathfrak{m}_2 \cdots$, stabilize, use that the $\mathfrak{m}_i$ are finitely-generated and apply Nakayama.
- If I is defined as the product at the minimal stabilized step, $I \subseteq J(A)$.
- Without loss of generality, assume $J(A) = 0$, so $I = 0$ since $\mathrm{mSpec}\, A/J(A) \cong \mathrm{mSpec}\, A$.
- $A/\mathfrak{m}_1 \cdots \mathfrak{m}_n = \prod A/\mathfrak{m}_i$ is a product of fields
- Corollary: $n > N$, $\mathfrak{m}_N$ will have empty support.

■

17 | Tuesday, March 15

17.1 Artin Rings

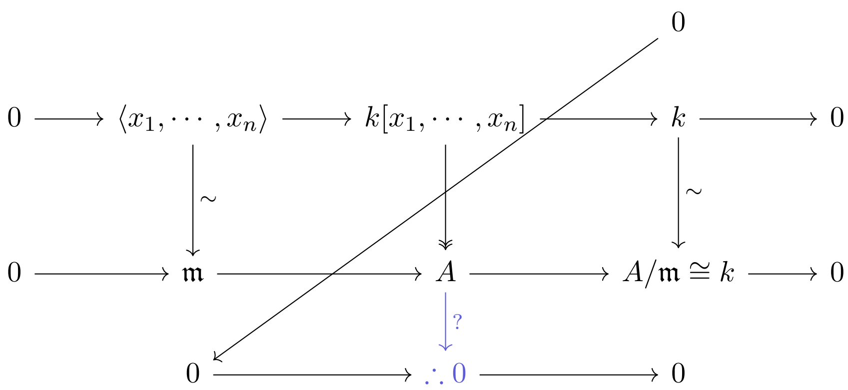
Remark 17.1.1: Last time: $A/J(A)$ is a product of Artin local rings.

Exercise 17.1.2 (?)
Show that if A is Artin local, then $\mathfrak{m}_A^n = 0$ for some n.

Hint: use that $\{\mathfrak{m}^k\}_{k \geq 0}$ stabilizes, so $\mathfrak{m}^n/\mathfrak{m}^{n+1} \cong \mathfrak{m}^n/\mathfrak{m}\mathfrak{m}^n = 0$ so $\mathfrak{m}^n = 0$ since A is Noetherian.

Exercise 17.1.3 (?)
Show that if A is an Artin local ring that is finitely generated over an algebraically closed field k. Then A is a quotient of $k[x_1, \cdots, x_n]/\langle x_1, \cdots, x_n\rangle^N$ for some n and N.

Solution:
Hint: use that $\mathfrak{m}$ is finitely generated, construct a surjection $k[x_1, \cdots, x_n] \to A$, and show $I_N \coloneqq \langle x_1, \cdots, x_n\rangle^N$ is in the kernel. Also use the corollary of the Nullstellensatz (EEKS) that finite extensions of k which are fields are necessarily k itself. Apply the snake lemma:



Link to Diagram

This shows surjectivity, and $\langle x_1, \cdots, x_n\rangle^N$ being in the kernel follows from the previous proposition.

17.2 DVRs

Exercise 17.2.1 (?)
Recall that a Noetherian local domain A is a DVR if $\mathfrak{m} \neq 0$ is principal. Show that $\dim_k \mathfrak{m}/\mathfrak{m}^2 = 1$ for $k = A/\mathfrak{m}$.

Hint: use Nakayama to bound the dimension by 1.

Example 17.2.2 *(?):* Examples of DVRs:

- $k[[t]]$
- $\mathbb{Z}_{\widehat{p}}$

- $\mathbb{Z}_{(p)} = \mathbb{Z}\left[\left\{ q^{-1} \;\middle|\; q \neq p \right\}\right]$.

A non-example: $k[[x, y]]$.

Exercise 17.2.3 (?)
Show that if A is a DVR with uniformizer π and $a \in A \setminus \{0\}$, then there is a unique $n \in \mathbb{Z}_{\geq 0}$ such that $a = \pi^n a_u$ with a_u a unit.

Hint: for existence, set $N := \max\left\{ N \;\middle|\; a \in \langle \pi^N \rangle \right\}$ which exists because $\cap_N \langle \pi^N \rangle = 0$. Use that $\pi I = I \implies I = 0$ by Nakayama, to write $a = \pi^n a_0$, and if a_0 is not a unit then $a_0 \in \langle \pi \rangle$ contradicting maximality.

Corollary 17.2.4 *(?).*
If A is a DVR, $\mathrm{Id}(A) = \{\langle 0 \rangle, \langle \pi^n \rangle\}_{n \geq 0}$ and $\operatorname{Spec} A = \{\langle \pi \rangle, \langle 0 \rangle\}$. This follows from writing

$$I = \langle a_1, \cdots, a_N \rangle = \langle \pi^{n_1} b_1, \cdots, \pi^{n_N} b_N \rangle = \langle \pi^m \rangle, \qquad m := \min \{n_j\}_{j \leq N}.$$

Exercise 17.2.5 (?)
Show that DVRs A biject with fields K equipped with a valuation $v : K \to \mathbb{Z} \cup \{\infty\}$ satisfying:

- $v(a+b) \geq \min(v(a), v(b))$,
- $v(ab) = v(a) + v(b)$,
- $v(a) = \infty \iff a = 0$,
- $v(K^\times) \neq \mathbb{Z}$.

Hint: for $A \in \mathsf{DVR}$, set $K = \mathrm{ff}(A)$ and $v(a/b) := v(a) - v(b)$ where $v(\pi^n a_0) = n$. Given (K, v), set $A = \left\{ x \in K \;\middle|\; v(x) \geq 0 \right\}$ with $\mathfrak{m} = \{v(x) > 0\}$, showing $\mathfrak{m}^c \subseteq A^\times$ and $\mathfrak{m}$ is generated by any x with $v(x) = 1$?

17.3 Classifying finitely-generated modules over a DVR

Remark 17.3.1: Recall that $M \in \mathsf{A\text{-}Mod}$ is torsionfree iff $\operatorname{Ann}_A(M) := \left\{ a \in A \;\middle|\; am = 0 \right\}$ contains only zero divisors iff $M \xrightarrow{\times a} M$ is injective for all nonzero $a \in A$.

Exercise 17.3.2 (?)
Show that if $A \in \mathsf{CRing}^{\mathrm{Noeth}}$ and $M \in \mathsf{A\text{-}Mod}^{\mathrm{fg}}$ then $0 \to M_{\mathrm{tors}} \to M \to M' \to 0$ is a SES

where M' is torsionfree, and moreover there exists some $a \in A$ such that $aM_{\text{tors}} = 0$.

Hint: for the latter statement, use that M_{tors} is finitely-generated and take a product of annihilators of generators. For the former, take $a \in \operatorname{Ann}_A(\tilde{m})$ for some $\tilde{m} \in M'$, lift to $m \in M$ and show $am \in M_{\text{tors}}$ for some a.

Exercise 17.3.3 (?)
Show that if A is a PID, then $M \in \mathsf{A\text{-}Mod}$ is flat iff M is torsionfree.

Hint: use $A \xrightarrow{\times a} A$ and apply $(-) \otimes_A M$. For the reverse, show $\operatorname{Tor}^1(M, A/I) = 0$ for any ideal in A and compute using the projective resolution $0 \to A \xrightarrow{\times a} A \to A/\langle a \rangle \to 0$. Note that $H_1(M \xrightarrow{\times a} M) = \ker(M \xrightarrow{\times a} M) = 0$ since M is torsionfree.

Exercise 17.3.4 (?)
Show that for A a DVR and $M \in \mathsf{A\text{-}Mod}^{\text{fg}}$, then M torsionfree implies M is free.

Hint: torsionfree $\implies$ flat $\implies$ free for finitely-generated Noetherian local rings.

Exercise 17.3.5 (?)
Show that if $M \in \mathsf{A\text{-}Mod}^{\text{fg}}$ for A a DVR then $M \cong M_{\text{tors}} \oplus A^n$ for some n.

Hint: use that the SES involving $M_{\text{tors}} \to M$ splits.

Exercise 17.3.6 (?)
Show that any finitely-generated torsion module over A is isomorphic to $\bigoplus_i A/\pi^{n_i} A$. Use this to classify all finitely-generated modules over $k[[t]]$.

18 | Tuesday, March 22

18.1 Classification of finitely-generated modules over a Dedekind domain

Definition 18.1.1 (Dedekind domain)
A ring A is a **Dedekind domain** iff

- $A \in \mathsf{NoethDomain}, \operatorname{krulldim} A = 1$,
- The local rings A_p are DVRs for all $p \in \operatorname{mSpec} A$.

Theorem 18.1.2 ***(Structure theorem for Dedekind domains).***
If $M \in \mathsf{A\text{-}Mod}$, $A \in \mathsf{DedekindDomain}$, then

$$M \underset{\mathsf{A\text{-}Mod}}{\cong} T \oplus F := \left(\bigoplus (A/\mathfrak{p}_i)^{n_i} \right) \oplus \left(\bigoplus \mathcal{L}_i \right)$$

where $\mathcal{L}_i \in \mathsf{A\text{-}Mod}^{\text{locfree,rank}=1}$ (and are in particular torsionfree) and the $A/\mathfrak{p}_i$ are torsion.

Exercise 18.1.3 (?)
Show that if $p \in \operatorname{Spec} A$ then $M_p \cong A_p^m$ where $m = \operatorname{rank} M$.

Hint: use that $M_p \oplus \operatorname{ff}(A_p) = \operatorname{ff}(A)^n = \operatorname{ff}(A)^n$

Exercise 18.1.4 (?)
Show that for A a Dedekind domain and $M \in \mathsf{A\text{-}Mod}^{\text{fg}}$, TFAE:

- $M \in \mathsf{A\text{-}Mod}^\flat$
- $M \in \mathsf{A\text{-}Mod}(\text{proj})$
- $M \in \mathsf{A\text{-}Mod}^{\text{locfree}}$
- M is torsionfree

Conclude that the following SES splits:

$$0 \to M_{\text{tors}} \to M \to M_{\text{torsfree}} \to 0.$$

Hint: the first three already hold for Noetherian domains. To show flat $\implies$ *torsionfree, that* $0 \to A \xrightarrow{\cdot a} A \to A/\langle a \rangle \to 0$ *for* $a \in \mathsf{Tors}(M)$ *and tensor with* M. *For the converse, show* M_p *is flat for all* p *and that* A_p *is a DVR – if* $\frac{a}{s'}\frac{m}{s} = 0 \implies \tilde{s}' sam = 0$ *for some* $\tilde{s}$, *then* m *is torsion.*

Remark 18.1.5: Note that torsionfree $\implies$ flat fails for most rings!

Exercise 18.1.6 (?)
Show that if $M \in \mathsf{A\text{-}Mod}^{\text{fg}}$ is torsion over a Dedekind domain A then $M \cong \bigoplus_i (A/p_i^{n_i})^{m_i}$.

Hint: use that $\operatorname{supp} M$ *is finite to produce a map* $M \to \bigoplus_i M_{p_i}$ *and show it is locally an isomorphism since* $(M_p)_q = M_q$ *when* $q \neq p$. *Also use that* $M_p = \bigoplus_j A_p/p^{i_j} \cong \oplus_j A/p^{i_j}$.

Exercise 18.1.7 (?)
Show that for $M \in \mathsf{A\text{-}Mod}^{\text{fg}}$ torsionfree over A a Dedekind domain, there is a (not necessarily

unique) decomposition $M \cong \bigoplus_i \mathcal{L}_i$ with $\mathcal{L}_i$ locally free of rank 1.

Induct on rank, where it's ETS there exists an $\mathcal{L}$ where $M/\mathcal{L}_i$ is torsionfree since $0 \to \mathcal{L} \to M \to M/\mathcal{L}_i \to 0$ splits (tensor to fraction field). To find such an $\mathcal{L}$, take any $m \neq 0$ and take $\mathcal{L}$ to be the preimage of $(M/\langle m \rangle)_{\text{tors}}$ under $M \to M/\langle m \rangle$; then $\mathcal{M}/\mathcal{L}$ will be torsionfree and is rank 1 since $\mathcal{L} \otimes \mathrm{ff}(A) \cong \langle m \rangle \otimes \mathrm{ff}(A)$.

18.2 Classification of locally free rank 1 modules over a Dedekind domain

Exercise 18.2.1 (?)
Show that if $I \trianglelefteq A$ is nonzero for A a Dedekind domain, then I is locally free of rank 1.

Hint: show that $I_p \leq A_p$ is nonzero, so $I_p = \langle \pi^n \rangle$ for π a uniformizer of A_p.

Definition 18.2.2 (Invertible modules)
A module $M \in \mathsf{A\text{-}Mod}$ is **invertible** iff M is locally free of rank 1. Equivalently, defining $M^\vee := \mathsf{A\text{-}Mod}(M, A)$ there is an evaluation isomorphism $M \otimes_A M^\vee \xrightarrow{\sim} A$. Define

$$\mathrm{Pic}(A) := (\mathsf{A\text{-}Mod}, \otimes_A)/\cong .$$

Note that for $A = \mathcal{O}_K, K \in \mathsf{Field}_{/\mathbb{Q}}$,

$$\mathrm{Pic}(A) \cong \mathrm{Cl}(K).$$

Definition 18.2.3 ((Weil) Divisors)
For A a Dedekind domain, a **divisor** is a formal linear combination

$$D = \sum_{\mathfrak{p} \neq 0 \in \operatorname{Spec} A} n_p \mathfrak{p} \in \mathsf{Free}_{\mathbb{Z}\text{-}\mathsf{Mod}}(\operatorname{Spec} A).$$

such that $n_p = 0$ for almost all p. These form a group $\mathrm{Div}(A)$. A divisor D is **effective** iff $n_p \geq 0$ for all p, yielding a submonoid $\mathrm{Div}^+(A)$ of effective divisors.

Remark 18.2.4: More generally, this will be a sum over height 1 primes.

Exercise 18.2.5 (?)

Show that there is a natural bijection

$$\begin{aligned} \mathrm{Div}^+(A) &\rightleftharpoons \mathrm{Id}(A) \setminus \{0\} \\ \sum n_p p &\rightleftharpoons \prod p^{n_p}. \end{aligned}$$

Hint: if $\operatorname{krulldim} A = 1$, *use that* $\mathrm{Id}(A) \cong \{I_p \trianglelefteq A_p\}_{p \in P}$ *where* $I_p = A_p$ *for almost all* p, *and each* $I_p \cong \langle \pi_p^{n_p} \rangle$.

Exercise 18.2.6 (?)
Show that if $\mathcal{L}$ is invertible then $\mathcal{L}^\vee$ is invertible.

Hint: show $\mathcal{L}^\vee$ *is locally free of rank 1, using* ${\mathcal{L}_p}^\vee \xrightarrow{\sim} \mathsf{A\text{-}Mod}(\mathcal{L}, A)_p \xrightarrow{\sim} \mathsf{A_p\text{-}Mod}(\mathcal{L}_p, A_p)$ *by post-composing with localization. Then check* $\mathcal{L} \otimes \mathsf{A\text{-}Mod}(\mathcal{L}, A) \to A$ *where* $(n, f) \mapsto f(n)$ *is an isomorphism by checking locally where everything is free.*

Proposition 18.2.7 ***(Picard group SES).***
Write $D \in \mathrm{Div}(A)$ as $D = D^+ - D^-$ where $D^\pm \in \mathrm{Div}^+(A)$ are effective. There is a SES

$$0 \longrightarrow \ker f \longrightarrow \mathrm{Div}(A) \overset{f}{\twoheadrightarrow} \mathrm{Pic}(A) \longrightarrow 0$$

$$D^+ - D^- \longmapsto \left(\prod_p p^{n_p^+}\right) \otimes \left(\prod_p p^{n_p^-}\right)^\vee$$

Link to Diagram

Remark 18.2.8: This is generally far from injective, and will instead biject with *fractional ideals*:

Definition 18.2.9 (Fractional ideals)
For $A \in \mathsf{Dedekind}$, a **fractional ideal** is a nonzero A-submodule $\mathcal{L} \leq A$ where $\mathcal{L} \in \mathsf{A\text{-}Mod}^{\mathrm{fg}}$.

Example 18.2.10 ***(?):***

- $I \trianglelefteq A$ or $I \trianglelefteq \mathrm{ff}(A)$ any ideal
- $\frac{1}{a} I \trianglelefteq \mathrm{ff}(A)$.

Exercise 18.2.11 (?)
Show that any fractional ideal is invertible.

Hint: use that any such I is torsionfree since it's a subset of $\operatorname{ff}(A)$, and $\mathcal{L} \otimes_A \operatorname{ff}(A) \cong \operatorname{ff}(A)$ implies rank 1.

19 | Tuesday, March 29

Remark 19.0.1: Last time: proving the following theorem.

Theorem 19.0.2 *(?).*
If $A \in \mathsf{NoethlocDomain}$ with $\operatorname{krulldim} A = 1$, then $A \in \mathsf{DVR} \iff A$ is integrally closed.

Proof (?).
Last time: DVR $\implies$ integrally closed.
$\impliedby$: It suffices to show that the maximal ideal is principal. Let $a \in \mathfrak{m} \setminus \{0\}$, then $A/\langle a \rangle$ is Artinian (which is true here for any quotient) and thus $\mathfrak{m}^k = 0$ in $A/\langle a \rangle$ for some k (chosen minimally), so $\mathfrak{m}^k \in \langle a \rangle$. By minimality, pick $b \in \mathfrak{m}^{k-1} \setminus \langle a \rangle$ – the claim is that a/b generates $\mathfrak{m}$. First, $\left(\frac{b}{a}\right) \mathfrak{m} \trianglelefteq A$ since $bm \in \mathfrak{m}^k \subseteq \langle a \rangle$. It's ETS $\left(\frac{b}{a}\right) m$ is not contained in $\mathfrak{m}$ – given this, $\left(\frac{b}{a}\right) \mathfrak{m} = A$ by maximality and this implies $\mathfrak{m} = \frac{b}{a} A$.
Why this last claim is true: the map $x \mapsto \frac{a}{b} x : \mathfrak{m} \to \mathfrak{m}$ is annihilated by some polynomial with coefficients in A by Cayley-Hamilton, and by integral closedness this implies $a/b \in A$. Then $a/b \in \mathfrak{m}$, a contradiction since a/b is a unit and $\mathfrak{m}$ is proper. ↯

■

Theorem 19.0.3 *(?).*
If $A \in \mathsf{NoethDomain}$ (not necessarily local) $\operatorname{krulldim} A = 1$, then $A \in \mathsf{DedekindDomain} \iff A$ is integrally closed.

Remark 19.0.4: A Dedekind means all local rings are DVRs, so it's ETS A is integrally closed iff $A\left[p^{-1}\right]$ is integrally closed (since we can apply the last theorem to $A\left[p^{-1}\right]$).

Lemma 19.0.5 *(Integral closure commutes with localization).*
Let $A \subseteq B$ be an inclusion of rings and let $\operatorname{cl}_B^{\text{int}} A$ be the integral closure of A in B. If $S \subseteq A$ is a multiplicative subset, then

$$\operatorname{cl}_B^{\text{int}}\left(A\left[S^{-1}\right]\right) \cong \left(\operatorname{cl}_B^{\text{int}} A\right)\left[S^{-1}\right].$$

Proof (of lemma).
$\supseteq$: Take $b \in A_{\text{int}}$ and $s \in S$, we then WTS b/s satisfies a monic polynomial over $A[S^{-1}]$. If $f(b) = b^n + a_1 b^{n-1} + \cdots = 0$, note that $s^n f(b) = 0$ and is of the form $\left(\frac{b}{s}\right)^n + sa_1 \frac{b^{n-1}}{s} + \cdots$ is a polynomial with coefficients in $A[S^{-1}]$.
$\subseteq$: Let b/s in the LHS, so $\left(\frac{b}{s}\right)^n + \frac{a_1}{s_1}\left(\frac{b}{s}\right)^{n-1} + \cdots = 0$. Multiply through by $s^n \left(\prod s_i\right)^n$ to get $\left(\prod s_i\right)^n b^n + a_1 s' b^{n-1} \left(\prod s_i\right)^{n-1} + \cdots$ by absorbing some factors into the coefficient s', so $b \prod s_i$ satisfies a monic polynomial. ■

Proof (of theorem).
By the lemma, A integrally closed $\implies$ $A[p^{-1}]$ is integrally closed because $(A[p^{-1}])_{\text{int}} = (A_{\text{int}})[p^{-1}] = A[p^{-1}]$. Suppose that $A[p^{-1}]$ is integrally closed. There is a map $A \to A_{\text{int}}$ which we want to show is an isomorphism. Check locally: $A[p^{-1}] \to (A_{\text{int}})[p^{-1}] \to (A[p^{-1}])_{\text{int}} = A_{[p^{-1}]}$. ■

Theorem 19.0.6 ***(Why Dedekind domains arise in nature: Dedekind is closed under certain integral closures).***
Let $A \in \mathsf{Dedekind}$ with $K = \text{ff}(A)$ and let K'/K be a finite separable extension. Then $\text{cl}^{\text{int}}_{K'} A \in \mathsf{Dedekind}$ is Dedekind.

Note that in characteristic zero, finite extensions are automatically separable.

Example 19.0.7 ***(?):***

- For $K/\mathbb{Q}$ a finite separable extension, then $\text{cl}^{\text{int}}_{K} \mathbb{Z}$ is Dedekind.
- For $L/k[t]$ finite separable, $\text{cl}^{\text{int}}_{L} k[t]$ is Dedekind.

Lemma 19.0.8 ***(?).***
Let $A' := \text{cl}^{\text{int}}_{K'} A$, then $A' \in \mathsf{A\text{-}Mod}^{\text{fg}}$.

Proof (of theorem, using the lemma).

- By the Hilbert basis theorem, A' is Noetherian
- A' is a domain since $A' \subseteq K'$
- A' is integrally closed: STS satisfying a monic polynomial over A' implies satisfying a monic polynomial over A. To check if $b \in K'$ is integral, one needs that $A'[b] \in \mathsf{A'\text{-}Mod}$ is finite, which implies $A'[b] \in \mathsf{A\text{-}Mod}$ is finite. This is true because $A[b] \leq A'[b]$ is a submodule and A is Noetherian, so b is integral over A.
- krulldim $A' = 1$: STS $\text{Spec}\, A' \subseteq \text{mSpec}\, A'$. Let p be prime in A and let $\tilde{p} := p \cap A \in \text{Id}(A)$. There is a finitely-generated extension $A/\tilde{p} \leq A'/p$, and it's ETS $A/\tilde{p} \in \mathsf{Field}$ by EEKS

(finitely-generated ring extensions of fields are fields). ETS $\tilde{p} \in \operatorname{mSpec} A$ – consider $\operatorname{Spec} A' \xrightarrow{\pi} \operatorname{Spec} A$, we WTS $\pi^{-1}(\langle 0 \rangle) = \{\langle 0 \rangle\}$. Fact: $\pi^{-1}(\langle 0 \rangle) = \operatorname{Spec} A'\left[A^{\times -1}\right] = \operatorname{Spec}(A' \otimes K) = \operatorname{Spec}(K')$. This uses the fact that $A' \otimes_A K \xrightarrow{\sim} K'$, which can be checked locally on DVRs.

■

Proof (of lemma).
Assume $\operatorname{ch} K = 0$. We know that K'/K is finite, so pick a basis $\{e_i\}$. We can scale the e_i by elements of A so that they are in A' – use that $K' = A' \otimes_A R$, so $e_i = \sum a_i'/s_i$ and one can scale by the s_i.
There is nondegenerate pairing $\langle x,\ y \rangle := \operatorname{Tr}_{K'/K}(xy)$ where $\operatorname{Tr}(z)$ is the trace of the K-linear map $K' \xrightarrow{\cdot z} K'$. Why this is nondegenerate: use the separability assumption. Given $x \in K' \setminus \{0\}$, produce a y such that $\langle x,\ y \rangle \neq 0$ by setting $y = x^{-1}$ (noting that this won't work in characteristic p).
Use this to define a dual basis $\{f_i\}$ such that $\langle e_i,\ f_j \rangle = \delta_{ij}$. We can then express $x = \sum_{1 \leq i \leq n} \langle x,\ f_i \rangle e_i$, and it's ETS that if $x \in A'$ then $\langle x,\ f_i \rangle \in A$. For $x \in A'$, we have $\operatorname{Tr}(x) \in A$, e.g. because it is a sum of Galois conjugates (after passing to a Galois extension), or that $\operatorname{Tr}(x) = \sum r_i$ the roots of a characteristic polynomial with coefficients in A. This exhausts the possible factors of the characteristic polynomial.

■

20 | Thursday, March 31

Remark 20.0.1: Last time: $A \in \mathsf{Dedekind}$ and $M \in \mathsf{A\text{-}Mod}^{\mathrm{fg}}$ implies $A \cong \bigoplus A/I_i \oplus \bigoplus_{\mathcal{L}_i \in \operatorname{Pic}(A)} \mathcal{L}_i$.

Exercise 20.0.2 (?)
Show that this implies the classification of finitely-generated modules over a PID: $M \cong \bigoplus A/p_i^{n_i} \oplus A^n$.

Solution:
Sketch:

- PIDs are Dedekind, so ETS $\operatorname{Pic}(A) = \{A\}$.
- Use the SES $K^\times \xrightarrow{x \mapsto \sum v_p(x) p} \operatorname{Div} A \to \operatorname{Pic} A \to 0$, ETS the given map is surjective.
- Reduce: ETS that for $g \in p \in \operatorname{Spec} A$ with p nonzero, there exists an $x \in K^\times$ such that $v_p(x) = 1$ and $v_{p'}(x) = 0$ for all $p' \neq p$.
  - If $p = \langle x \rangle$, this works: $v_p(x) = 1$ since this is the maximal k such that $x \in p^k$, but $x \in p^2 \implies p = p^2$, contradicting Nakayama. ↯

– If $v_{p'}(x) > 0$, then $x \in p'$, $p \subseteq p'$ contradicting maximality of p. ↯

Remark 20.0.3: What about rings of dimension $d \geq 2$? E.g. $\dim k[x_1, \cdots, x_n] = n$, regarded as functions on $\mathbb{A}^n_{/k}$, noting that we haven't quite defined dimension yet.

Conjecture 20.0.4 ***(Serre-Swan, proved by Quillen).***
Finitely-generated locally free modules over $k[x_1, \cdots, x_n]$ are free.

20.1 Toward dimension: filtered/graded rings

Remark 20.1.1: It is easy to slightly modify this statement to make such a classification impossible!

Question 20.1.2
How many homogeneous polynomials of degree d in n variables are there? Counting the dimension as a k-vector space is yields $\binom{n+d}{d} \sim d^n$.

Remark 20.1.3:
Recall:

- An increasing filtration $\{M_i\}$ of $M \in \mathsf{A}\text{-}\mathsf{Mod}$ satisfies $M_i \subseteq M_{i+1}$ and $\cup_i M_i = M$ and there exists some i_0 with $M_{i_k} = 0$ for all $i_k \leq i_0$.
- A module morphism $M \xrightarrow{f} N$ respects filtrations iff $f(M_i) \subseteq N_i$.
- For $N \leq M$ there is an induced filtrations $N_i := M_i \cap N$.
- For $M \xrightarrow{f} \twoheadrightarrow Q$ a *surjective* morphism, there is an induced filtration $Q_i := f(M_i)$.
- A **graded abelian group** is a group $M \cong \bigoplus_{i \in \mathbb{Z}} M_i$ with $M_i = 0$ for $i \ll 0$. Typical example: $k[x_1, \cdots, x_n] = \bigoplus_d k[x_1, \cdots, x_n]_d$ is graded by homogeneous degree d parts.
- Passing between filtrations and gradings: given a grading $\oplus_i M_i$, set $\mathrm{Fil}^i M = \bigoplus_{k \leq i} M_i$. Given a filtration, take the associated graded $\bigoplus_i M_i / M_{i-1}$.
- Taking the filtration is innocuous and doesn't change the module, but taking the associated graded does. Example: $M = \mathbb{Z}$, take the filtration $\langle 0 \rangle \subseteq \langle 2 \rangle \subseteq \mathbb{Z}$ whose associated graded is $\langle 2 \rangle \bigoplus \mathbb{Z}/2\mathbb{Z} \cong \mathbb{Z} \oplus C_2$, which now has torsion.
- A morphism of filtered modules induces a well-defined map on the associated graded modules.

Exercise 20.1.4 (?)
Show that if $f : M \to N$ is a morphism of filtered modules, then $\operatorname{gr} f$ is injective/surjective $\implies$ f is injective/surjective respectively. Show that the converse is not necessarily true unless f is injective/surjective and the filtrations are induced.

Solution:
For injectivity, if $\operatorname{gr} f$ is injective choose i minimally so that M_i intersects $\ker f$, and contradict $i > -\infty$. By minimality, $x \notin M_{i-1}$ so $\bar{x} \neq 0 \in \operatorname{gr}^i(M) = M_i/M_{i-1}$, but $\operatorname{gr}(f)(\bar{x}) = 0$.
For surjectivity, ETS $f : M_i \to N_i$ is surjective for all i. Induct on i, using that $M_i = N_i = 0$ for $i \ll 0$. Given $x \in N_i$, take $\bar{x} \in N_i/N_{i-1}$ and lift to $y \in M_i$ such that $\operatorname{gr}(f)(\bar{y}) = \bar{x}$; take $x - f(y) \in N_{i-1}$.

Remark 20.1.5: Definitions:

- A **graded ring** is a ring $A = \bigoplus_i A_i$ with $A_i A_j \to A_{i+j}$, a filtered ring has a filtration $\{A_i\}$ with $A_i A_j \to A_{i+j}$.
- A **graded module** M **over a graded ring** A satisfies $A_i M_j \subseteq M_{i+j}$
- A **filtered module over a filtered ring** satisfies $A_i M_j \subseteq M_{i+j}$.

Exercise 20.1.6 (?)
Show that if A is filtered then $\operatorname{gr} A$ is naturally graded.

Solution:
Show

$$\frac{A_i}{A_{i-1}} \cdot \frac{A_j}{A_{j-1}} \to \frac{A_{i+j}}{A_{i+j-1}}.$$

Exercise 20.1.7
Show that if M is a filtered module over A a filtered ring, if $\operatorname{gr}(M) \in \operatorname{gr}(\mathsf{A})\text{-}\mathsf{Mod}^{\mathrm{fg}}$ then $M \in \mathsf{A}\text{-}\mathsf{Mod}^{\mathrm{fg}}$.

Solution:
Pick homogeneous generators $\{\bar{x}_i\}$ for $\operatorname{gr} M$ and show there is a surjection

$$\begin{aligned} f : A^r &\to M \\ e_i &\to x_i. \end{aligned}$$

Reduce to showing $\operatorname{gr} f$ is surjective.

Exercise 20.1.8 (?)
Show that if A is a filtered ring and $\operatorname{gr} A$ is Noetherian then A is Noetherian.

Solution:
Use that $I \in \mathrm{Id}(A)$ is a submodule with an induced filtration and $\mathsf{gr}\, I \subseteq \mathsf{gr}\, A$ is finitely-generated to show that $I \subseteq A$ is finitely-generated.

Definition 20.1.9 (Good filtrations)
If $A \in \mathsf{CRing}$ is filtered and $M \in \mathsf{A}\text{-}\mathsf{Mod}$ is filtered, then $\mathrm{Fil} M$ is a **good filtration** if $\mathsf{gr}\,(M) \in \mathsf{gr}\,(\mathsf{A})\text{-}\mathsf{Mod}^{\mathrm{fg}}$.

Example 20.1.10 ***(?):*** Letting $M \in \mathsf{A}\text{-}\mathsf{Mod}$ for $A = k[x_1, \cdots, x_n]$, if $\left\{\mathrm{Fil}^i M\right\}$ is a good filtration then $P(i) := \dim_k \mathrm{Fil}^i M$ will be a polynomial for $i \gg 0$ and one can define $\dim M = \deg P$. To be justified:

- A good filtration exists,
- P is asymptotically polynomial,
- P is independent of good filtration chosen.

In the free rank 1 case: $\dim k[x_1, \cdots, x_n]_d = \binom{d+n}{d} \sim d^n$, so $\dim k[x_1, \cdots, x_n]_{\leq d} = \sum_{k \leq n} d^k \sim d^n$.

21 | Tuesday, April 05

21.1 Hilbert dimension

Remark 21.1.1: Preliminary definitions of dimension:

- For $M \in \mathsf{A}\text{-}\mathsf{Mod}^{\mathrm{fg}}$, find a *good filtration* $\{\mathrm{Fil}_i M\}$ where $\dim_k M_i$ is eventually a polynomial p in i, so define $\dim M := \deg p$.
- For $A_{/k}$ a finitely-generated domain, define $\dim A := \mathrm{trdeg}_k A$ as the transcendence degree.
- Krull dimension: $\dim A = n$ iff the longest chain of prime ideals $\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n$ (note that n is the number of *inclusions*!)

When they're all defined, they all agree.

Definition 21.1.2 (Good filtrations)
Let A be a filtered ring and $M \in \mathsf{A}\text{-}\mathsf{Mod}^{\mathrm{fg}}$. A filtration $\mathrm{Fil}_\bullet M$ is a **good filtration** iff

- The pair $(M, \mathrm{Fil}_\bullet M)$ is a filtered module, i.e. $A_i M_j \subseteq M_{i+j}$,
- The more contentful condition: $\mathsf{gr}\,(M) \in \mathsf{gr}\,(\mathsf{A})\text{-}\mathsf{Mod}^{\mathrm{fg}}$.

Exercise 21.1.3 (?)
Show that every such M admits a good filtration.

Solution:
Use that $A^n \xrightarrow{f} M$ and A^n has a good filtration, and take its image. Show that $M_i := f(A_i^n)$ is good using that $\operatorname{gr}(A)^n \twoheadrightarrow \operatorname{gr}(M)$, which generally won't stay surjective but will in this case because M receives the induced filtration.

Theorem 21.1.4 ***(Artin-Rees lemma (extremely important for any arguments involving filtrations!!)).***
Suppose $\operatorname{gr} A$ is Noetherian and let (M, M_i) be an A-module with a good filtration and let $N \leq M$. Then the induced filtration $N_i := N \cap M_i$ is a good filtration.

Proof (?).
Since $N \hookrightarrow M$ and we're taking an induced filtration, $\operatorname{gr}(N) \hookrightarrow \operatorname{gr}(M)$ remains injective. Since $\operatorname{gr}(M)$ is finitely-generated over $\operatorname{gr}(A)$ which is Noetherian, $\operatorname{gr}(N)$ is finitely-generated. ■

Theorem 21.1.5 ***(?).***
Let A be a filtered ring, $M \in \mathsf{A\text{-}Mod}^{\mathrm{fg}}$, and let $F_\bullet, G_\bullet$ be two good filtrations on M. Then there exists a k such that

$$F_{i-k} M \subseteq G_i M \subseteq F_{i+k} M.$$

Remark 21.1.6: There is a notion of a topology on M induced by a filtration, and this theorem says all good filtrations induce the same topology. We'll need the following to prove this theorem:

Lemma 21.1.7 ***(?).***
Let (M, F) be a module with a good filtration. Then there exists some n and i_0 such that

$$i \geq i_0 \implies F_{i+n} \subseteq A_{i+i_0} F_n.$$

Proof (of theorem, assuming lemma).
Choose n as in the lemma, choose m such that $F_n \subseteq G_{n+m}$, then

$$F_{i+n} \subseteq A_{i+i_0} F_n \subseteq A_{i+i_0} G_{n+m} \subseteq G_{i+n+m}.$$

Now run the same argument on G. ■

Proof (of lemma).
Since $\operatorname{gr}(M)$ is finitely-generated over $\operatorname{gr}(A)$, take a finite set of homogeneous generators

m_i. Choose n, i_0 such that $n - i_0 < \deg m_i < n$, the claim is that these work. Induct on i: suppose $F_{i+n} \subseteq A_{i+i_0} F_n$ and we WTS this still holds when $i \mapsto i+1$. Letting $m \in F_{i+n+1}$, if $m \in F_{i+n}$ we're done. Otherwise $\overline{m} \neq 0 \in \mathsf{gr}_{i+n+1} M$, so $\overline{m} = \sum \bar{c}_i m_i$ and picking lifts, $\overline{m} - \sum c_i m_i \in F_{i+n}$.

■

Theorem 21.1.8 *(?)*.
Let $A := k[x_1, \cdots, x_n]$, $(M, M_i) \in \mathsf{A\text{-}Mod}^{\mathrm{fg}}$ have a good filtration, and let $\Phi(i) := \dim_k M_i$. Then

- There exists a polynomial p with $\deg p \leq n$ such that $\Phi(i) = p(i)$ for $i \gg 0$.
- The degree and leading coefficient of p are independent of the choice of good filtration

We then define

$$\dim_A M = \deg p.$$

Exercise 21.1.9 (Method of finite differences)
Let $\Phi : \mathbb{Z} \to \mathbb{Z}$ such that $\Phi(i+1) - \Phi(i)$ is eventually polynomial of degree $d' \leq n-1$. Show that Φ is eventually polynomial of some bounded degree $d \leq m$.

Solution:
Write $\Phi(i) = C + \sum_{i \geq 1} \Phi(i) - \Phi(i-1) = C + \sum_{i \geq 1} q(i)$ for some polynomial q with $\deg q \leq n-1$. So it's ETS $\sum_{0 \leq n \leq i} n^a$ polynomial in i for $a \geq 0$.

Proof (of theorem, part 1).
Proceed by induction on the number of variables. Since $\dim_k(M_i) = \sum_i \dim_k(\mathsf{gr}_i M)$ where $\mathsf{gr}_i M = M_i / M_{i+1}$, it's ETS $\dim M_i - \dim M_{i-1} = \dim \mathsf{gr}_i M$. An awesome maneuver: take a regrading to get a LES

$$0 \to K := \ker f \to \mathsf{gr}\, M \xrightarrow{f} \Sigma \mathsf{gr}\, M \to C := \operatorname{coker} f \to 0.$$

Note that $K, C \in k[x_1, \cdots, x_{n-1}]$ has fewer variables, and by alternating additivity in exact sequences,

$$\dim \mathsf{gr}_i K - \dim \mathsf{gr}_i M + \dim \mathsf{gr}_{i+1} M - \dim \mathsf{gr}_i C = 0.$$

Thus

$$\dim \mathsf{gr}_{i+1} M - \dim \mathsf{gr}_i M = \dim \mathsf{gr}_i C - \dim \mathsf{gr}_i K,$$

and since the RHS involves good filtrations, these are eventually polynomial, thus so is the LHS.

■

Proof (of theorem, part 2).
Let F, G be good filtrations with $F_{i-k} \subseteq G_i \subseteq F_{i+k}$ with $\Phi_F(i) = \dim F_i, \Phi_G(i) = \dim G_i$. Then

$$\Phi_F(i-k) \leq \Phi_G(i) \leq \Phi_G(i+k),$$

and if Φ_G, Φ_G have different degrees or the same degrees but different leading terms, it would violate this inequality for large i.

■

21.2 Properties of dimension

Exercise 21.2.1 (?)
Let $M \in \mathsf{A\text{-}Mod}^{\mathrm{fg}}$.

- Show that if $N \leq M$ then $\dim N \leq \dim M$.
- Show that if $M \twoheadrightarrow N$ then $\dim N \leq \dim M$.

Solution:

- Restrict the good filtration on M, this is good by Artin-Rees.
- Take an induced good filtration. Use that quotients of finitely-generated are finitely-generated and $\mathsf{gr}\,(A)^n \twoheadrightarrow \mathsf{gr}\,(M) \twoheadrightarrow \mathsf{gr}\,(N)$.

Exercise 21.2.2 (?)
Show that if $0 \to M_1 \to M_2 \to M_3 \to 0$ is a SES in $\mathsf{A\text{-}Mod}$ for $A = k[x_1, \cdots, x_n]$, then

$$\dim M_2 = \max(\dim M_1, \dim M_3).$$

Solution:
Choose good filtrations F_i, M_i, G_i on M_1, M_2, M_3 respectively, so $\dim M_i = \dim F_i + \dim G_i$. The RHS involves polynomials in i with non-negative leading terms, and their sum has the max of the 2 degrees and again has a nonnegative leading term.

Exercise 21.2.3 (?)
Show that if $A = k[x_1, \cdots, x_n]$ then $\dim_A A = n$ and $\dim_A A^n = n$.

22 | Thursday, April 07

22.1 Hilbert dimension of finitely generated algebras over a field

Remark 22.1.1: Prove:

- $f : M \hookrightarrow M$ implies $\dim \operatorname{coker} f < \dim M$.
- If $A = k[x_1, \cdots, x_n]$ and $f \in A \setminus \{0\}$ then $\dim A/f < \dim A$.
- If $A \in \mathsf{k}[\mathsf{x}_1, \cdots, \mathsf{x}_\mathsf{n}]\text{-}\mathsf{Mod}^{\mathrm{fg}}$ and $f \in A$ is a nonzero divisor, then $\dim A/F < \dim A$.

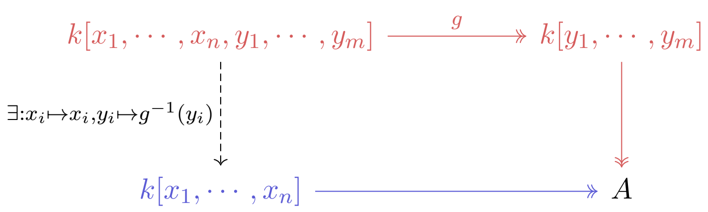
Definition 22.1.2 (Dimension)
If $A \in \mathsf{Alg}_{/k}{}^{\mathrm{fg}}$, choosing a surjection $k[x_1, \cdots, x_n] \twoheadrightarrow A$, define $\dim A := \dim_{k[x_1, \cdots, x_n]} A$.

Exercise 22.1.3 (?)
Show this is well-defined. It follows from the following:

Theorem 22.1.4 *(?).*
Let $M \in \mathsf{k}[\mathsf{x}_1, \cdots, \mathsf{x}_\mathsf{n}, \mathsf{y}_1, \cdots, \mathsf{x}_\mathsf{n}]\text{-}\mathsf{Mod}$ with $M \in \mathsf{k}[\mathsf{x}_1, \cdots, \mathsf{x}_\mathsf{n}]\text{-}\mathsf{Mod}^{\mathrm{fg}}$. Then

$$\dim_{k[x_1, \cdots, x_n]} M = \dim_{k[x_1, \cdots, x_n, y_1, \cdots, y_n]} .$$

Solution:
Why the theorem implies the exercise:



Link to Diagram

This can be filled in to a commutative diagram, so

$$\dim_{k[x_1, \cdots, x_n]} A = \dim_{k[x_1, \cdots, x_n, y_1, \cdots, y_m]} A = \dim_{k[y_1, \cdots, y_m]} A.$$

Proof (of theorem).
It's ETS this when $m = 1$, so suppose $M \in \mathsf{k}[\mathsf{x}_1, \cdots, \mathsf{x}_\mathsf{n}][\mathsf{y}]\text{-}\mathsf{Mod}$ with $M \in \mathsf{k}[\mathsf{x}_1, \cdots, \mathsf{x}_\mathsf{n}]\text{-}\mathsf{Mod}^{\mathrm{fg}}$. Take the surjection $k[x_1, \cdots, x_n]^r \twoheadrightarrow M$ where $e_i \mapsto m_i$ for generators m_i, and let M_i be the induced filtration. Similarly take $k[x_1, \cdots, x_n, y]^r \twoheadrightarrow M$ with $e_i \mapsto m_i$ and let $\tilde{M}_i$ be the induced filtration. Since $M_i \subseteq \tilde{M}_i$, $\dim_{k[x_1, \cdots, x_n]} M \leq \dim_{k[x_1, \cdots, x_n][y]} M$.
The claim is that one can choose k such that $yM_0 \subseteq M_k$ and $\tilde{M}_i \subseteq M_{ik}$. Why: $\tilde{M}_i = \left\{ m \in M \;\middle|\; m \text{ is in the image of } (f_1, \cdots, f_r) \text{ of degree at most } i \right\}$. So write $f_j = \sum_n f_{j,n}(x_1, \cdots, x_n) y^n$, then $\tilde{M}_i \subseteq \operatorname{span}(M_i + yM_{i-1} + y^2 M_{i-2} + \cdots)$.
Why this finishes the proof: $\dim_{k[x_1, \cdots, x_n]} M = \dim_{k[x_1, \cdots, x_n][y]} M$, so

- $\Phi(i) = \dim M_i$
- $\tilde{\Phi}(i) = \dim \tilde{M}_i$
- $\Phi(i) \leq \tilde{\Phi}(i) \leq \Phi(ik)$,

which forces $\deg \Phi = \deg \tilde{\Phi}$.

■

Definition 22.1.5 (Dimensions of modules)
If $A \in \mathsf{Alg}_{/k}{}^{\mathrm{fg}}$ and $M \in A\text{-}\mathsf{Mod}^{\mathrm{fg}}$, choose $k[x_1, \cdots, x_n] \twoheadrightarrow A$ to write $\dim_A M := \dim_{k[x_1, \cdots, x_n]} M$.

Remark 22.1.6: Upshot: we have a notion of dimension for $M \in \mathsf{Alg}_{/k}{}^{\mathrm{fg}}$ when $k \in \mathsf{Field}$ which doesn't depend on choices.

Exercise 22.1.7
Show the following properties:

- $\dim k[x_1, \cdots, x_n] = n$.
- If $A, B \in \mathsf{Alg}_{/k}{}^{\mathrm{fg}}$ and $A \to B$ with $B \in A\text{-}\mathsf{Mod}^{\mathrm{fg}}$, then $\dim_A B = \dim B$ (this just involves adding variables to $k[x_1, \cdots, x_n]$).
- If $M \in A\text{-}\mathsf{Mod}^{\mathrm{fg}}$ then $\dim_A M \leq \dim A$. Hint: $A^n \twoheadrightarrow M$.
- If $A, B \in \mathsf{Alg}_{/k}{}^{\mathrm{fg}}$ and $A \to B$ with $B \in A\text{-}\mathsf{Mod}^{\mathrm{fg}}$, then $\dim B \leq \dim A$.
- If $A, B \in \mathsf{Alg}_{/k}{}^{\mathrm{fg}}$ with $A \hookrightarrow B$ then $\dim A < \dim B$. Hint: consider

$$
\begin{array}{ccc}
k[x_1, \cdots, x_n] & \twoheadrightarrow & A \\
\downarrow & & \downarrow \\
k[x_1, \cdots, x_n][y_1, \cdots, y_m] & \twoheadrightarrow & B
\end{array}
$$

Link to Diagram

Take the induced filtrations A_i, B_i induced by degree on $k[x_1, \cdots, x_n]$, then $\iota(A_i) \subseteq B_i$.

- If $I \trianglelefteq A \in \mathsf{Alg}_{/k}{}^{\mathrm{fg}}$ then $\dim A/I \leq \dim A$ (geometrically this is passing to a closed subspace).
- If $A \in \mathsf{Alg}_{/k}{}^{\mathrm{fg}}$ is a domain with $I \trianglelefteq A$ then $\dim A/I < \dim A$. Hint: it's ETS $\dim A/f < \dim A$, so use the SES $0 \to A \xrightarrow{f} A \to A/f \to 0$.
- If A is a domain and $f \in A \setminus \{0\}$ then $\dim A/f = \dim A - 1$.

Example 22.1.8 *(?):* Using these properties:

- $A = k[x, y]$ satisfies $\dim A = 2$, while $B = k[x, y]/y = k[x]$ satisfies $\dim B = 1$.
- For $A = k[x, y]/xy$ and $B = A/y = k[x]$, both are dimension 1. Geometrically, A is the union of the x and y axes, and B sets $y = 0$ which results in the x axis.
- If $f : A \to B$ yields $B \in \mathsf{A\text{-}Mod}^{\mathrm{fg}}$, the induced map $f^* : \operatorname{Spec} B \to \operatorname{Spec} A$ has finite fibers.
  - If additional f is an injection, the dimensions are equal.

Injection means dense image, and dense image + finite fibers preserves dimension.

22.2 Other notions of dimension: transcendence degree

Remark 22.2.1: Setup: let

- $k \subseteq K$ be an inclusion of fields (with no finiteness conditions)
- $\alpha \in K$ is **integral over** k iff $f(\alpha) = 0$ for some monic $f \in k[x]$.
- K is **algebraic over** k iff every $\alpha \in K$ is integral over k.
- K is **transcendental over** k if it is not algebraic.

Example 22.2.2 *(?):* Some examples:

- $\mathbb{Q}(\sqrt{5})$ is algebraic over $\mathbb{Q}$.
- $\mathbb{Q}(t)$ is transcendental over $\mathbb{Q}$.

Definition 22.2.3 (Algebraically independent elements)
A subset $\{x_a\} \subseteq K$ is **algebraically independent over** k iff there does not exist a nonzero polynomial $p \in k[t_1, \cdots, t_n]$ with $p(\cdots, x_a, \cdots) = 0$. Such a subset is **maximal** if it is not strictly contained in another such subset.

Definition 22.2.4 (Transcendence degree)
If K/k admits a maximal finite algebraically independent subset S, define $\operatorname{trdeg}_k K = \sharp S$.

Claim: This is well-defined. To be justified next time!

Definition 22.2.5 (?)
If A is a finitely generated domain over k, then $\dim A = \operatorname{trdeg}_k \operatorname{ff}(A)$.

23 | Thursday, April 14

Remark 23.0.1: Recall that if $A \in \mathsf{Alg}_{/k}$, the algebraic differentials are constructed as

$$\Omega_{A_{/k}} = \mathsf{Free}\left\{dA \;\middle|\; a \in A\right\} / \left\langle d(ab) = ad(b) + d(a)b \right\rangle.$$

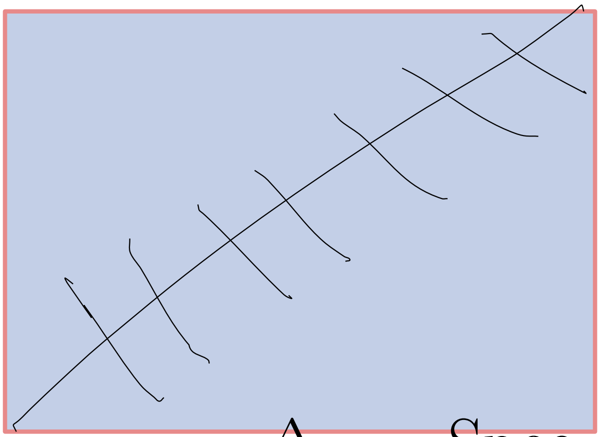
Note that $\underset{\mathsf{A-Mod}}{\operatorname{Hom}}(\Omega_{A_{/k}}, M) \cong \underset{\mathsf{k-Mod}}{\operatorname{Der}}(A, M)$, where derivations are importantly not A-linear! This is meant to emulate a cotangent bundle.

There is a SES

$$0 \to I \to A \otimes_k A \xrightarrow{m} A \to 0.$$

Example 23.0.2 ***(?):*** For $A = k[x]$, check that $x, y \mapsto x$ and $I = \langle x - y \rangle$, and moreover $\Omega_{k[x]_{/k}} \cong I/I^2 = \langle x - y \rangle / \left\langle x^2 + xy + y^2 \right\rangle$.

Slogan 23.0.3
The cotangent bundle is the conormal bundles of the diagonal:



Spec A $\Delta_X = \operatorname{Spec} A \otimes_k A$

Exercise 23.0.4 (?)
Show that $\Omega_{A_{/k}} \cong I/I^2$.

Solution:
Hints: a map $\Omega_{A_{/k}} \xrightarrow{f} I/I^2$ is equivalent to a derivation $A \to I/I^2$. Show that if $\psi(a) := a \otimes 1 - 1 \otimes a \in I$, then $m(\psi(a)) = 0$ and ψ is a derivation, i.e. $\psi(ab) - a\psi(b) - \psi(b)a \in I^2$. Check

$$\begin{aligned} ab \otimes 1 - 1 \otimes ab - a(b \otimes 1 - 1 \otimes b) - b(a \otimes 1 - 1 \otimes a) &= ab \otimes 1 - 1 \otimes ab - ab \otimes 1 + a \otimes b - ba \otimes 1 + b \otimes a \\ &= -1 \otimes ab + a \otimes b - ab \otimes 1 + b \otimes a \\ &= -(a \otimes 1 - 1 \otimes a)(b \otimes 1 - 1 \otimes b) \in I^2. \end{aligned}$$

For an inverse, take

$$\begin{aligned} g : I/I^2 &\to \Omega_{A_{/k}} \\ \sum a_i \otimes b_i &\mapsto \sum b_i d(a_i) \end{aligned}$$

where $\sum a_i b_i = 0$ and check

- $g(I)^2 = 0$
- $f \circ g = \mathrm{id}, g \circ f = \mathrm{id}$.

Exercise 23.0.5 (?)
Show that $(1 \circ x - x \circ 1) \to d1 + 1dx$ where $x - y = -dx$.

Exercise 23.0.6 (General algebra)
Show that $M \to N \to L \to 0 \in \mathsf{A\text{-}Mod}$ is exact iff $0 \to [L, S]_A \to [N, S]_A \to [M, S]_A$ is exact for all $S \in \mathsf{A\text{-}Mod}$.

Solution:
Hint: $\Longleftarrow$ is the nontrivial direction. Toward a contradiction take $S := \operatorname{coker}(A \to L)$.

Exercise 23.0.7 (?)
Show that if $A, B \in \mathsf{Alg}_{/k}$ and $0 \to I \to A \xrightarrow{f} B \to 0$ is exact then there is an exact sequence

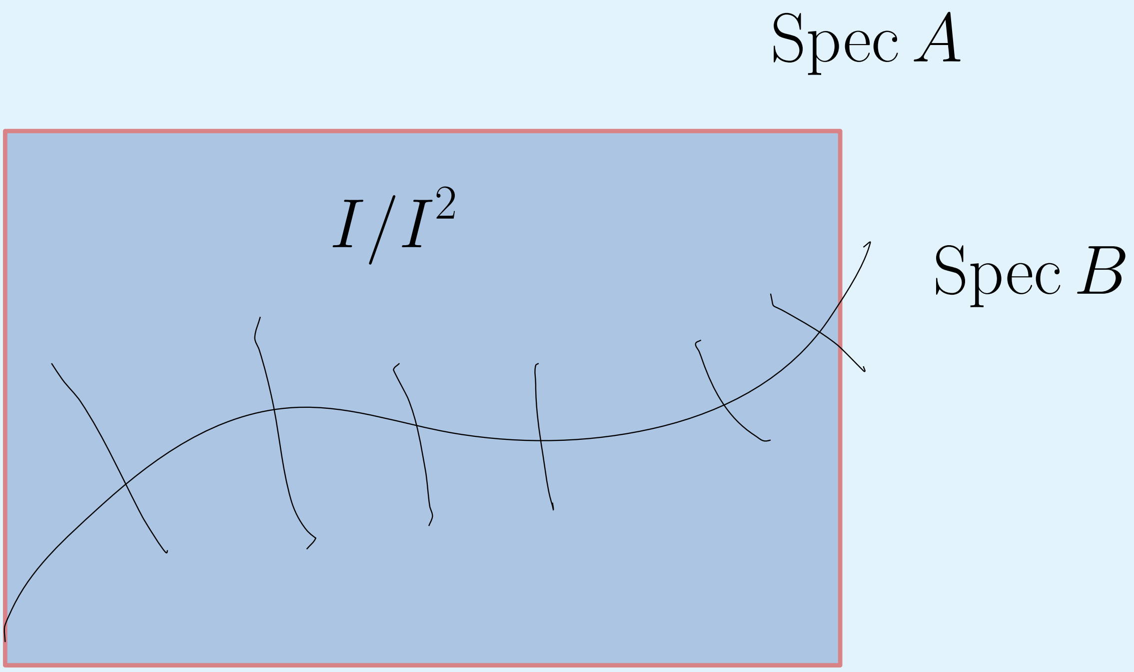
$$I/I^2 \longrightarrow \Omega_{A/_{/k}} \longrightarrow \Omega_{B/_{/k}} \longrightarrow 0 \qquad \in \mathsf{B\text{-}Mod}$$

$$x \longmapsto dx$$

$$da \longmapsto d(f(a))$$

Link to Diagram

Idea: $\operatorname{Spec} B \hookrightarrow \operatorname{Spec} A$ is like an embedded submanifold, and I/I^2 is the conormal bundle.



Solution:
Identify

$$\begin{array}{ccccccc}
0 & \xrightarrow{?} & [\Omega_{B_{/k}}, S]_B & \longrightarrow & [\Omega_{A_{/k}} \otimes_A B, S]_B \cong [\Omega_{A_{/k}}, S]_A & \longrightarrow & [I/I^2, S]_B \\
 & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\
0 & \xrightarrow{\therefore} & \underset{k}{\mathrm{Der}}[B, S] & \longrightarrow & \underset{k}{\mathrm{Der}}[A, S] & \longrightarrow & [I/I^2, S]_B
\end{array}$$

Link to Diagram

Check:

- $\psi \in \underset{k}{\mathrm{Der}}(B, S)$ with $\psi \circ f$ surjective implies $\psi = 0$ when f is surjective.
- $\psi \in \underset{k}{\mathrm{Der}}(A, S)$ with $\rho\psi I = 0$ implies $\psi \in \mathrm{im}(\underset{K}{\mathrm{Der}}(B, S) \to \underset{k}{\mathrm{Der}}(A, S))$

Definition 23.0.8 (?)
For $f \in \mathsf{CRing}(A, B)$, $\Omega_{B/A}$ is the unique object in $\mathsf{B\text{-}Mod}$ such that

$$[\Omega_{B/A}, M] = \underset{A}{\mathrm{Der}}(B, M).$$

Explicitly,

$$\Omega_{B/A} = \mathsf{Free}\,\{b \in B\} \,/\, \left\langle d(b_1 b_2) = b_1 d(b_2) + d(b_1) b_2, da \;\middle|\; a \in A, b_i \in B \right\rangle.$$

Exercise 23.0.9 (?)
Show that for $A, B \in \mathsf{Alg}_{/k}$, there is a SES

$$\Omega_{A_{/k}} \otimes_A B \to \Omega_{B_{/k}} \to \Omega_{B/A} \to 0 \qquad \in \mathsf{B\text{-}Mod}.$$

Remark 23.0.10: If $A \in \mathsf{Alg}_{/k}{}^{\mathrm{fg}}$ and $k = \bar{k}$ and $\mathfrak{m} \in \operatorname{mSpec} A$, then $0 \to \mathfrak{m} \to A \to k \to 0$ (by EEKS) and there is a SES

$$0 \to \mathfrak{m}/\mathfrak{m}^2 \to \Omega_{A_{/k}} \otimes_A k \to \Omega_{k_{/k}} = 0 \to 0.$$

Definition 23.0.11 (Tangent/cotangent spaces)
Define $\mathbf{T}^\vee A \coloneqq \mathfrak{m}/\mathfrak{m}^2$ and $\mathbf{T}A \coloneqq (\mathfrak{m}/\mathfrak{m}^2)^\vee \coloneqq [\mathfrak{m}/\mathfrak{m}^2, k]_k$.

Exercise 23.0.12 (?)
Let $A = k[x]$ and $\mathfrak{m} = \langle x \rangle$, then for $f \in \mathfrak{m}$ define $\tau \in \mathbf{T}^\vee A$, so $\tau(\bar{f}) \in k$, by $\tau = \dfrac{\partial}{\partial x}$. Then $\bar{f} \to f'(0)$.
Similarly for $A = k[x_1, \cdots, x_n]$ and $\mathfrak{m} = \langle x_1, \cdots, x_n \rangle$, check $\mathbf{T}_0 = \underset{\mathbb{C}}{\mathrm{span}} \left\langle \dfrac{\partial}{\partial x_1}, \cdots, \dfrac{\partial}{\partial x_n} \right\rangle$.

Theorem 23.0.13 *(?)*.
For $k = \bar{k}$ and $A \in \mathsf{Alg}_{/k}{}^{\mathrm{fg}}$ a domain,

- For any $m \in \operatorname{mSpec} A$,

$$\dim_k \Omega_{A_{/k}} \otimes_k A/m = \dim_k m/m^2 \geq \dim A.$$

- There exists a nonempty open $U \subseteq \operatorname{Spec} A$ such that for some $m \in U$ this is an equality, so

$$\dim_k m/m^2 = \dim_{\mathrm{ff}(A)} \mathrm{ff}(A) \otimes_A \Omega_{A_{/k}}.$$

Remark 23.0.14: What goes into a proof:

- Find $k[x_1, \cdots, x_n] \hookrightarrow A$ with $n = \dim A$ making $A \in \mathsf{k[x_1, \cdots, x_n]\text{-}Mod}^{\mathrm{fg}}$.
- Check $\Omega_{A/k[x_1, \cdots, x_n]}$ is torsion.

Definition 23.0.15 (Smooth points)
A point $m \in \operatorname{mSpec} A$ is **smooth** if

$$\operatorname{rank} \mathbf{T}^\vee A = \dim A.$$

Example 23.0.16 *(?)*: A non-example: $\operatorname{krulldim} A = 0$ for $A = k[x]/x^2$, since A is Artin. Check $\Omega_{A_{/k}} = A\,dx/\left\langle d(x)^2 = 0 \right\rangle \cong \dfrac{k[x]/x^2\,dx}{2x\,dx}$, which is $k\,dx$ if $\operatorname{ch} k \neq 2$ and $k[x]/x^2\,dx$ if $\operatorname{ch} k = 2$.

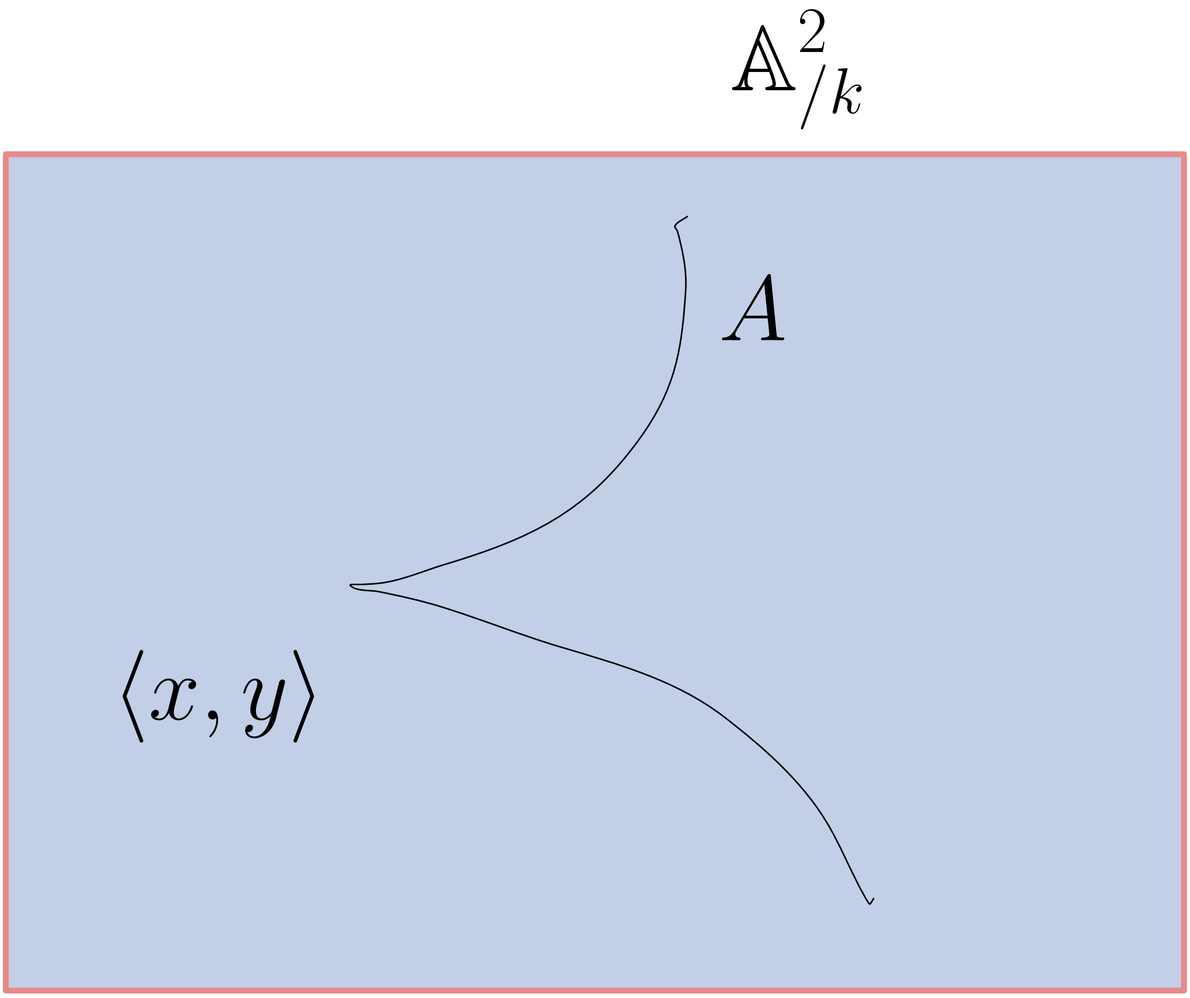
Example 23.0.17 *(?)*: For $A = k[x,y]/\left\langle y^2 - x^3 \right\rangle$ with $\operatorname{ch} k = 0$, check

$$\Omega_{A_{/k}} = \frac{A\,dx \oplus A\,dy}{d(y^2 - x^3)} = \frac{A\,dx \oplus A\,dy}{dy\,dy - 3x^2\,dx}.$$

Given $m \in \operatorname{mSpec} A$,

$$\dim_{A/m} \Omega_{A/k} \otimes_A A/m = \begin{cases} 1 & m \notin \langle x, y \rangle \\ 2 & m \in \langle x, y \rangle. \end{cases}$$

Note that this records the nodal singularity:



Example 23.0.18 *(?):* Over $k = \mathbb{F}_p(t)$ and $A = \mathbb{F}_p\left(t^{\frac{1}{p}}\right) = k[x]/\left\langle x^p - t \right\rangle$, check $\dim A = 0$ and

$$\Omega_{A_{/k}} = \frac{A\,dx}{d(x^p - t)} = \frac{A\,dx}{d(x^p)} = A\,dx.$$

So the algebraic differentials detect when a field fails to be separable.

24 | Tuesday, April 19

24.1 Completion

Remark 24.1.1: Recall that $\varprojlim_{i\in I} M_i \in \mathsf{A\text{-}Mod}$ is the universal A-module living above all of the M_i, such that for any other N above the M_i there is a morphism $N \to \varprojlim_i M_i$. This can be realized as

$$\varprojlim M_i = \left\{ (m_i)_{i\in I} \in \prod_{i\in I} M_i \;\middle|\; \varphi(m_i) = m_{i+1}\ \forall i \right\}.$$

Note that the inverse limit has a mapping ***in*** *property, as does* $\prod$. *Limits can be constructed out of equalizers and products:*

$$\varprojlim F \cong = \left(\prod_{X\in \mathsf{C}} F(X) \to \prod_{f, Y\in \mathsf{C}(X,Y)} F(Y) \right).$$

Exercise 24.1.2 (?)
Show that this satisfies the correct universal property.

Exercise 24.1.3 (?)
Show the following:

$$\varprojlim_i k[t]/\left\langle t^i \right\rangle \cong k[[t]]$$

$$\varprojlim(\cdots \to \mathbb{Z}/p^2\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z} \to 0) \cong \mathbb{Z}_{\widehat{p}}$$

$$\varprojlim_n \mathbb{Z}/n!\mathbb{Z} \cong \widehat{\mathbb{Z}} = \prod_p \mathbb{Z}_{\widehat{p}}.$$

Exercise 24.1.4 (?)
Show that the functor $\varprojlim_i(-)$ is left-exact and $\underset{i}{\underrightarrow{\mathrm{colim}}}(-)$ is exact, i.e. if $0 \to (M_i) \to (N_i) \to (L_i) \to 0$ is a SES of inverse systems, then $0 \to \varprojlim_i M_i \to \varprojlim_i N_i \to \varprojlim_i L_i$.

Solution:
Hint: for injectivity, use the product definition to realize $f : \varprojlim_i M_i \to \varprojlim_i N_i$ as $f = \prod_i f_i : M_i \to N_i$.

Remark 24.1.5: Show that $\tau_{\geq 2}\mathbb{R}\varprojlim_i(-) = 0$, so there is always a 6-term exact sequence.

Theorem 24.1.6 ***(Mittag-Leffler, sufficient conditions for*** $\varprojlim^1$ ***vanishing).***

If $M_i \to M_{i+1}$ is surjective, then $\lim^1 = 0$ and $\varprojlim N_i \twoheadrightarrow \varprojlim L_i$.

Exercise 24.1.7 (?)
Prove this!
Hints:

- Given $(n_i) \in \varprojlim N_i$, we want to produce (l_i) with $(g_i(n_i)) = \ell_i$.
- Lift ℓ_0 to n_0 and induct on i:
- Choose an approximate lift $\tilde{n}_i$ of ℓ_i.
- Use commutativity of the diagrams of inverse systems to correct this choice, using the surjectivity assumption.

Definition 24.1.8 (Completion)
Recall that given a sequence of submodules M_i of a module M, so $M \geq M_1 \geq M_2 \geq \cdots$, one can form an inverse system $\cdots \to M/M_2 \to M/M_1 \to 0$. The **completion** of M is $\varprojlim_i M/M_i$.

Remark 24.1.9: Topological interpretation: define a subset $U \subseteq M$ iff for each $m \in U$ there is some M_i such that $m + M_i \subseteq U$; equivalently $\{M_i\}$ forms a basis of neighborhoods of zero.

Exercise 24.1.10 (?)
Show that $\varprojlim M/M_i$ is Hausdorff iff $\cap_i M_i = \{0\}$ (i.e. $\widehat{M}$ is separated).

Solution:
Hints:
$\implies$: If $m \neq 0 \in \cap M_i$, show every open containing zero contains m.
$\impliedby$: Pick $m_1 \neq m_2$ and find M_i with $(m_1 + M_i) \cap (m_2 + M_i) = \emptyset$.

Remark 24.1.11: Call a sequence (m_i) **Cauchy** iff $m_n - m_{n'} \in M_i$ for all $n, n' > N_i$. Note that one can define a metric this way and take the Cauchy completion, defined as $\widehat{M}$, which is canonically isomorphic to $\widehat{M}$.

Exercise 24.1.12 (?)
Produce the isomorphism from Cauchy sequences modulo equivalence to $\widehat{M}$.

Solution:
For $\widehat{M} \to M/M_i$, take $N_i \gg 0$ and project (i.e. send the sequence to its limit). For $\varprojlim M/M_i \to \widehat{M}$, send $(m_i) \mapsto (\tilde{m}_i)$ by choosing arbitrary lifts.

Exercise 24.1.13 (?)
Show that if $0 \to M_1 \to M_2 \to M_3 \to 0$ and M_2 admits a filtration, then there is a SES of the

completions at *induced* filtrations $0 \to \widehat{M_1} \to \widehat{M_2} \to \widehat{M_3} \to 0$.

Solution:
Start with the SES

$$0 \to \frac{M_1}{M_1 \cap \mathrm{Fil}_i M_2} \to \frac{M_2}{\mathrm{Fil}_i M_2} \to \frac{M_3}{f(\mathrm{Fil}_i M_2)} \to 0,$$

and letting i vary yields a SES of inverse systems. By Mittag-Leffler, the limits are exact.

Exercise 24.1.14 (?)
Show that completion is idempotent, so $\widehat{\widehat{M}} \cong \widehat{M}$.

Definition 24.1.15 (adic filtration)
For $A \in \mathsf{CRing}, M \in \mathsf{A\text{-}Mod}, \alpha \in \mathrm{Id}(A)$, there is an α-adic filtration $\mathrm{Fil}_i M := \alpha^i M$, and the corresponding completion $\widehat{M}$ is the **α-adic completion**.

Example 24.1.16 ***(?):*** Some examples:

- $k[t], \langle t \rangle$
- $\mathbb{Z}, \langle p \rangle$
- $k[x, y], \langle x, y \rangle$

Slogan 24.1.17
Completions are inverse limits of Artin rings and are local, and thus mediate between the two.

Exercise 24.1.18 (?)
Show that if $A \in \mathsf{CRing}, m \in \mathrm{mSpec}\, A$, then $\widehat{A}$ completed with respect to m is a local ring.

25 | Thursday, April 21

25.1 Artin Rees

Exercise 25.1.1 (?)
Show that if $A \in \mathsf{CRing}$ and $m \in \mathrm{mSpec}\, A$ then $A_{\widehat{m}}$ is local.

Solution:
Show that $x \in A_{\widehat{m}} \setminus \ker(A_{\widehat{m}} \xrightarrow{\pi} A/m)$ is invertible. If $\pi(x) = 1$ then $1 - x \in \ker \pi$ and $y = \dfrac{1}{1-(1-x)} = \sum_k (1-x)^k$ is an inverse.

Remark 25.1.2: Note that α-adic completion is not generally exact, but *is* exact in most cases of interest, e.g. for $M \in \mathsf{A\text{-}Mod}^{\mathrm{fg}}$. There is a derived functor used in e.g. Bhatt-Scholze.

Theorem 25.1.3 *(Artin-Rees).*
For $M \in \mathsf{A\text{-}Mod}^{\mathrm{fg}}$ for $A \in \mathrm{Noeth}\mathsf{CRing}$ with $M' \leq M$ and $\alpha \in \mathrm{Id}(A)$, the α-adic topology on M' coincides with the topology on M' induced by the α-adic topology on M.

Corollary 25.1.4 *(?).*
Taking the α-adic completion for finitely-generated modules on Noetherian rings is exact.

Proof (?).
This coincides with the induced topology, and we showed that taking the *induced* completions is exact. ∎

Definition 25.1.5 (?)
For $M \in \mathsf{A\text{-}Mod}$ and $\mathrm{Fil}M$ a descending filtration,

- $\mathrm{Fil}M$ is **compatible** with α if $\alpha\mathrm{Fil}_i M \subseteq \mathrm{Fil}_{i+1} M$.
  - Equivalently, $\mathsf{gr}_i M \in \mathsf{gr}\,\mathsf{A\text{-}Mod}$.
- $\mathrm{Fil}M$ is **α-good** if for $i \gg 0$ this is an equality.
  - Equivalently, $\mathsf{gr}_i M \in \mathsf{gr}\,\mathsf{A\text{-}Mod}^{\mathrm{fg}}$.

Observation 25.1.6
Recall that for Hilbert dimension, we took a good filtration and used the eventual degree. If $\mathrm{Fil}M, \mathrm{Fil}'M$ are two α-good filtrations,

$$\mathrm{Fil}'_{i-k} M \subseteq \mathrm{Fil}_i M \subseteq \mathrm{Fil}'_{i-k} M.$$

As a corollary, any two α-good filtrations induce the same topology, since this yields a containment of basis elements.

Theorem 25.1.7 *(Artin-Rees reprised).*
If M, M', A, α as in the original statement above, $M' \cap \alpha^i M$ is α-good.

Remark 25.1.8: Why this implies the previous version: the induced filtration is α-good and induces the same topology as the α-adic filtration.

Definition 25.1.9 (Rees algebra/blowup algebra)
For $\alpha \in \mathrm{Id}(A)$, define the **Rees algebra**

$$\mathrm{Rees}A := A \oplus \alpha[1] \oplus \alpha^2[2] \oplus \cdots .$$

Exercise 25.1.10 (?)
Show that if A is Noetherian then $\mathrm{Rees}A$ is Noetherian.

Solution:
Show that if $\{a_i\}$ generate α, there is a surjection $A[x_1, \cdots, x_n] \twoheadrightarrow (\mathrm{Rees}A)[1]$ where $x_i \mapsto a_i$ and apply the Hilbert basis theorem.

Definition 25.1.11 (Rees modules)
Let (M, Fil) be an α-compatible filtered A-module. Define

$$\mathrm{Rees}M := M \oplus (\mathrm{Fil}_1 M)[1] \oplus (\mathrm{Fil}_2 M)[2] \oplus \cdots \qquad \in \mathrm{Rees}\mathsf{A}\text{-}\mathsf{Mod}.$$

Proposition 25.1.12*(?)*.
Let $A \in \mathrm{Noeth}\mathsf{CRing}$, then TFAE:

- $M \in \mathsf{A}\text{-}\mathsf{Mod}^{\mathrm{fg}}$ with an α-good filtration,
- $\mathrm{Rees}M \in \mathrm{Rees}\mathsf{A}\text{-}\mathsf{Mod}^{\mathrm{fg}}$.

Proof (?).
$2 \implies 1$: restrict generators to degree zero, then use that finitely generated implies that generators are in bounded degree to get α-goodness.
$1 \implies 1$: find i_0 such that $\alpha M_i = M_{i+1}$ for $i \geq i_0$, so $\mathrm{Rees}M$ is generated by $\tau_{\leq i_0}\mathrm{Rees}M$. Each M_i is finitely generated since it's a submodule of a finitely-generated module over a Noetherian ring.

■

Proof (of Artin-Rees reprised).
Let $\mathrm{Rees}M'$ be the induced filtration, then $\mathrm{Rees}M$ is α-good and finitely-generated over A^* and we want to show $\mathrm{Rees}M' \subseteq \mathrm{Rees}M$. Conclude using that $\mathrm{Rees}A$ is Noetherian.

■

25.2 Injections into completions

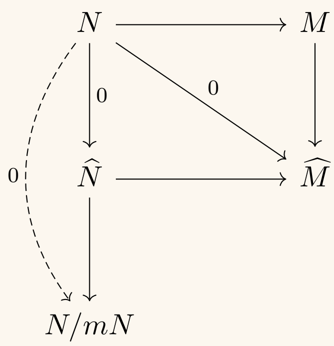
25 Thursday, April 21

Corollary 25.2.1 ***(?)*.**
If $A \in \mathsf{LocNoethCRing}$ with $m \in \operatorname{mSpec} A$ and $M \in A\text{-}\mathsf{Mod}^{\mathrm{fg}}$,

$$M \hookrightarrow M_{\widehat{m}}.$$

Note that this is not true for arbitrary modules!

Proof (?).
Let N be the kernel, use:



Link to Diagram

So $N/mn = 0 \implies N = 0$ by Nakayama.

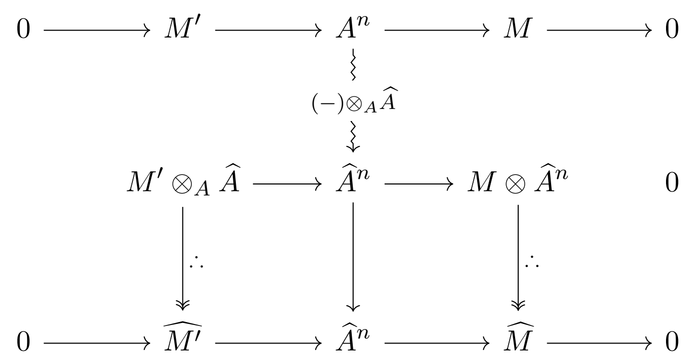
■

Example 25.2.2 ***(of when injectivity fails):*** Let $A = k[t]$ and $M = k[t]/\langle t-1\rangle$ with $m = \langle t\rangle$. Then $M/t^i M = 0$ for all i, so $M_{\widehat{m}} = 0$.

Exercise 25.2.3 (?)
Show that for $A \in \mathsf{NoethCRing}, M \in A\text{-}\mathsf{Mod}^{\mathrm{fg}}, \alpha \in \operatorname{Id}(A)$,

$$\widehat{A} \otimes_A M \xrightarrow{\sim} \widehat{M}.$$

Solution:
Hint: $(a_i)_{i\in I} \otimes m \mapsto (a_i m)_{i\in I}$. Now carry out a diagram chase and use the 5 lemma (or snake lemma) on the following diagram:



Link to Diagram

Exercise 25.2.4 (?)
Show that if $A \in \mathsf{NoethCRing}$ and $\alpha \in \mathrm{Id}(A)$ then $\widehat{A} \in \mathsf{A\text{-}Mod}^\flat$.

Solution:
Hint: take $0 \to M_1 \to M_2 \to M_3 \to 0$, tensor and use the proposition, then it STS $0 \to \widehat{M_1} \to \widehat{M_2} \to \widehat{M_3} \to 0$ is exact. It suffices to check flatness for $M_i \in \mathsf{A\text{-}Mod}^{\mathrm{fg}}$ by a previous HW exercise.

25.3 Motivations

Theorem 25.3.1 ***(Cohen structure theorem).***
If $k = \bar{k} \in \mathsf{Field}$ and $A \in \mathrm{Noeth}\mathsf{Alg}_{/k}$ is *regular* which is local and complete with respect to its maximal ideal, then $A \xrightarrow{\sim} k[[x_1, \cdots, x_n]]$.

Example 25.3.2 ***(?):*** A non-regular example: take a complete local ring like $k[[x,y]]/\langle xy \rangle$ and localize to zoom in on $\mathbf{0} \in \mathbb{A}^2_{/k}$.

An important example in algebraic curves: if $A_{/k}$ for A Dedekind and $m \in \mathrm{mSpec}\, A$, the completion is $\widehat{A} \cong k[[x]]$.

Slogan 25.3.3
Here *regular* means smooth, so ${\mathbf{T}_m}^\vee = \dim A$.

Remark 25.3.4: A general pattern for studying rings:

- Start with a ring A.

- Localize to achieve smoothness.
- Complete to reduce to questions about power series.

Remark 25.3.5: What would show up in a 2nd course on commutative algebra: singularity theory. See Grothendieck duality, Cohen–Macaulay rings.

26 | Thursday, April 28

26.1 Dimension theory

Remark 26.1.1: Given an arbitrary grid, can you tile it with 2×1 dominoes? What dominoes corresponding to Young's diagrams for partitions $\lambda = (2, 1)$?

A principled way of approaching such problems: consider labeling the grid with monomials:

$$\begin{array}{cccc} \vdots & & & \\ y^2 & xy^2 & x^2y^2 & \\ y & xy & x^2y & \\ 1 & x & x^2 & \cdots \end{array}$$

Link to Diagram

Now labeling the 2×1 tile with $1, x$ and the 1×2 tiles with $1, y$. Note that there is a polynomial $f(x, y) = 1 + x + y + x^2 + xy + y^2 + \cdots$ associated to the grid; if there admits a tiling then $f \in \langle 1 + x, 1 + y \rangle \subseteq k[x, y]$. One can then check that $\overline{f} \in k[x, y]/\langle 1 + x, 1 + y \rangle = k$ satisfies $\overline{f}(-1, -1) = 2$, so $\overline{f} \neq 0$. Note that $\overline{f} = 0$ is a necessary but not sufficient condition.

Remark 26.1.2: Last few topics: toward the Cohen structure theorem. Setup: $A \in \text{Noeth}\mathsf{LocRing}$ and let $\mathfrak{m} \in \operatorname{mSpec} A$, e.g.

- $A = k[[x_1, \cdots, x_n]]$,
- $A = (k[x_1, \cdots, x_n]/I)_{\mathfrak{p}}$
- $A = \mathbb{Z}_{\widehat{p}}[[x_1, \cdots, x_n]]/I$.

We don't have a good dimension theory for these, since they aren't finitely generated algebras. Some approaches:

- Hilbert dimension,
- Krull dimension,
- Generator dimension.

Definition 26.1.3 (Hilbert dimension)
Setup:

- Let $M \in \mathsf{A\text{-}Mod}^{\mathrm{fg}}$ and choose an $\mathfrak{m}$-good filtration $\mathrm{Fil}_i M$ on M, so $\mathfrak{m} M_i \subseteq M_{i+1}$ for $i \gg 0$. Equivalently, $\mathrm{Rees} M$ is finitely-generated over $\mathrm{Rees} A$.
- Let $\Phi(i) = \mathrm{len}(M/\mathrm{Fil}_i M)$, which is eventually polynomial.
- Define $\dim_A M = \deg \Phi$.

This defines a dimension for any finitely generated module.

Remark 26.1.4: There is a naturally good filtration: $\mathrm{Fil}_i M := \mathfrak{m}^i M$, then

$$\Phi(i) = \mathrm{len}(M/\mathrm{Fil}_i M) = \sum_{0 \leq j \leq i-1} \dim_{A/\mathfrak{m}} \mathsf{gr}_j M$$

where $\mathsf{gr}_j M = \mathfrak{m}^j M / \mathfrak{m}^{j+1} M$.

Lemma 26.1.5 *(?)*.

- Given a SES $A \to B \to C$, $\mathrm{hilbdim}\, B = \max(\mathrm{hilbdim}\, A, \mathrm{hilbdim}\, C)$.
- For $\Phi : M \hookrightarrow M$, $\mathrm{hilbdim}\, M/m\varphi \leq \mathrm{hilbdim}\, M - 1$

Example 26.1.6 *(?)*: Check $\mathrm{hilbdim}\, k[[x_1, \cdots, x_n]] = n$ using $\mathfrak{m} = \langle x_1, \cdots, x_n \rangle$; then $\Phi(i) = \mathrm{len}\, k[[x_1, \cdots, x_n]]/\mathfrak{m}^i = \sum_{0 \leq j \leq i-1} \binom{n+j}{j}$ by counting monomials.

Exercise 26.1.7 (?)
Show $\mathrm{hilbdim}\, \mathbb{Z}_{\widehat{p}}[[x_1, \cdots, x_n]] = n + 1$ using $\mathfrak{m} = \langle p, x_1, \cdots, x_n \rangle$.

Exercise 26.1.8 (?)
Find $\mathrm{krulldim}\, k[[x_1, \cdots, x_n]]$ by finding a maximal chain of prime ideals.

Definition 26.1.9 (Generator dimension)
Define $\mathrm{gendim}\, A$ to be the minimal d such that there exist $x_1, \cdots, x_d \in \mathfrak{m} = \sqrt{\langle x_1, \cdots, x_d \rangle}$.

Theorem 26.1.10 *(?)*.
If $A \in \mathsf{Alg}_{/k}{}^{\mathrm{fg}}$ then $\dim A = \max_{m \in \mathrm{mSpec}\, A} \dim A_m$.

Exercise 26.1.11 (Dimension $\neq$ minimal number of generators)
Show that for $A = k[[x_1, x_2]]/\left\langle x_2^2 \right\rangle$ that $\langle x_1, x_2 \rangle$ is a minimal set of generators for the maximal ideal but $\operatorname{gendim} A = 1$ since $m = \sqrt{x_i}$.

Exercise 26.1.12 (?)
Show if $A \in \mathsf{LocNoethCRing}$ with $\mathfrak{m} \in \mathrm{mSpec}\, A$, then

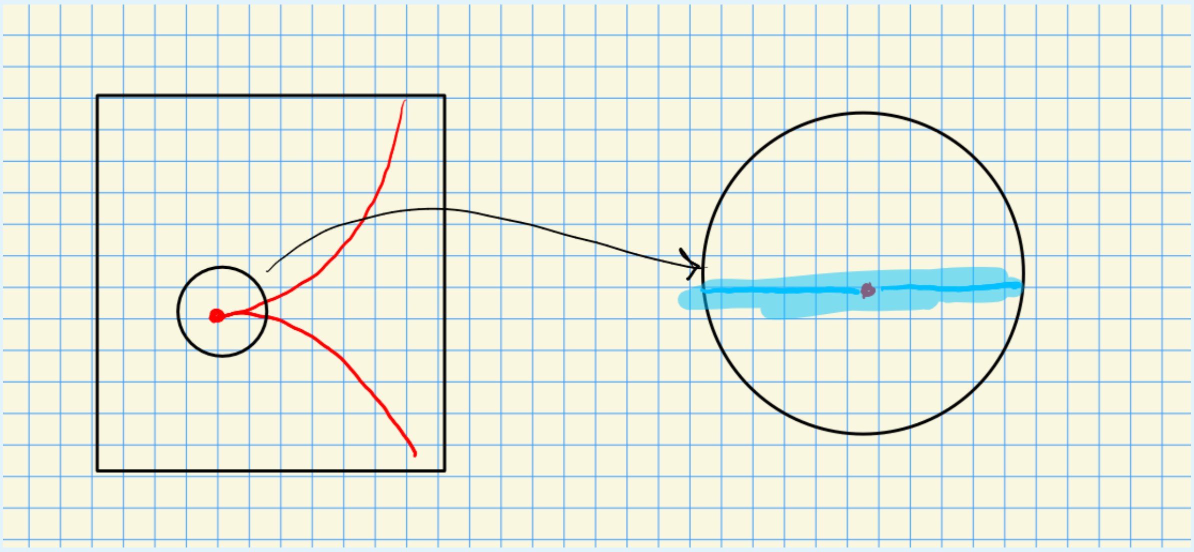
$$\dim A \leq \dim_{A/\mathfrak{m}} \mathfrak{m}/\mathfrak{m}^2.$$

Solution:
Hint for using generator dimension: pick a basis $\{\bar{x}_i\}_{1 \leq i \leq n}$ of $\mathfrak{m}/\mathfrak{m}^2$. Lift to $\{x_i\}$ and take $\mathfrak{m} := \langle x_1, \cdots, x_n \rangle$ and conclude by Nakayama.
Hint for using Hilbert dimension: show $\operatorname{hilbdim} A = \operatorname{hilbdim} \oplus m^i/m^{i+1}$, since the LHS is $\deg(i \mapsto \operatorname{len} A/m_i)$ and the RHS is $\deg(i \mapsto \sum_{0 \leq j \leq i-1} \dim_k m^j/m^{j+1})$. Then show there is an inequality $\operatorname{hilbdim} \bigoplus_i m^i/m^{i+1} \leq \dim_k m/m^2$ by showing there is an important multiplication map $\mathrm{Sym}\, m/m^2 \twoheadrightarrow \oplus_i m^i/m^{i+1}$. Conclude using that the LHS is isomorphic to a power series rings $k[[\bar{x}_1, \cdots, \bar{x}_n]]$ which is dimension n.

Remark 26.1.13: $\oplus m^i/m^{i+1}$ is the tangent cone and $\mathrm{Sym}\, m/m^2$ are functions on the tangent space, and the tangent cone is a subvariety of the tangent space.

Remark 26.1.14: Let $M = k[[x,y]]/\langle x, y \rangle$, $m = \langle x, y \rangle$, and $m/m^2 = \langle \bar{x}, \bar{y} \rangle$. Then $\mathrm{Spec}\,\mathrm{Sym}\, m/m^2$ is a curve (?) and $\oplus_i m^i/m^{i+1} = k[[x,y]]/\langle xy \rangle$.

Exercise 26.1.15 (?)
For $k[[x,y]]/\left\langle y^2 - x^3 \right\rangle$, $m/m^2 = \langle \bar{x}, \bar{y} \rangle$, $m^2/m^3 = \left\langle \bar{x}^2, \bar{x}\bar{y} \right\rangle$, so $\oplus_i k[[x,y]]/\left\langle y^2 \right\rangle$. This yields a thickened line along $y = 0$:



Slogan 26.1.16
The dimension of a variety is at most the dimension of its tangent spaces.

26.2 Smoothness and regularity

Remark 26.2.1: Idea: regularity is "almost smooth".

Definition 26.2.2 (Regularity)
A ring $(A, m) \in$ NoethLocCRing is **regular** iff the natural multiplication map $\operatorname{Sym} m/m^2 \twoheadrightarrow \oplus_i m^i/m^{i+1}$ is an isomorphism. For arbitrary $A \in$ NoethCRing (not necessarily local), A is *regular* iff A_m is regular for all $m \in \operatorname{mSpec} A$.

Example 26.2.3*(?)*:

- $k[x_1, \cdots, x_n]$ is regular.
- $k[[x_1, \cdots, x_n]]$ is regular.

Exercise 26.2.4 (?)
Any field is regular local, but regularity is not preserved under extensions. Take $\mathbb{F}_p\left(t^{\frac{1}{p}}\right) \supseteq \mathbb{F}_p(t)$ and show

$$A := \mathbb{F}_p\left(t^{\frac{1}{p}}\right) \otimes_{\mathbb{F}_p(t)} \mathbb{F}_p\left(t^{\frac{1}{p}}\right) \cong \mathbb{F}_p\left(t^{\frac{1}{p}}\right)[s]/\left\langle s^p\right\rangle,$$

which is not reduced. Use that $\mathbb{F}_p(t^{\frac{1}{p}}) = \mathbb{F}_p(t)[x]/\langle x^p - t\rangle$, so

$$\begin{aligned} A &= \mathbb{F}_p(t)[x,y]/\langle x^p - t, y^p - t\rangle \\ &= \mathbb{F}_p(t^{\frac{1}{p}})[y]/\left\langle y^p - (t^{\frac{1}{p}})^p\right\rangle \\ &= \mathbb{F}_p(t^{\frac{1}{p}})[y]/\left\langle y - t^{\frac{1}{p}}\right\rangle^p. \end{aligned}$$

Definition 26.2.5 (Smoothness)
If $A \in \mathsf{Alg}_{/k}{}^{\mathrm{fg}}$ and $\bar{k} = \mathrm{cl}^{\mathrm{alg}}\, k$, then A is **smooth** iff $A \otimes_k \bar{k}$ is regular.

Exercise 26.2.6 (?)
Show that a ring $A \in \mathrm{Noeth}\mathsf{LocCRing}$ is regular iff $\dim A = \dim_{A/m} m/m^2$.

Solution:
Hint: use that regularity implies equality implies that the map $\pi : \operatorname{Sym} m/m^2 \to \oplus m^i/m^{i+1}$ is an isomorphism. Assume this is not an equality, let $f \in \ker \pi$, then $\dim A = \dim \oplus m^i/m^{i+1} \leq \dim \operatorname{Sym} m/m^2/f < \dim \operatorname{Sym} m/m^2 = \dim m/m^2$, a contradiction.

Exercise 26.2.7 (?)
Show that regular local rings are domains.

Solution:
Pick nonzero zero divisors, a, b with $ab = 0$. Then $\bar{a}, \bar{b} \in m^i/m^{i+1}$ with $\bar{a}\bar{b} = 0$, a contradiction.

27 Problem Set 1

AM Ch. 1, 1, 8, 10, 13, 15, 16, and 19.

28 Problem Set 2

Problem 28.0.1 (AM 2.1)
Show that

$$m\mathbb{Z} + n\mathbb{Z} = 1 \implies C_m \otimes_{\mathbb{Z}} C_n = 0,$$

and more generally

$$C_m \otimes_{\mathbb{Z}} C_n \cong C_d, \qquad d = \gcd(m, n).$$

Solution:
To fix notation, set $C_m := \left\langle x \mid x^m \right\rangle = \left\{1 = x^0, x, x^2, \cdots, x^{m-1}\right\}$, written multiplicatively. The nth power map $x \mapsto x^n$ induces a SES

$$0 \to \mathbb{Z} \xrightarrow{(-)^n} \mathbb{Z} \to C_n \to 0 \quad \in \mathbb{Z}\text{-}\mathsf{Mod}.$$

Apply the right-exact functor $(-) \otimes_{\mathbb{Z}} C_m$ and use that $\mathbb{Z} \otimes_{\mathbb{Z}} (-) \simeq \mathrm{id}$ to obtain

$$\cdots \to C_m \xrightarrow{(-)^n} C_m \to \operatorname{coker}((-)^n) \cong C_n \otimes_{\mathbb{Z}} C_m \to 0,$$

so it suffices to show surjectivity, that every element in C_m has an nth root – i.e that if $y \in C_m$ then $y = z^n$ for some $z \in C_m$. This immediately reduces to finding nth roots of the generator x, since if $y = z^n \in C_m$, writing $y = x^k$ for some k, we have

$$y = x^k = z^n \implies z = x^{\frac{k}{n}} = (x^{\frac{1}{n}})^k,$$

and thus z can be expressed as a power of an nth root of x. That such a root can always be found follows from Bezout's identity: since m, n are coprime, there are solutions (a, b) to $1 = am + bn$, so

$$x = x^1 = x^{am+bn} = x^{am} x^{bn} = (x^b)^n,$$

using that $(-)^m$ annihilates every element in C_m, making x^b an nth root of x.
More generally, using the same resolution and tensoring with any $A \in \mathbb{Z}\text{-}\mathsf{Mod}$ yields

$$\cdots \to A \xrightarrow{(-)^n} A \to \operatorname{coker}((-)^n) \cong \frac{A}{nA} \to 0,$$

the submodule of n-divisible elements, and take $A = C_m$ to get

$$\frac{A}{nA} = \frac{C_m}{nC_m} \cong \frac{\mathbb{Z}}{m\mathbb{Z} + n\mathbb{Z}} \cong \frac{\mathbb{Z}}{d\mathbb{Z}} \cong C_d.$$

Remark 28.0.1: Note that similarly applying $\underset{\mathbb{Z}\text{-}\mathsf{Mod}}{\operatorname{Hom}}(C_n, -)$ yields

$$\underset{\mathbb{Z}\text{-}\mathsf{Mod}}{\operatorname{Hom}}(C_n, C_m) \cong \ker((-)^n) \cong C_d.$$

Problem 28.0.2 (AM 2.2)
Let $A \in \mathsf{CRing}, \mathfrak{a} \in \operatorname{Id}(A), M \in \mathsf{A\text{-}Mod}$. Show that $(A/\mathfrak{a}) \otimes_A M \cong M/\mathfrak{a}M \in \mathsf{A\text{-}Mod}$.

Tensor the exact sequence $0 \to \mathfrak{a} \to A \to A/\mathfrak{a} \to 0$ with M.

Solution:
Applying the hint yields the following:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathfrak{a} & \xhookrightarrow{\iota} & A & \longrightarrow & A/\mathfrak{a} & \longrightarrow & 0 \\ & & & \Downarrow (-)\otimes_A M & & & & & \\ \cdots & \longrightarrow & \mathfrak{a} \otimes_A M & \xhookrightarrow{\iota_*} & A \otimes_A M \cong M & \longrightarrow & A/\mathfrak{a} \otimes_A M & \longrightarrow & 0 \end{array}$$

Link to Diagram

Thus it suffices to show $\operatorname{im} \iota_* \cong \mathfrak{a}M$. This is clear since $\mathfrak{a} \hookrightarrow A$ is an inclusion, and the natural map $A \otimes_A M \xrightarrow{\sim} M$ is given by $(a, m) \mapsto am$.

Remark 28.0.2: Note that there is a map

$$\begin{aligned} f : \mathfrak{a} \times M &\to \mathfrak{a}M \\ (a, m) &\mapsto am, \end{aligned}$$

which is clearly surjective and bilinear, lifting to map out of the tensor product by the universal property. However, it is *not* always an isomorphism, and it being an isomorphism for all ideals is equivalent to M being flat as an A-module. In other words,

$$\mathfrak{a} \otimes_A M \cong \mathfrak{a}M \quad \forall \mathfrak{a} \in \mathrm{Id}(A) \iff M \in \mathsf{A\text{-}Mod}^\flat.$$

An easy counterexample:

- $A = k[\varepsilon]/\left\langle \varepsilon^2 \right\rangle$
- $\mathfrak{a} = \langle \varepsilon \rangle \in \mathrm{Id}(A)$
- $M = \mathfrak{a}$

Then $\mathfrak{a}^{\otimes_A^2} \to \mathfrak{a}^2$ is not injective. Note that $\mathfrak{a}^2 = 0$ in A, so it STS $\mathfrak{a}^{\otimes_A^2} \neq 0$. The claim is that

$$\mathfrak{a}^{\otimes_A^2} \cong \mathfrak{a}^{\otimes_{A/\mathfrak{a}}^2} \cong \mathfrak{a}^{\otimes_k^2},$$

which is the tensor product of two 1-dimensional k-vector spaces, and is thus 1-dimensional over k.

Problem 28.0.3 (AM 2.9)
Let

$$0 \to A \xrightarrow{d_1} B \xrightarrow{d_2} C \to 0 \in \mathsf{A\text{-}Mod}$$

with $A, C \in \mathsf{A\text{-}Mod}^{\mathrm{fg}}$, and show $B \in \mathsf{A\text{-}Mod}^{\mathrm{fg}}$ (i.e. B is finitely generated as an A-module).

Solution:
Let $\mathcal{C}, \mathcal{A}$ be generators for C and A respectively, and consider

$$\mathcal{B} := \left\{ d_1(a) \;\middle|\; a \in \mathcal{A} \right\} \cup \left\{ \tilde{c} \in d_2^{-1}(c) \;\middle|\; c \in \mathcal{C} \right\},$$

where the $\tilde{c}$ are arbitrarily chosen lifts of the generators $c \in \mathcal{C}$. Then $\mathcal{B}$ is a finite set, and the claim is that it generates B as an A-module.

Problem 28.0.4 (AM 2.11)
Let A be a ring $\neq 0$.

- Show that $A^m \cong A^n \Rightarrow m = n$.
- If $\varphi : A^m \to A^n$ is surjective, then $m \geq n$.
- If $\varphi : A^m \to A^n$ is injective, is it always the case that $m \leq n$?

Hint: Let m be a maximal ideal of A and let $\varphi : A^m \to A^n$ be an isomorphism. Then $1 \otimes \varphi : (A/m) \otimes A^m \to (A/m) \otimes A^n$ is an isomorphism between vector spaces of dimensions m and n over the field $k = A/m$. Hence $m = n$. (Cf. Chapter 3, Exercise 15.)

Problem 28.0.5 (AM 2.14)
A partially ordered set I is said to be a directed set if for each pair i, j in I there exists $k \in I$ such that $i \leq k$ and $j \leq k$.
Let A be a ring, let I be a directed set and let $(M_i)_{i \in I}$ be a family of A-modules indexed by $\cdot I$. For each pair i, j in I such that $i \leq j$, let $\mu_{ij} : M_i \to M$, be an A-homomorphism, and suppose that the following axioms are satisfied:

(1) μ_{lt} is the identity mapping of M_i, for all $i \in I$;
(2) $\mu_{ik} = \mu_{jk} \circ \mu_{ly}$ whenever $i \leq j \leq k$.

Then the modules M_1 and homomorphisms μ_{if} are said to form a direct system $\mathbf{M} = (M_t, \mu_{tj})$ over the directed set I.
We shall construct an A-module M called the direct limit of the direct system M. Let C be the direct sum of the M_1, and identify each module M_i with its canonical image in C. Let D be the submodule of C generated by all elements of the form $x_i - \mu_{tj}(x_i)$ where $i \leq j$ and $x_i \in M_i$. Let $M = C/D$, let $\mu : C \to M$ be the projection and let μ_i be the restriction of μ to M_i.
The module M, or more correctly the pair consisting of M and the family of homomorphisms $\mu_i : M_1 \to M$, is called the direct limit of the direct system M, and is written $\underrightarrow{\operatorname{colim}}_i M_i$. From the construction it is clear that $\mu_t = \mu_j \circ \mu_{if}$ whenever $i \leq j$.

Problem 28.0.6 (AM 2.16)
Show that the direct limit is characterized (up to isomorphism) by the following property. Let N be an A-module and for each $i \in I$ let $\alpha_i : M_t \to N$ be an $A-$ module homomorphism such that $\alpha_t = \alpha_j \circ \mu_{ts}$ whenever $i \leq j$. Then there exists a unique homomorphism $\alpha : M \to N$ such that $\alpha_1 = \alpha \circ \mu_t$ for all $i \in I$.

Problem 28.0.7 (AM 2.20)
Keeping the same notation as in Exercise 14 , let N be any A-module. Then $(M_1 \otimes N, \mu_{ij} \otimes 1)$ is a direct system; let $P = \underrightarrow{\operatorname{colim}}_i (M_i \otimes N)$ be its direct limit.
For each $i \in I$ we have a homomorphism

$$\mu_i \otimes 1 : M_i \otimes N \to M \otimes N,$$

hence by Exercise 16 a homomorphism $\psi : P \to M \otimes N$. Show that ψ is an isomorphism, so that

$$\underrightarrow{\operatorname{colim}}_i (M_i \otimes N) \cong \left(\underrightarrow{\operatorname{colim}}_i M_i \right) \otimes N.$$

Hint: For each $i \in I$, let

$$g_i : M_t \times N \to M_t \otimes N$$

be the canonical bilinear mapping. Passing to the limit we obtain a mapping $g : M \times N \to P$. Show that g is A-bilinear and hence define a homomorphism $\varphi : M \otimes N \to P$. Verify that $\varphi \circ \psi$ and $\psi \circ \varphi$ are identity mappings.

29 Problem Set 3

Problem 29.0.1 (AM 3.1)
Let S be a multiplicatively closed subset of a ring A, and let M be a finitely generated A-module. Prove that $S^{-1}M = 0$ if and only if there exists $s \in S$ such that $sM = 0$.

Problem 29.0.2 (AM 3.4)
Let $f : A \to B$ be a homomorphism of rings and let S be a multiplicatively closed subset of A. Let $T = f(S)$. Show that $S^{-1}B$ and $T^{-1}B$ are isomorphic as $S^{-1}A$-modules.

Problem 29.0.3 (AM 3.12)
Let A be an integral domain and M an A-module. An element $x \in M$ is a torsion element of

M if Ann $(x) \neq 0$, that is if x is killed by some non-zero element of A. Show that the torsion elements of M form a submodule of M. This submodule is called the torsion submodule of M and is denoted by $T(M)$. If $T(M)=0$, the module M is said to be torsion-free. Show that

i) If M is any A-module, then $M/T(M)$ is torsion-free.

ii) If $f : M \to N$ is a module homomorphism, then $f(T(M)) \subseteq T(N)$.

iii) If $0 \to M' \to M \to M''$ is an exact sequence, then the sequence $0 \to T\left(M'\right) \to T(M) \to T\left(M''\right)$ is exact.

iv) If M is any A-module, then $T(M)$ is the kernel of the mapping $x \mapsto 1 \otimes x$ of M into $K \otimes_1 M$, where K is the field of fractions of A.

For iv), show that K may be regarded as the direct limit of its submodules $A\xi (\xi \in K)$; using Chapter 1, Exercise 15 and Exercise 20, show that if $1 \otimes x = 0$ in $K \otimes M$ then $1 \otimes x = 0$ in $A\xi \otimes M$ for some $\xi \neq 0$. Deduce that $\xi^{-1} x = 0$.

Problem 29.0.4 (Ex. 1)
Show that $\mathbb{Q}$ is a flat $\mathbb{Z}$-module which is not free

Problem 29.0.5 (Ex. 2)
Prove that if B, C are A algebras, the tensor product algebra $B \otimes_A C$ has the following universal property: an algebra homomorphism $B \otimes_A C \to S$ is the same as a pair of algebra homomorphisms $B \to S, C \to S$.

30 | Problem Set 4

Problem 30.0.1 (?)
Let $f : M \to N$ be a map of modules over a local ring A, with N finitely generated. Show that if the induced map $M/\mathfrak{m}M \to N/\mathfrak{m}N$ is surjective, the same is true for f. Does a similar statement hold for injectivity?

Problem 30.0.2 (?)
Let M be a finitely-generated module over a ring A, and let p be a prime ideal of A. Suppose $M_p = 0$. Show that there exists a finite set $x_1, \cdots, x_n$ of elements of $A \backslash \mathfrak{p}$ such that the localization of M at the multiplicative set generated by the x_i is zero.

Problem 30.0.3 (?)
Let M be a finitely-generated module over a Noetherian ring A (that means any submodule of a finitely-generated module is finitely generated), and let p be a prime ideal of A. Suppose M_p is free. Show that there exists a finite set $x_1, \cdots, x_n$ of elements of $A \backslash p$ such that the localization of M at the multiplicative set generated by the x_i is free.

Problem 30.0.4 (?)
Give an example of a module M over a ring A such that M_p is free for each prime ideal p of A, but M itself is not free. Such modules are called locally free.

Problem 30.0.5 (?)
Give an example of a flat module which is not projective.

Problem 30.0.6 (?)
Write a careful proof of the Cayley-Hamilton theorem over an arbitrary field.

31 | Problem Set 5

HW 5 (due Feb 17): AM Chapter 2, exercises 24, 25, 26

1. Let M be an A-module and let $F_\bullet : \cdots \to F_2 \to F_1 \to F_0$ be a flat resolution of M, i.e. a chain complex with each F_i flat, and such that $H_i(F_\bullet) = 0$ for $i > 0$ and $H_0(F_\bullet) = M$. Show that for any A-module N, $H_i(F_\bullet \otimes_A N) = \mathrm{Tor}_i^A(M, N)$.
2. Let M be a finitely-generated flat module over a Noetherian local ring A. Show that M is free.
3. Carefully check that Ext is well-defined, i.e. independent of the choice of injective resolution in the definition.
4. Compute $\mathrm{Ext}^i_{\mathbb{Z}}(\mathbb{Z}/p\mathbb{Z}, \mathbb{Z})$ for each prime p.

32 | Problem Set 6

HW 6 (due Feb 24): AM Chapter 5, exercises 1, 2, 10, 12, 14, and Chapter 6, exercises 2 and 5

33 | Problem Set 7

HW 7 (due Thursday, March 17, after Spring Break):

1. Let M be an A-module, and $f \in A$. Construct an isomorphism between M_f and $\varinjlim M \xrightarrow{\cdot f} M \xrightarrow{\cdot f} M \xrightarrow{\cdot f} \cdots$.
2. Construct a module M over a ring A such that for each prime ideal $\mathfrak{p}$ of $A, M_{\mathfrak{p}}$ is finitely generated, but M is not finitely-generated.
3. Let M be a finitely-generated module over a Noetherian ring. Show that $M = 0$ if and only if the support of M is empty.
4. Suppose $\operatorname{Spec}(A) = V_1 \sqcup V_2$, where V_1, V_2 are clopen disjoint subsets. Show that there exists a direct sum decomposition $A = A_1 \oplus A_2$ such that the natural quotient maps $A \to A_i$ induce isomorphisms $\operatorname{Spec}(A_i) \to V_i$ for $i = 1, 2$.
5. Show that exactness of a long exact sequence is a local property.
6. Let A be a Noetherian local domain with residue field k and fraction field K, and M a finitely-generated A-module. Show that the following are equivalent:
7. M is free
8. $\dim_k M \otimes_A k = \dim_K M \otimes_A K$.
9. Let A be a Noetherian ring and M finitely-generated. Show that the following are equivalent:
10. M is locally free
11. M is projective
12. M is flat.
13. Show that any Artinian ring is Noetherian
14. Show that if A is a Noetherian ring such that $\operatorname{Spec}(A)$ is Hausdorff, then A is Artinian.

34 | Problem Set 8

HW 8 (due March 31)

1. Give an example (with proof) of a rank one locally free module over a Dedekind domain that is not free.
2. Give an example (with proof) of a Noetherian domain of Krull dimension one which is not a Dedekind domain.
3. Let M be a finitely generated module over a Dedekind domain A. Show that M has a projective resolution of length two. Conclude that $\operatorname{Tor}_i^A(M, -)$ and $\operatorname{Ext}_A^i(M, -)$ equal zero for $i > 1$.
4. Let M, N be a finitely generated modules over a Dedekind domain A. Show that $\operatorname{Tor}_1^A(M, N)$ is a torsion A-module. Can you identify its support?
5. Give an example of a Dedekind domain with uncountable Picard group.
6. Let k be a field. Show that the Picard group of $k[t]$ is trivial, i.e. any rank one locally free sheaf over $k[t]$ is free.

7. Give an example of a domain with a maximal non-zero ideal $\mathfrak{m}$ such that $\mathfrak{m}^2 = \mathfrak{m}$.

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