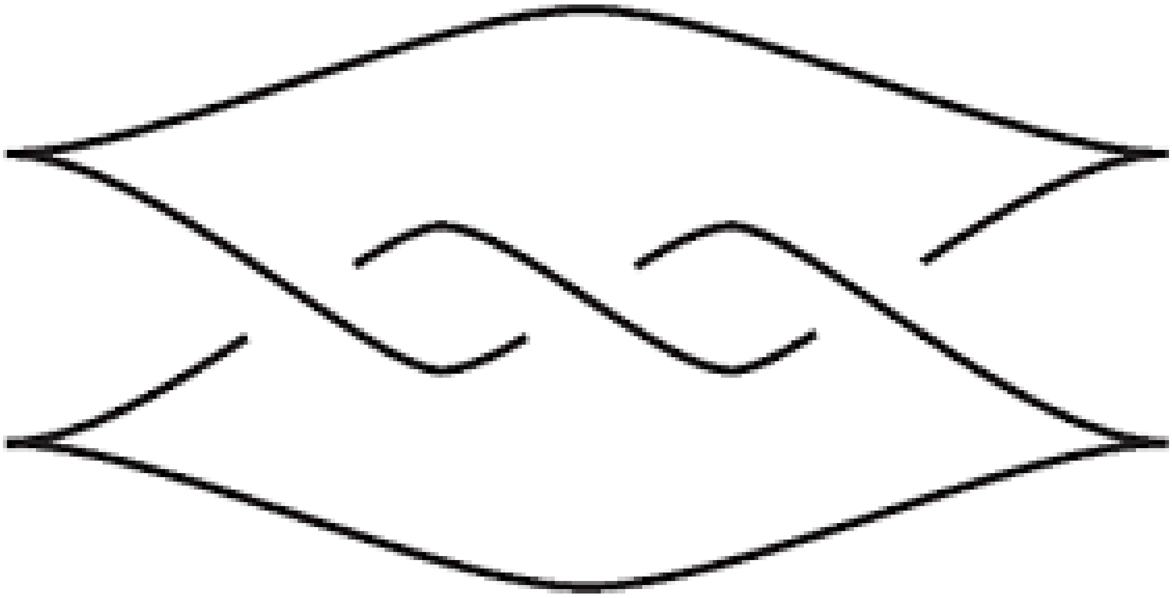

Notes: These are notes live-tex'd from a graduate course in Contact Topology taught by Peter Lambert-Cole at the University of Georgia in Spring 2022. As such, any errors or inaccuracies are almost certainly my own.

Contact Topology

Lectures by Peter Lambert-Cole. University of Georgia, Spring 2022

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Remark 1.0.1: References:

- https://www.ma.imperial.ac.uk/~ssivek/courses/12s-math273.php
- https://web.ma.utexas.edu/users/gdavtor/notes/contact_notes.pdf
- PLC's Notes

Emphasis for the course: applications to low-dimensional topology, lots of examples, and ways to construct contact structures. The first application is critical to 4-manifold theory:

1.1 Application 1

Theorem 1.1.1 ***(Cerf's Theorem).***
Every diffeomorphism $f : S^3 \to S^3$ extends to a diffeomorphism $\mathbb{B}^4 \to \mathbb{B}^4$.

Remark 1.1.2: This isn't true in all dimensions! This is essentially what makes Kirby calculus on 4-manifolds possible without needing to track certain attaching data.

Remark 1.1.3: There is a standard contact structure on S^3: regard $\mathbb{C}^2 \cong \mathbb{R}^4$ and suppose $f : S^3 \to S^3$. There is an intrinsic property of contact structures called *tightness* which doesn't change under diffeomorphisms and is fundamental to 3-manifold topology.

Theorem 1.1.4 ***(Eliashberg).***
There is a unique tight contact structure ξ_{std} on S^3.

So up to isotopy, f fixes ξ_{std}.

Remark 1.1.5: A useful idea: tiling by holomorphic discs. This involves taking S^1 and foliating the bounded disc by geodesics – by the magic of elliptic PDEs, this is unobstructed and can be continued throughout the disc just using convexity near the boundary. In higher dimensions: $\mathbb{B}^4$ is foliated by a 2-dimensional family of holomorphic discs.

1.2 Application 2

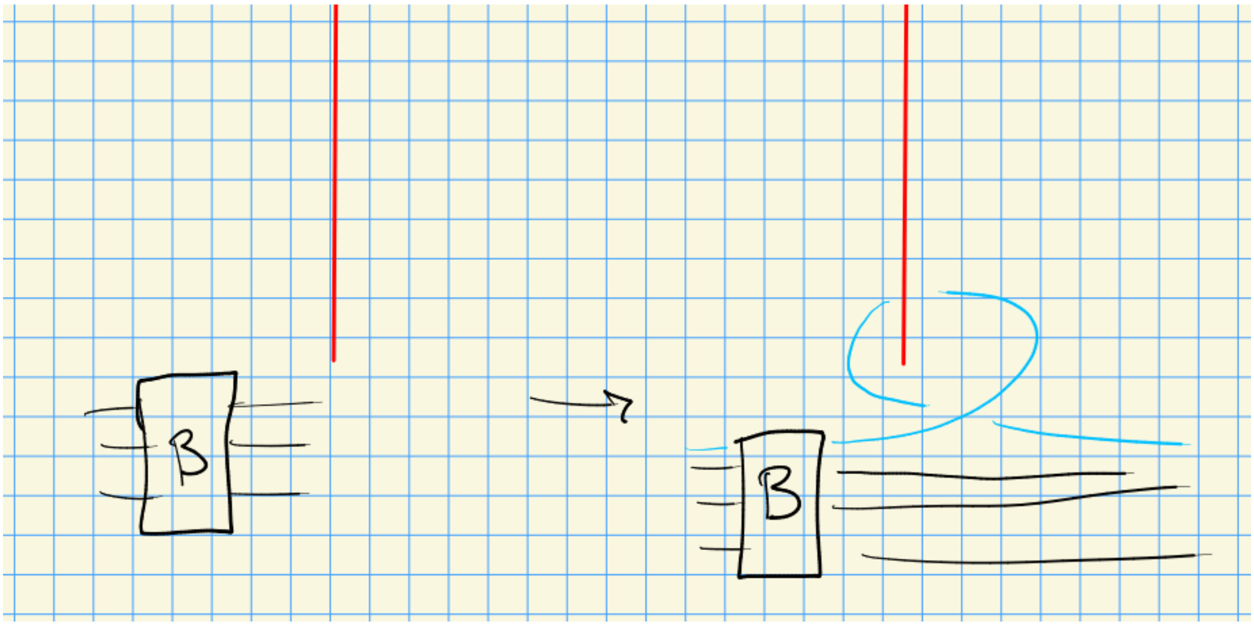
Remark 1.2.1: Another application: monotonic simplification (?) of the unknot. Given a knot $K \hookrightarrow S^3$, a theorem of Alexander says K can be braided about the z-axis, which can be described by a word $w \in B_n$, the braid group

$$B_n = \left\{ \sigma_1, \cdots, \sigma_{n-1} \,\middle|\, [\sigma_i, \sigma_j] = 1 \, |i-j| \geq 2, \, \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \, i = 1, \cdots, n-2 \right\}.$$

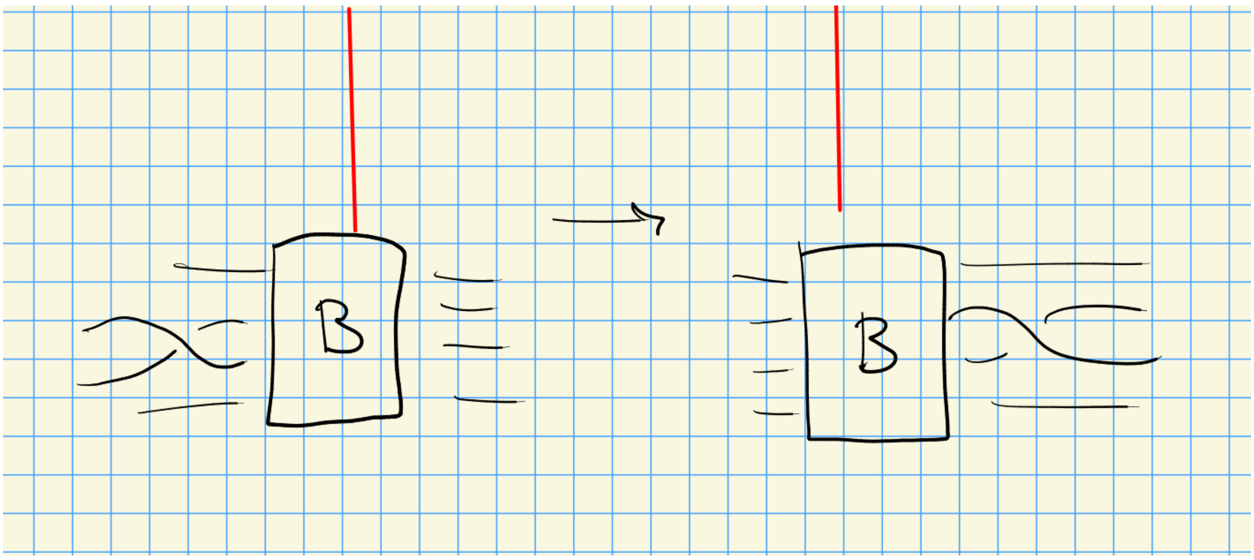
This captures positive vs negative braiding on nearby strands, commuting of strands that are far apart, and the Reidemeister 3 move. Write $K = K(\beta)$ for β a braid for the braid closure.

Remark 1.2.2: Markov's theorem: if $K = K(\beta_1), K(\beta_2)$ where $\beta_1 \in B_n$ and $\beta_2 \in B_m$ with m, n not necessarily equal, then there is a sequence of Markov moves β_1 to β_2. The moves are:

- Stabilization and destabilization:



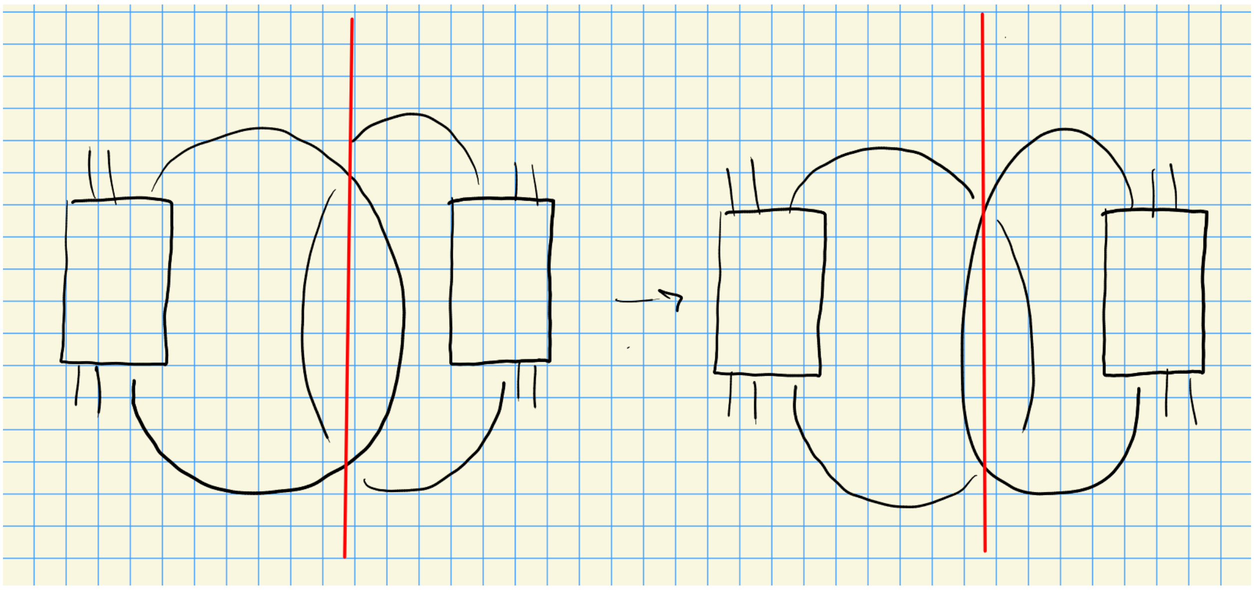
- Conjugation in B_n:



- Braid isotopy, which preserves braid words in B_n.

Remark 1.2.3: A theorem of Birman-Menasco: if $K(\beta) = U$ is the unknot for $\beta \in B_n$, then there is a sequence of braids $\{\beta_i\}_{i \leq k}$ with $\beta_k = 1 \in B_1$ such that

- $\beta_i \in B_{n_i}$
- $K(\beta_i) = U$
- $n_1 \geq n_2 \geq \cdots \geq n_k = 1$
- $\beta_i \to \beta_{i+1}$ is either a Markov move or an exchange move.



1.3 Application 3

Remark 1.3.1: Genus bounds. A theorem due to Thurston-Eliashberg: if ξ is either a taut foliation or a tight contact structure on a 3-manifold Y and $\Sigma \neq S^2$ is an embedded orientable surface in Y, then there is an Euler class $e(\xi) \in H^2(Y)$. Then

$$|\langle e(\xi),\ \Sigma \rangle| \leq -g(\Sigma),$$

which after juggling signs is a lower bound on the genus of any embedded surface.

Remark 1.3.2: Taut foliations: the basic example is $F \times S^1$ for F a surface. The foliation carries a co-orientation, and the tangencies at critical points of an embedded surface will have tangent planes tangent to the foliation, so one can compare the co-orientation to the outward normal of the surface to see if they agree or disagree and obtain a sign at each critical point. Write $c_\pm$ for the number of positive/negative elliptics and $h_\pm$ for the hyperbolics. Then

$$\chi = (e_+ + e_-) - (h_+ + h_-),$$

by Poincaré-Hopf. On the other hand, $\langle e(\xi),\ \Sigma\rangle = (e_+ - h_+) - (e_- - h_-)$, so adding this yields

$$\langle e(\xi),\ \Sigma\rangle + \chi = 2(e_+ - h_+) \leq 0.$$

Isotope the surface to cancel critical points in pairs to get rid of caps/cups so that only saddles remain.

1.4 Contact Geometry

Definition 1.4.1 (?)
A contract structure on Y^{2n+1} is a hyperplane field (a codimension 1 subbundle of the tangent bundle) $\xi = \ker \alpha$ such that $\alpha \wedge (d\alpha)^{\wedge^n} > 0$ is a positive volume form.

Example 1.4.2*(?):* On $\mathbb{R}^3$,

$$\alpha = dz - ydx \implies d\alpha = -dy \wedge dx = dx \wedge dy,$$

so

$$\alpha \wedge d\alpha = (dz - ydx) \wedge (dx \wedge dy) = dz \wedge dx \wedge dy = dx \wedge dy \wedge dz.$$

Exercise 1.4.3 (?)
On $\mathbb{R}^5$, set $\alpha = dz - y_1 dx_1 - y_2 dx_2$. Check that

$$\alpha \wedge (d\alpha)^2 = 2(dz \wedge dx_1 \wedge dy_1 \wedge dx_2 \wedge dy_2).$$

2 | Contact Forms and Structures (Thursday, January 13)

Definition 2.0.1 (Contact form)
A **contact form** on Y^3 is a 1-form α with $\alpha \wedge d\alpha > 0$. A **contact structure** is a 2-plane field $\xi = \ker \alpha$ for some contact form.

Remark 2.0.2: Forms are more rigid than structures: if $f > 0$ and α is contact, then $f \cdot \alpha$ is also contact with $\ker(\alpha) = \ker(f\alpha)$.

2.1 Examples of Contact Structures

Example 2.1.1*(Standard contact structure):* On $\mathbb{R}^3$, a local model is $\alpha := dz - y\,dx$.

Exercise 2.1.2 *(?)*
Show $\alpha \wedge d\alpha = dz \wedge dx \wedge dy$.

Write $\xi = \operatorname{span}_{\mathbb{R}}(\partial y, y\partial z + \partial x)$, which yields planes with a corkscrew twisting. Verify this by writing $\alpha = 0 \implies \dfrac{\partial z}{\partial x} = y$, so the slope depends on the y-coordinate.

Example 2.1.3 ***(Rotation of the standard structure):*** On $\mathbb{R}^3$, take $\alpha_2 := dz + x\,dy$ and check $\alpha_2 \wedge d\alpha_2 = dz \wedge dx \wedge dy$. This is a rigid rotation by $\pi/2$ of the previous α, so doesn't change the essential geometry.

Example 2.1.4 ***(Radially symmetric contact structure):*** Again on $\mathbb{R}^3$, take $\alpha_3 = dz + \frac{1}{2}r^2 d\theta$. Check that $d\alpha_3 = r\,dr \wedge d\theta$ and $\alpha_3 \wedge d\alpha_3 = r\,dz \wedge dr \wedge d\theta$. Then $\xi = \operatorname{span}_{\mathbb{R}}(\partial r, \frac{1}{2}r^2\partial z + \partial\theta)$. Note that as $r \to \infty$, the slope of these planes goes to infinity, but doesn't depend on z or θ.

Example 2.1.5 ***(Rectangular version of radially symmetric structure):*** Set $\alpha_4 = dz + \frac{1}{2}(x\,dy - y\,dx)$, then this is equal to α_3 in rectangular coordinates.

Example 2.1.6 ***(Overtwisted):*** Set

$$\alpha_4 = \cos(r^2)\,dz + \sin(r^2)\,d\theta.$$

Exercise 2.1.7 *(?)*
Compute the exterior derivative and check that this yields a contact structure.

Now note that

$$\alpha = 0 \implies \frac{\partial z}{\partial \theta} = -\frac{\sin(r^2)}{\cos(r^2)} = -\tan(r^2),$$

which is periodic in r. So a fixed plane does infinitely many barrel rolls along a ray at a constant angle θ_0.

This is far too twisty – to see the twisting, consider the graph of $(r, \tan(r^2))$ and note that it flips over completely at odd multiples of $\pi/2$. In the previous examples, the total twist for $r \in (-\infty, \infty)$ was less than π.

Definition 2.1.8 (Contactomorphisms)
A **contactomorphism** is a diffeomorphism

$$\psi : (Y_1^3, \xi_1) \to (Y_2^3, \xi_2)$$

such that $\varphi_*(\xi_1) = \xi_2$ (tangent vectors push forward).

A strict contactomorphism is a diffeomorphism

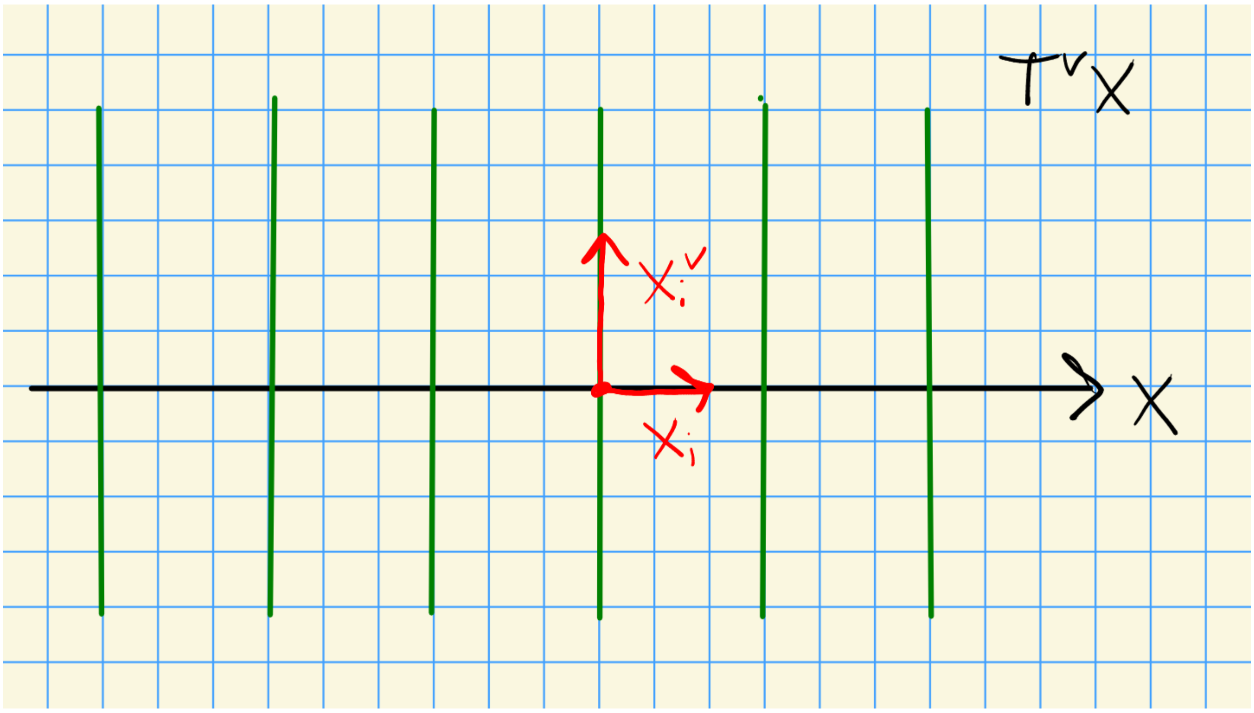
$$\varphi : (Y_1^3, \ker \alpha_1) \to (Y_2^3, \ker \alpha_2).$$

such that $\varphi^*(\alpha_2) = \alpha_1$ (forms pull back).

Remark 2.1.9: Strict contactomorphisms are more important for dynamics or geometric applications.

Exercise 2.1.10 (?)
Prove that $\alpha_1, \cdots, \alpha_4$ are all contactomorphic.

Remark 2.1.11: Recall that X has a cotangent bundle $\mathbf{T}^\vee X \xrightarrow{\pi} X$ of dimension $2 \dim X$. There is a canonical 1-form $\lambda \in \Omega^1(\mathbf{T}^\vee X)$, i.e. a section of $T^\vee(T^\vee X)$. Given any smooth section $\beta \in \Gamma(\mathbf{T}^\vee X_{/X})$ there is a unique 1-form λ on $\mathbf{T}^\vee X$ such that $\beta^*(\lambda) = \beta$, regarding β as a smooth map on the left and a 1-form on the right. In local coordinates $(x_1, \cdots, x_n)$ on X, write $y_i = dx_i$ on the fiber of $\mathbf{T}^\vee X$. Why this works: the fibers are collections of covectors, so if x_i are horizontal coordinates there is a dual vertical coordinate in the fiber:



So we can write

$$\lambda = \sum y_i \, dx_i \in \Omega^1(\mathbf{T}^\vee X),$$

regarding the y_i as functions on $\mathbf{T}^\vee X$ and dx_i as 1-forms on $\mathbf{T}^\vee X$.

Exercise 2.1.12 *(?)*
Find out what $\beta = \sum a_i \, dx_i$ is equal to as a section of $\mathbf{T}^\vee X$.

Remark 2.1.13: To get a contact manifold of dimension $2n+1$, consider the 1-jet space $J^1(X) \coloneqq T^\vee X \times \mathbb{R}$. Write the coordinates as $(x, y) \in \mathbf{T}^\vee X$ and $z \in \mathbb{R}$ and define $\alpha = dz - \lambda$, the claim is that this is contact.

For dimension $2n-1$, choose a cometric on X and take $\mathbb{S}\mathbf{T}^\vee X$ the unit cotangent bundle of unit-length covectors. Then $\alpha \coloneqq -\lambda|_{\mathbb{S}\mathbf{T}^\vee X}$ is contact.

Exercise 2.1.14 (?)
Check that $\mathbb{R}^3 = J^1(\mathbb{R})$ and $\mathbb{S}\mathbf{T}^\vee(\mathbb{R}^2) = \mathbb{R}^2 \times S^1$.

Remark 2.1.15: A neat theorem: the contact geometry of $\mathbb{S}\mathbf{T}^\vee \mathbb{R}^3$ is a perfect knot invariant. This involves assigning to knots unique Legendrian submanifolds.

2.2 Perturbing Foliation

Example 2.2.1 *(?)*: Define

$$\alpha_t = dz - ty \, dx \qquad t \in \mathbb{R}$$

to get a 1-parameter family of 1-forms. Check that $\alpha_t \wedge d\alpha_t = t(\, dz \wedge \, dx \wedge \, dy)$. Consider $t \in (-\varepsilon, \varepsilon)$:

- $t > 0 \implies \alpha = dz - y \, dx$ yields a positive contact structure,
- $t > 0 \implies \alpha = dz$ is a foliation,
- $t < 0 \implies \alpha = dz + y \, dx$ is a negative contact structure.

Remark 2.2.2: What is a (codimension r) foliation on an n-manifold? A local diffeomorphism $U \cong \mathbb{R}^n \times \mathbb{R}^{n-r}$ with *leaves* pt $\times \mathbb{R}^{n-r}$. For example, $\mathbb{R}^3 \cong \mathbb{R} \times \mathbb{R}^2$ with coordinates t and (x, y). We're leaving out a lot about how many derivatives one needs here!

For a fiber bundle or vector bundle to admit an interesting foliation, one needs a flat connection.

Definition 2.2.3 (Integrability)
Any $\xi \coloneqq \ker \alpha$ is **integrable** iff for all vector fields $X, Y \subseteq \xi$, their Lie bracket $[X, Y] \subseteq \xi$.

Theorem 2.2.4 ***(Frobenius Integrability).***
For α nonvanishing on Y^3, $\ker \alpha$ is tangent to a foliation by surfaces iff $\alpha \wedge d\alpha = 0$.

Example 2.2.5 ***(?):*** Consider $\alpha = dz - y\,dx$, so $\ker \alpha = \operatorname{span}_{\mathbb{R}} \{\partial y, y\partial z + \partial x\}$ which bracket to $\partial z \notin \ker \alpha$. This yields a non-integrable contact structure.

On the other hand, for $\alpha = dz$, $\ker \alpha = \operatorname{span}_{\mathbb{R}} \{\partial x, \partial y\}$ which bracket to zero. So this yields a foliation.

Remark 2.2.6: A theorem of Eliashberg and Thurston: taut foliations can be perturbed to a (tight) positive contact structure.

3 | Tuesday, January 18

Remark 3.0.1: Refs:

- Geiges, Intro to Contact
- Ozbogi-Stipsicz
- Etnyre lecture notes
- Massot
- Sivck

Definition 3.0.2 (Standard contact structure)
For $S^3 \subseteq \mathbb{C}^2$, define a form on $\mathbb{R}^4$ as

$$\alpha := -y_1 dx_1 + x_1 dy_1 - y_2 dx_2 + x_2 dy_2.$$

Then then **standard contact form** on S^3 is

$$\xi_{\text{std}} := \ker \alpha|_{S^3}.$$

Exercise 3.0.3 (?)
Show that α defines a contact form.

Solution:
Write $f = x_1^2 + y_1^2 + x_2^2 + y_2^2$, then

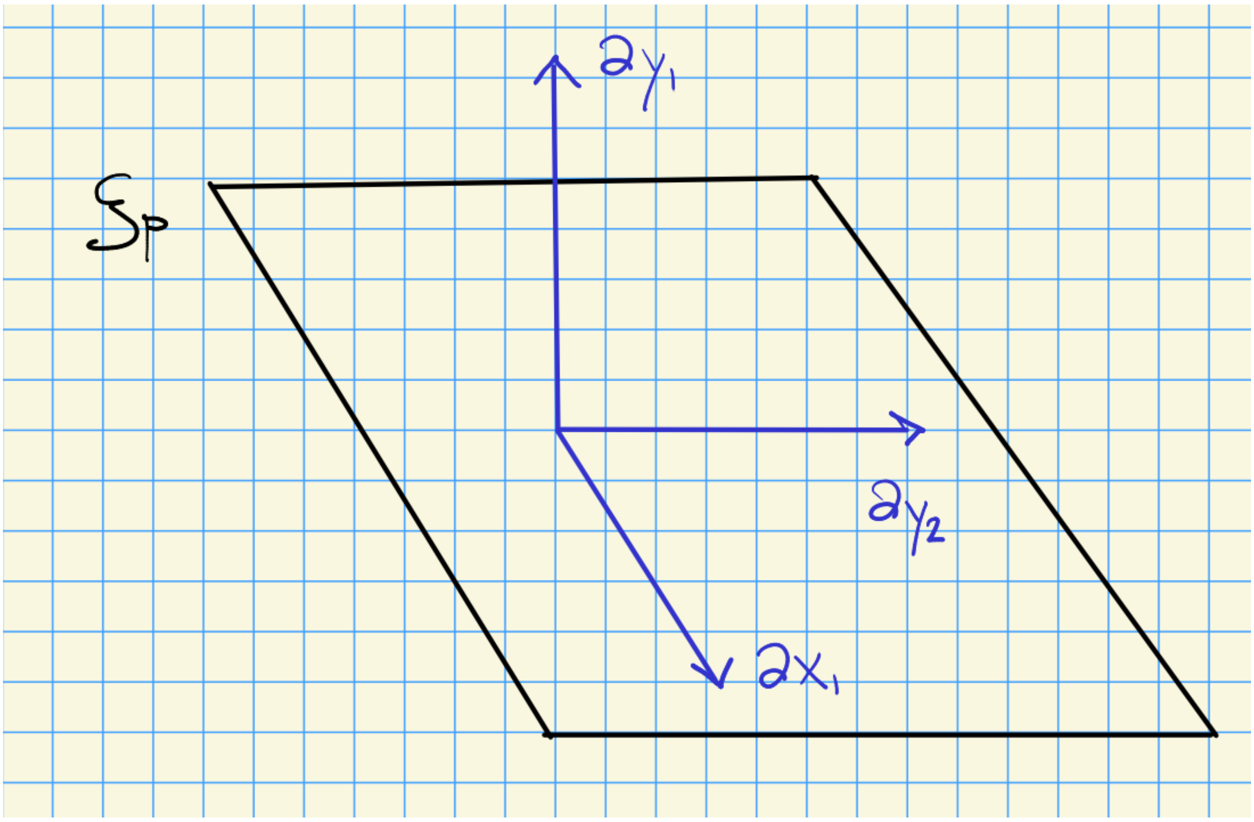
$$\alpha|_{S^3} \wedge d\alpha|_{S^3} > 0 \iff df \wedge d\alpha \wedge d\alpha > 0.$$

Check that

- $d\alpha = 2(dx_1 \wedge dy_1) + 2(dx_2 + dy_2)$
- $df = 2(x_1 dx_1 + y_1 dy_1) + 2(x_2 dx_2 + y_2 dy_2)$.

Remark 3.0.4: Note that at $p = [1, 0, 0, 0] \subseteq S^3$, $\mathbf{T}_p S^3 = \operatorname{span}\{\partial y_1, \partial x_2, \partial y_2\}$. and $\alpha_p = -0dx_1 +$

$1dy_1 - 0dx_2 + 0dy_2 = dy_1$ and $\xi_p = \ker dy_1 = \text{span}\{\partial x_1, \partial y_2\}$.



Then $\xi_p \leq \mathbf{T}_p\mathbb{C}^2 = \text{span}\{\partial x_1, \partial y_1, \partial x_2, \partial y_2\} \cong \mathbb{C}^4$ is a distinguished complex line.

Definition 3.0.5 (Almost complex structures)
An **almost complex structure** on X is a bundle automorphism $J : \mathbf{T}X \circlearrowleft$ with $J^2 = -\text{id}$.

Example 3.0.6 *(?):* For $X = \mathbb{C}^2$, take

$$\begin{aligned} \partial x_1 &\mapsto \partial y_1 \\ \partial y_1 &\mapsto -\partial x_1 \\ \partial x_2 &\mapsto \partial y_2 \\ \partial y_2 &\mapsto -\partial x_2. \end{aligned}$$

Exercise 3.0.7 (?)
Show that $f : \mathbb{C} \to \mathbb{C}$ is holomorphic if $df \circ J = J \circ df$, which corresponds to the Cauchy-Riemann equations.

Lemma 3.0.8 *(?).*
Given $J : W \to W$, an $\mathbb{R}$-subspace $V \leq W$ is a $\mathbb{C}$-subspace iff $J(V) = V$.

Definition 3.0.9 (?)
The field of J-complex tangents is the hyperplane field

$$\xi_p := \mathbf{T}S^3 \cap J(\mathbf{T}S^3).$$

Example 3.0.10 ***(?):*** Consider $\mathbf{T}_pS^3$ for $p = [1,0,0,0]$, then

$$J(\operatorname{span}\{\partial y_1, \partial x_1, \partial y_2\}) = \operatorname{span}\{-\partial x_1, \partial y_2, -\partial x_2\},$$

so $\xi_p = \operatorname{span} \partial x_1, \partial y_2$ is the intersection and coincides ξ_{std}.

Question 3.0.11
Where does α come from?

Let $\rho = \sum x_i \partial x_i + \sum y_i \partial y_i$ be the radial vector field, so $\rho = \frac{1}{2} \operatorname{grad}\left[\sum x_i^2 + \sum y_i^2\right]$. Setting $\omega := \bigwedge dx_i \wedge \bigwedge dy_i$, then $\alpha = \iota_p \omega := \omega(p, -)$ is the interior product of ω. Then

$$\alpha = dx_1 \wedge dy_1(x_1 \partial x_1 + y_1 \partial y_1 + \cdots) + \cdots = x_1 dy_1 - y_1 dx_1 + \cdots .$$

So the contact form comes from pairing the symplectic form against a radial vector field.

Remark 3.0.12: Recall $f := \sum x_i^2 + \sum y_i^2$ satisfies $df = 2\sum x_i dx_i + 2\sum y_i dy_i$. Note that J acts on 1-forms by $J^*(dx)(-) = dx(J(-))$. For $J = i$,

- $\delta x : dx(J\partial x) = dx(\partial y) = 0$,
- $\partial y : dx(J\partial y) = dx(-\partial x) = -1$.

So $J^*(dx) = -dy$, and

$$J^*(df) = 2x_1(-dy_1) + 2y_1(dx_1) + 2x_2(-dy_2) + 2y_2(dx_2) = -2\alpha.$$

Thus $J^*(df)$ is a rotation of df by $\pi/2$.

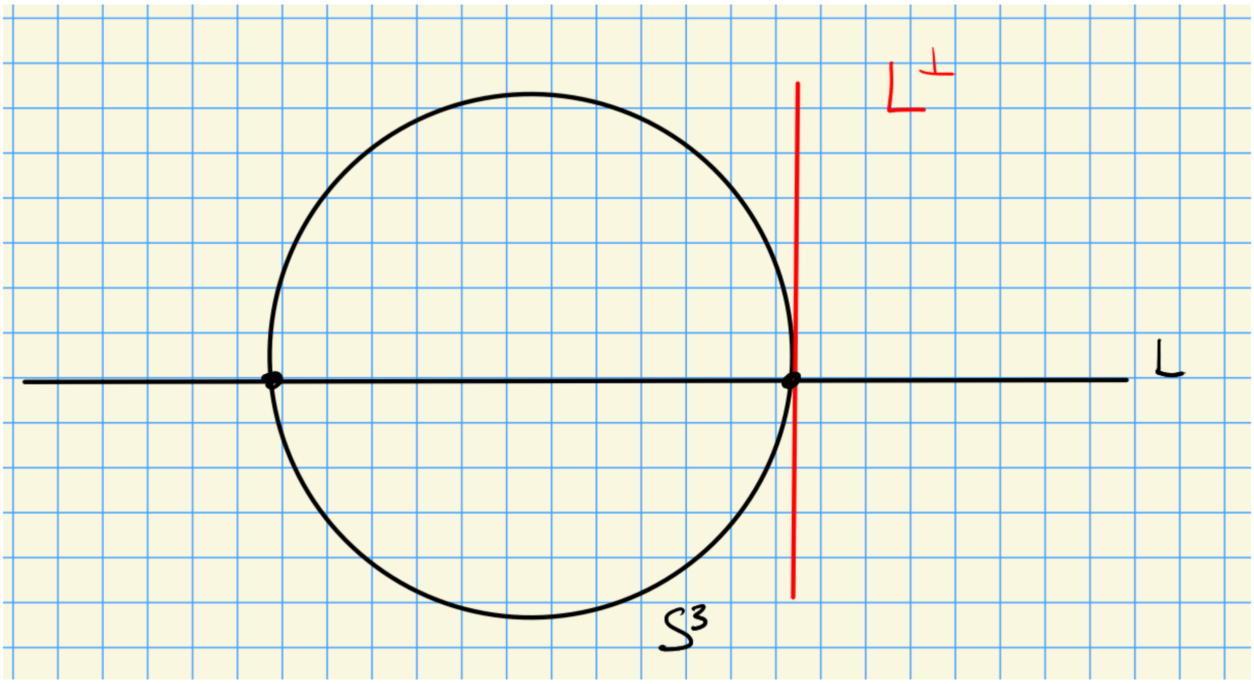
Example 3.0.13 ***(?):*** The field of complex tangencies along $Y = f^{-1}(0)$ is the kernel of $df(J(-))|_Y$.

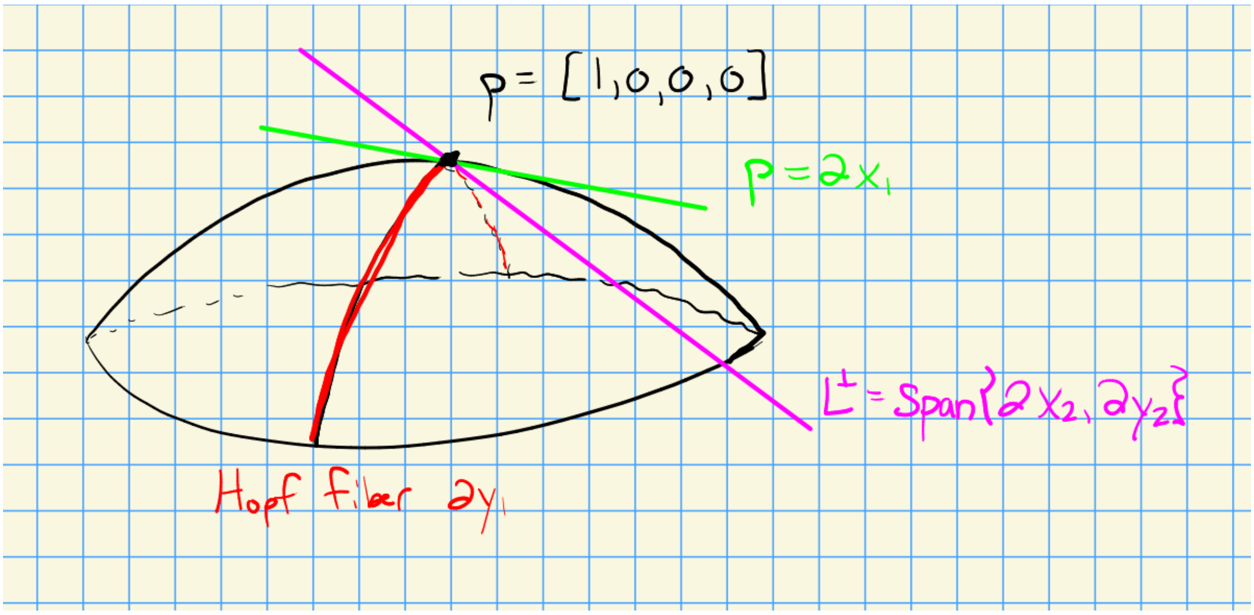
Remark 3.0.14: Methods of getting contact structures: for a vector field X, being contact comes from $\mathcal{L}_X \omega = \omega$. For functions $f : \mathbb{C}^2 \to \mathbb{R}$, being contact comes from $\alpha = d^{\mathbb{C}} f$ being contact. See strictly plurisubharmonic functions and Levi pseudoconvex subspaces.

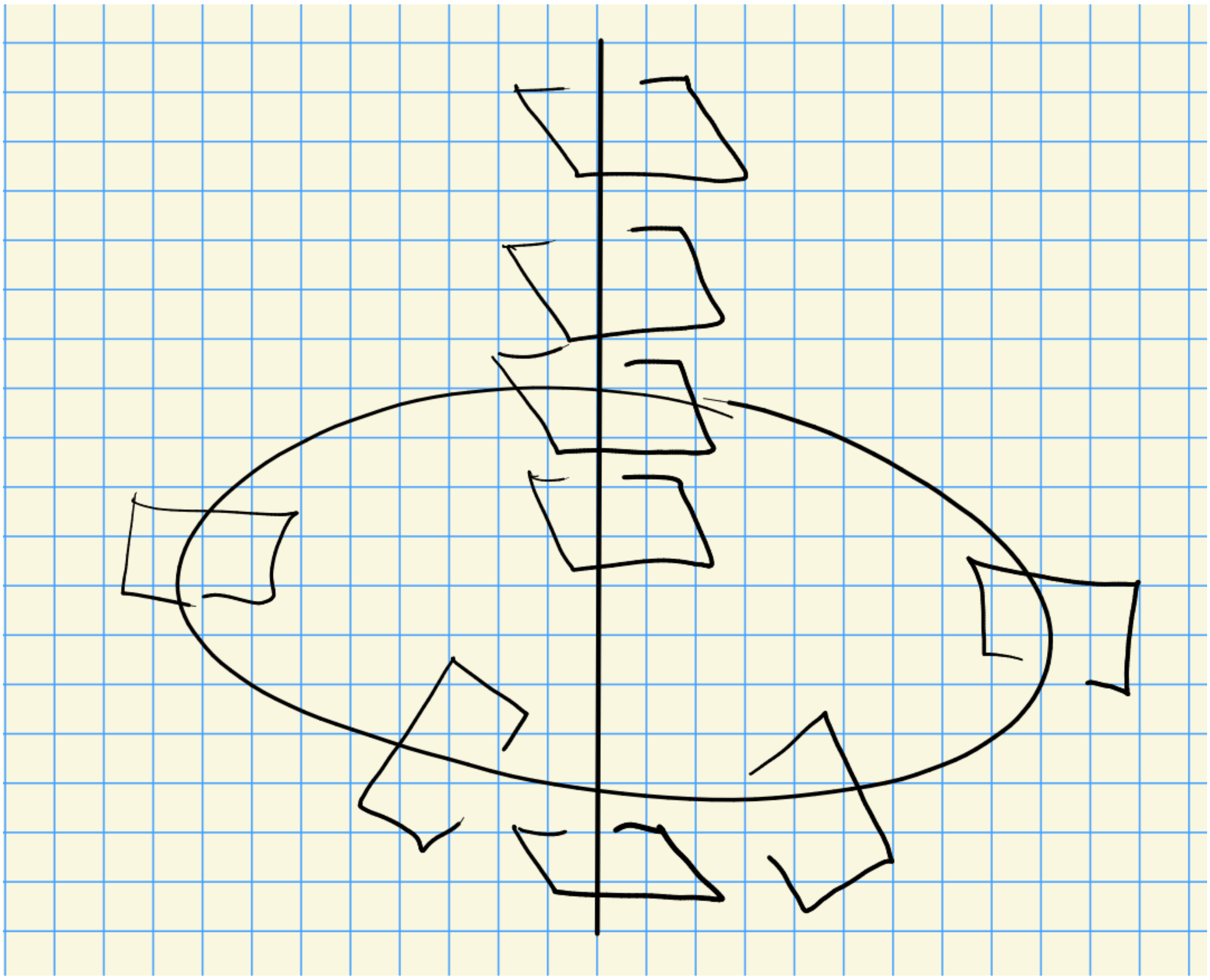
Example 3.0.15 ***(?):*** The standard contact structure is orthogonal to the Hopf fibration: define a map

$$\begin{aligned} \mathbb{C}^2 \setminus \{0\} &\to \mathbb{CP}^1 \cong S^2 \\ [z, w] &\mapsto [z : w], \end{aligned}$$

which restricts to a map $S^3 \to S^2$ defining the Hopf fibration. If L is a complex line through 0, then $L \cap S^3$ is a Hopf fiber that is homeomorphic to S^1.



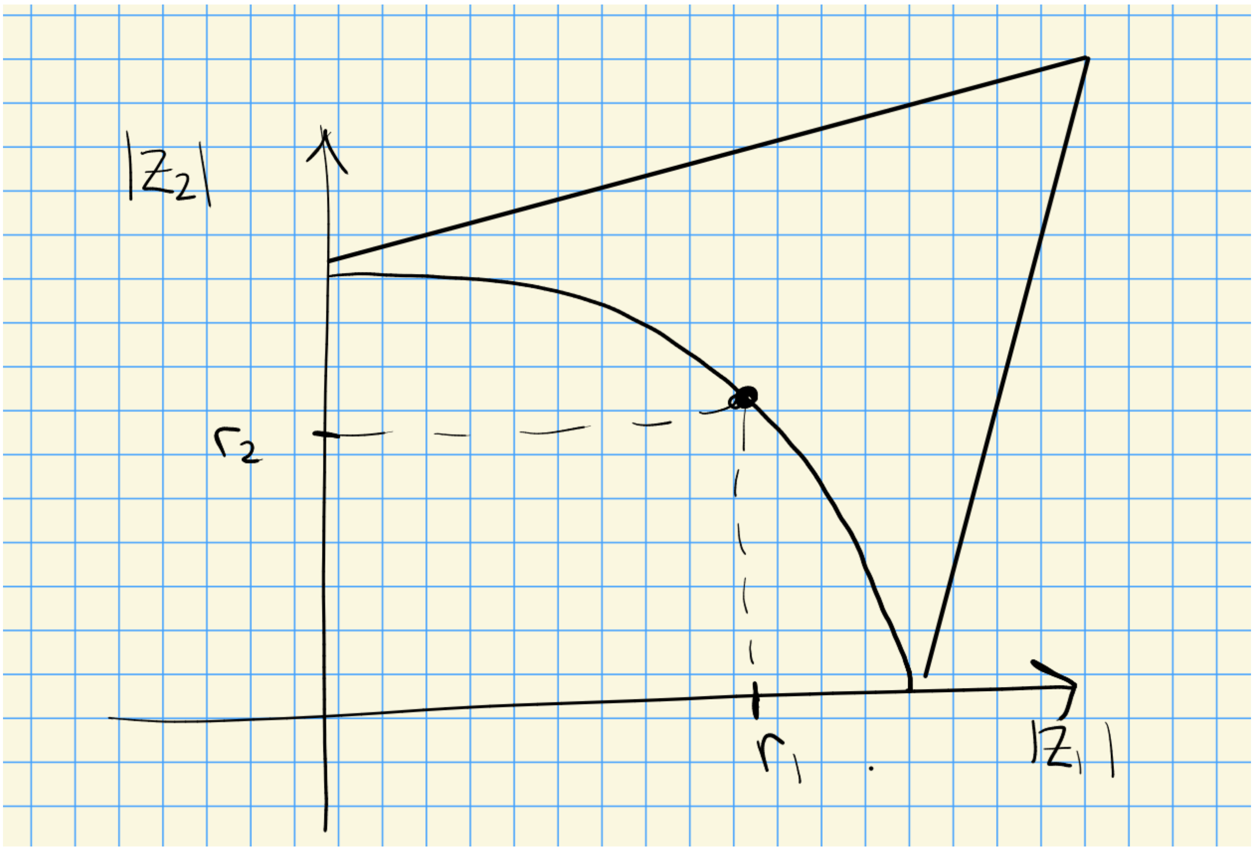




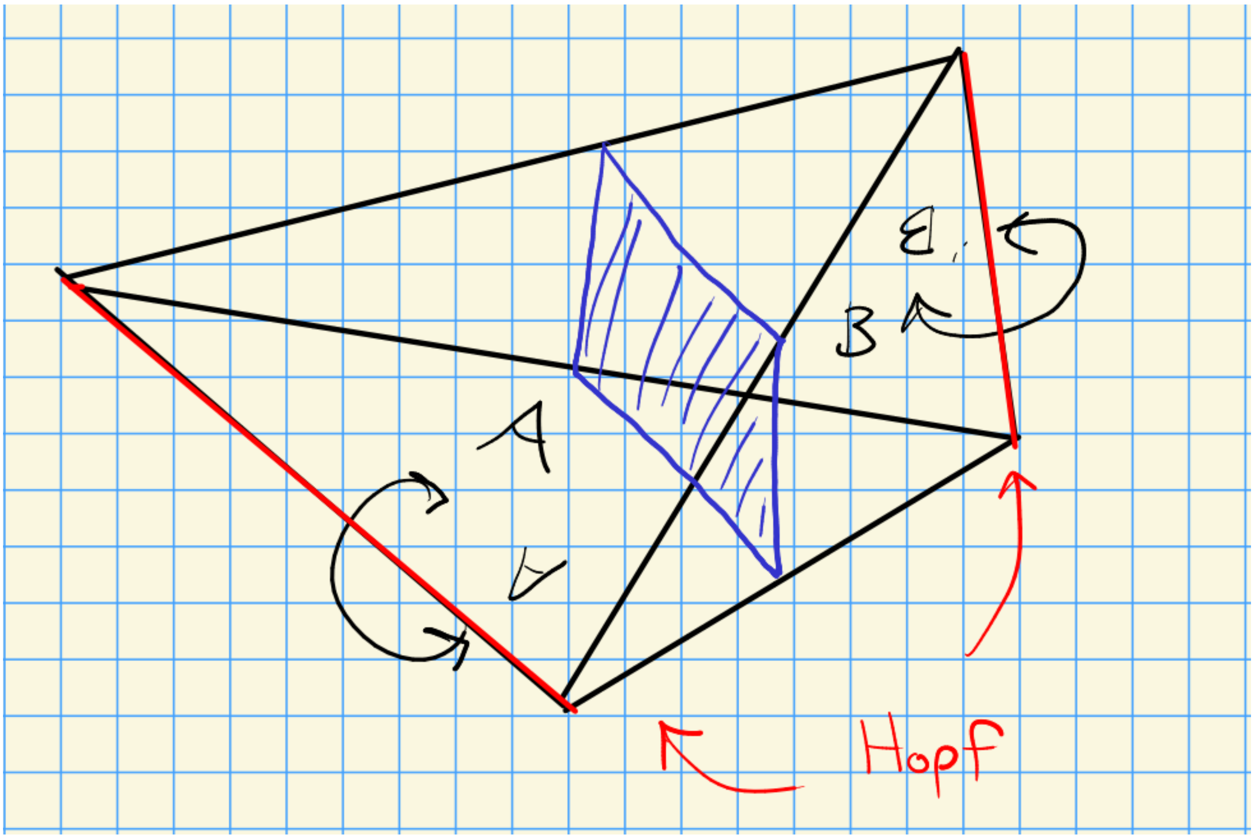
Take

$$\mathbb{C}^2 \to \mathbb{R}^2$$
$$(z_1, z_2) \mapsto (|z_1|, |z_2|).$$

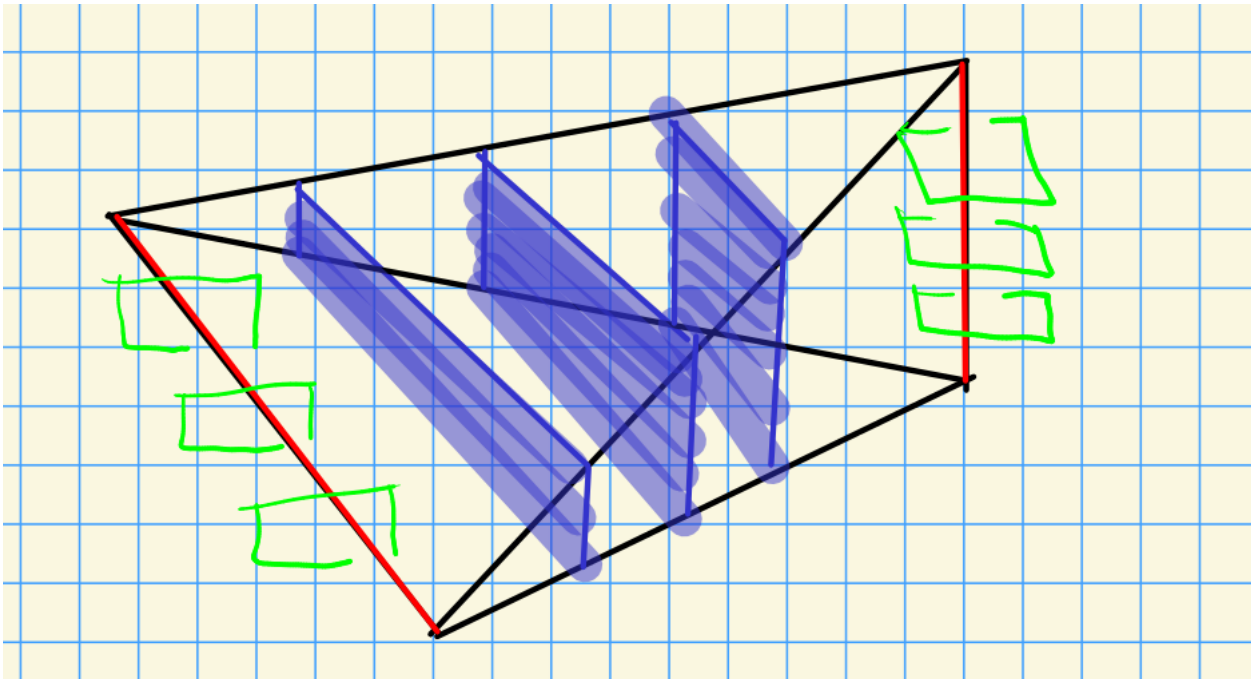
Consider the image of $S^2 = \left\{ |z_1|^2 + |z_2|^2 = 1 \right\}$:



The preimage is $S^1 \times S^1$. This can be realized as a tetrahedron with sides identified:



There are Hopf fibers on the ends, and undergo a $\pi/2$ twist as you move through the tetrahedron.



4 Darboux and Gromov Stability (Thursday, January 20)

Remark 4.0.1: Almost-complex structures: weaker than an actual complex structure, but not necessarily integrable. Useful for studying pseudoholomorphic curves. A necessary and sufficient condition for integrability: the Nijenhuis tensor $N_J = 0$ iff J is integrable. In real dimension 2, all J are integrable.

Theorem 4.0.2 ***(Darboux).***
If (Y^3, ξ) is contact then for every point p there is a chart U with coordinates x, y, z where $\xi = \ker(\,dz - y\,dx) = \ker(\alpha_{\text{std}})$.

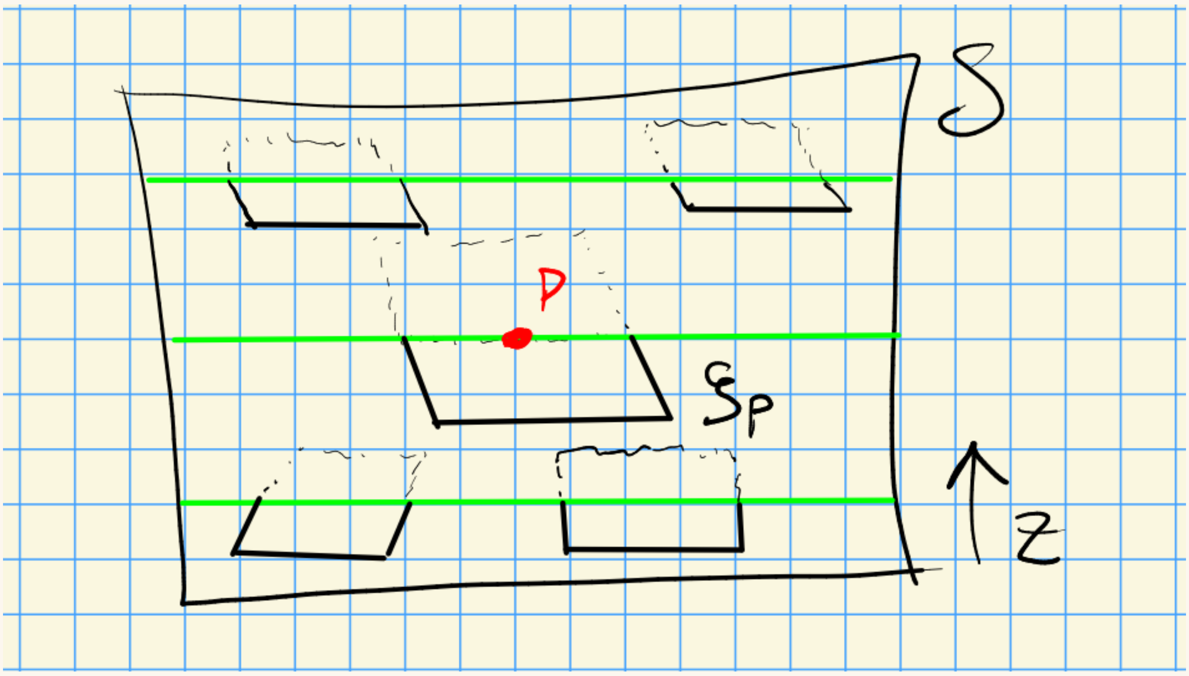
Slogan 4.0.3
Locally, all contact *structures* (not necessarily forms) look the same. The mantra: local flexibility vs global rigidity.

4.1 Proof of Darboux

Remark 4.1.1: Two proofs:

- Geometric, due to Giroux
- PDEs, which generalizes. This uses Moser's trick.

Proof (1).
Locally write $\xi = \ker \alpha$ with $\alpha \wedge d\alpha > 0$. Pick a contact plane ξ_p and let S be a transverse surface, so $\mathbf{T}_p S \pitchfork \xi_p$. This produces a set of curves in S which are tangent to ξ_p everywhere, called the *characteristic foliation*.



Then $\alpha|_S = dz$, which is a 1-form that is nonvanishing near p and is locally integrable. Sending $\alpha \to X$ a vector field along S yields a set of integral curves tracing out the characteristic foliation. This yields an x direction and a z direction on S by flowing $t \in (-\varepsilon, \varepsilon)$ around p along X.
Choose a vector field ∂t which is transverse to S and contained in ξ. Then $\alpha(\partial t) = 0$, so we can write

$$\alpha = f\,dx + g\,dz + h\,dt = f\,dx + g\,dz.$$

Since $g(p) = 1$, replace α with $\frac{1}{g}\alpha$ which is positive near p and doesn't change the contact structure ξ. So write

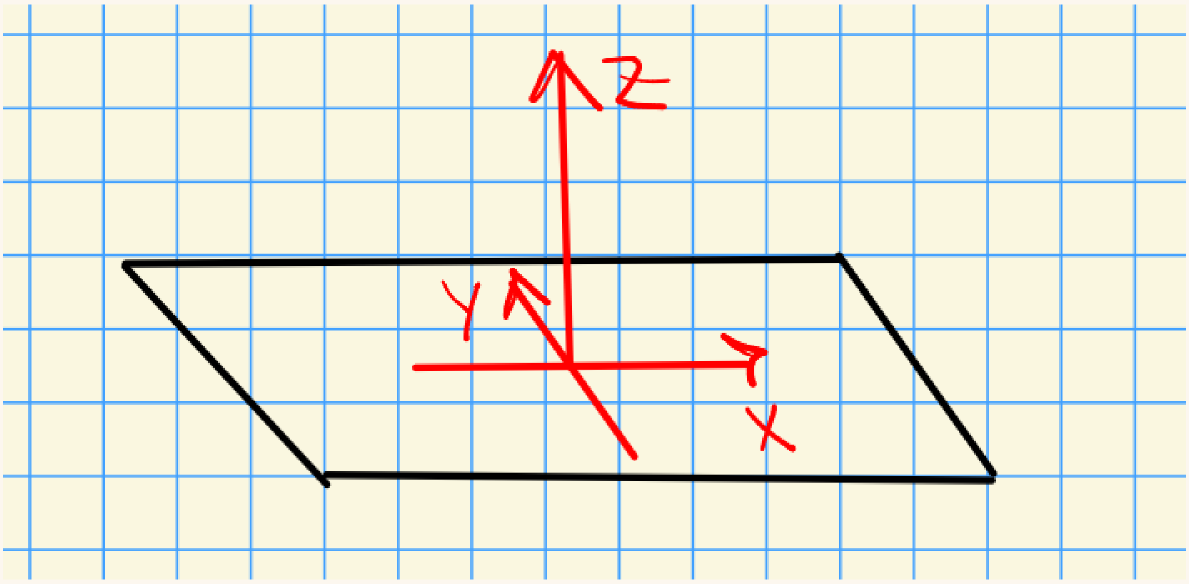
$$\alpha = f\,dx + dz \implies \alpha \wedge d\alpha = \alpha \wedge (f_t\,dy \wedge dx + f_z\,dz \wedge dx) = -f_t\,dx \wedge dt \wedge dz > 0,$$

meaning $f_t < 0$ and we can set $y = f(x, z, t)$. This yields

$$\alpha = dz + f\,dx = dz - y\,dx.$$

■

Proof (2, Moser's Trick).
By a linear change of coordinates, choose x, y along ξ to write $\alpha_p = dz$ and $\xi_p = \operatorname{span} \partial x, \partial y$:



Write $(\alpha_0)_p$ for the original form and $\alpha_1 = dz - y\,dx$ the standard form, then the claim is that $\alpha_0 \simeq \alpha_1$ through a path of contact forms.

Lemma 4.1.2 *(?)*.
In a neighborhood of p, there is a family α_t for $t \in [0,1]$.
To obtain this, interpolate:

$$d\alpha_t = t d\alpha_1 + (1-t) d\alpha_0 \implies \alpha_t \wedge d\alpha_t = t^2 \alpha_1 \wedge d\alpha_1 + t(1-t)(\alpha_0 \wedge d\alpha_1 + \alpha_1 \wedge d\alpha_0) + (1-t)^2 \alpha_0 \wedge d\alpha_0.$$

The first and last terms are positive since the α_i are contact. For the middle term, $\alpha_0 = \alpha_1$ near p, so by continuity this is positive in some neighborhood of p.

Remark 4.1.3: Note that $\dot{\alpha}_t := \frac{\partial}{\partial t} \alpha_t$, so

$$\frac{\partial}{\partial t}(t\alpha_1 + (1-t)\alpha_0) = \alpha_1 - \alpha_0.$$

We'll assume that there is a time-dependent vector field $V_t \in \xi_t$ with flow Φ_t such that $(\Phi_t)_*(\xi_t) = \xi_0$. We'll also require $\xi_t = \ker \alpha_t$, so this is a contactomorphism for each t. The goal is to show $(\Phi_1)_*(\xi_0) = \xi_1$, or equivalently $\Phi_t^* \alpha_t = f_t \alpha_0$ with $f_t > 0$. Take $\frac{\partial}{\partial t}$ of both sides here to get

$$\Phi_t^*(\dot{\alpha}_t + \mathcal{L}_{V_t} \alpha_t) = \dot{f}_t \alpha_0.$$

See Prop 6.4 in Cannas da Silva.

Remark 4.1.4: ✨Cartan's magic formula✨:

$$\mathcal{L}_V(\alpha) = d(\iota_V \alpha) + \iota_V(d\alpha),$$

so

$$\mathcal{L}_{V_t}(\alpha_t) = d(\alpha_t(V_t)) + d\alpha_t(V_t, -) = 0 + d\alpha_t(V_t, -).$$

We can thus write this equation as

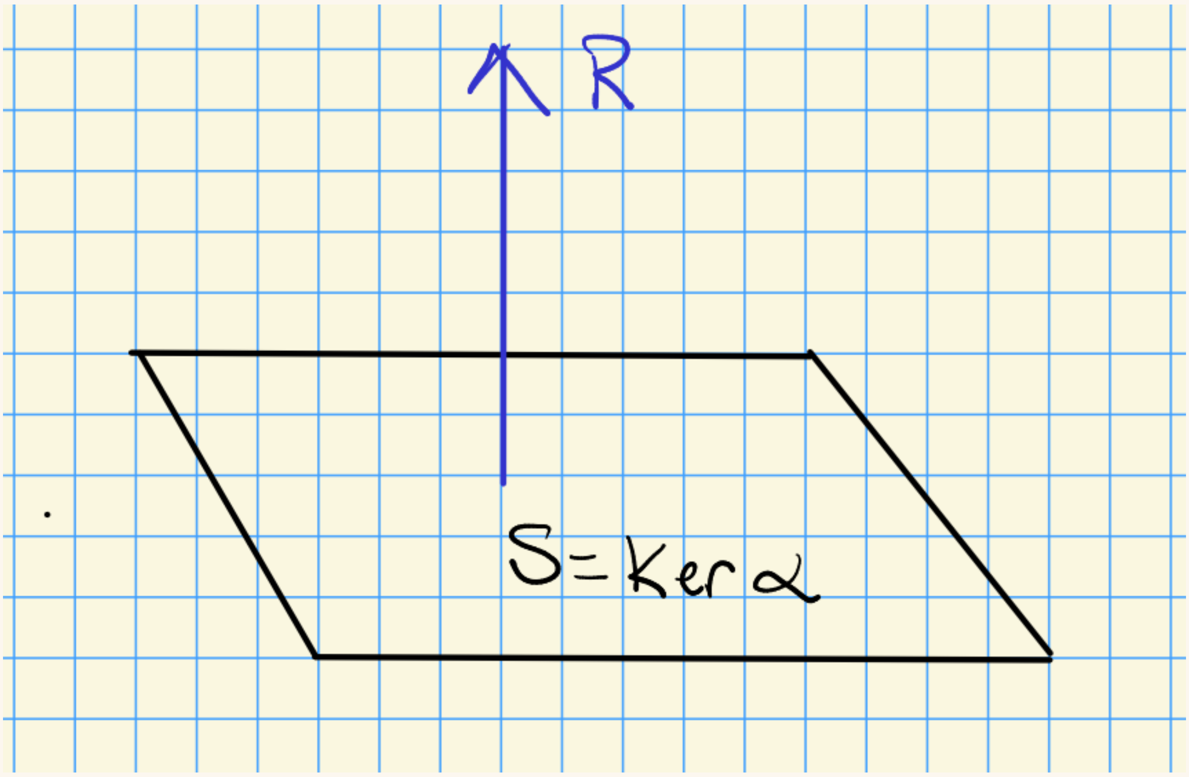
$$\Phi_t^*(\dot{\alpha}_t + d\alpha_t(V_t, -)) = \dot{f}_t\alpha_0 = \dot{f}_t\left(\frac{\Phi_t^*(\alpha_t)}{f_t}\right).$$

Applying $(\Phi_t^*)^{-1}$ yields

$$\dot{\alpha}_t + d\alpha_t(V_t, -) = \frac{\dot{f}_t}{f_t}\alpha_t.$$

Now try to solve this for V_t. Let R_t be the **Reeb vector field** of α_t, which satisfies

- $\alpha_t(R_t) = 1$
- $d\alpha_t(R_t, -) = 0$.



Then

$$\dot{\alpha}_t(R_t) = \frac{\dot{f}_t}{f_t} = \frac{\partial}{\partial t}\log(f_t) := \mu_t,$$

so $\dot{\alpha}_t(R_t)$ determines f_t by first integrating and exponentiating.
We now need to solve

$$d\alpha_t(V_t, -)|_{\xi_t} = \mu_t\alpha_t - \dot{\alpha}_t|_{\xi_t}.$$

Since $d\alpha_t$ is a volume form on ξ_t, it identifies vector fields in ξ_t with 1-forms on ξ_t using the happy coincidence that $n = 2$ so $1 \mapsto n - 1 = 1$. So V_t is uniquely determined by the solution to the above equation.

■

5 | Gray Stability (Tuesday, January 25)

Remark 5.0.1: A homotopy of contact structures o Y^3 is a smooth family $\{\varphi_t\}$ of contact structures. Similarly, an **isotopy** of structures such that $\{D\varphi_t(\xi_0)\}$ for an isotopy $\varphi_t : Y \to Y$ with $\varphi_0 = \text{id}$. If Y^3 is closed then every homotopy of contact structures is an isotopy. Theorem: contact structures mod isotopy is discrete, which critically uses closedness.

Lemma 5.0.2 *(?).*
For φ_t an isotopy generated by the flow of X_t and α_t a family of 1-forms,

$$\frac{\partial}{\partial t}\varphi_t^*(\alpha_t)\Big|_{t=t_0} = \varphi_{t_0}^*(\dot{\alpha}_{t_0} + \mathcal{L}_{X_{t_0}}\alpha_{t_0}).$$

Proof (?).
Write

$$\varphi_x^*(\alpha_y) = \frac{\partial}{\partial x}\,? + \frac{\partial}{\partial y}\,? = \varphi_{x_0}^*\mathcal{L}_{X_0}\mathcal{L}_X\alpha_{y_0} + \varphi_{x_0}^*\alpha_y,$$

and proceed similarly to the proof of Darboux's theorem.
Pick $\{\varphi_t\}$ a homotopy, one can choose α_t with $\xi_t = \ker\alpha_t$ for all t. Apply Moser's trick: assume there exists a φ_t with $\varphi_t^*(\alpha_t) = \lambda_t\alpha_0$ and try to find v_t generating it, where $\lambda_t : Y \to \mathbb{R}_+$. What does φ_t need to look like? Differentiate in t:

$$\varphi_{t_0}^*(\cdot\alpha_{t_0} + \mathcal{L}_{V_{t_0}}\alpha_{t_0}) = \dot{\lambda}_t\alpha_0 = \dot{\lambda}_t\left(\frac{\varphi_{t_0}^*(\alpha_t)}{\lambda_t}\right).$$

Apply $(\varphi_{t_0}^*)^{-1}$:

$$\frac{\cdot\alpha_t + \mathcal{L}_{V_t}\alpha_t = \mu_t\alpha_t \qquad \mu_t = (\varphi_{t_0}^*)^{-1}(\dot{\lambda}_t}{\lambda_t).}$$

Use that V_t is always tangent to the contact structure, so $V_t \in \xi_t$, to assume $\alpha_t(V_t) = 0$. Apply Cartan:

$$\dot{\alpha}_t + d\alpha_t(V_t) + \iota_{V_t}d\alpha_t = \mu_t\alpha_t,$$

and $d\alpha_t(V_t) = 0$, so

$$\iota_{V_t}d\alpha_t = \mu_t\alpha_t - \dot{\alpha}_t.$$

Plug in the Reeb vector field R_t, then $\alpha_t(R_t) = 0$ so $\mu_t = \dot{\alpha}_t(R_t)$.

■

Corollary 5.0.3 *(?).*
Let Y be n $S^3 \subseteq \mathbb{C}^2$ that is transverse to the radial vector field. Then

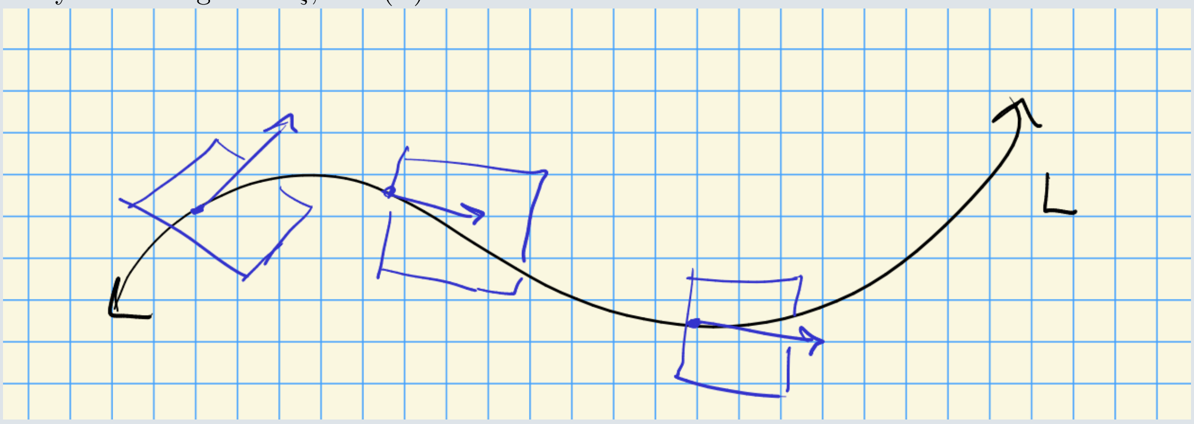
$$\alpha = x_1 dy_1 - y_1 dx_1 + x_2 dy_2 - y_2 dx_2 \Big|_y$$

defines the standard tight contact structure.

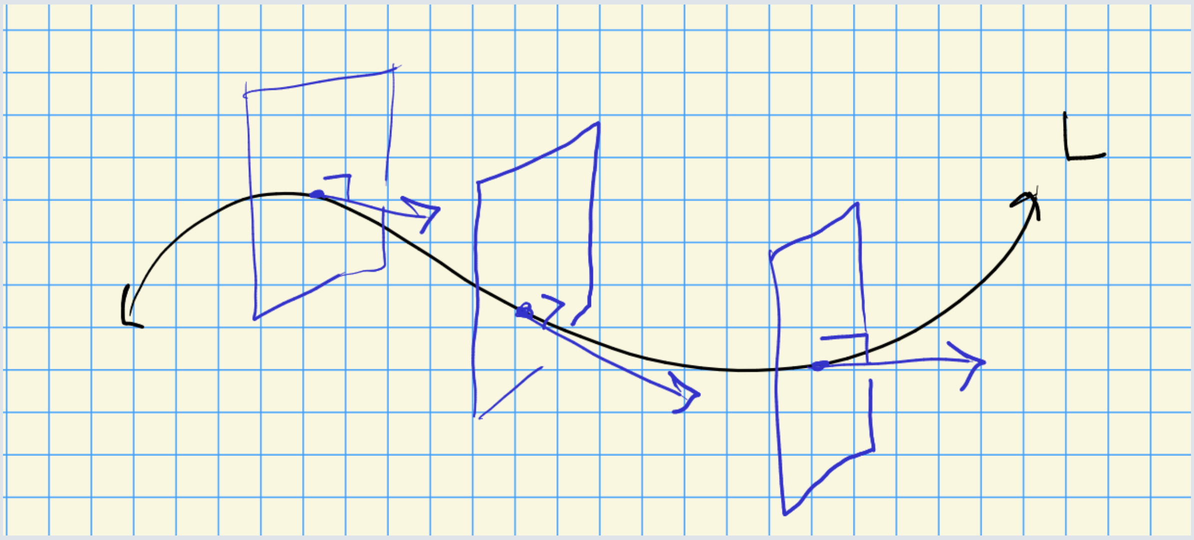
Proof (?).
Write $Y \subseteq \mathbb{R} \times S^3$ in coordinates $(f(x), x)$ as the graph of a function $f : S^3 \to \mathbb{R}$. Take an isotopy $Y_t = (tf(x), x) \subseteq \mathbb{R} \times S^3$ to get a family of contact forms where $\alpha_0 = \alpha_{\text{std}}$ and α_1 is some unknown form. By Gray stability, the contact structures are isotopic. ■

5.1 Legendrian Links

Definition 5.1.1 (Legendrian and transverse knots)
Let Y be a contact 3-manifold and $L \hookrightarrow Y$ a link. Then L is a **Legendrian knot** iff it is everywhere tangent to ξ, so $\alpha(L) = 0$:



This is a closed condition.
L is **transverse** if it is everywhere transverse to ξ, so $\alpha(L) > 0$:



This is an open condition.

Remark 5.1.2: Every Legendrian knot has a transverse pushoff (up to transverse isotopy). Every transverse knot has a Legendrian approximation.

Example 5.1.3*(?):* Take $\mathbb{R}^3$ and $\alpha_{\text{std}} = dz - ydx$, then the y-axis $L_1 := \{[0, t, 0]\}$ is Legendrian. Similarly the x-axis L_2 is Legendrian, checking that $\mathbf{T}L_2 = \operatorname{span}\{[1, 0, 0]\}$. However the slight pushoff $L_3 := \{[t, -\varepsilon, 0]\}$ is transverse since $\alpha|_{L_3} = \varepsilon dx > 0$.

Theorem 5.1.4*(Neighborhood theorem, Darboux for Legendrian/transverse knots).* *Every Legendrian has a neighborhood contactomorphic to the zero section in $J_1 S^1 = \mathbf{T}S^1 \times \mathbb{R}$. Every transverse has a neighborhood contactomorphic to the z-axis in $\mathbb{R} \times S^1$ with $\alpha := dz + r^2\, d\theta$.*

6 | Thursday, January 27

Remark 6.0.1: Goal: classify Legendrian knots up to (Legendrian) isotopy. Recall a knot $\gamma : S^1 \hookrightarrow Y$ satisfies $\gamma^*(\alpha) = 0$, and a Legendrian isotopy is a 1-parameter family γ_t which are Legendrian for all t.

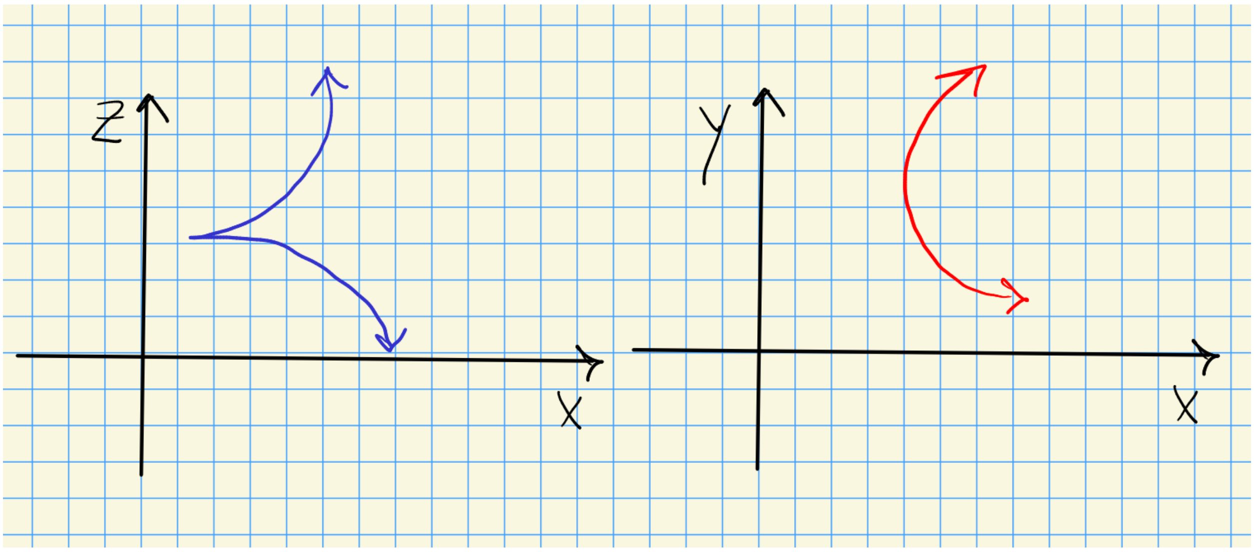
Example 6.0.2*(?):* $\gamma(s) = [x(s), y(s), z(s)]$ and $\xi = \ker \alpha, \alpha = dz - y\, dx$. Then $\gamma^*(\alpha) = z'\, ds - yx^1\, ds = (z' - yx')\, ds$, which is Legendrian iff $y = z'/x'$.

Example 6.0.3*(?):* Let $f : \mathbb{R} \to \mathbb{R}$ and take the 1-jet $\gamma(s) = [s, f'(s), f(s)]$ of the graph of f – this is like the graph of the 1st order Taylor expansion. This is Legendrian since $s' = 1$ implies $z'/x' = f'/s' = f'$.

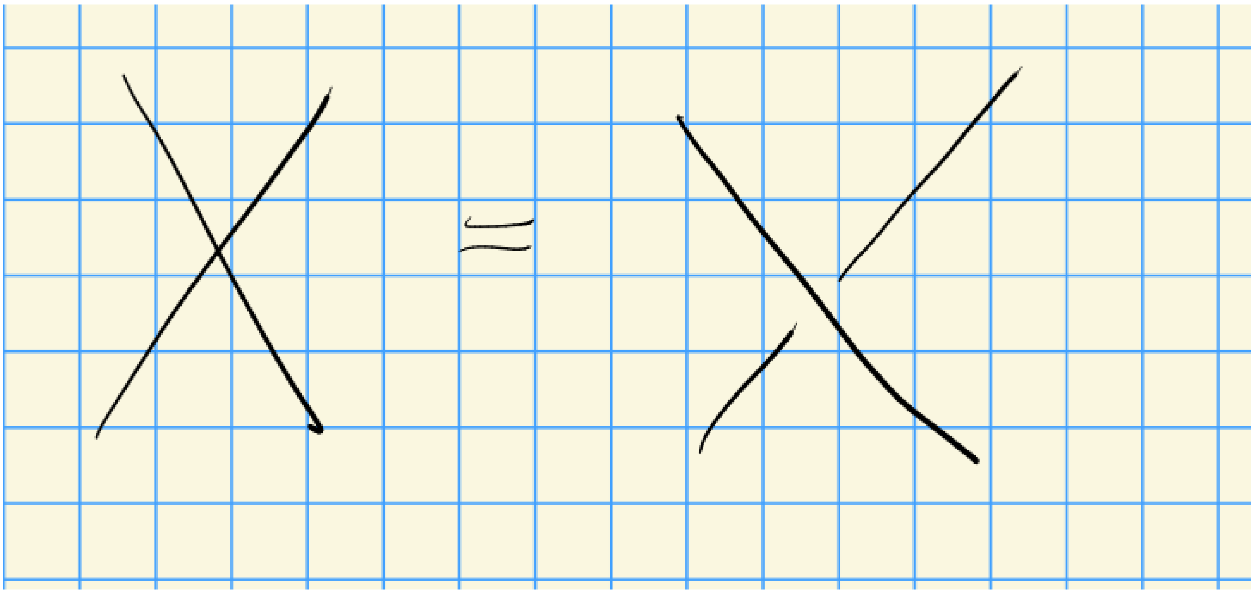
Remark 6.0.4: There are two projections:

- $[x, y, z] \to [x, z]$, a wave front projection, plotted with y into the board,
- $[x, y, z] \to [x, y]$, Lagrangian projection.

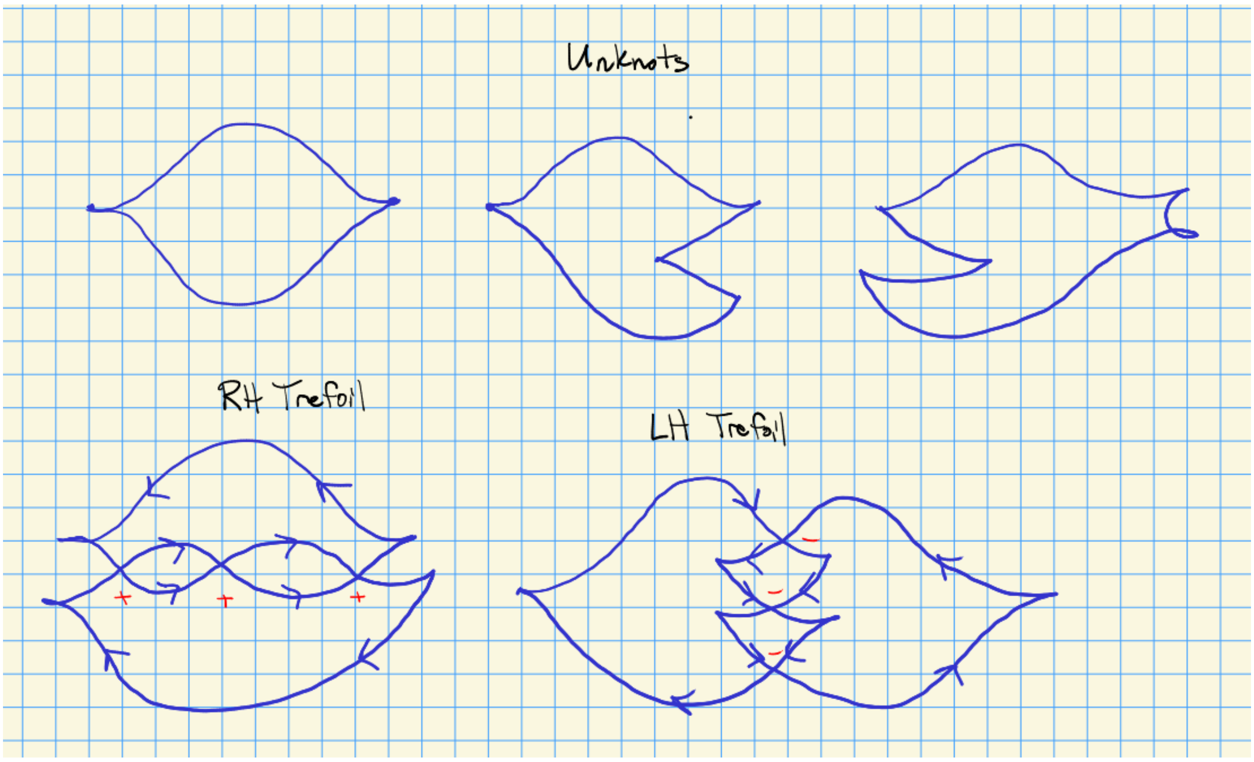
Example 6.0.5 ***(?):*** Let $\gamma(s) = \left[s^2, \frac{3}{2}s, s^3\right]$, then the two projections are as follows:



Remark 6.0.6: The front projection uniquely determines L, since the y coordinate can be recovered as $y = z'/x'$. So for example, there is no ambiguity about crossing order: the more negatively sloped line in a diagram is the over-crossing:

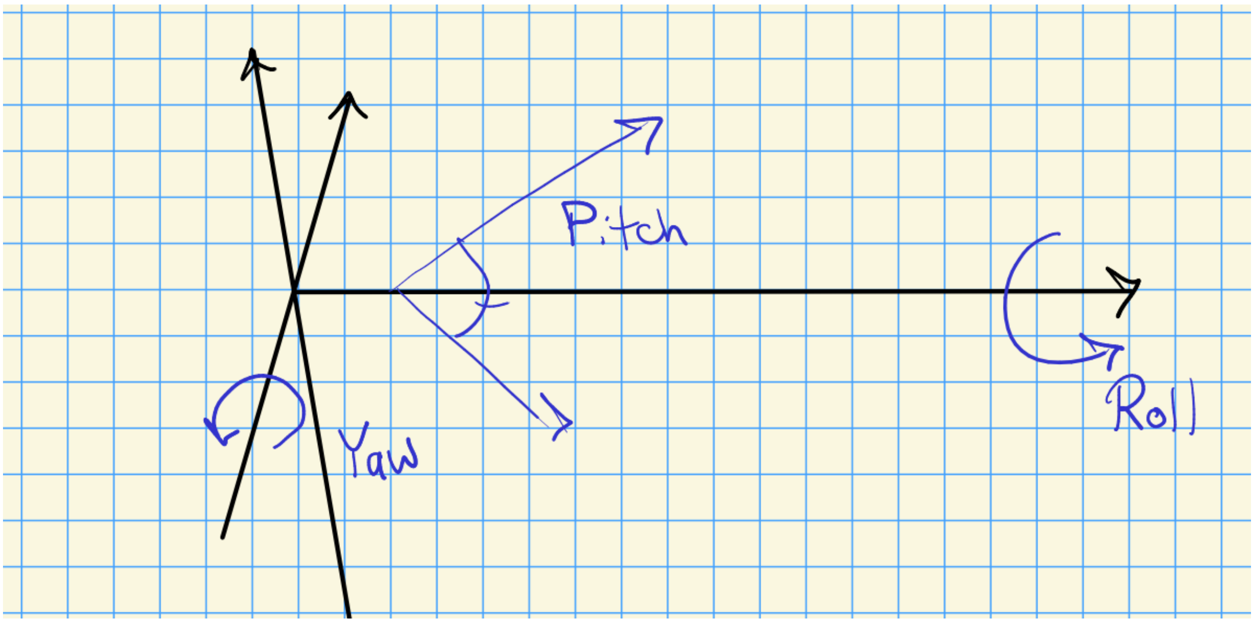


Example 6.0.7 ***(?):*** A front diagram of the unknot:



Theorem 6.0.8 *(?)*.
Every knot $K \hookrightarrow \mathbb{R}^3$ can be C^0 approximation by a Legendrian knot L.
Idea: zigzags in an ε tube in the knot diagram, which will be Legendrian. How to measure: $\sup_{s\in I} |\gamma_1(s) - \gamma_2(s)| \leq \varepsilon$?

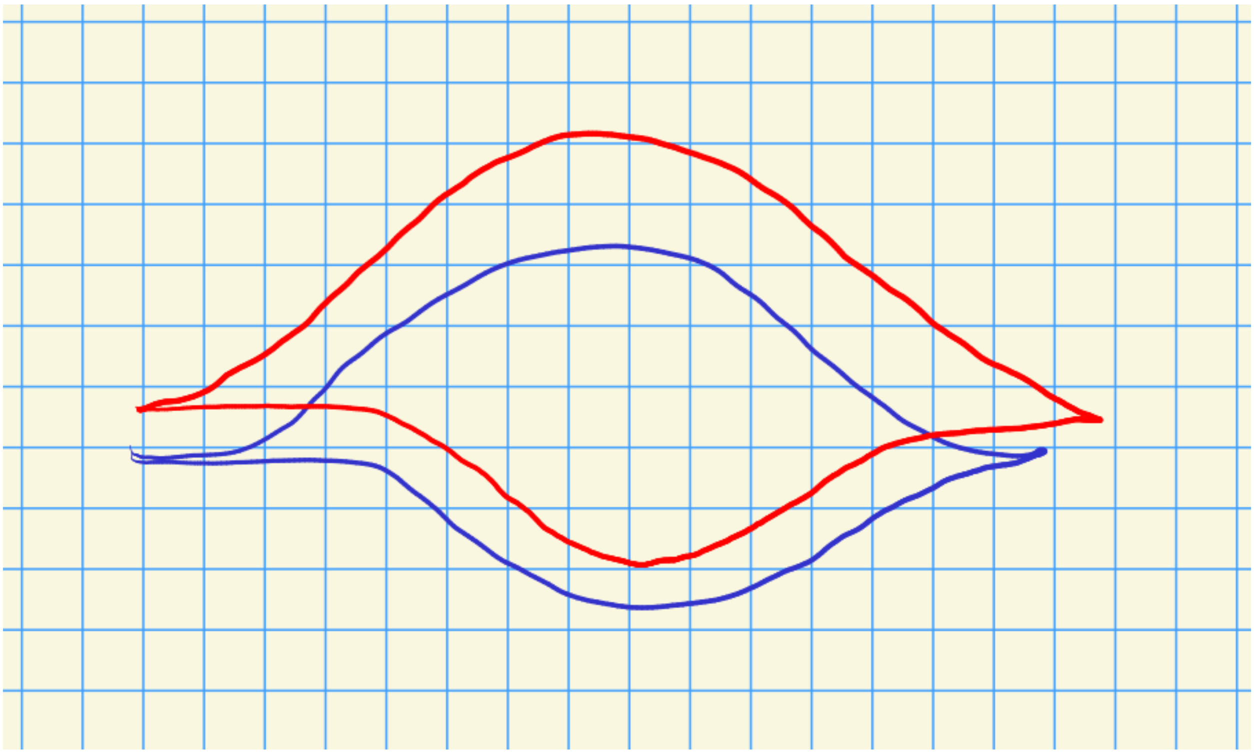
Remark 6.0.9: Note that $\mathsf{Lie}(\mathrm{SO}_3) \coloneqq \mathbf{T}_e(\mathrm{SO}_3) = \mathfrak{su}_2$, spanned by roll, pitch, and yaw generators:



So measuring the number of rotations along each generator after traversing L in a full loop yields integer invariants.

Definition 6.0.10 (The Thurston–Bennequin number)
A **framing** of a knot K is a trivialization of its normal bundle, so an identification of $\nu(K) \cong S^1 \times \mathbb{D}^2$. The potential framings are in $\pi_1(\mathrm{SO}_2) \cong \pi_1(S^1) \cong \mathbb{Z}$, since a single vector field normal (?) to the knot determines the framing by completing to an orthonormal basis. The Reeb vector field is never tangent to a Legendrian knot, so this determines a framing called the **contact framing**. The **Thurston–Bennequin number** is the different between the 0-framing and the contact framing. The 0-framing comes from a Seifert surface. This is an invariant of Legendrian knots, since Legendrian isotopy transports frames. Note that adding zigzags adds cusps, and thus decreases this number.

Remark 6.0.11: How to compute: take a pushoff and compute the linking number:



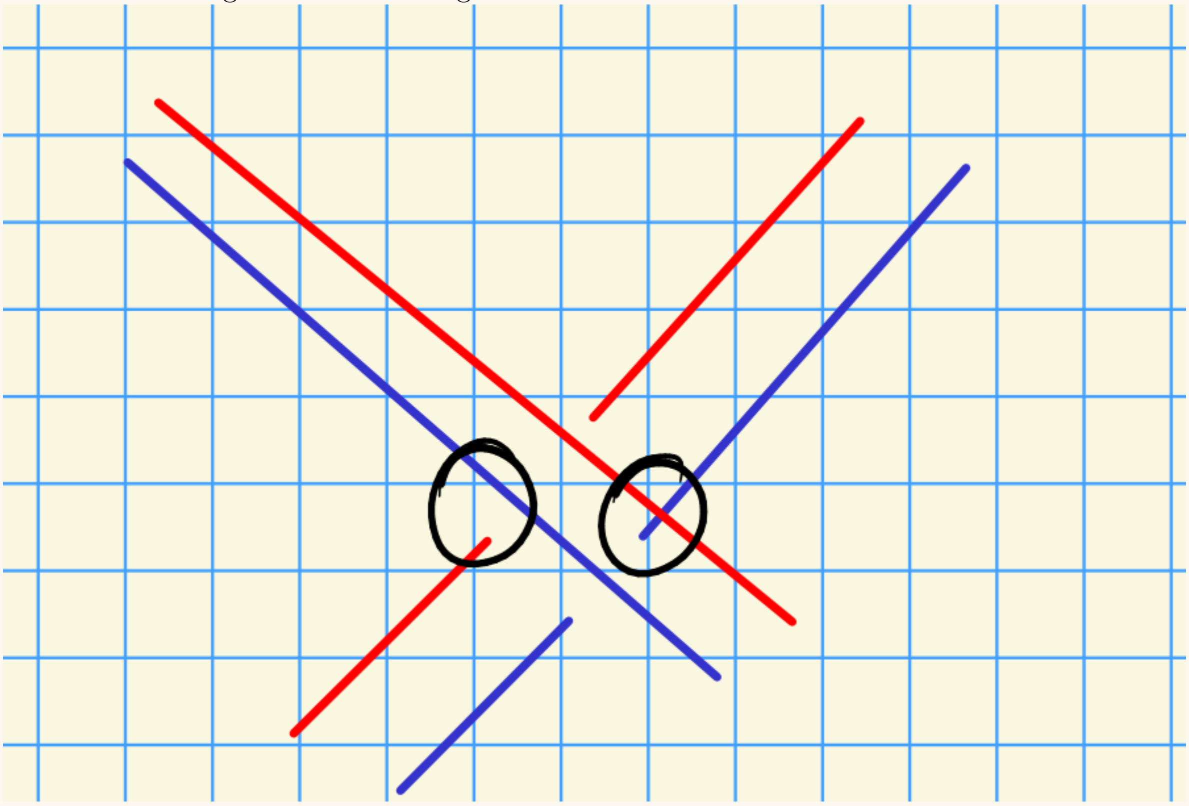
Proposition 6.0.12 *(?).*

$$\mathrm{tb}(L) = w(L) - \frac{1}{2}C(L),$$

where $w(L)$ is the writhe and $C(L)$ is the number of cusps.

Proof (?).

The linking number is $\frac{1}{2}(c_+(L) - c_-(L))$, half of the signed number of crossings.
Here all 4 crossing have the same sign:



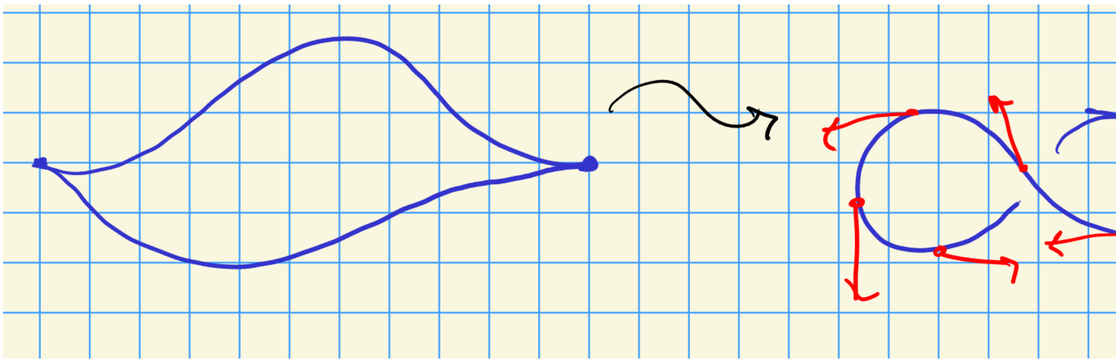
■

Example 6.0.13 ***(?):*** TB for the knots from before:

- The 3 unknots:
  - 2 cusps, so -1
  - 4 cusps, so -2
  - 4 cusps, so -2
- The 2 trefoils:
  - $3 - \frac{1}{2}4 = 1$
  - $3 - \frac{1}{2}6 = -6.$

Remark 6.0.14: Since adding zigzags decreases tb, define TB to be the max over all Legendrian representatives of K. This distinguishes mirror knots. In fact $\text{tb}(L) \leq 2g_3(L) - 1$ (the Bennequin bound), involving the 3-genus.

Definition 6.0.15 (Rotation number)
The **rotation number** of L is the *turning number* $\mathrm{rot}(L)$ in the Lagrangian projection, i.e. how many times a tangent vector spins after traversing the knot.



Example 6.0.16 *(?):*

It turns out that

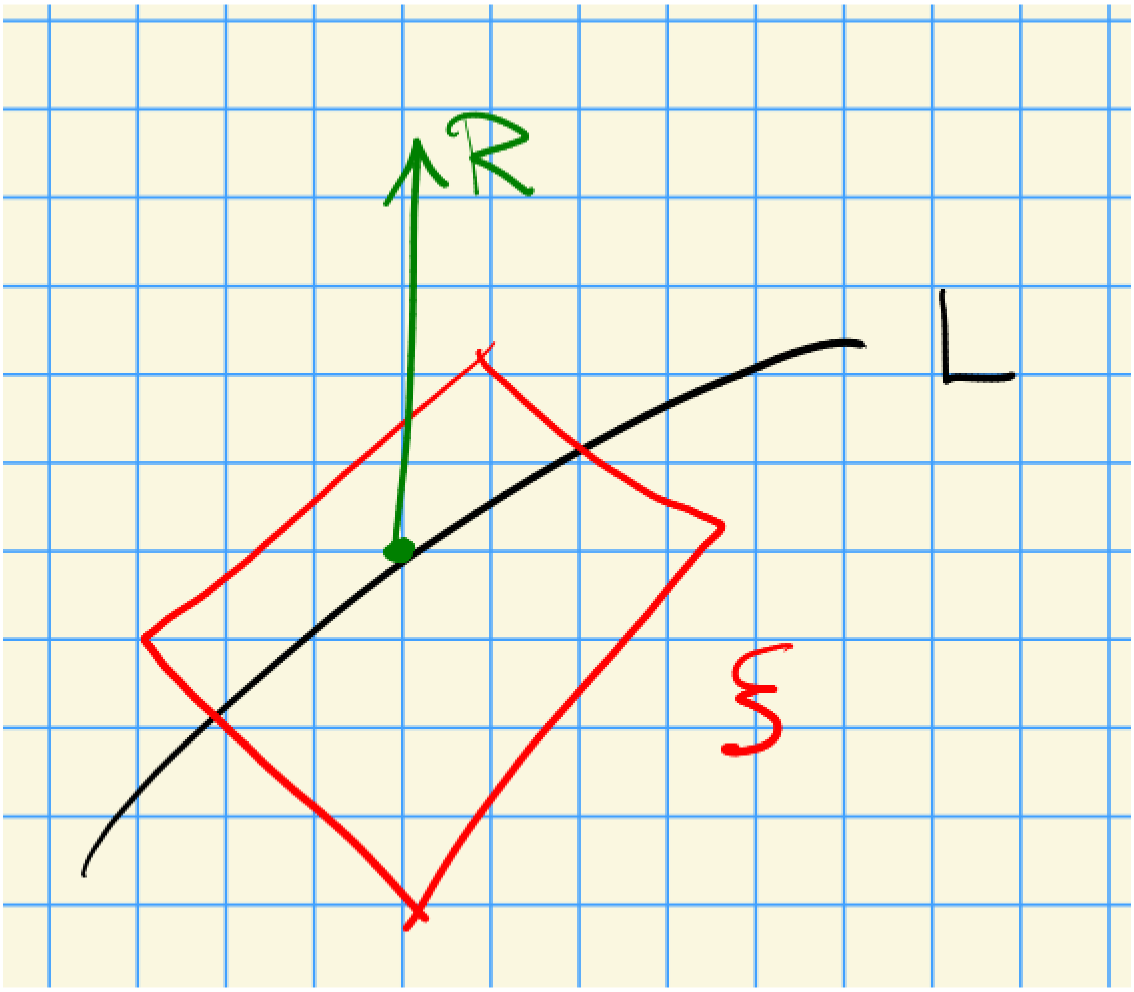
$$\mathrm{rot}(L) = \frac{1}{2}\left(\sharp\text{down cusps} - \sharp\text{up cusps}\right).$$

7 | Tuesday, February 01

Remark 7.0.1: Last time: front diagrams $[x, y, z] \mapsto [x, z]$, where $\alpha = dz - y\,dx$ forces $y = ds/\,dx$ can be recovered as the slope in the projection. Note that we can also recover crossing information from the Legendrian condition, since y always points into the board, so more negative slopes go on top.

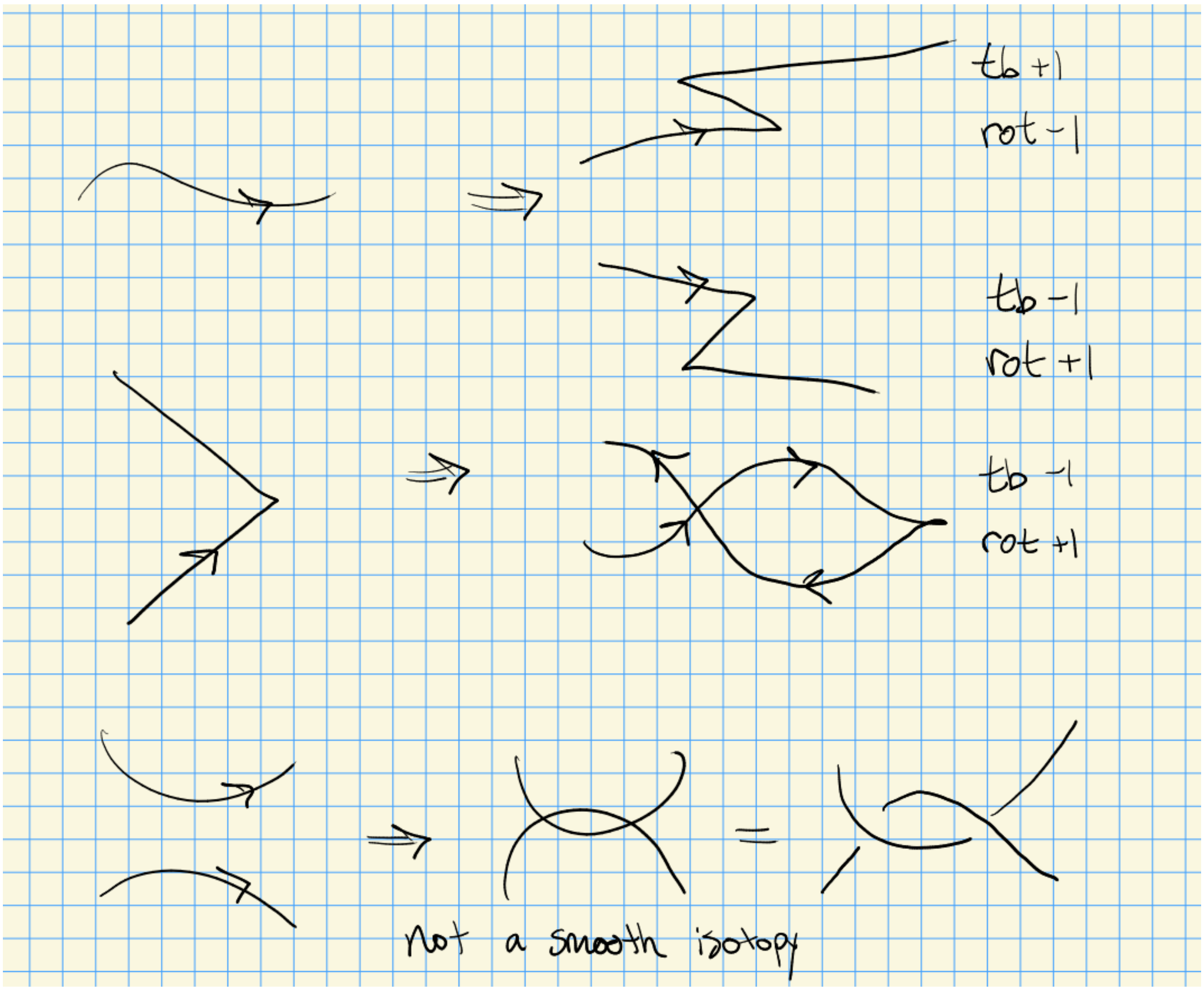
Some invariants:

- Thurston-Bennequin invariant: a contact framing with respect to the Reeb vector field

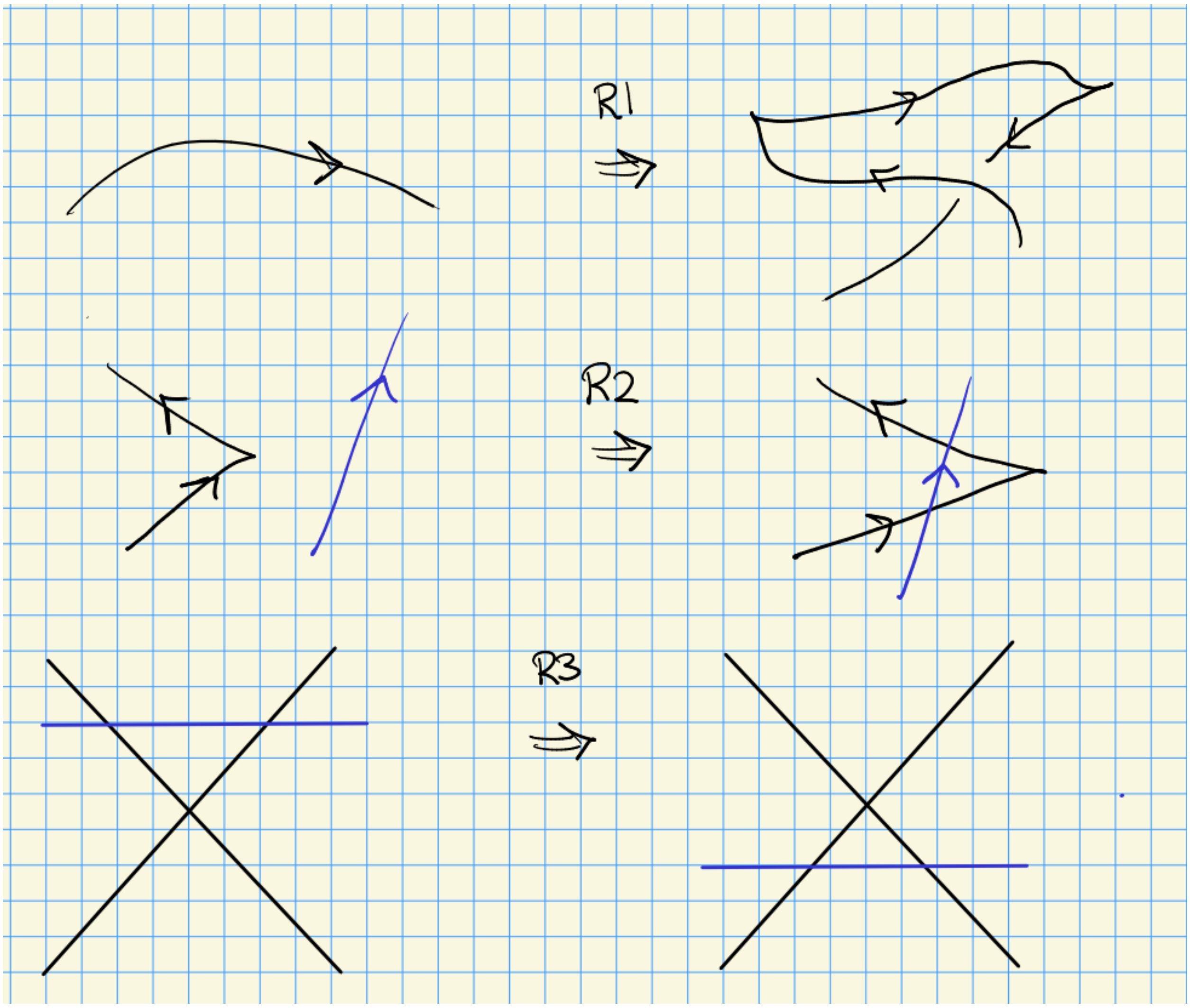


- Equal to writhe minus half the number of cusps.
- Rotation numbers: Turning number of L with respect to ξ, after fixing a trivialization of ξ. Equal to $\frac{1}{2}(D - U)$, the number of down/up cusps respectively.

Remark 7.0.2: Disallowed moves:



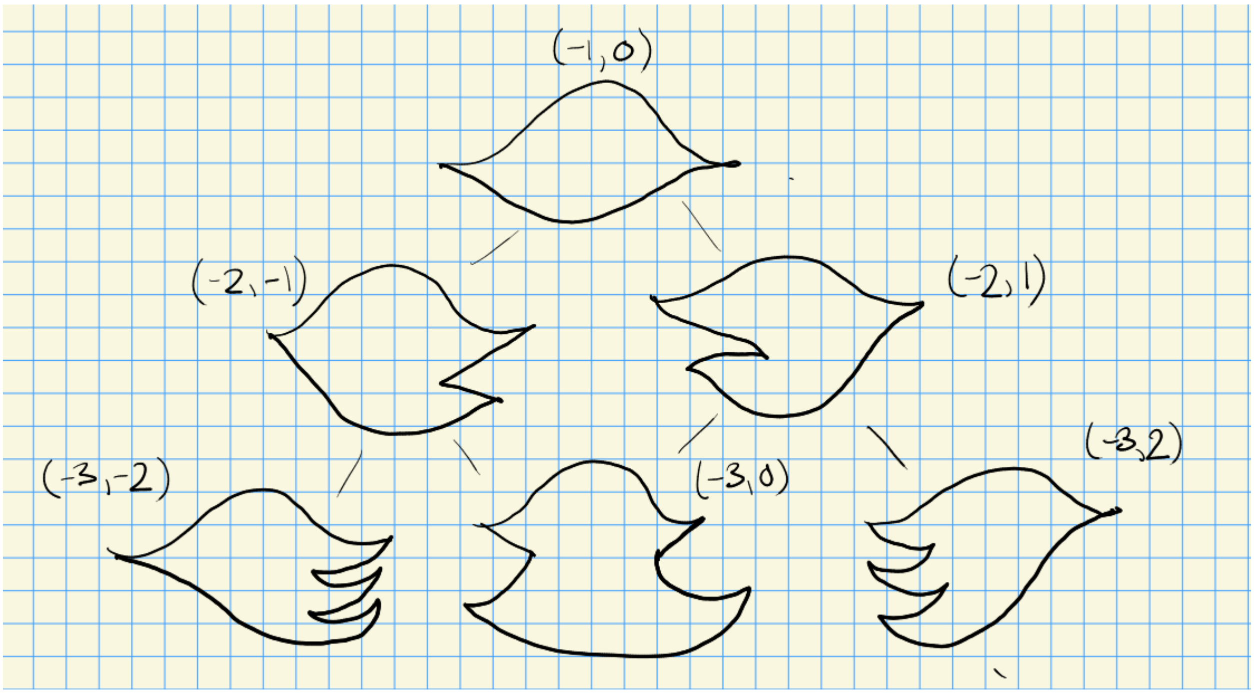
Allowed moves:



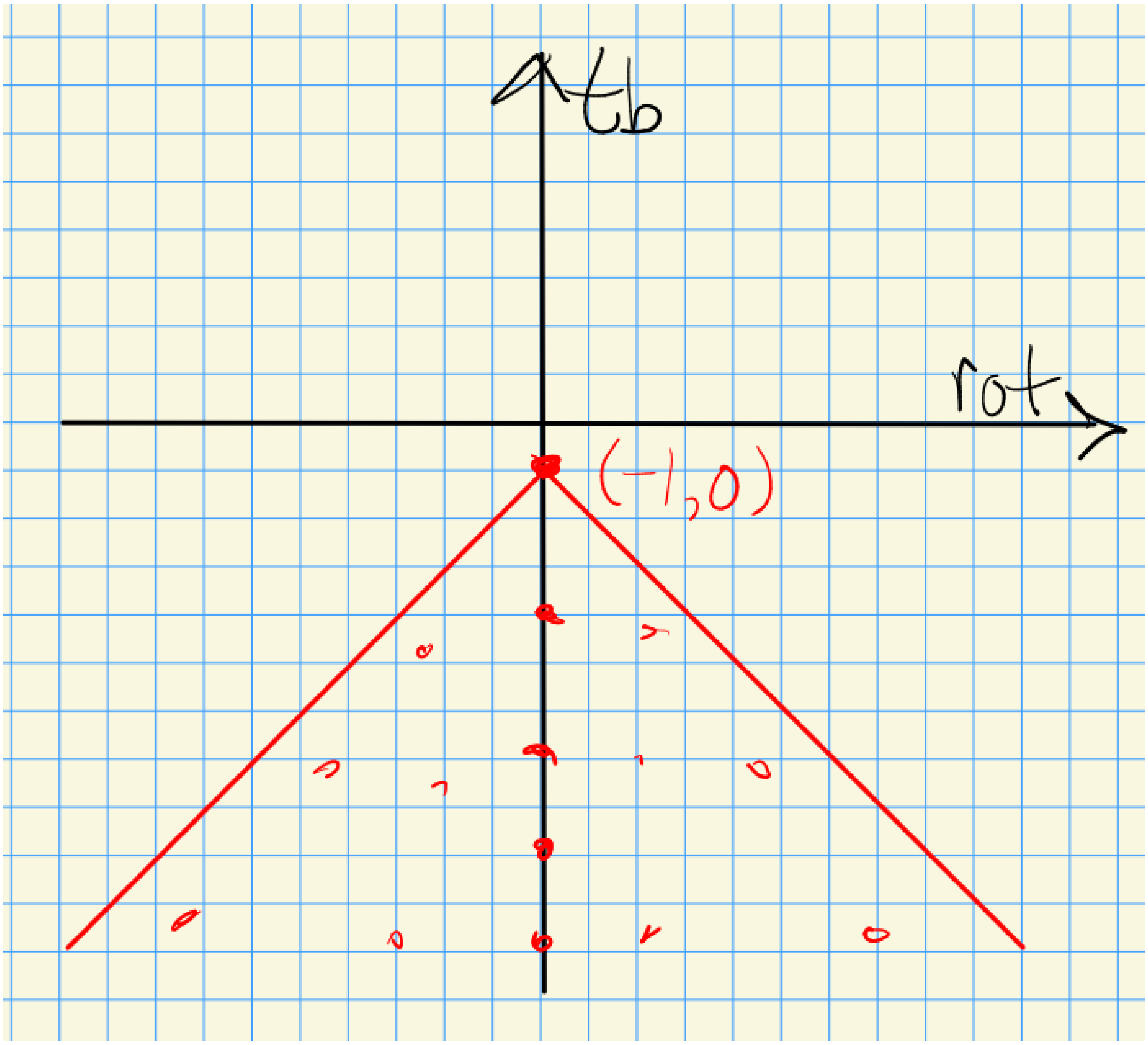
Remark 7.0.3: Geography problem: given a smooth knot K, which pairs $(t, r) \in \mathbb{Z}^2$ are realized as $(\mathrm{tb}(L), \mathrm{rot}(L))$ for L a Legendrian representative of K?

Botany problem: given $(t, r) \in \mathbb{Z}^2$, how many inequivalent L representing K realize $(t, r) = (\mathrm{tb}(L), \mathrm{rot}(L))$?

Example 7.0.4*(?):* For K the unknot:



So these numerical pairs fall into a cone:



Proposition 7.0.5 *(?)*.
For $L \subseteq \mathbb{R}^3$ a Legendrian knot,

$$\text{tb}(L) + \text{rot}(L) \equiv 1 \bmod 2.$$

Remark 7.0.6: Note that $\chi(S) \equiv 1 \bmod 2$ for S a Seifert surface.

Theorem 7.0.7 *(Bennequin-Thurston inequality)*.
For any Seifert surface S,

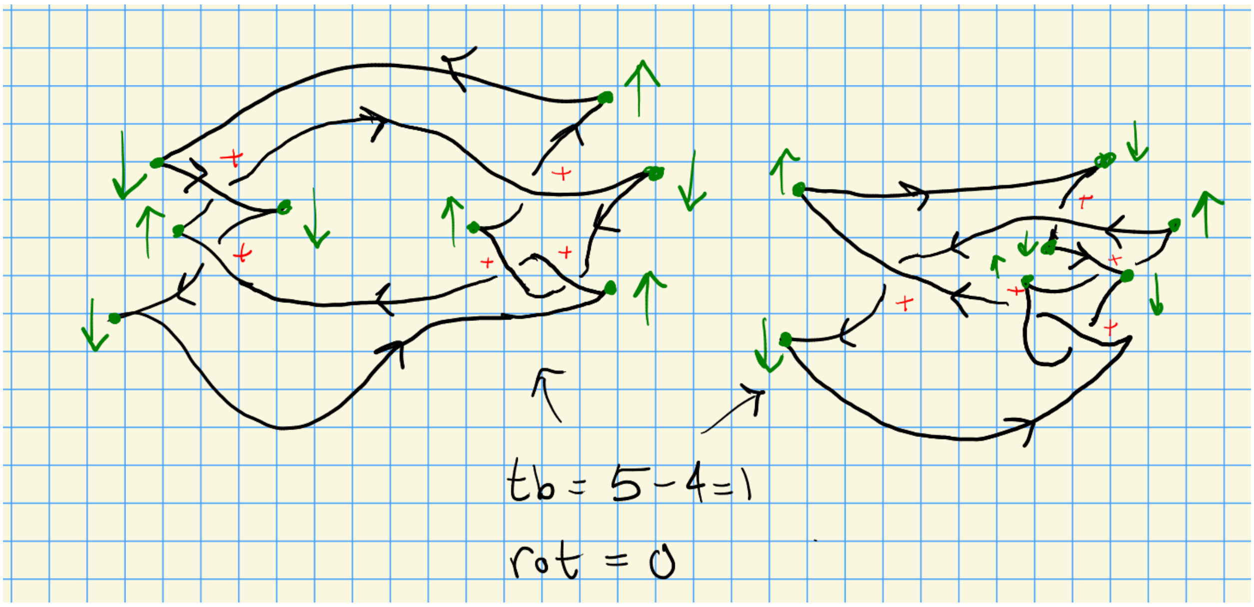
$$\text{tb}(L) + |\text{rot}(L)| \leq -\chi(S).$$

Remark 7.0.8: This solves the geography problem: this cone contains all of the possible pairs.

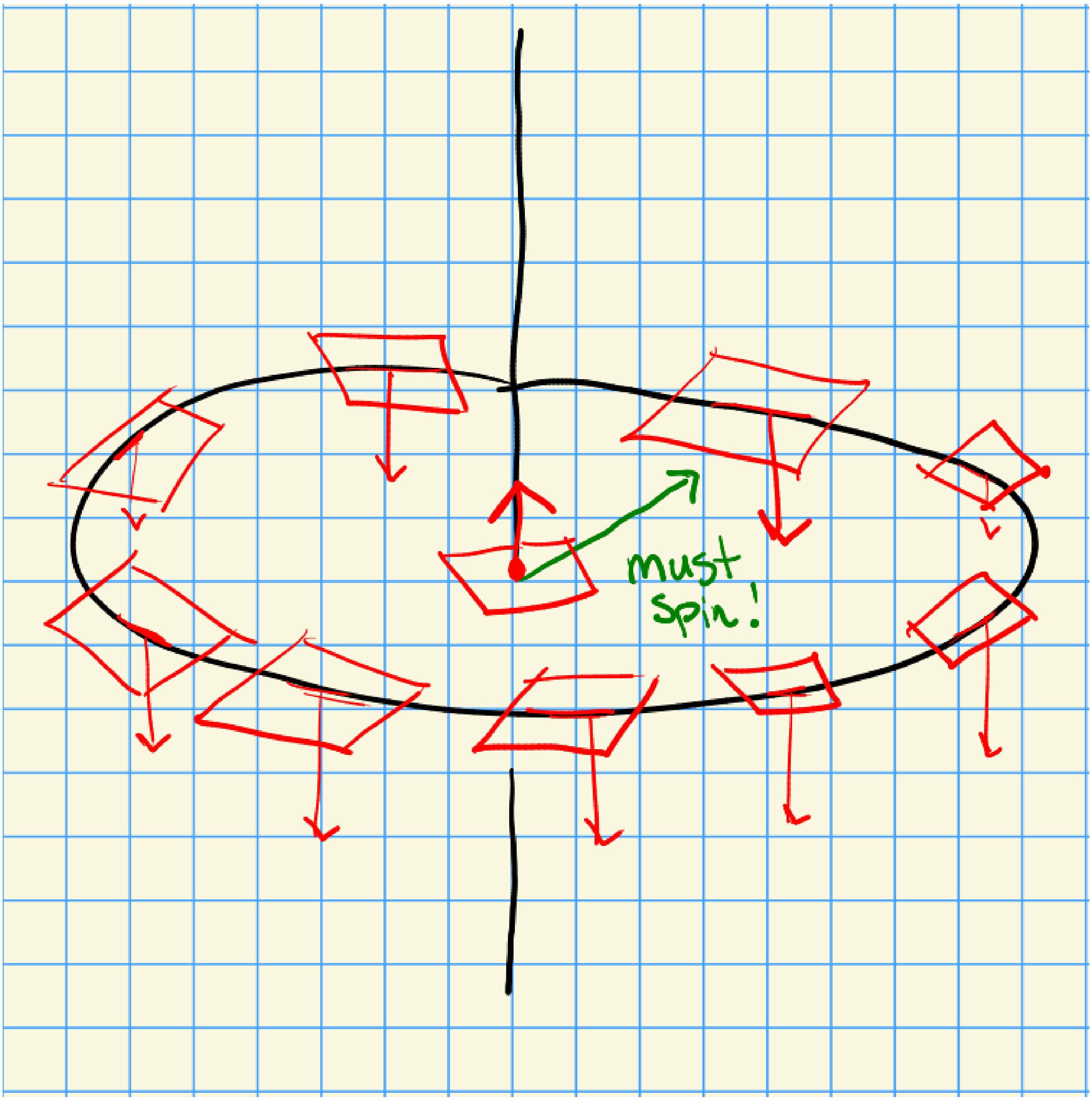
Theorem 7.0.9 ***(Eliashberg-Fraser).***
The unknot is **Legendrian simple**: if $\operatorname{tb}(L_1) = \operatorname{tb}(L_2)$ and $\operatorname{rot}(L_1) = \operatorname{rot}(L_2)$, then L_1 is isotopic to L_2.

Remark 7.0.10: This solves the botany problem: every red dot has exactly one representative.

Remark 7.0.11: Other knots are Legendrian simple, e.g. the trefoil. A theorem of Checkanov says the following 5_2 knots are not Legendrian isotopic:



Remark 7.0.12: This all depended on the standard contact form. Consider instead the overtwisted disc: take $\mathbb{R}^3$ with $\alpha = \cos(r)\,dz + \sin(r)\,d\theta$. Take the curve $[r, \theta, z] = \gamma(t) := [1, t, 0]$, a copy of S^1 in the x, y-plane. Then $\gamma' = [0, 1, 0]$, and at $\theta = \pi, \alpha = \cos(\pi)\,dz + \sin(\pi)\,d\theta = -\,dz$, but at $r = 0$ $\alpha = dz$, so traversing a ray from 0 to -1 in the x, y-plane forces the contact plane to flip:



One can check that tb is given my $\text{lk}(L, L') = 0$ where L' is a pushoff of L, and can be made totally disjoint from L in this case by moving in the z-plane.

Definition 7.0.13 (Overtwisted discs)
An **overtwisted disc** in (Y^3, ξ) that is locally contactomorphic to this local model. Y is **overtwisted** if it contains an overtwisted disc, and is tight otherwise.

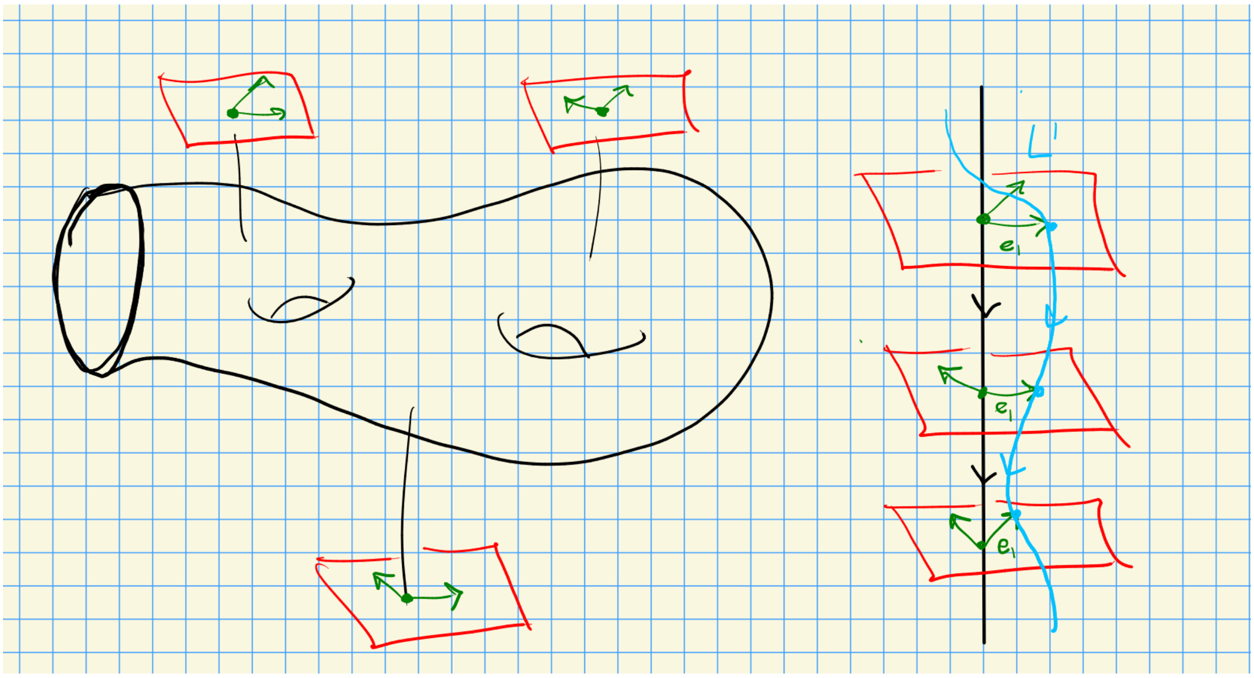
Theorem 7.0.14 ***(Bennequin).***
$(\mathbb{R}^3, \xi_{\text{std}})$ is a tight contact structure.

Theorem 7.0.15 ***(Eliashberg).***
For every closed oriented Y^3, every homotopy class of 2-plane fields on Y contains a unique (up to isotopy) overtwisted contact structure.

7.1 Transverse Knots

Definition 7.1.1 (Self-linking)
The **self-linking number** $\mathrm{sl}(T, S)$ of a transverse knot rel a Seifert surface S is $\mathrm{lk}(T, T')$ for T' a pushoff of T determined by a trivialization of $\xi|_S$.

Remark 7.1.2: In this case, ξ restricts to an $\mathbb{R}^2$ bundle over Σ, which is trivial since Σ is closed with boundary and $e(\xi) \in H^2(S) = 0$. To see this, use $H^2(S) \cong H_0(S, \partial S) = 0$ by Lefschetz duality. This yields a section of the frame bundle over S, which gives a pushoff direction along the first basis vector:

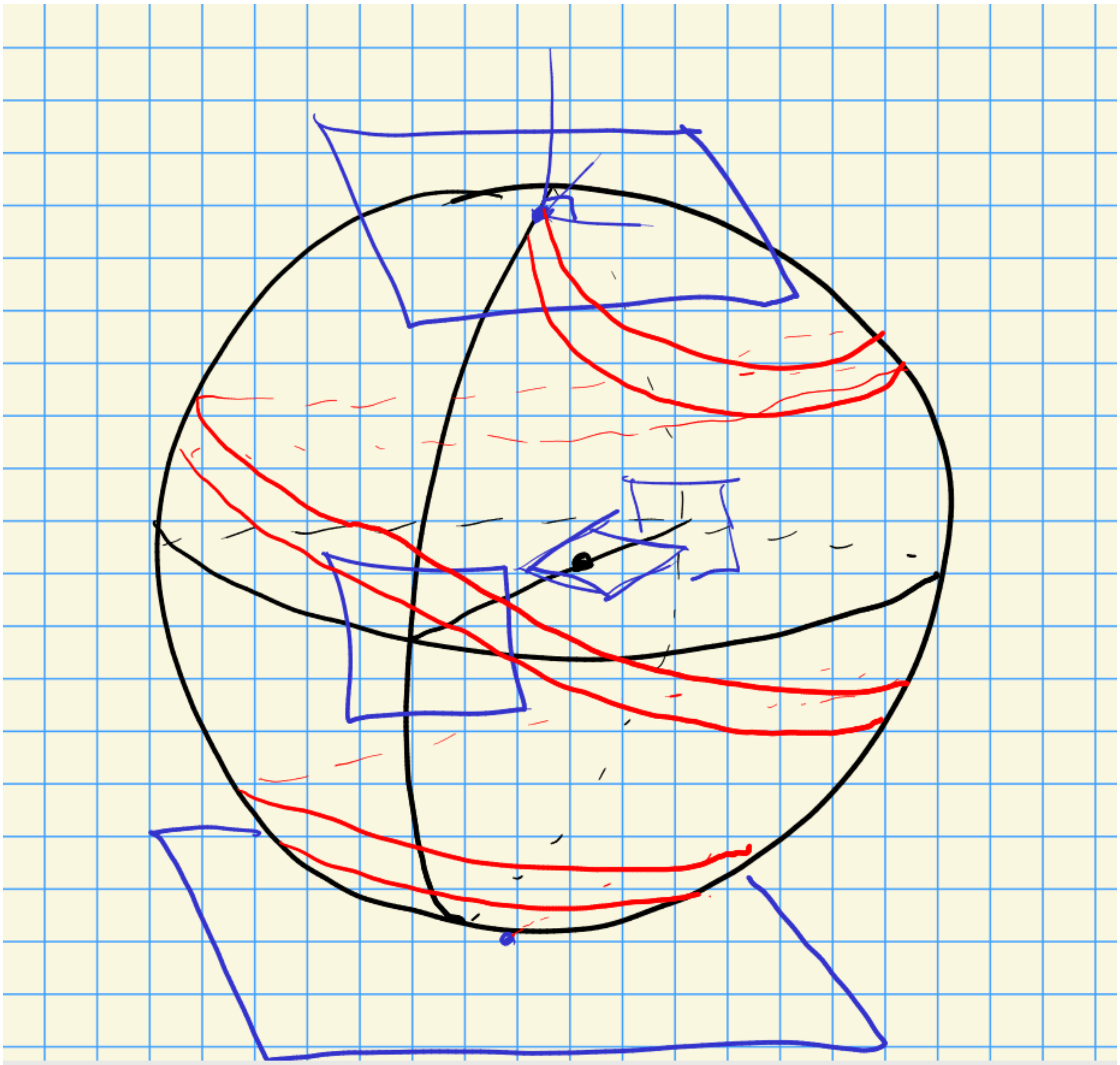


This turns out to be well-defined: it's independent of the surface S chosen and the trivialization of ξ. The difference of two trivializations gives a map $\pi_1(S) \to \mathbb{Z}$, which factors through $\pi_1(S)^{\mathrm{ab}} = H_1(S)$. The difference in surfaces is measured by $\langle e(S),\ \Sigma_1 \coprod_T \Sigma_2 \rangle$, which is a glued surface.

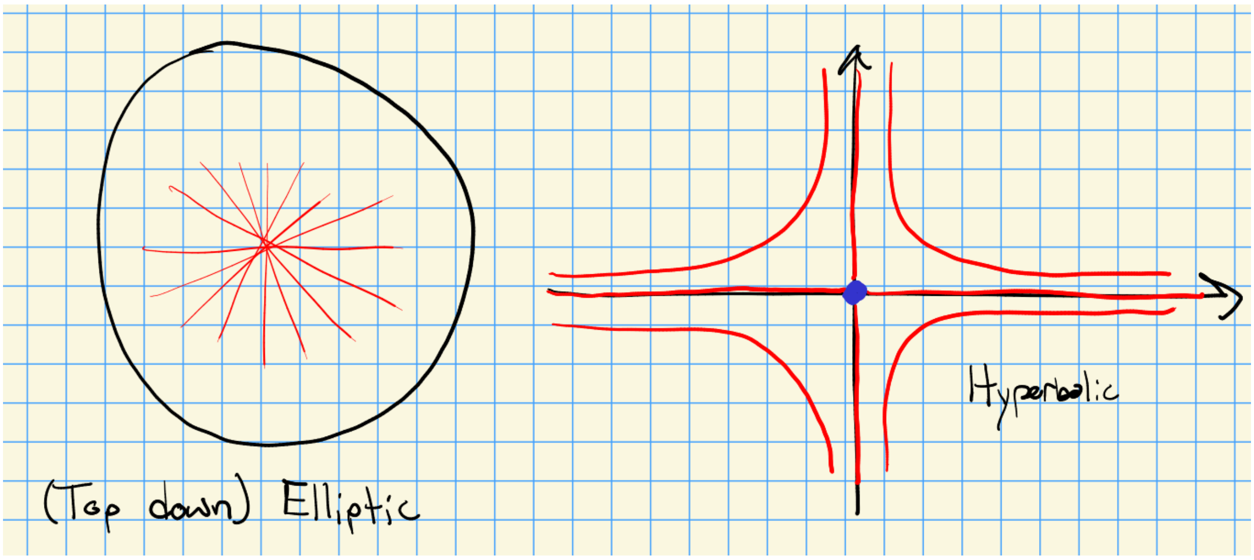
8 | Thursday, February 03

Remark 8.0.1: Last time: self-linking of transverse knots. Today: surfaces with transverse boundary. Let Σ be a surface embedded in (Y, ξ) with $\partial\Sigma$ transverse to ξ. Let F be the characteristic foliation, the singular foliation on Σ induced by $\xi|_\Sigma$. Equivalently, if $\xi = \ker \alpha$, consider the 1-form $\alpha|_\Sigma$. Generically, $\ker \alpha|_{\mathbf{T}\Sigma}$ is 1-dimensional except at finitely many points where $\alpha_p = 0$, i.e. ξ is tangent to Σ. This line field integrates to a singular foliation. Recall that $\mathrm{sl}(L)$ is the self-linking number.

Example 8.0.2 *(?):* Take $\alpha = dz + x\,dy - y\,dx$ and $\Sigma = S^2$, then the singular foliation is given by



Remark 8.0.3: Two possible types of singularities, the local models:



There are also two numerical invariants:

- $e_{\pm}$: the number of positive (resp. negative) elliptics
- $h_{\pm}$: the number of positive (resp. negative) hyperbolics

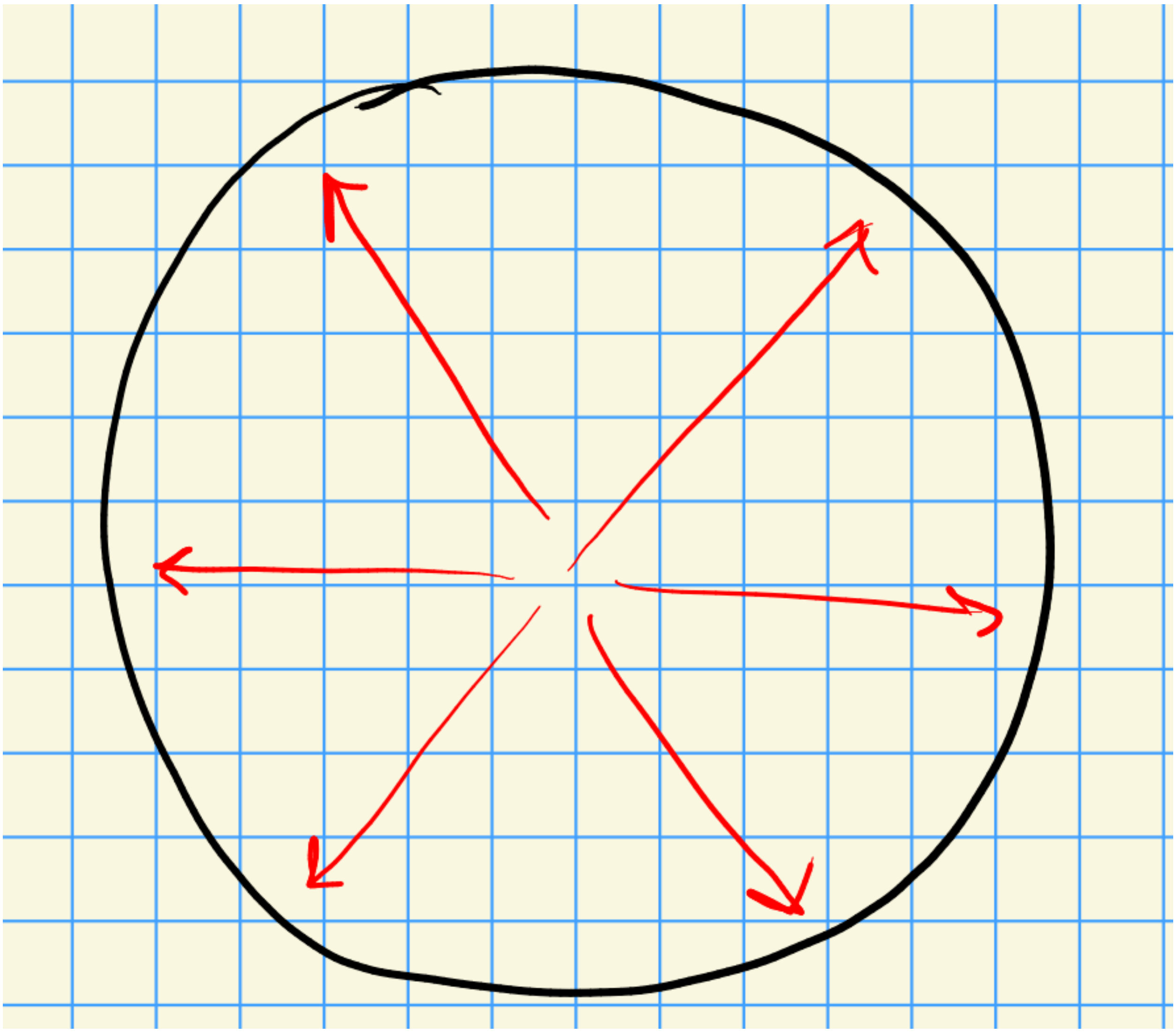
A theorem

$$\langle c(\Sigma),\ \Sigma\rangle = (e_+ - h_+) - (e_- - h_-).$$

If Σ is transverse, $\mathrm{sl}(\partial\Sigma, \Sigma) = -(e_+ - h_+) + (e_- - h_-)$.

8.1 Local Model 1: Elliptic

Remark 8.1.1: σ is the x, y-plane and $\xi = \ker(\,dz + x\,dy - y\,dx)$ with $\alpha|_\Sigma = x\,dy - y\,dx$. Set $V : x\partial_x + y\partial_y$ and $L' = \langle x\partial_y - y\partial_x\rangle$, and $\alpha(i) = x^2 + y^2 = 1 > 0$.



Here sl = 1. To compute sl:

- Trivialize $\xi|_\Sigma$ to get $\tau = \langle e_1, e_2 \rangle$ a fiberwise basis for ξ.
- Let $\tilde{L}$ be a pushoff in the e_1 direction.
- Compute $\mathrm{sl} = \mathrm{lk}(L, \tilde{L})$.

Set

- $e_1 = \partial_x + y\partial_z$
- $e_2 = \partial_y - x\partial_z$
- $\rho = x\partial_x + y\partial_y$
- $\theta = x\partial_y - y\partial_x$.

Then

$$x\rho - y\theta = x(x\partial_x + y\partial_y) - y(-y\partial_x + x\partial_y) = (x^2 + y^2)\, dx.$$

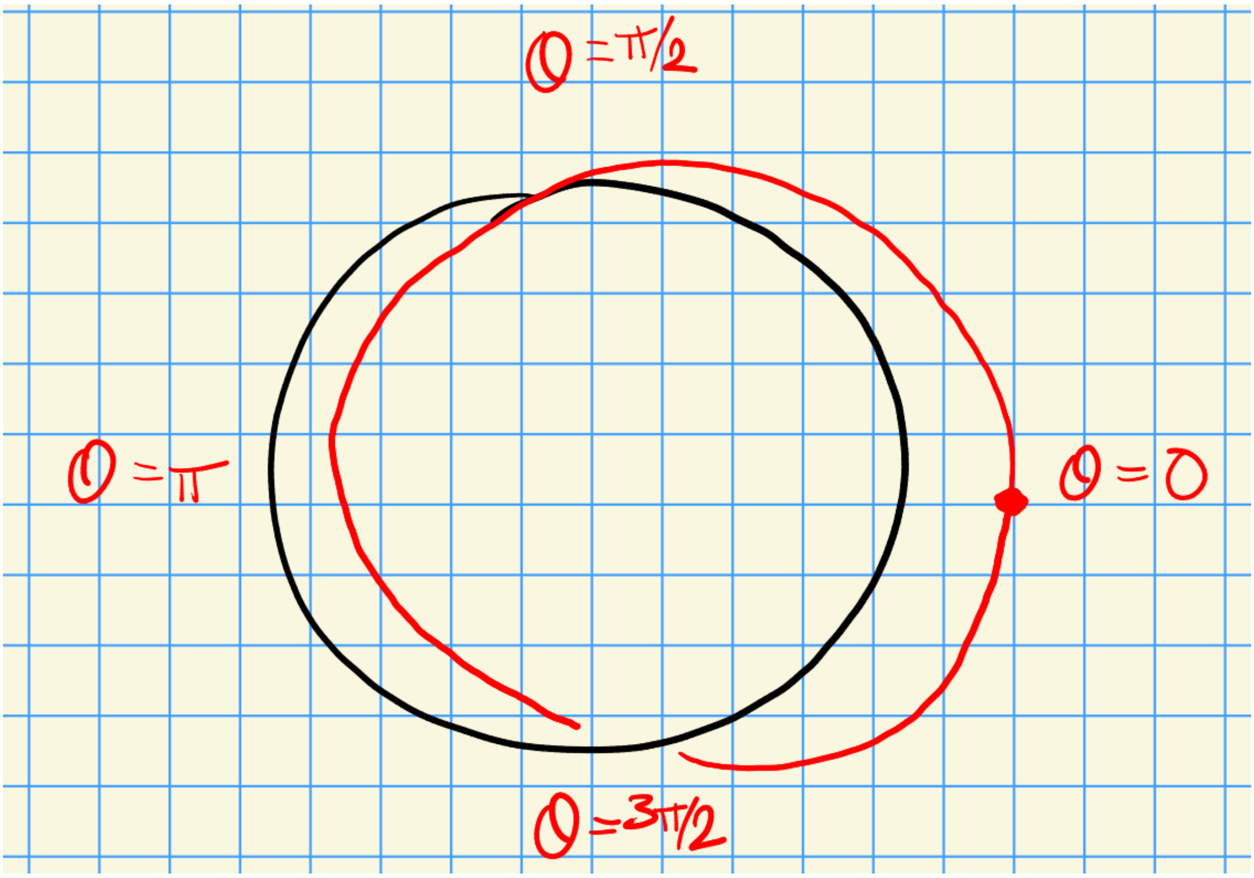
Then

- $c_1 = x\rho - y\theta + y\partial_z$
- $\overline{c_1} = x\rho + y\partial_z = \cos(\rho) + \sin(\theta)\partial_z$.

Example:

- $\theta = 0 \implies e_1 = \rho$
- $\theta = \pi/4 \implies e_1 = \frac{\sqrt{2}}{2}(\rho + \partial_z)$
- $\theta = \pi/2 \implies e_1 = \partial_z$

So here $\operatorname{lk}(U, \tilde{U}) = -1$:



8.2 Local Model 2: Hyperbolic

Remark 8.2.1: Here ξ is the x, y-plane, so $\xi = \ker(dz + 2x\,dy + y\,dx)$ with $\alpha|_\Sigma = 2x\,dy + y\,dx$ and $V = y\partial_y + dx\partial_x \in \ker(\alpha|_\Sigma)$.

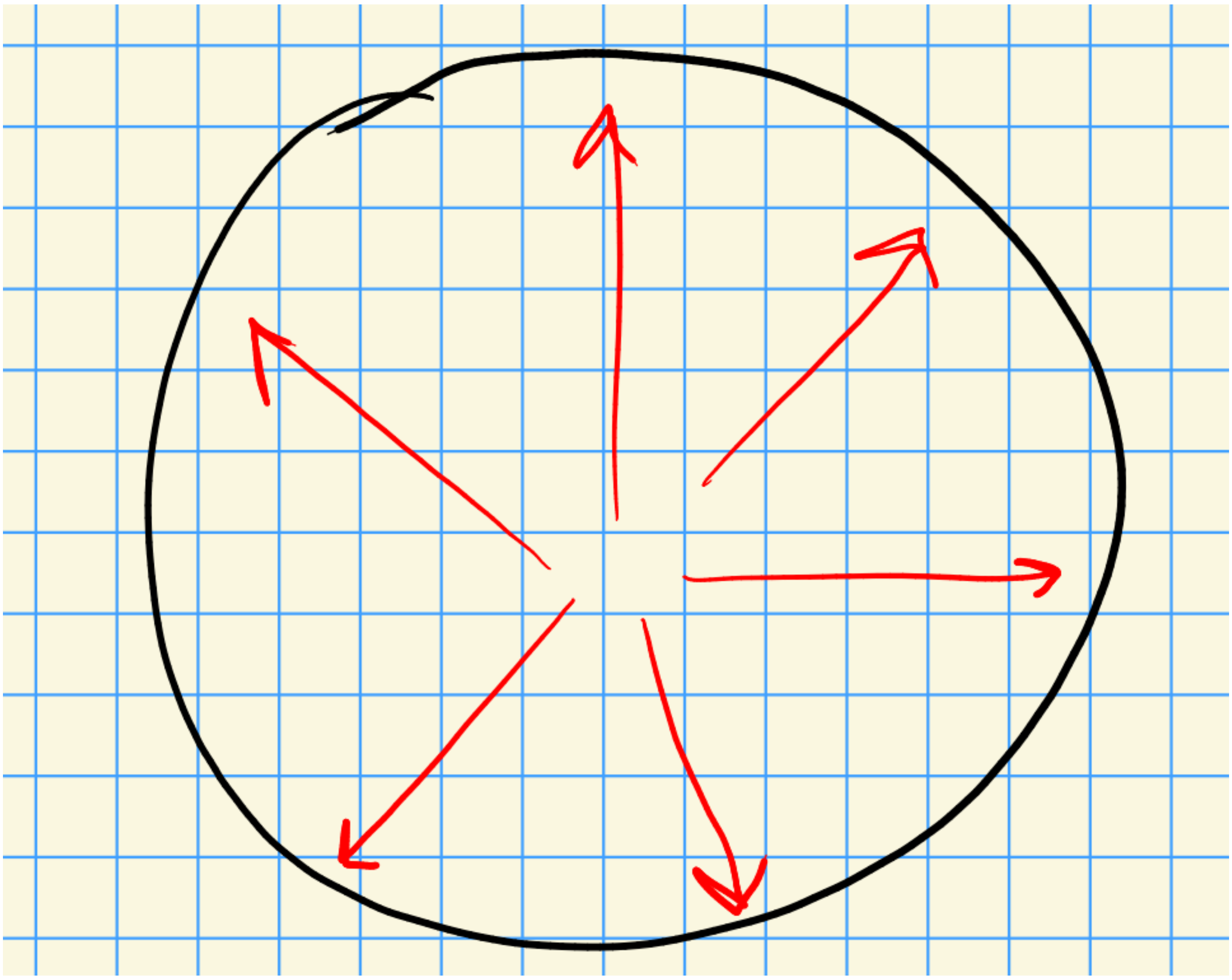
Remark 8.2.2: The Euler class of a real vector bundle $E \xrightarrow{\pi} X$ is the obstruction to finding a nonvanishing section s of E, given by $e(E) \in H^k(X)$. It is Poincare dual to $[s^{-1}(0)] \in H_{n-k}(X, \partial X)$. For the tangent bundle, $e(\mathbf{T}X) \in H^n(X)$, and

$$\langle e(\mathbf{T}X),\ [X] \rangle = \chi(X).$$

Since a section of $\mathbf{T}X$ is a vector field, $e(\mathbf{T}X)$ is an obstruction to finding a nonvanishing vector field. If $\partial X \neq \emptyset$ and t is a section of $E|_{\partial X}$, there is a relative Euler class $e(E,t) \in H^k(X, \partial X) \cong H_{n-k}(X)$. Similarly,

$$\langle e(\mathbf{T}X, t),\ [X] \rangle = \chi(X).$$

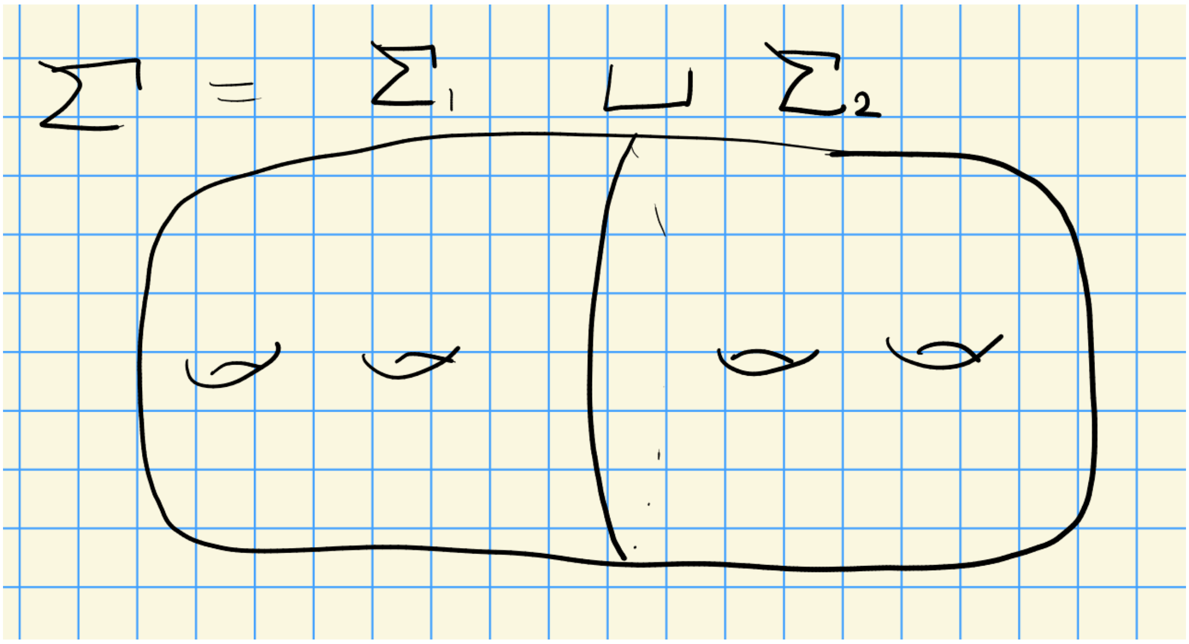
Example 8.2.3 *(?)***:** Note $\chi(\mathbb{D}) = 1$, so any vector field has a singularity?



Proposition 8.2.4 *(?)***.**
The total class is the sum of the relative obstructions. If $\sigma = \Sigma_1 \coprod_{\partial} \Sigma_2$ and τ is a nonvanishing

section of $\Sigma|_{\partial\Sigma_1} = \Sigma|_{\partial\Sigma_2}$, then

$$c(E) = e(E|_{\Sigma_1}, \tau) + c(E|_{\Sigma_2}, \tau).$$



8.3 More Contact Geometry

Remark 8.3.1: Let Σ have transverse boundary with characteristic foliation F, and let V be the vector field directing F, so $V \in \xi \cap \mathbf{T}\Sigma$. We can assume V is outward-pointing along $\partial\Sigma$.

Check that

- $\chi(\Sigma) = e(\mathbf{T}\Sigma, V) \in H^2(\Sigma, \partial\Sigma) \cong H_0(\Sigma)$
- $\mathrm{sl}(\partial\Sigma, \Sigma) = e(\xi, V) \in H^2(\Sigma, \partial\Sigma)$

Fact 8.3.2

- $e_+ + e_-$ correspond to $+1$ in $e(\mathbf{T}\Sigma, V)$,
- h_+, h_- correspond to -1 in $e(\mathbf{T}\Sigma, V)$.

Proof: near a zero, V determines a map $S^1 \to S^1$ and the contribution to e is the degree of this map.

- e_+ contributes -1 to $e(\xi, V)$, by the same computation of $\mathrm{sl}(U)$ for U the unknot.

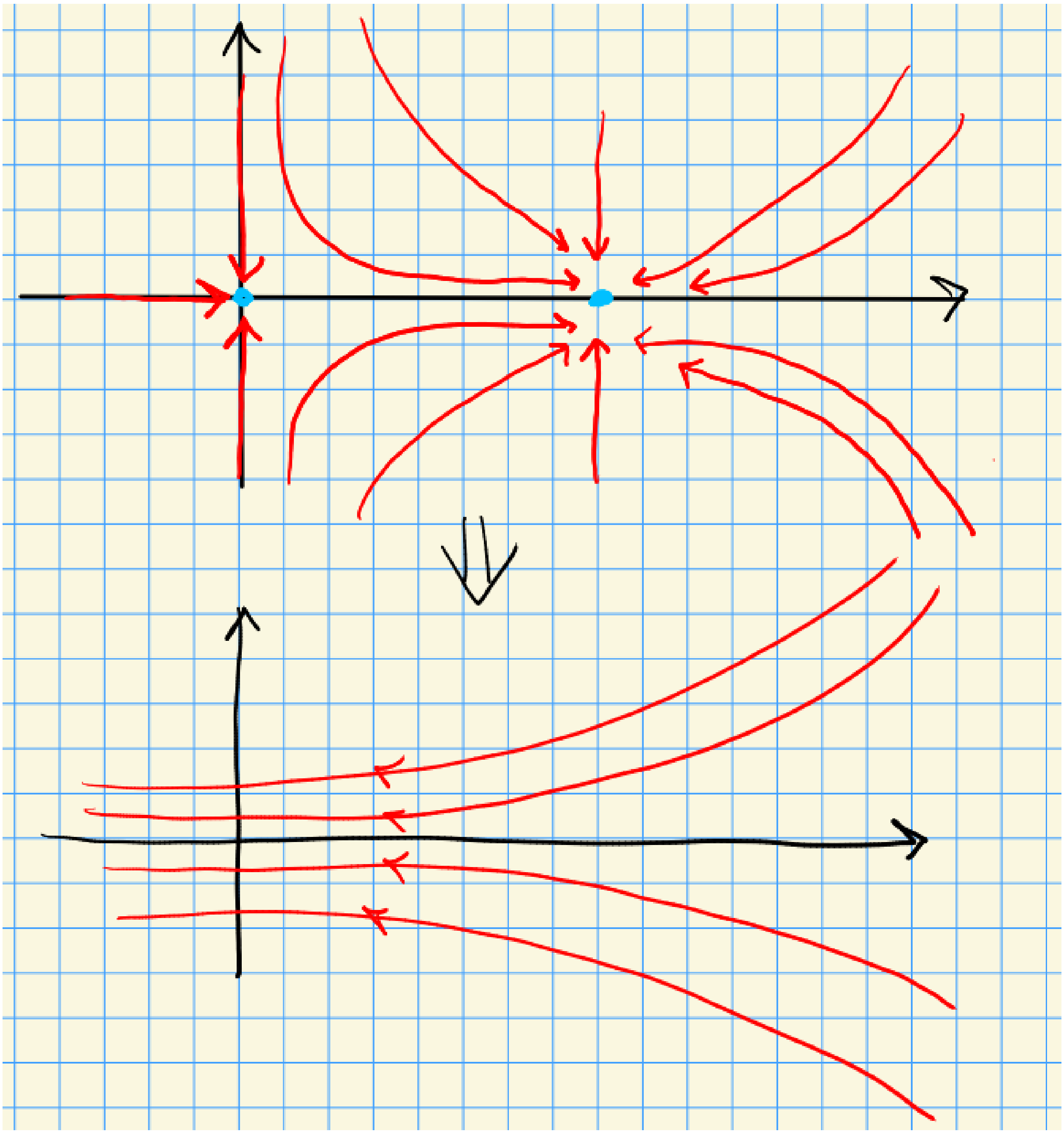
- e_- contributes $-(-1) = +1$ to $e(\xi, V)$.
- h_+ contributes $+1$ to $e(\xi, V)$
- h_- contributes $-(+1) = -1$ to $e(\xi, V)$.

Proof: exercise.

Remark 8.3.3: Bennequin inequality:

$$\mathrm{sl}(T, \Sigma) \leq -\chi(\Sigma) \implies e_+ + h_+ + e_- + h_- \leq -(e_+ + e_- - h_+ - h_-) \iff e_- \leq h_-.$$

Try to cancel in pairs:



The inequality follows if we can cancel every e_- with some h_-.

9 | Tuesday, February 08

Remark 9.0.1: Topics for talks:

- Thom-Pontryagin

- Brieskorn spheres
- Milnor fibrations
- Lens spaces

Theorem 9.0.2 *(?).*
Every closed oriented 3-manifold Y admits a (positive) contact form.

Remark 9.0.3: Three proofs:

- Lickorish-Wallace, using that Y is Dehn surgery on a link in S^3,
- Birman-Hildon, using that Y is a branched cover of S^3,
- Alexander, using that Y admits an open book decomposition.

Remark 9.0.4: Dehn surgery for slope p/q: for $K \hookrightarrow S^3$, cut out $\nu(K) \cong S^1 \times \mathbb{D}$ and re-glue by a map $\partial(S^1 \times \mathbb{D}) \to \partial\nu(K)$ such that $[\{0\} \times \partial\mathbb{D}] = p[m] + q[\ell] \in H^1(\partial\nu(K))$. Use that $\nu(K) \cong S^1 \times \mathbb{D}$ and $\partial\nu(K) \cong S^1 \times S^1 = T^2$. Idea: wrapped p times longitudinally, q times around the meridian.

Remark 9.0.5: Recall:

- Every knot K can be C^0 approximated by a transverse knot
- Every link L can be C^0 approximated by a transverse link
- Neighborhood theorem: for every transverse knot K, there is a $w(K)$ and a contactomorphism to a standard model: $S^1 \times \mathbb{D}$ in coordinates (φ, r, θ) with $0 \leq r \leq \delta$ and $\alpha = d\varphi + r^2\, d\theta$. Re-gluing corresponds to the map $[0, \delta, \theta] \mapsto [q\theta, \delta, p\theta]$.

$$\left[0, \delta, \bar{\theta}\right] \mapsto \left[q\bar{\theta}, \delta, p\bar{\theta}\right]$$
$$\left[\overline{\pi}, \bar{r}, \bar{\theta}\right] \mapsto [\varphi, r, \theta].$$

If p, q are coprime there exist m, n with $pm - qn = 1$. So define

$$\psi : \left[\overline{\pi}, \bar{r}, \bar{\theta}\right] \mapsto [\varphi, r, \theta],$$

so

$$\psi^*(\alpha) = d(\alpha\bar{\theta} + m\overline{\varphi}) + r^2 d(p\bar{\theta} + n\overline{\varphi}) = (q + r^2 p)d\bar{\theta} + (m + r^2 n)d\overline{\varphi}.$$

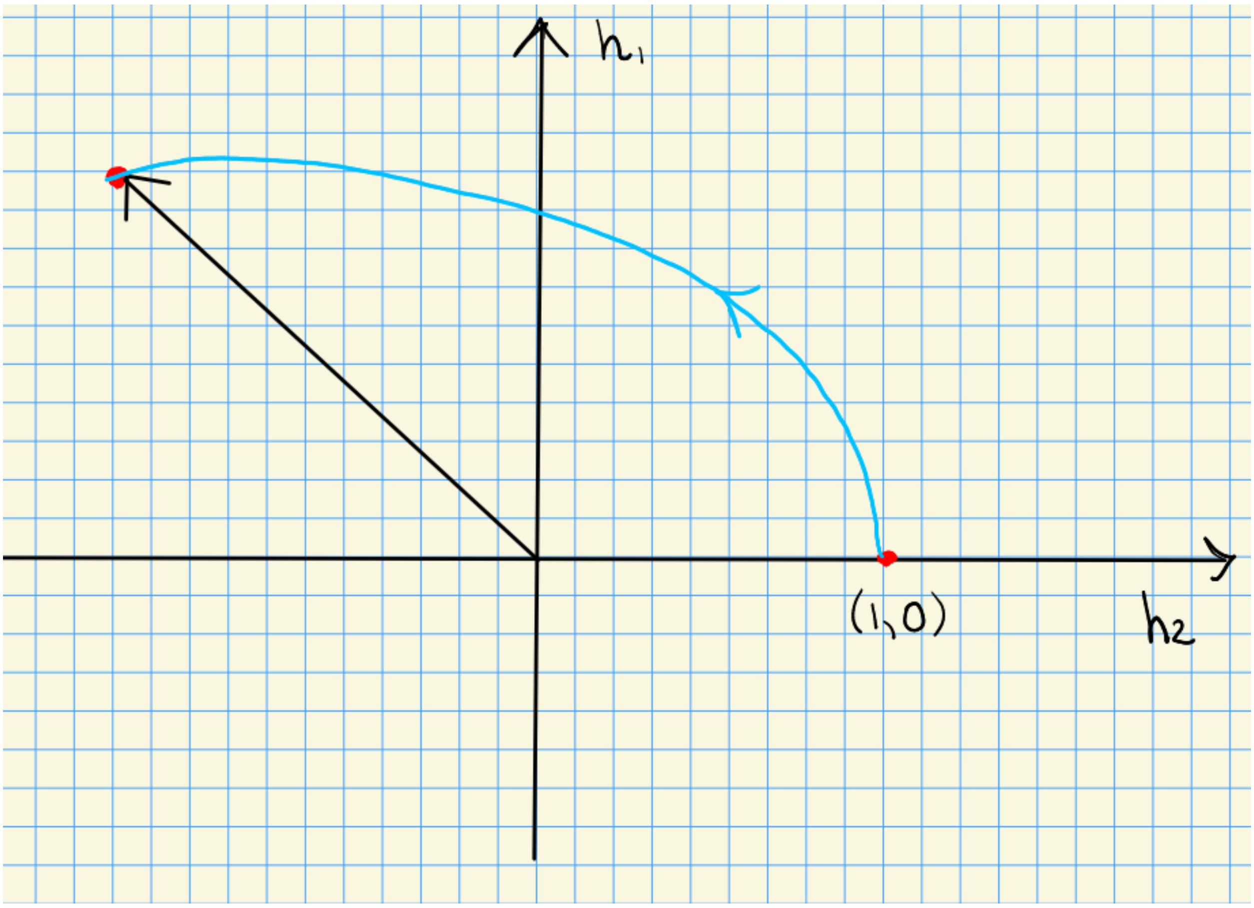
We want $\alpha = h_1(r)d\bar{\theta} + h_2(r)d\bar{\theta}$ to be contact and satisfy $(h_1, h_2) = (r^2, 1)$ near $r = 0$ and $(q + r^2 p, m + r^2 n)$ near $r = \delta$. This requires

$$d\alpha = h_1'\, dr \wedge d\overline{\varphi} + h_2'\, dr \wedge d\bar{\theta} = (h_2 h_1' - h_1 h_2')dr \wedge d\bar{\theta} \wedge d\overline{\varphi},$$

which happens iff

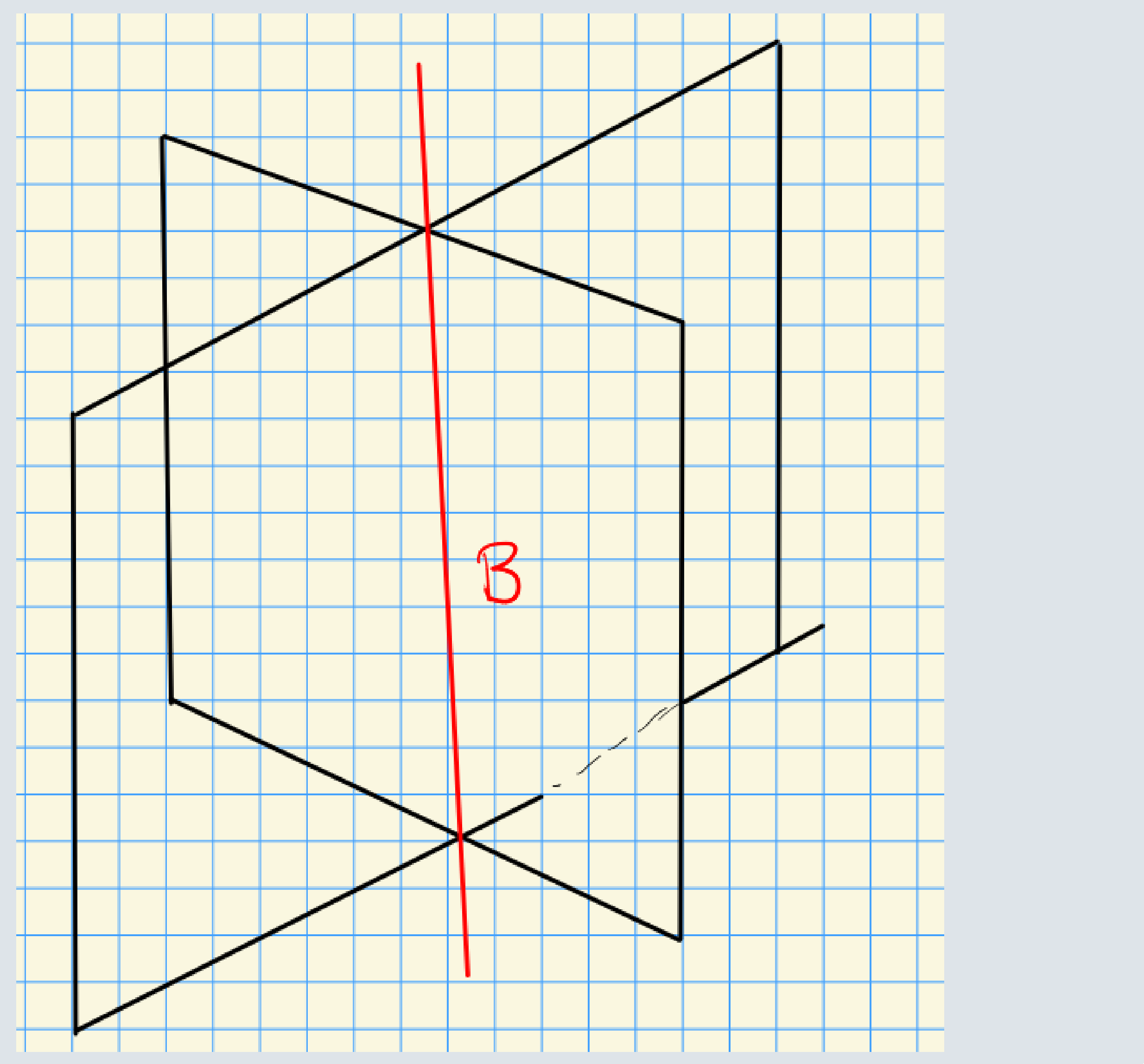
$$\det \begin{bmatrix} h_2 & h_2' \\ h_1 & h_1' \end{bmatrix} > 0.$$

Think of $[h_2, h_1]$ as a path with tangent vector $[h_2', h_1']$. This requires moving counterclockwise.



Definition 9.0.6 (?)
An **open book decomposition** of Y is a pair (B, π) where

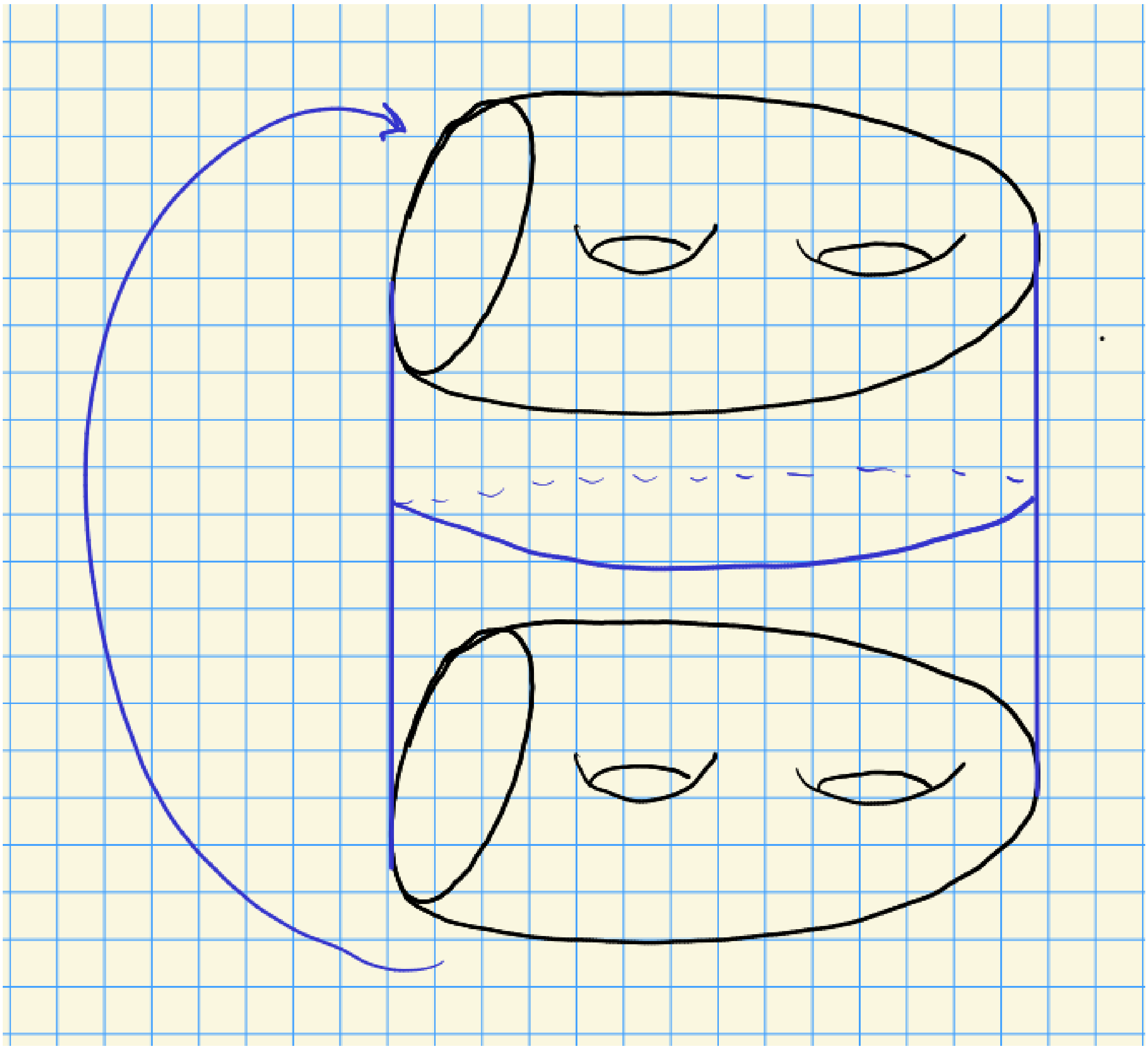
- B is a link in Y, called the **binding**
- $\pi : Y \setminus B \to S^1$ is a locally trivial fibration of relatively compact fibers **pages**



Remark 9.0.7: An open book decomposition is determined by its monodromy map $\varphi : \Sigma_0 \to \Sigma_0$, which determines a class $[\varphi] \in \mathrm{MCG}(\Sigma_0)$. Form

$$Y \setminus \nu(B) \cong \frac{\Sigma \times I}{\varphi(x) \times \{0\} \sim x \times \{1\}},$$

which is a glued cylinder:



Definition 9.0.8 (Open book decompositions supporting a contact structure)
An open book decomposition **supports** a contact structure ξ iff there exists a contact form α such that $d\alpha$ is an area form on each page and B is a transverse link in (B, ξ).

Theorem 9.0.9 ***(Thurston-Winkelnkemper).***
Every open book decomposition admits a contact structure.

Theorem 9.0.10 ***(Giroux).***
Every (Y^3, ξ) with Y closed has a supporting open book decomposition.

Proposition 9.0.11 ***(?).***
If an open book decomposition supports ξ_1 and ξ_2, then ξ_1 is isotopic to ξ_2.

Proof (?).
Two steps:

- Form a mapping cylinder of the monodromy map φ,
- Extend over the binding, using the same idea as in Dehn surgery.

Choose an area form ω on Σ and a primitive β with $d\beta = \omega$. Let $\beta_1 \coloneqq \varphi^* \beta$ and $\beta_0 = \beta$, then set

$$\beta_t = t\beta_1 + (1-t)\beta_0.$$

This yields a 1-form on $\Sigma \times I$ that extends to the mapping cylinder. Moreover $d\beta_t = td\beta_1 + (1-t)d\beta_0$ is an area form on $\sigma \times \{t\}$ and $\alpha = dt + \varepsilon\beta_t$ is a contact form for small $\varepsilon > 0$. Then $d\alpha = \varepsilon d\beta_t + \varepsilon\, dt \wedge \dot{\beta}_t$ and $\alpha \wedge d\alpha = \varepsilon dt \wedge d\beta_t + O(\varepsilon^2)$.

■

10 | Tuesday, February 15

Missed due to orthodontic appointment! Please send me notes. :)

11 | Thursday, February 17

Remark 11.0.1: Let $\Sigma \subseteq (Y^3, \xi)$.

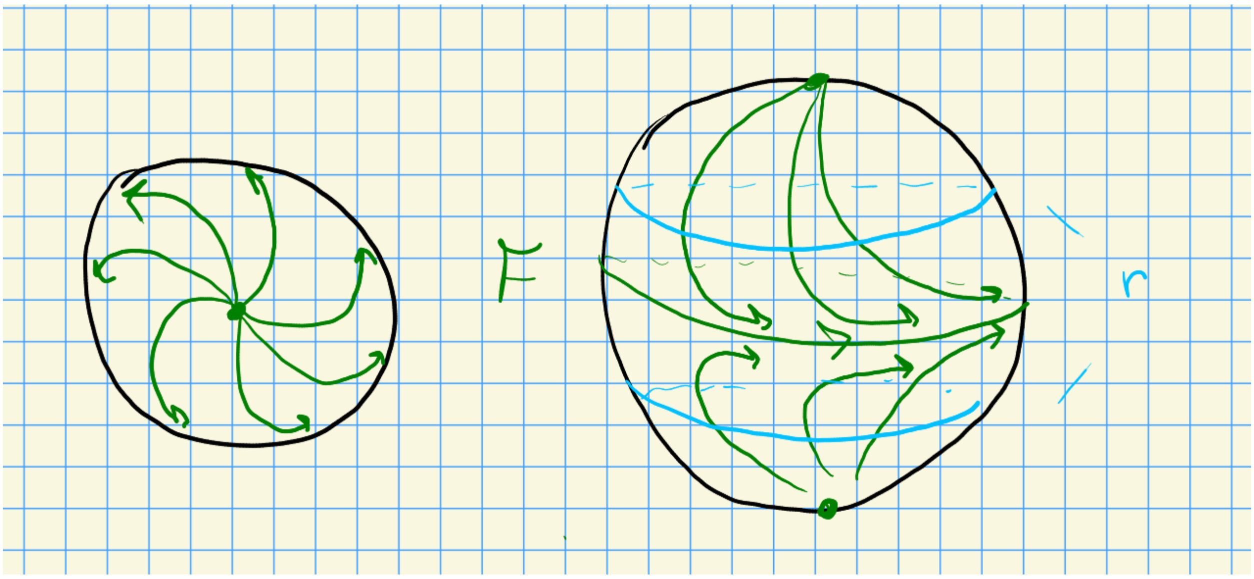
- Characteristic foliation: $F = \xi \cap \mathbf{T}\Sigma$, complicated but necessary
- Dividing set: a multicurve, simpler

Theorem 11.0.2 *(?).*
If Σ is convex with a dividing set Γ and F is any foliation divided by Γ, there is a C^0-small isotopy φ_t wt

- $\varphi_0(\Sigma) = \Sigma, \varphi_t(\Gamma) = \Gamma$
- $\varphi_t(\Sigma)$ is convex for all $t \in [0,1]$
- The characteristic foliation of $\varphi_1(\Sigma)$ is F.

Remark 11.0.3: Idea: dividing sets give ways to detect overtwisted contact structures.

Remark 11.0.4: If $\Sigma = S^2$ and $\sharp\Gamma \geq 2$, then (Y, ξ) is overtwisted. Recall that an overtwisted disc is an embedded D^2 with Legendrian boundary such that $\mathrm{tb}(\partial D) = 0$ and $\mathrm{tw}(\xi, \partial D)$.



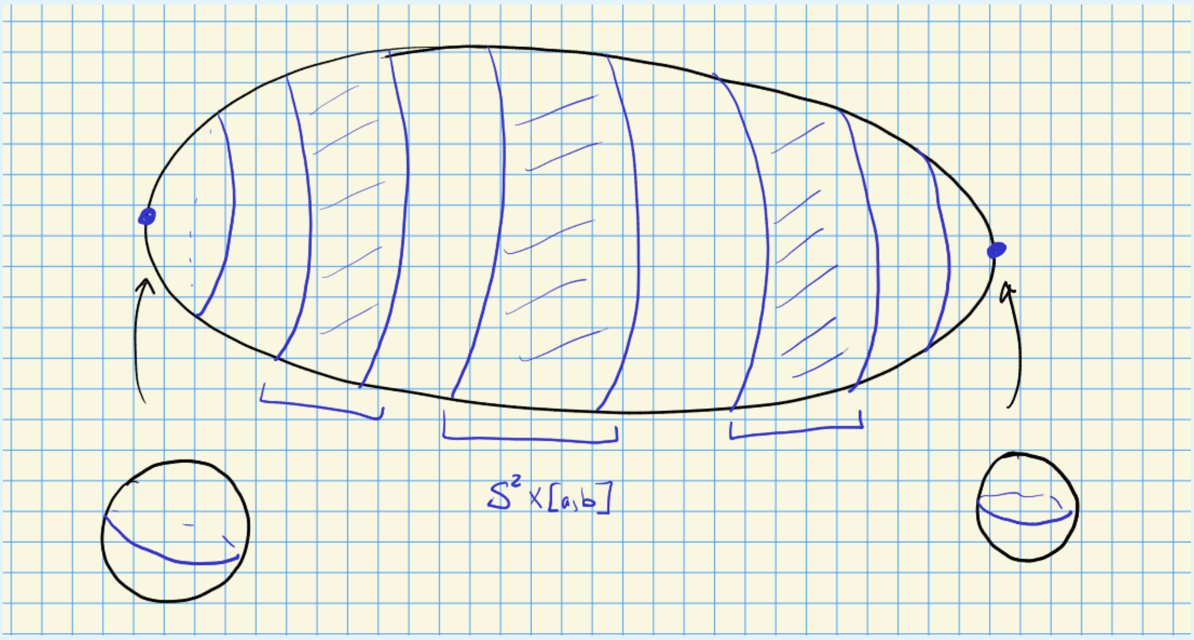
Spheres can have exactly one dividing component.

Exercise 11.0.5 (?)
Generalize to an arbitrary number of components $\sharp\Gamma = n$.

Remark 11.0.6: Same if $\Sigma \neq S^2$ and Γ contains a contractible curve. Contrapositively, if (Y, ξ) is tight, then either

- $\Sigma = S^2$ and Γ is connected, or
- $\Sigma \neq S^2$ and Γ has no contractible components.

Exercise 11.0.7 (?)
Consider tight contact structures on S^3. Choose Darboux B^3 neighborhoods at the ends, and note the interior is $S^2 \times [0, 1]$:



The $S^3 \times \{t_0\}$ slices can be perturbed to be complex. So there is only one tight contact structure on S^3.

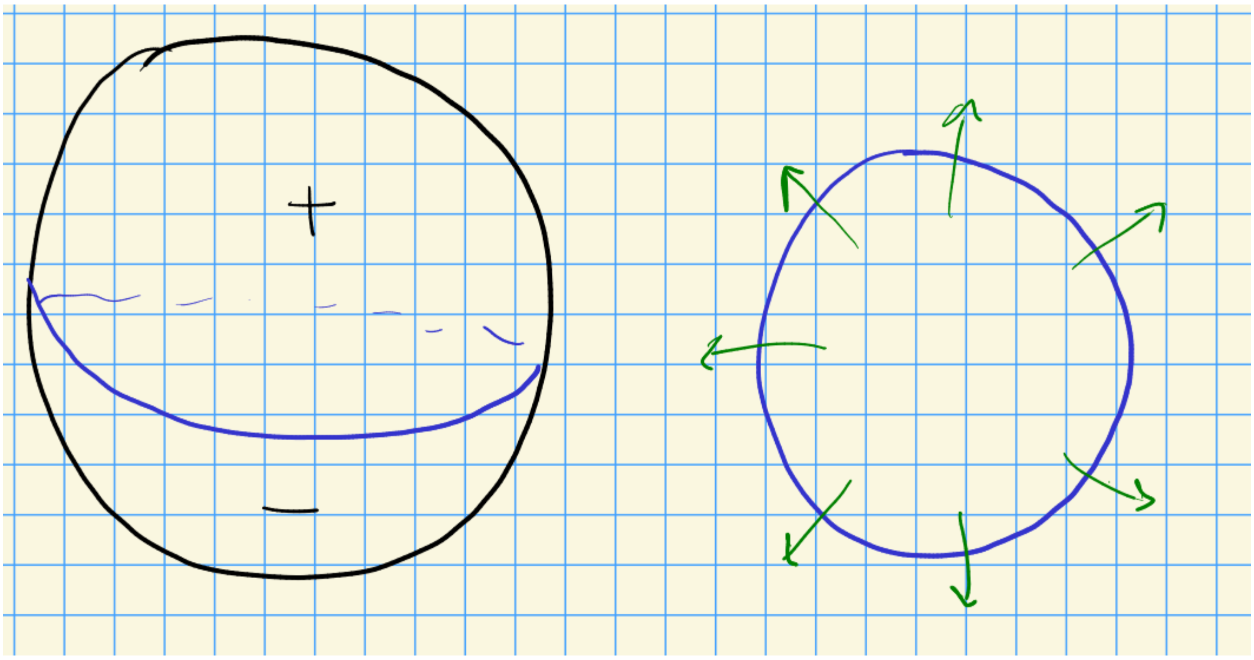
Remark 11.0.8: What can F look like on an S^2 in a tight (Y, ξ)? F can be perturbed to be Morse-Smale.

- There are a finite number of elliptic/hyperbolic singularities
- There are nondegenerate periodic orbits, either attracting or repelling
- There are no saddle-saddle arcs
- The limit sets are singularities or periodic orbits

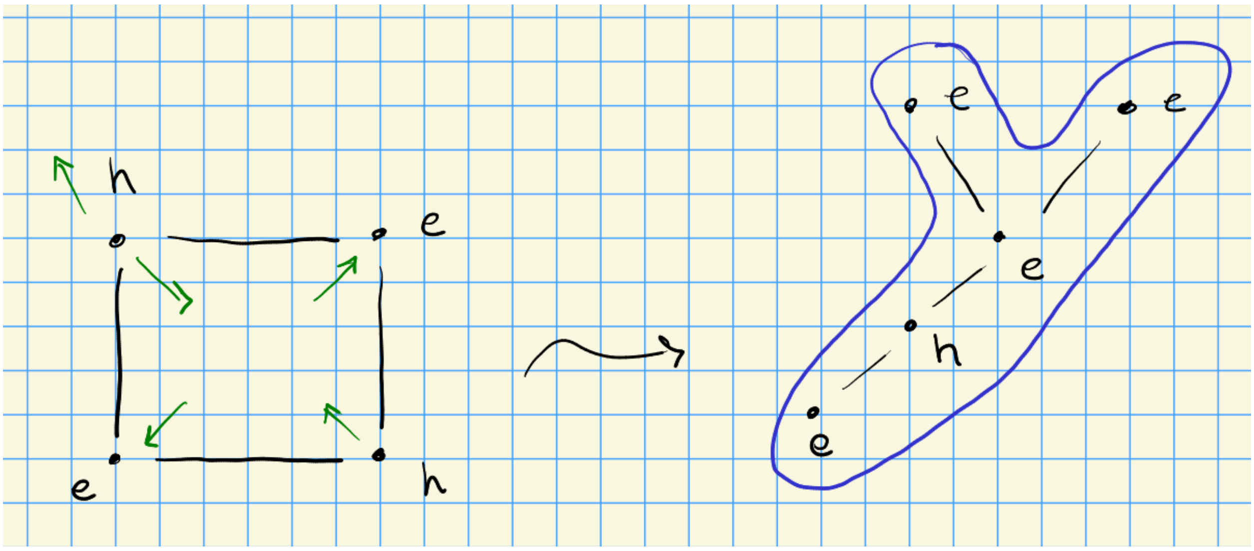
Dimension 3: strange attractors! Two types of limit sets:

- ω limit sets: $x \in Y$ where there exists a sequence $\{t_1 < \cdots\}$ with $\varphi(t_k) \to x$.
- α limit sets: $x \in Y$ where there exists a sequence $\{t_1 > \cdots\}$ with $\varphi(t_k) \to x$.

Remark 11.0.9: For S^2, take S^+ with an outward pointing vector field.



There are no periodic orbits since (Y, ξ) is tight. The only limit sets are singular points. $\chi(D) = 1 = \sharp e - \sharp h$. Stable manifold of h: Stab_h are $x \in D^2$ such that there exists a flow like with $\varphi(0) = x$ and $\varphi(t) \to h$ Form a 1-complex $\bigcup_h \mathrm{cl}_X(\mathrm{Stab}_h)$ – this contains no cycles, thus this is a tree, and the dividing set is a neighborhood of the tree.



Proposition 11.0.10 *(?)*.
If F on Σ is Morse-Smale, then it admits dividing curves.

Proof (?).
Let $G = \bigcup_h \mathrm{cl}(\mathrm{Stab}_h) \cup \bigcup_e e_t$ along with all of the repelling periodic orbits. Then $\Gamma = \partial\nu(G)$ divides F.

■

Theorem 11.0.11 ***(?).***
If Σ is orientable, then there is a C^∞ small perturbation of F such that it is Morse-Smale.

Proposition 11.0.12 ***(?).***
Every oriented $\Sigma \subseteq (Y, \xi)$ can be perturbed to be convex.

Proof (?).
Near Σ, $\alpha = \beta_t + \alpha_t\, dt$ and β_0 define F. By Peixoto there exists $\tilde{\beta}_t$ such that$\tilde{\beta}_t$ defines a Morse-Smale F. For $\left\|\beta - \tilde{\beta}\right\|_{C^\infty} \ll \varepsilon$, $\tilde{\alpha} = \tilde{\beta}_t + \alpha_t\, dt$ is contact. Then $\alpha_s = s\tilde{\alpha} + (1-s)\alpha$ is a path of contact forms, so by Gray stability there is an isotopy φ_s such that $\varphi_s^*(\alpha_s) = \lambda_s \alpha$ and we can take $\varphi_1(\Sigma)$ to be our surface.

■

Proposition 11.0.13 ***(?).***
If $(\Sigma, \tilde{F})$ admits dividing curves, then it is convex.

12 | Thursday, February 24

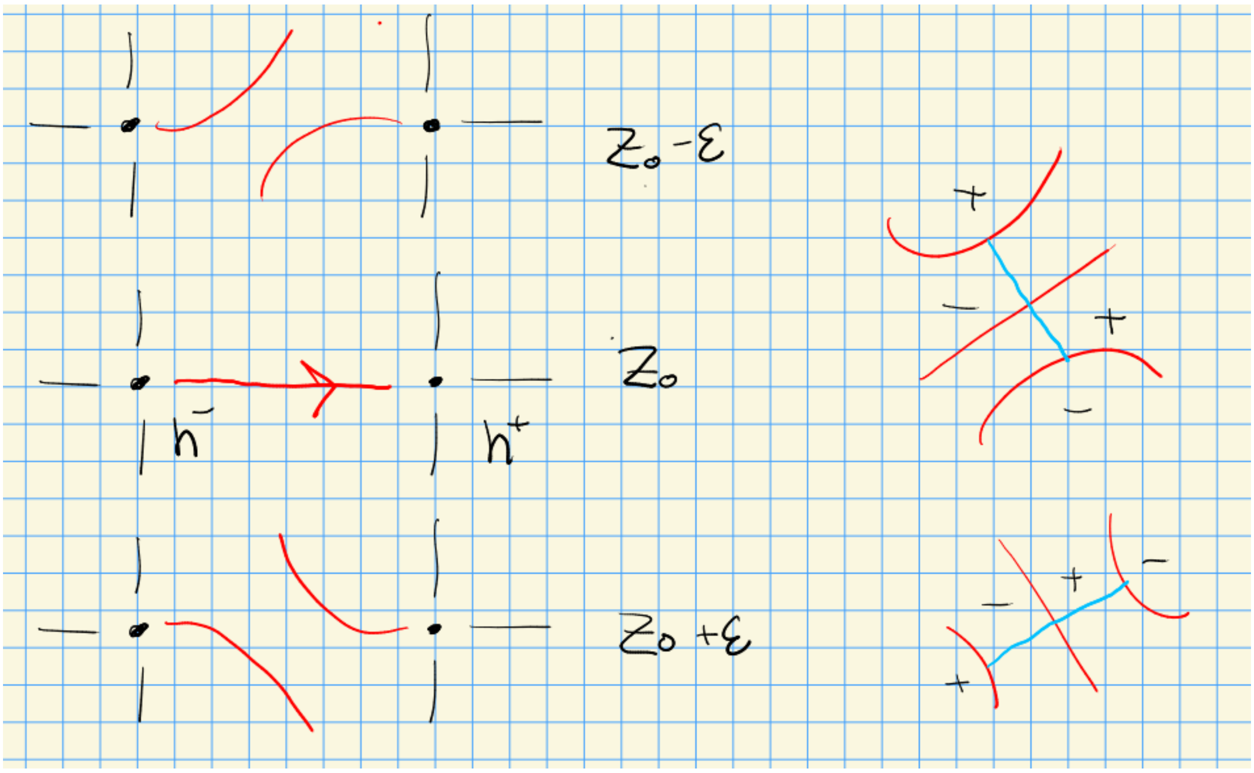
Remark 12.0.1: Last time: there is a unique tight contact structure on S^3, using the existence of a contact structure on $S^3 \times I$. Next: tight contact structures on

- $T^2 \times I$
- $S^1 \times \mathbb{D}^2$
- $L(p, q)$
- T^3

Given dividing sets of $\Gamma_0, \Gamma_1 \in T^2 \times I$, how can contact structures vary in a family. Tightness implies no contractible components in Γ, so Γ consists of $2n$ embedded curves of slow p/q. So the dividing set is governed by two parameters.

Remark 12.0.2: The only change to the dividing set in a generic family can be:

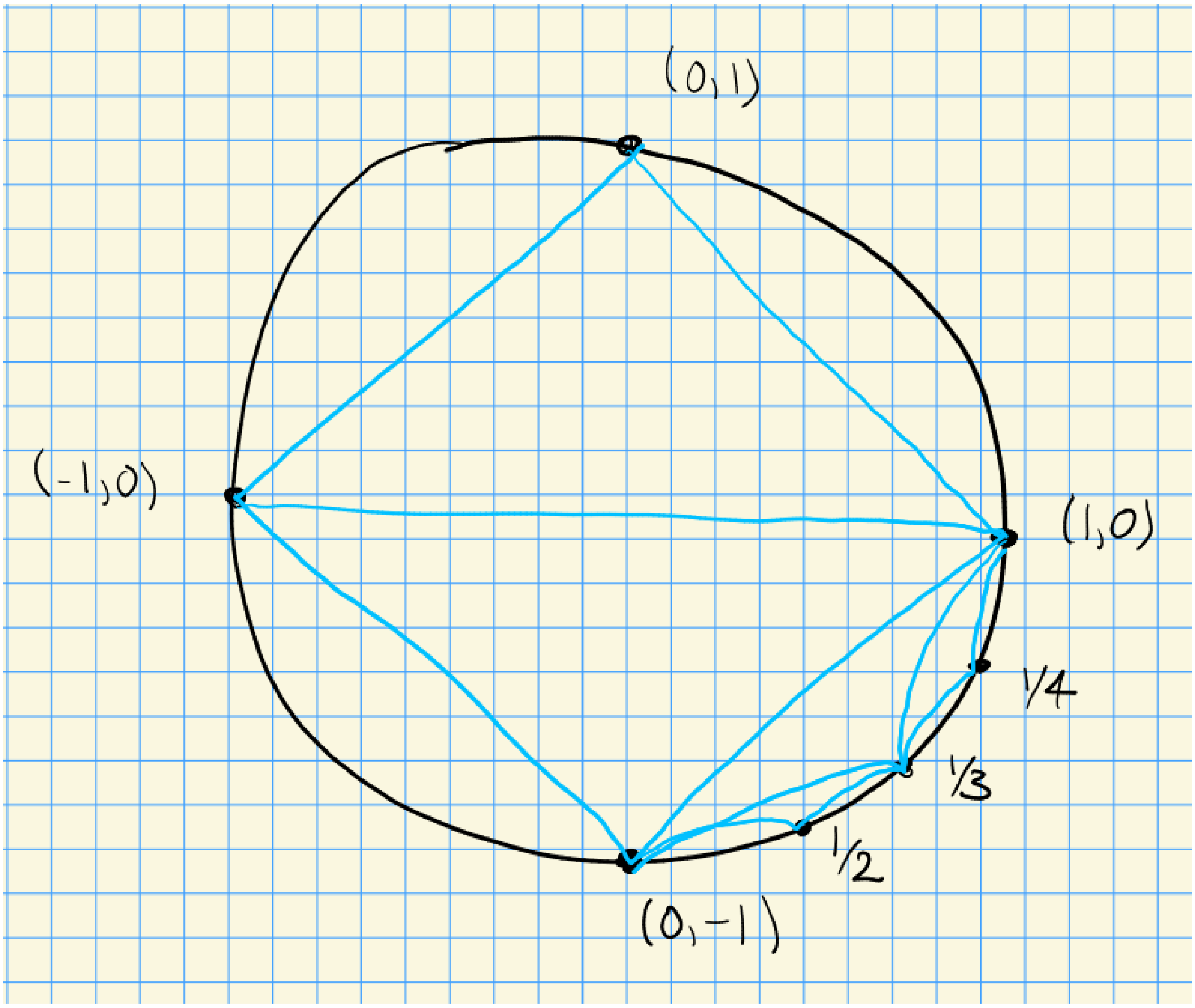
- Retrograde saddle-saddle, yielding by pass moves.



> **Proposition 12.0.3.**
> Given any contact structure on $\Sigma \times I$ with dividing sets Γ_0, Γ_1, ξ is determined by a finite number of bypass moves.

> *Proof (?).*
> Diagrams?
> ⋮
> ■

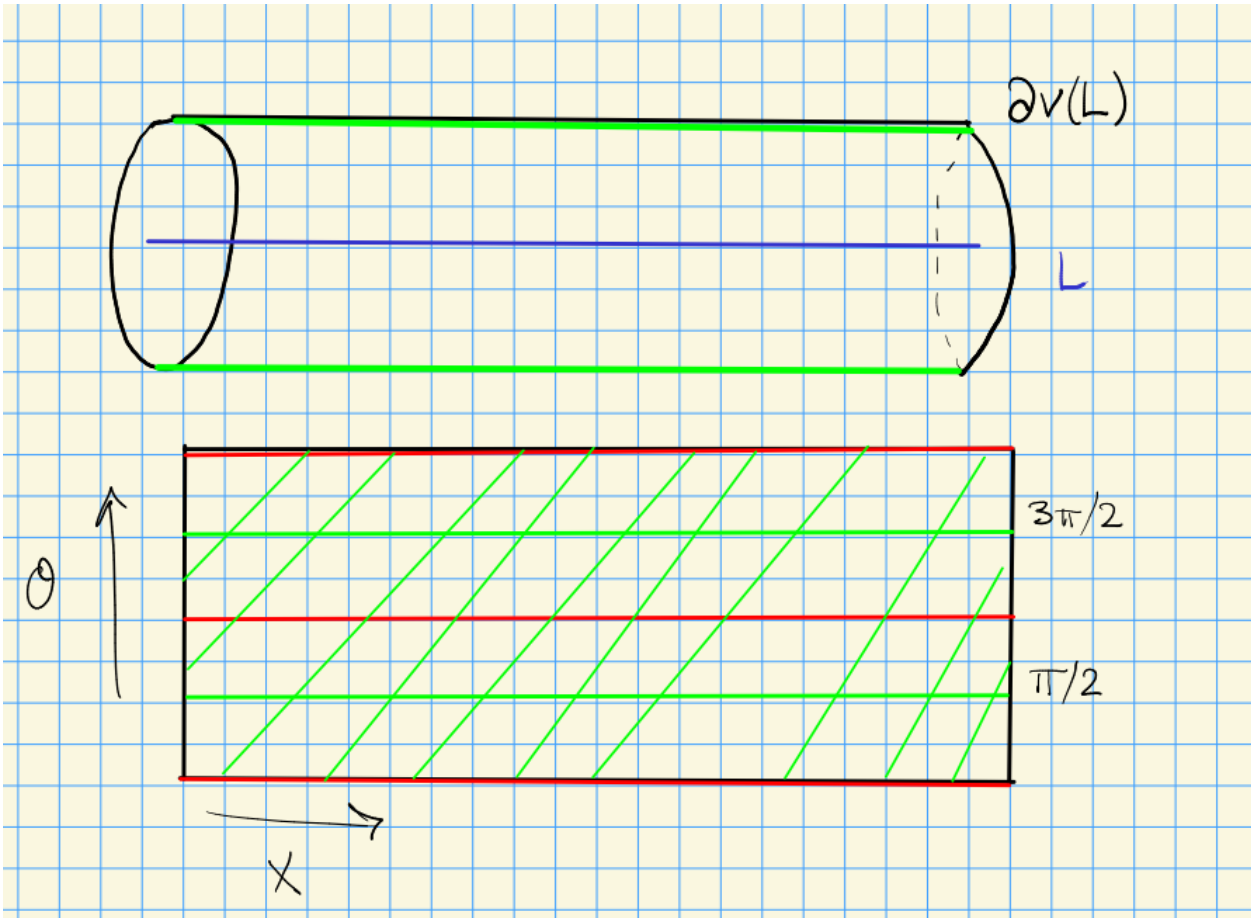
Remark 12.0.4: Given Γ_0 with slope p/q and Γ_1 with slope r/s, form a Farey graph:



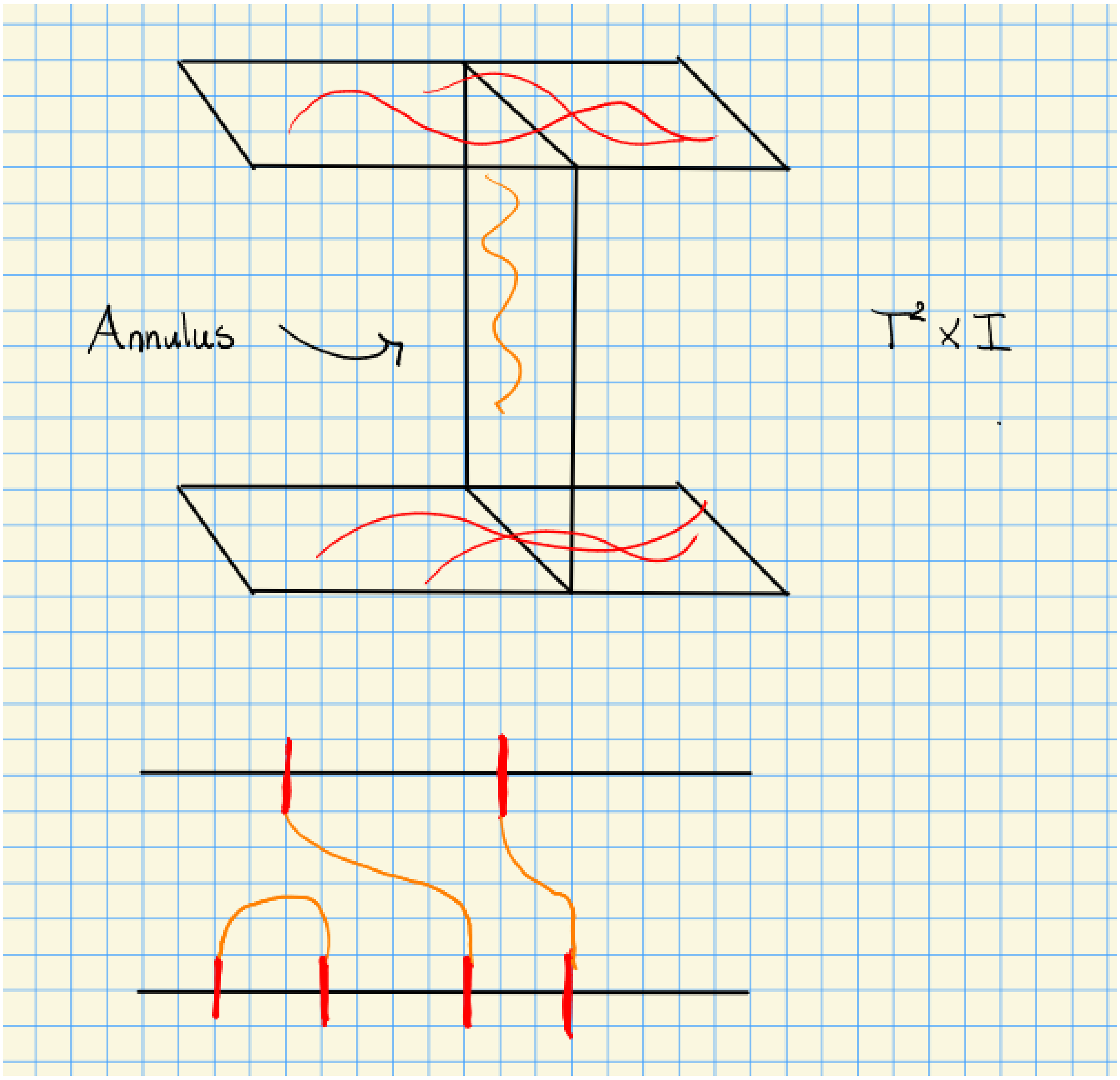
⋮

Proposition 12.0.5 ***(Legendrian Darboux).***
If L is a Legendrian knot in M, then a neighborhood of L is contactomorphic to a neighborhood of a zero section in $J(S^1) \cong \mathbb{R} \times \mathbf{T}^\vee S^1 \cong S^1 \times \mathbb{R}^2$..

Remark 12.0.6: Write this in coordinates as $(z, (x, y))$, so $\alpha = dz - y\,dx$ with $x \in \mathbb{R}/\mathbb{Z}$. Then $v(L) = \left\{y^2 + z^2 \leq \varepsilon\right\}$, $y = r\cos\theta, z = r\sin\theta$. $T^2 = \{x, \theta\}$, $\alpha|_{T^2} = y\,d\theta - y\,dx = \varepsilon\cos\theta(\,d\theta - \,dx)$. Unwrap:



Note that $d\alpha > 0$ at $\pi/2$ and $d\alpha < 0$ at $3\pi/2$. Idea: given two unrelated surfaces with their own foliations, how do they interact at the boundary? Dividing sets on each can be extended into the annulus, and this reduces to a combinatorial problem of how to connected arcs:



13 | Tuesday, March 15

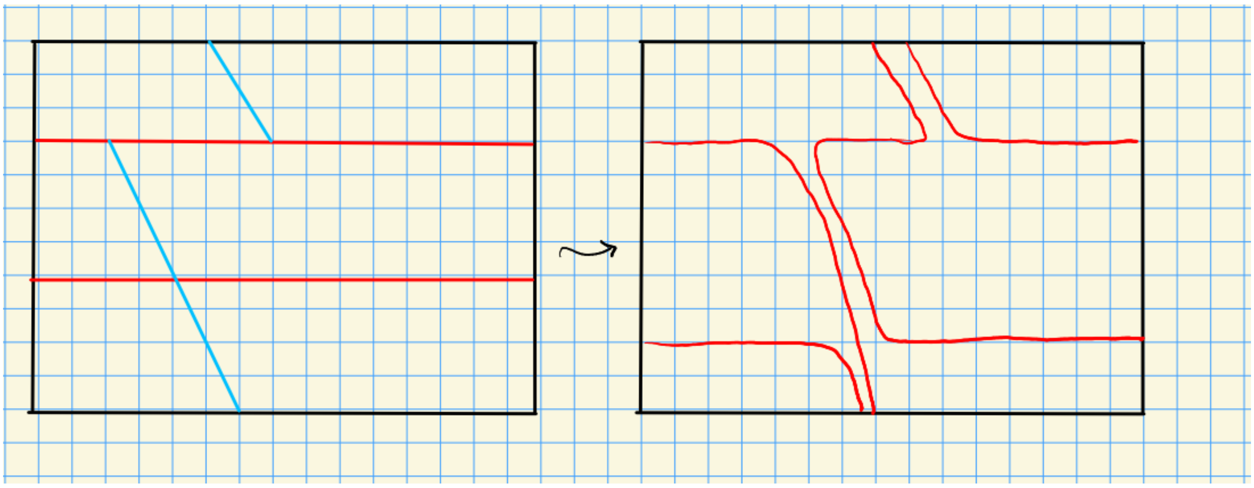
> *See* https://arxiv.org/pdf/math/9910127.pdf

Remark 13.0.1: Last time: classifying tight contact structures on T^3. Some contact structure:

$$\xi_n = \ker(\cos(2\pi nz)\, dx - \sin(2\pi nz)\, dy).$$

Realize T^3 as a cube with faces glued, then moving in the z direction twists n times as you traverse the cube. We can reduce this to ξ_1 using $[x, y, z] \mapsto [x, y, nz]$.

Remark 13.0.2: Goal: classify tight contact structures on lens spaces $L_{p,q} = T^2 \times I/\sim$. We can discretize the contact structure on $\Sigma \times I$ into a finite number of *bypass moves* on the dividing sets. The basic move:



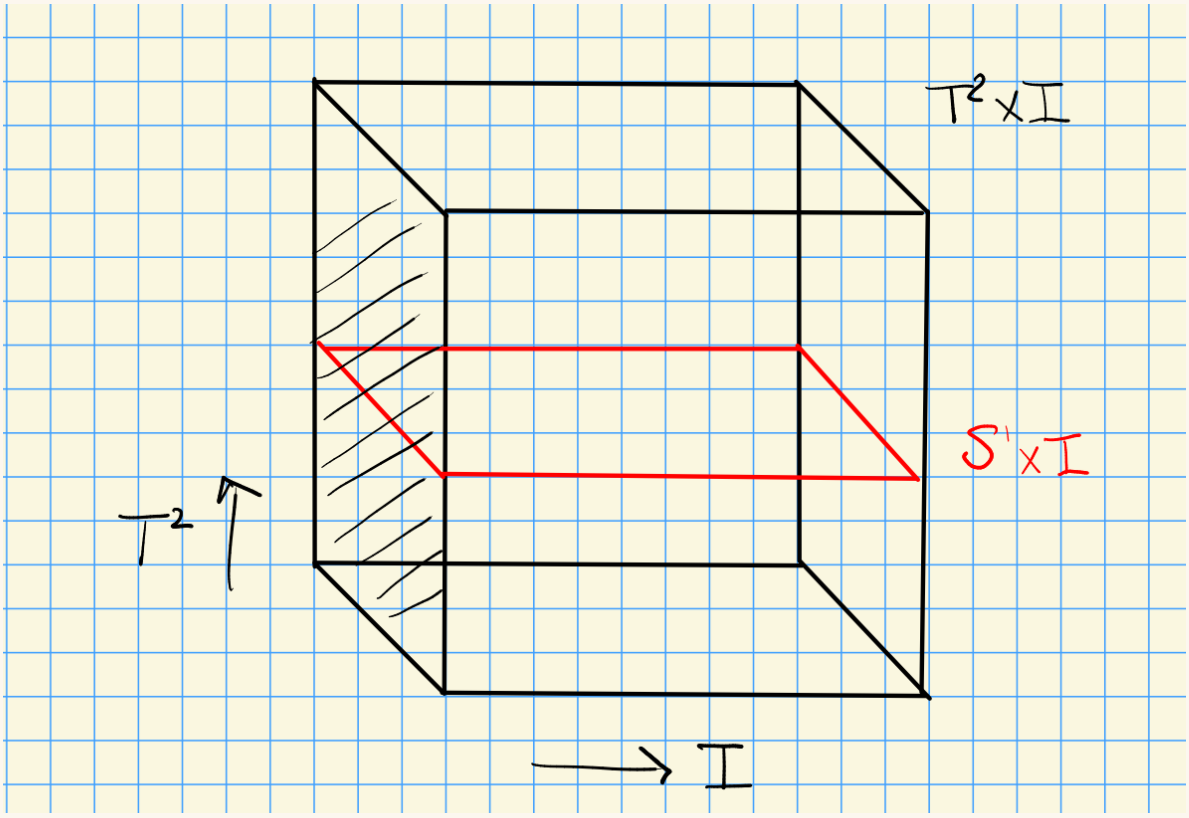
Definition 13.0.3 (Basic slice)
A **basic slice** is a contact structure on T^2 such that

- $T^2 \times \{0\}$ is convex with 2 dividing curves of slope 0
- $T^2 \times \{1\}$ is convex with 2 dividing curves of slope -1
- ξ is tight
- ξ is minimally twisting, so if $T^2 \subseteq T^2 \times I$ is convex then $\operatorname{slope}(r) \in [-1, 0]$.

Proposition 13.0.4*(?)*.
There are exactly 2 basic slices. Both embed in $(T^3, \xi_1) = \ker(\cos(2\pi z)\, dx - \sin(2\pi z)\, dy) = T^2 \times I/\sim$, and are given by

- $(T_2 \times [0, 1/8], \xi_1)$
- $(T_2 \times [1/2, 5/8], \xi_1)$

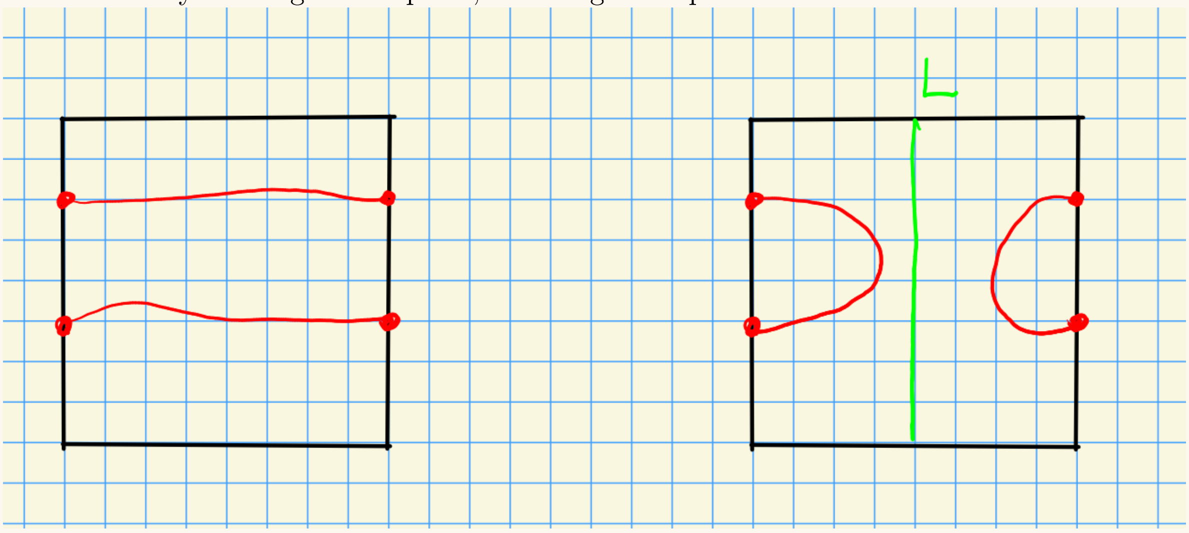
Proof (?).
Step 1: There are at most 2 basic slices. Reduce to $S^1 \times D^2$ by removing a convex annulus. Note that $T^2 \times I \setminus (S^1 \times I) \cong S^1 \times I^2 \cong S^1 \times D^2$.



Since the boundary is convex, we can make the foliations on both of the ruling curves of slope ∞.

?

Take an annulus A with some condition on ∂A, perturb to be convex? Something contradicts the "minimally twisting" assumption, involving these pics:



Smooth corners?

?

■

Definition 13.0.5 (Relative Euler class)
Let (M, ξ) be a contact 3-manifold with $\xi|_{\partial M}$ trivial. Let s be a nonvanishing section of $\xi|_{\partial M}$, then the **relative Euler class** $e(\xi, s) \in H^2(M, \partial M; \mathbb{Z}) \cong H_1(M)$ (by Lefschetz duality) is the dual of the vanishing set of an extension of s to a section of ξ on M.

Remark 13.0.6: In this case $\dim s^{-1} = \dim M - \dim \xi$.

Lemma 13.0.7 *(?).*
If $\Sigma \hookrightarrow (M, \xi)$ is a properly embedded convex surface and s is a section of $\xi|_{\partial M}$ that is tangent to $\partial\Sigma$ with the correct orientation, then

$$\langle e(\xi, s),\ \Sigma\rangle = \chi(\Sigma_+) - \chi(\Sigma_-).$$

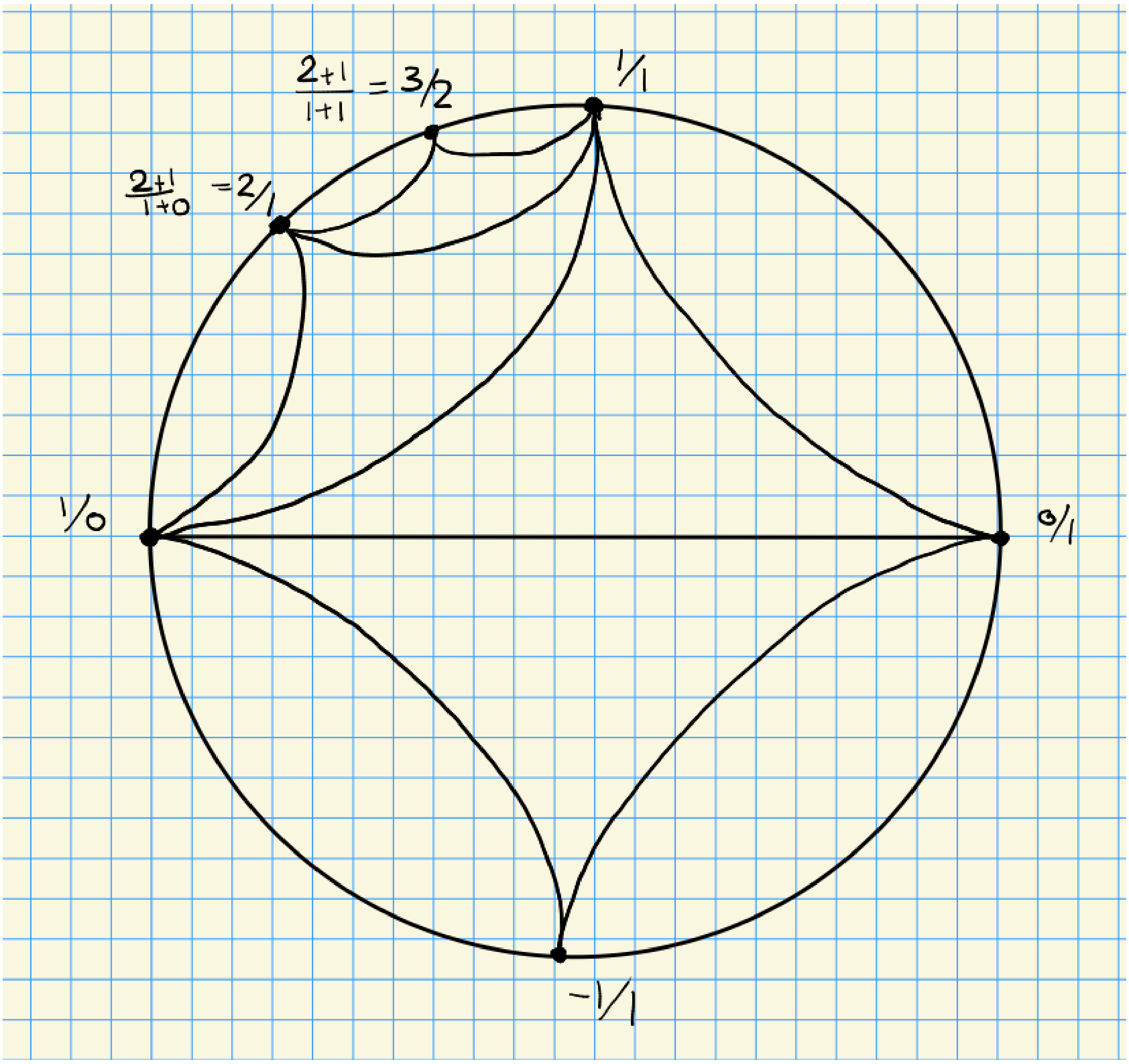
where $\langle -,\ -\rangle : H^2(M, \partial M; \mathbb{Z}) \times H_2(M, \partial M; \mathbb{Z}) \to \mathbb{Z}$.

Remark 13.0.8: Note $H_2(T^2 \times I, \partial; \mathbb{Z}) = \langle [\alpha \times I], [\beta \times I]\rangle$ where $H_2(T^2) = \langle \alpha, \beta\rangle$.

14 | Tuesday, March 22

14.1 Farey Graphs

Remark 14.1.1: Build a graph on the hyperbolic plane in the Poincare disc model:



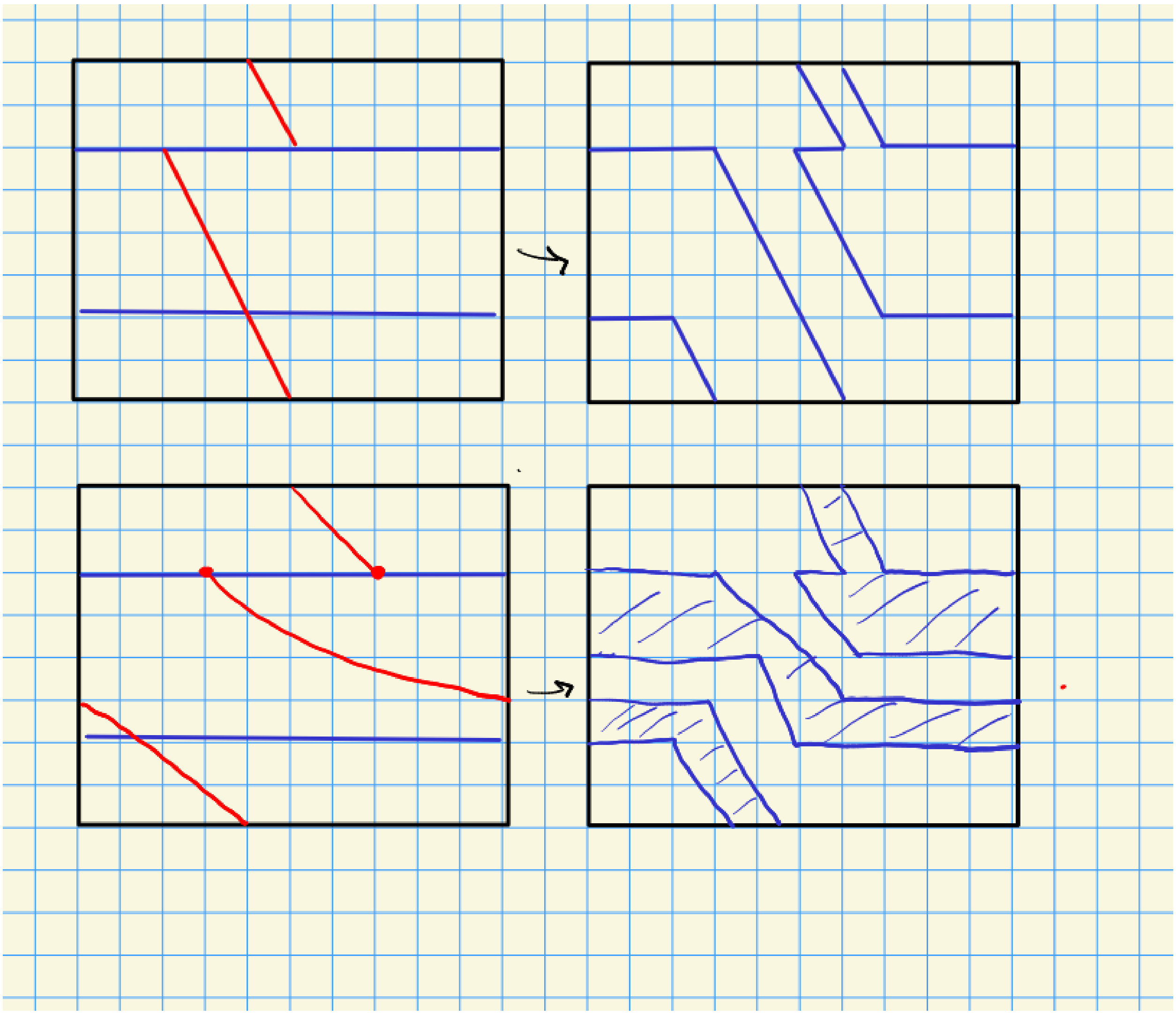
Here every midpoint corresponds to adding numerators and denominators respectively.

Associate slopes:

- $0/1 \rightsquigarrow 1\alpha + 0\beta$
- $1/0 \rightsquigarrow 0\alpha + 1\beta$
- $1/1 \rightsquigarrow 1\alpha + 1\beta$

Any pair of these is a $\mathbb{Z}$-basis for $H^1(T^2; \mathbb{Z}) \cong \mathbb{Z}^{\times 2}$. Use $\mathrm{SL}_2(\mathbb{Z}) \hookrightarrow \mathrm{PSL}_2(\mathbb{C})$ to realize any change of basis as an isometry of $\mathfrak{h}$. This makes the interior/exterior of any tile isometric to the full upper/lower half-disc.

Remark 14.1.2: Basic moves: bypasses



The first case corresponds to slopes $r \in (-\infty, -1)$ and the second to $r \in (-1, -1/2)$. Idea: the resulting dividing set is locally constant in perturbations of r, provided one doesn't cross the endpoints of the curve for the bypass move. This produces a continued fraction defined inductively by $r_0 = \left\lfloor -\frac{p}{q} \right\rfloor$, writing $-\frac{p}{q} = r_0 - \frac{1}{p'/q'} = -\frac{q'}{p'}$ with $-p/q < -p'/q' < -1$ and thus $0 < -\frac{p}{q} - r_0 < 1$, so set $r_1 = \left\lfloor -\frac{p'}{q'} \right\rfloor$. This yields

$$-\frac{p}{q} = r_0 - \frac{1}{r_1 - \frac{1}{r_2 - \cdots}} = [r_0, r_1, \cdots, r_m],$$

which terminates in finitely many steps since p/q is rational. Note that $r_i \leq -1 \implies \lfloor r_i \rfloor \leq -2$.

Proposition 14.1.3 *(?).*
If $r = -p/q = [r_0, \cdots, r_k]$ in a continued fraction expansion and $s = a/b$ is the first point connected to p/q while moving counterclockwise from $0/1$ on the Farey graph, then $-a/b = [r_0, \cdots, r_k + 1]$.

Remark 14.1.4: This gives the minimal graph path from p/q back to $0/1$ by jumping the maximal distance along the circle to a/b. Noting that $[r_1, \cdots, r_{k-1}, -1] = [r_1, \cdots, r_{k-1}+1]$ which is a shorter continued fraction.

Example 14.1.5*(?):* Let $p/q = 53/17$, then

- $r_0 = -4 = -68/17$
- $r_1 = -2 = -30/15$
- $r_2 = -2 = -26/13$
- $r_3 = -2 = \cdots$

So this yields $[-4, -2, \cdots_7, -2, -3]$.

Remark 14.1.6: Idea: decompose $p/q = [r_0, \cdots, r_k]$ surgery into integer surgeries on a link with k components.

15 | Tuesday, March 29

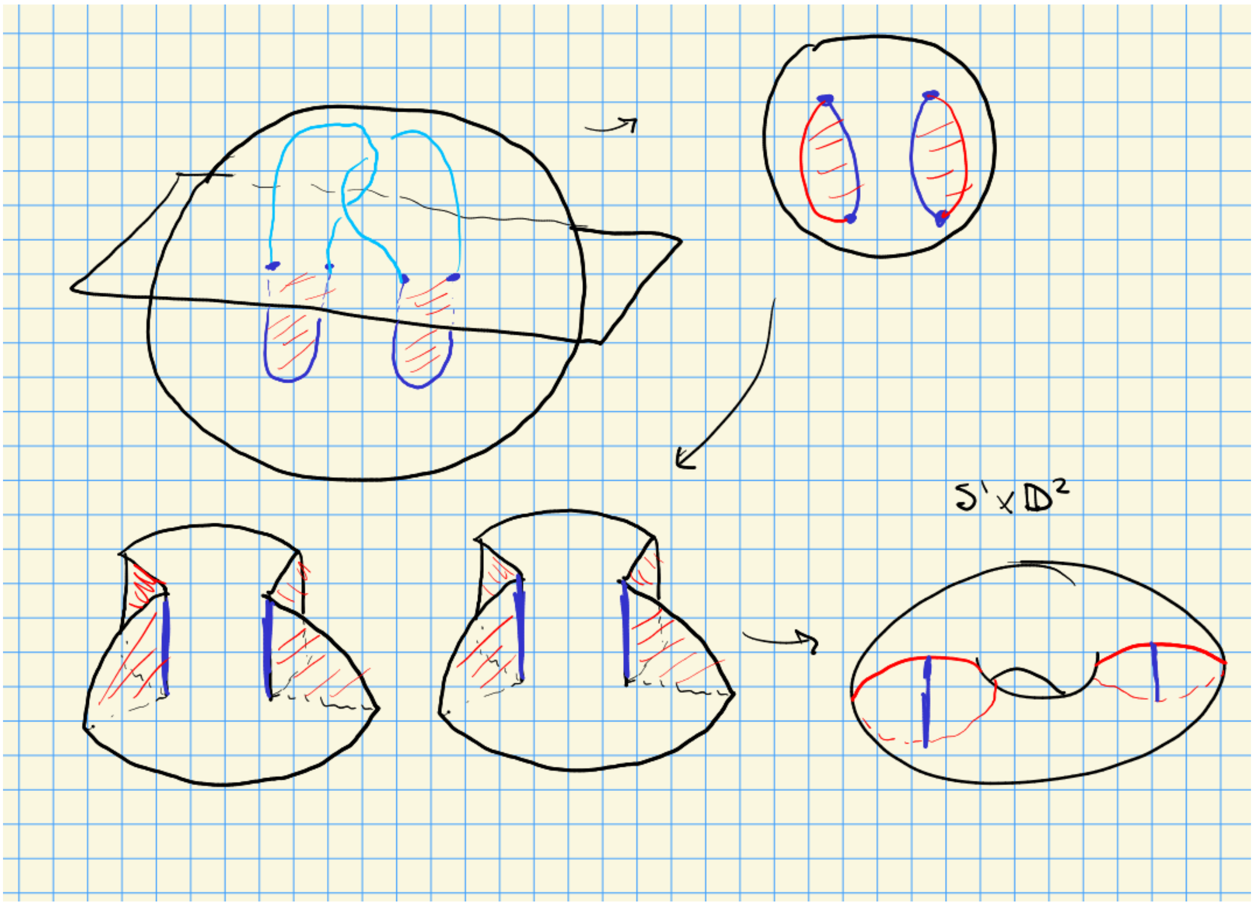
Remark 15.0.1: Goal: classification of tight contact structures on lens spaces.

Lens spaces: $L_{p,q} = S^3/C_p$ where the action is $[z_1, z_2] \mapsto \left[e^{\frac{2\pi i}{p}}, e^{\frac{2\pi i q}{p}}\right]$ which has order p. Note $L_{p,q} \cong L_{p,q'}$ when $q \equiv q' \mod p$, so we can assume $-p < q \leq 0$, so $p/q < -1$.

Some examples:

- $L_{1,q} \cong S^3$
- $L_{2,1} \cong \mathbb{RP}^3 \cong \mathrm{SO}_3(\mathbb{R})$.

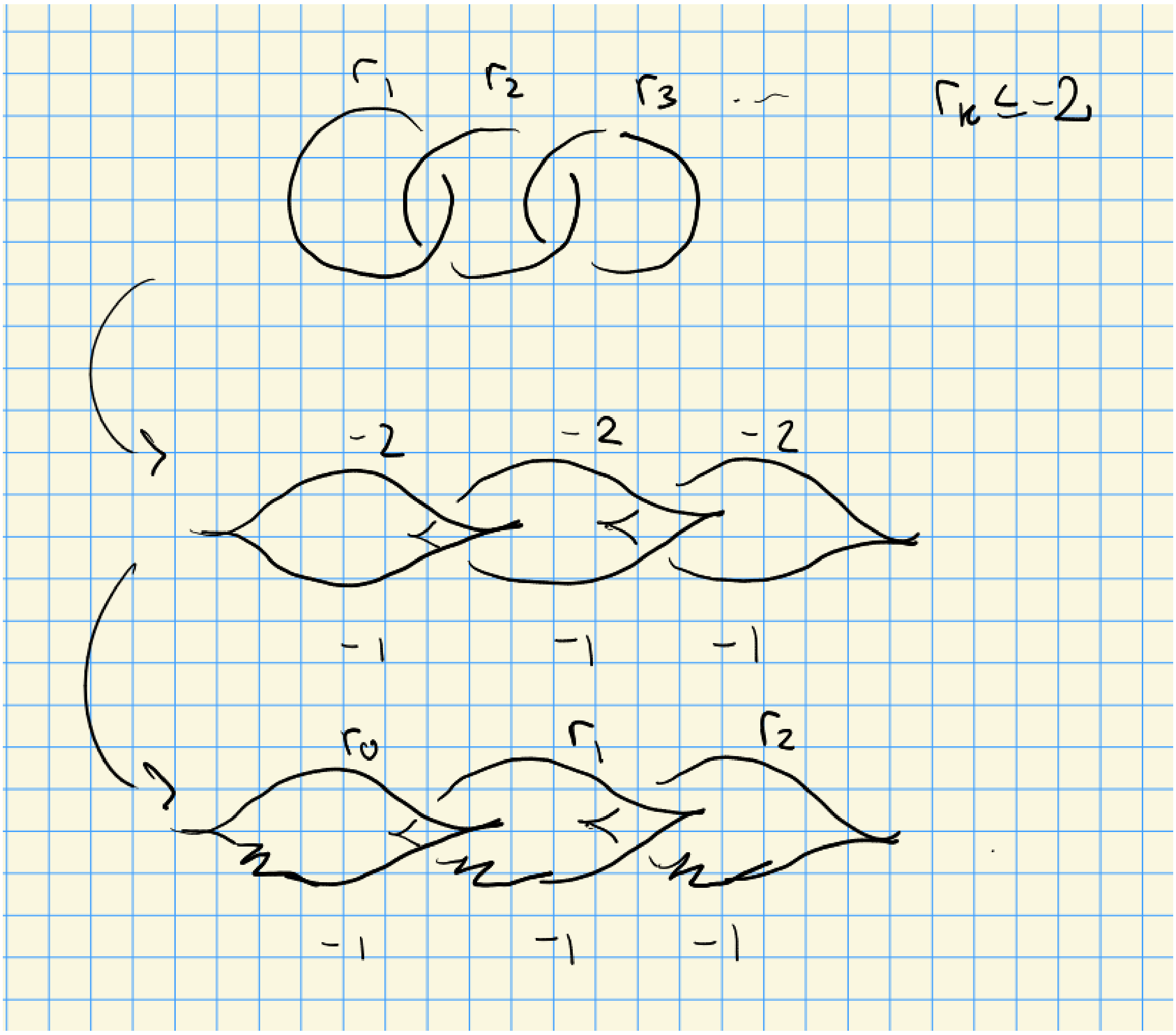
There is a genus 1 Heegaard splitting. The double branched cover of a 2-bridge link is a lens space:



All lens spaces can be generated by genus 1 Heegaard splittings?

Remark 15.0.2: $-p/q$ Dehn surgery is equivalent to a sequence of linked unknots with numbers $r_1, \cdots, r_k$. When can this be done in a way that preserves the contact structure? Idea: Legendrian surgery, which removes a Legendrian knot and reglues.

Remark 15.0.3: Let L be Legendrian and $\nu(L)$ is a standard neighborhood (so standard contact structure). Then $\partial\nu(L) \cong T^2$ is convex with 2 dividing curves, where "slope" is the contact framing. For $[\theta, x, y] \in S^1 \times \mathbb{R}^2$, set $\alpha = dx + y\, d\theta$. Then $[\theta, 0, 0]$ is Legendrian. When can we extend ξ uniquely across surgery $S^1 \times \mathbb{D}^2$? Need to attach handles along integer framing (choice of integer in $\pi_1 \mathrm{SO}_2(\mathbb{R}) \cong \mathbb{Z}$ corresponding to trivializing the normal bundle $\nu(K)$ in an embedding). Need good surgery slopes: $\{n\}_{n\in\mathbb{Z}} \cap \left\{\frac{1}{k}\right\}_{k\in\mathbb{Z}} = \{\pm 1\}$, relative to the tb-framing. So $\mathrm{tb} - 1$ is the best framing.:



Stabilize up to $r_K + 1$ on each Legendrian knot. Fact: yields a Stein fillable thing, implies tight contact structure.

Remark 15.0.4: There are $-r_0 - 1$ ways to perform $-r_0 - 2$ stabilizations. E.g. for $-r_0 - 2 = 3$, break into positive and negative stabilizations:

- $(3, 0)$
- $(2, 1)$
- $(1, 2)$
- $(0, 3)$

So there are $\prod_{1 \leq i \leq k} (-r_k - 1)$ tight contact structures on $-p/q = [r_0, \cdots, r_k]$.

16 | Tuesday, April 05

16.1 Symplectic Fillings

Example 16.1.1 ***(Properties of the standard contact structure on S^3):*** Consider $(S^3, \xi_{\text{std}}) \subseteq \mathbb{C}^2$; some things that are true:

- There is a symplectic form $\omega = dx_1 \wedge dy_1 + dx_2 \wedge dy_2$ where $d\omega = 0$ and $\omega \wedge \omega = 2d\text{Vol} > 0$. Write $\varphi = \sum x_i^2 + \sum y_i^2$, then $S^3 = \varphi^{-1}(1)$.
- Letting $\rho = \sum x_i \partial x_i + \sum y_i \partial y_i = \frac{1}{2} \operatorname{grad} \varphi$ in the standard metric yields a contact form $\alpha = \omega(\rho, -)$.
- Since $\omega|_{\xi_{\text{std}}} > 0$, this yields an area form on contact planes.
- There is also a complex structure $J : \mathbf{T}_p\mathbb{C}^2 \to \mathbf{T}_p\mathbb{C}^2$ where $J(\partial x_i) = \partial y_i$ and $J(\partial y_i) = -\partial x_i$ with a compatibility $g(x, y) = \omega(x, Jy)$.

Definition 16.1.2 (Fillings)
A complex symplectic manifold (X^4, ω, J) is a filling of (Y^3, ξ) if $Y = \partial X$,

- **Stein filling**: (X^4, J) is a Stein manifold, and $\xi = \mathbf{T}Y \cap J(\mathbf{T}Y)$.
- **Strong filling**: if there is an outward pointing (Liouville) vector field ρ with $\mathcal{L}_\rho \omega = \omega$ with $\xi \ker(\omega(\rho, -))$ (which is always contact). Note $\mathcal{L}_\rho \omega = d(\iota_\rho \omega) + \iota_\rho(d\omega)$ where the 2nd term vanishes for a symplectic form.
- **Weak filling**: $\omega|_\xi > 0$.

Note that we aren't defining what "Stein" means here.

Theorem 16.1.3 ***(?).***
There are strict implications

- Stein $\implies$
- Strong $\implies$
- Weak $\implies$
- Tight.

Note that the last implication is the harder part of the theorem.

Problem 16.1.1 (?)
Given (Y, ξ), classify all fillings.

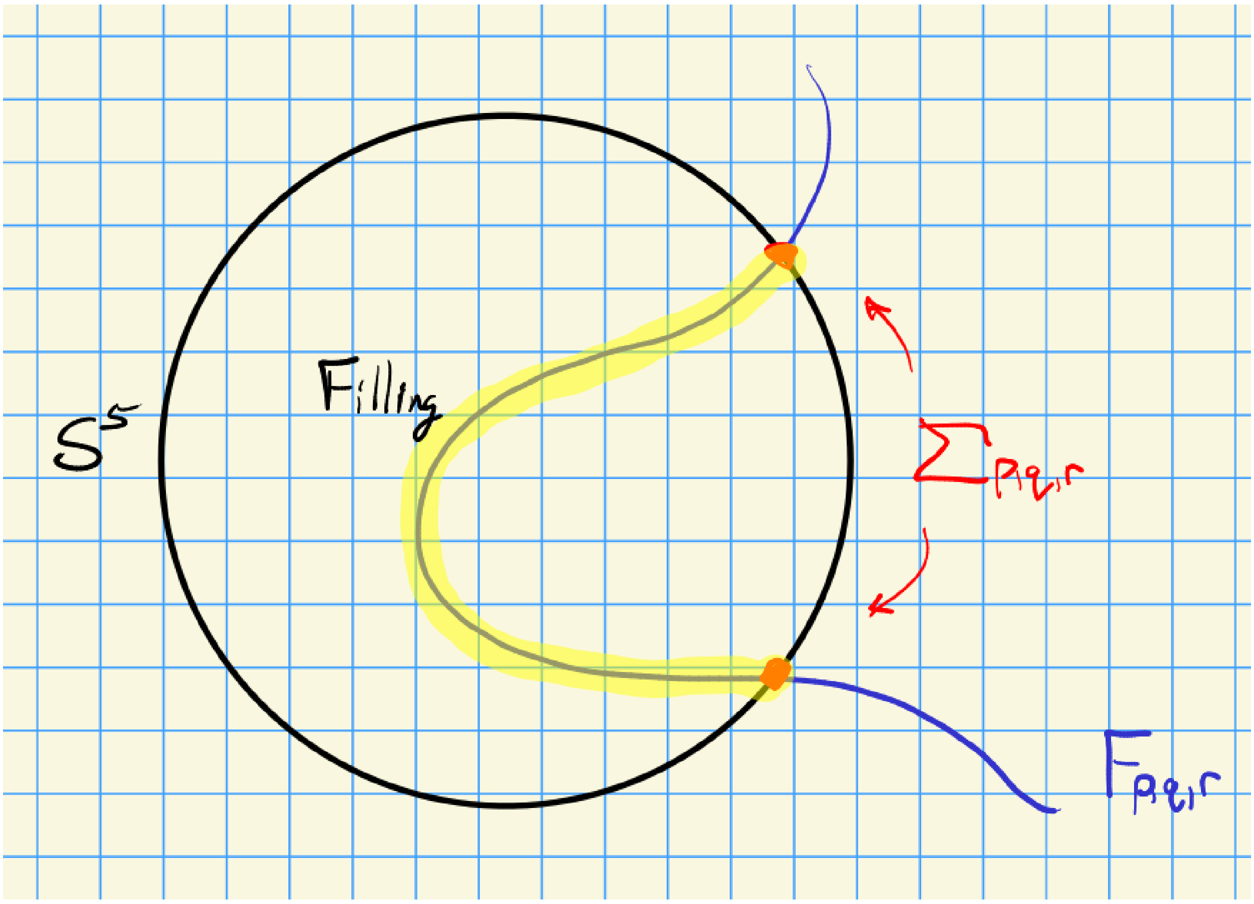
Example 16.1.4 ***(?):*** Consider (T^3, ξ_n) – if $n = 1$, this is Stein fillable, and for $n \geq 2$ these are weakly fillable but not strongly fillable. In this case, all of the filling manifolds are $T^2 \times \mathbb{B}^2$.

Example 16.1.5 ***(?):*** For lens spaces $L_{p,q}$ all tight contact structures are Stein fillable with the same smooth filling. Take the linear plumbing X of copies of S^2 corresponding to $-p/q = [r_1, r_2, \cdots, r_k]$ as a continued fraction expansion. They're distinguished by Chern classes $c_1(T_X, J)$.

Example 16.1.6 ***(?):*** Brieskorn spheres are examples of fillings, related to Milnor fibers. For $p, q, r \geq 2$, define

$$\Sigma(p,q,r) \coloneqq \{F_{p,q,r}(x,y,z) = x^p + y^q + z^r = \varepsilon\} \cap S^5 \subseteq \mathbb{C}^3 = \operatorname*{span}_{\mathbb{C}} \{x, y, z\}.$$

In this case, we have:



Note that $\varepsilon = 0$ yields a singular variety, while $\varepsilon > 0$ small yields a smooth manifold.

Exercise 16.1.7 (?)
Show $\Sigma_{p,q,r}$ is the r-fold cyclic branched cover of S^3 over the torus knot $T_{p,q}$.

Remark 16.1.8: Let $J : \mathbf{T}X \to \mathbf{T}X$ with $J^2 = -\operatorname{id}$, so the eigenvalues are $\pm i$. So consider complexifying to $\mathbf{T}_{\mathbb{C}}X \coloneqq \mathbf{T}X \otimes_{\mathbb{R}} \mathbb{C}$, so e.g. $\partial x_k \mapsto (a_k + ib_k)\partial x_k$. This splits into positive (holomorphic) and negative (antiholomorphic) eigenspaces $\mathbf{T}_{\mathbb{C}}^{1,0}X \oplus \mathbf{T}_{\mathbb{C}}^{0,1}X$. Take a change of basis $[x_1, y_1, x_2, y_2] \mapsto [z_1, \bar{z}_1, z_2, \bar{z}_2]$ which yields $\partial z = \frac{1}{2}(\partial x - i\partial y)$ and $\bar{\partial} z = \frac{1}{2}(\partial x + i\partial y)$.

Exercise 16.1.9 (?)
Let $f(z) = |z|^2$ and check

- $\partial\bar{\partial} f = \partial(z d\bar{z}) = dz \wedge d\bar{z} = -2i(dx \wedge dy)$.
- $d = \partial + \bar{\partial}$

Practicing this type of change of variables is important!

Definition 16.1.10 (Levi forms and plurisubharmonicity)
Let $\varphi : X \to \mathbb{R}$ for X a complex manifold, then the **Levi form** of φ is

$$\mathcal{L}\varphi = \partial\bar{\partial}\varphi = \sum_{i,j} \frac{\partial^2}{\partial z_j \bar{\partial} \bar{z}_k} dz_j \wedge d\bar{z}_k,$$

generalizing the Hessian. The function φ is **plurisubharmonic** if $\mathcal{L}\varphi$ is positive semidefinite at every point.

Example 16.1.11 ***(?):*** Consider $\varphi : \mathbb{C} \to \mathbb{R}$, then

$$\begin{aligned}
\mathcal{L}\varphi &= \partial\bar{\partial}\varphi \\
&= 2\left(\frac{1}{2}(\varphi_x + i\varphi_y)\right) \\
&= \frac{1}{2}\left(\frac{1}{2}(\varphi_{xx} - \varphi_{xy}) + \frac{1}{2}(\varphi_{yx} - i\varphi_{yy})\right) \\
&= \frac{1}{4}(\varphi_{xx} + \varphi_{yy}) \\
&= \frac{1}{4}\Delta\varphi,
\end{aligned}$$

so plurisubharmonic implies positive Laplacian. Note that in 1 dimension, $\Delta f = 0 \implies f'' = 0$, so $(x, f(x))$ is a straight line. In higher dimensions, $f'' > 0$ forces convexity, so secant lines are under the straight lines, hence the "sub" in subharmonic.

Proposition 16.1.12 ***(?).***
If $\varphi : X \to \mathbb{R}$ is plurisubharmonic and 0 is a regular value, then $(\varphi^{-1}(0), \xi)$ (where ξ is its complex tangencies) forms a contact structure and the sub-level set $\varphi^{-1}(-\infty, 0]$ is a Stein filling.

Example 16.1.13 ***(A basic example of a plurisubharmonic function):*** The radical function $\varphi : \mathbb{C}^3 \to \mathbb{R}$ where $\varphi(z_1, z_2, z_3) = \sum |z_i|^2$ is plurisubharmonic, as is its restriction to any submanifold of $\mathbb{C}^3$, including any filling of $\Sigma_{p,q,r}$. Hard theorem: any Stein manifold and any Stein filling essentially comes from this construction.

17 | Thursday, April 21

Note: student talks in previous weeks!

Remark 17.0.1: Possible topics for the remainder of the class:

- Open book decompositions
- Every (Y^3, ξ) is homotopic to a contact structure.
- Seifert fibered spaces

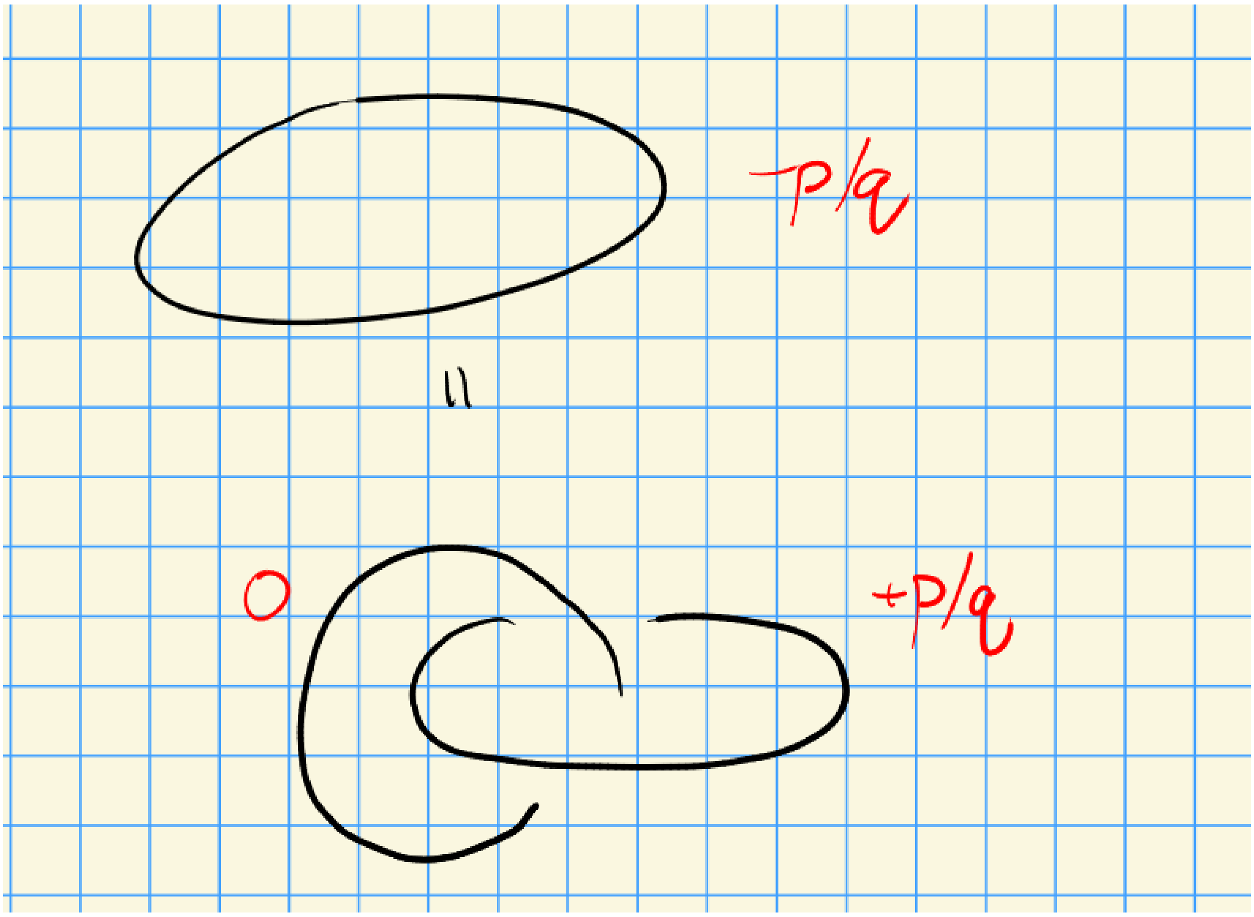
17.1 Seifert Fibered Spaces

Remark 17.1.1: Brieskorn spheres $\Sigma(p, r, q) \coloneqq \{x^p + y^q + z^r = 0\} \cap S^5_\varepsilon \subseteq \mathbb{C}^3$ are 3-manifolds *foliated* by S^1. Note that $S_1 \to X \to S^2$ for $X = S^3$ or $L(p, q)$ are actual fibrations. Idea: a foliation by F's is a decomposition $X = \coprod F \to B$ which is a fibration with ramification in some fibers.

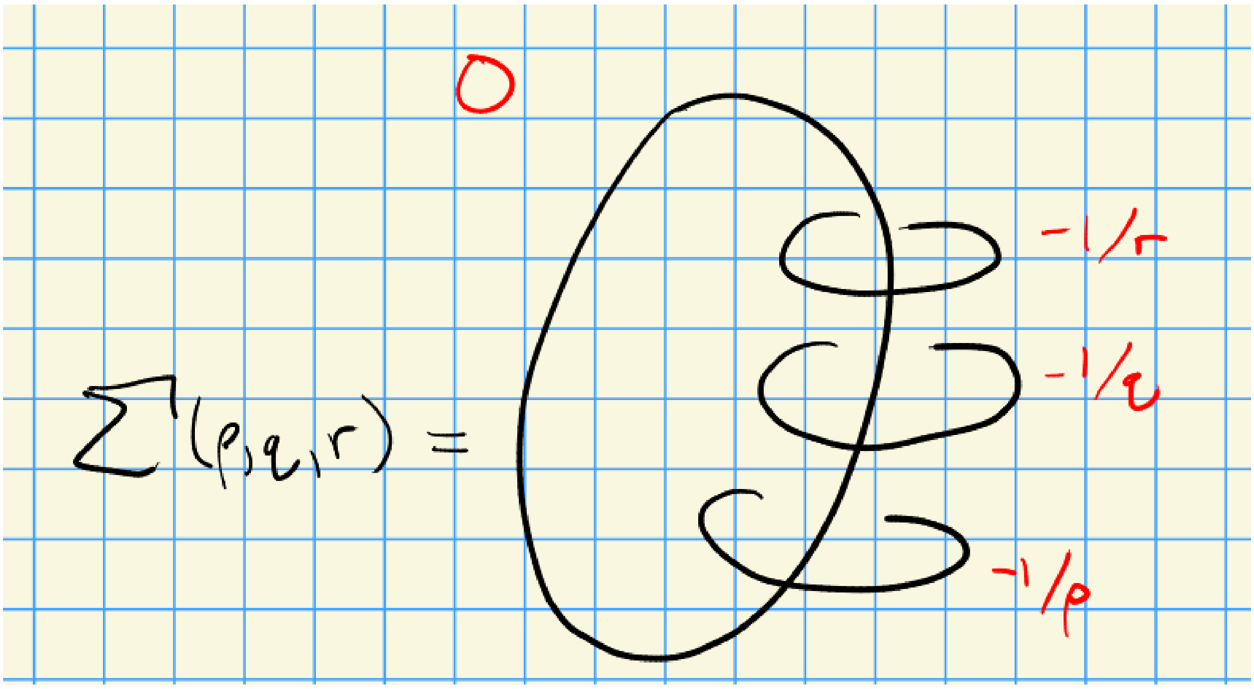
Definition 17.1.2 (Seifert fibered spaces)
A **Seifert fibered space** associated to $(\Sigma, (p_1/q_1, \cdots, p_n/q_n))$ with $p_i/q_i \in \mathbb{Q}$ and Σ an orbifold surface is a 3-manifold Y and knots $L_1, \cdots, L_n$ with neighborhoods νL_i such that

- $Y \setminus \cup_i \nu(L_i) = (\Sigma \setminus \{\mathrm{pt}_1, \cdots, \mathrm{pt}_n\}) \times S^1$
- $\nu L_i = S^1 \times \mathbb{D}^2$ is glued in by p_i/q_i Dehn surgery.

Example 17.1.3*(?)*: $L(p, q)$ is $-p/q$ surgery on S^1, or by a slam-dunk move:



$\Sigma(p, q, r)$:



Exercise 17.1.4 (?)
Show that for $\Sigma(p, q, r)$, removing the axes in $\mathbb{C}^3$ yields a trivial fibration by copies of S^1 over $S^2 \setminus \mathrm{pt}_1, \mathrm{pt}_2, \mathrm{pt}_3$ and check the surgery slopes.

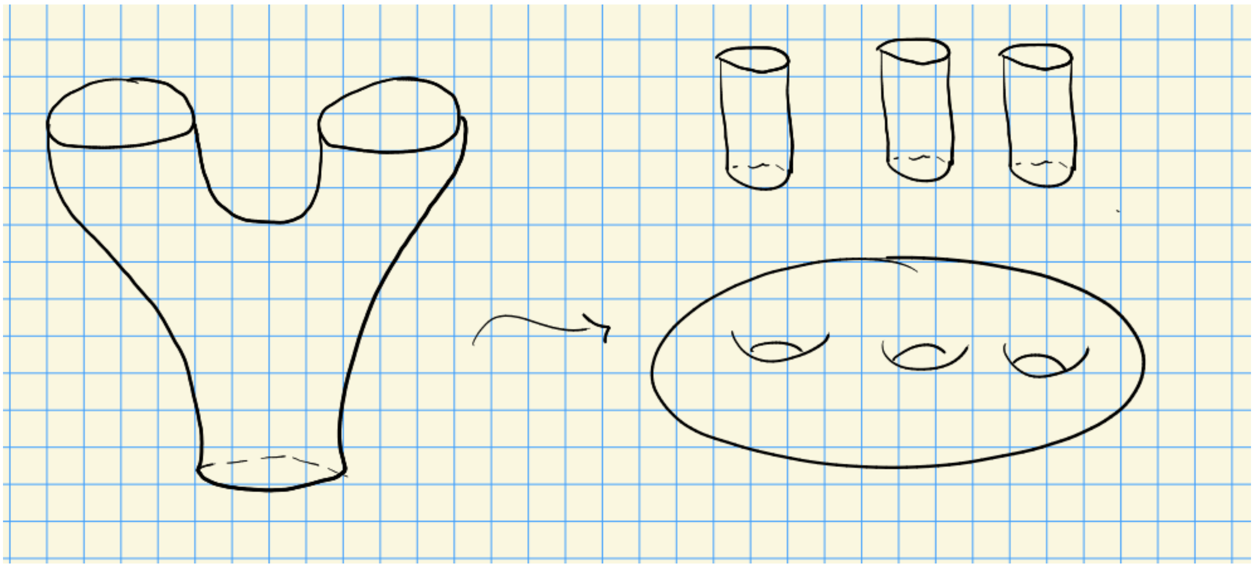
Exercise 17.1.5 (?)
Prove that $\Sigma(p, q, r)$ comes from the plumbing diagram for the Milnor fibration using Kirby calculus.

18 | Tuesday, April 26

Remark 18.0.1: Recall that $\mathrm{PHS}^3 = \Sigma(2, 3, 5)$ has a Stein-fillable (and hence tight) contact structure.

Theorem 18.0.2 *(?)*.
The negative $-\Sigma(2, 3, 5)$ admits no tight contact structures.

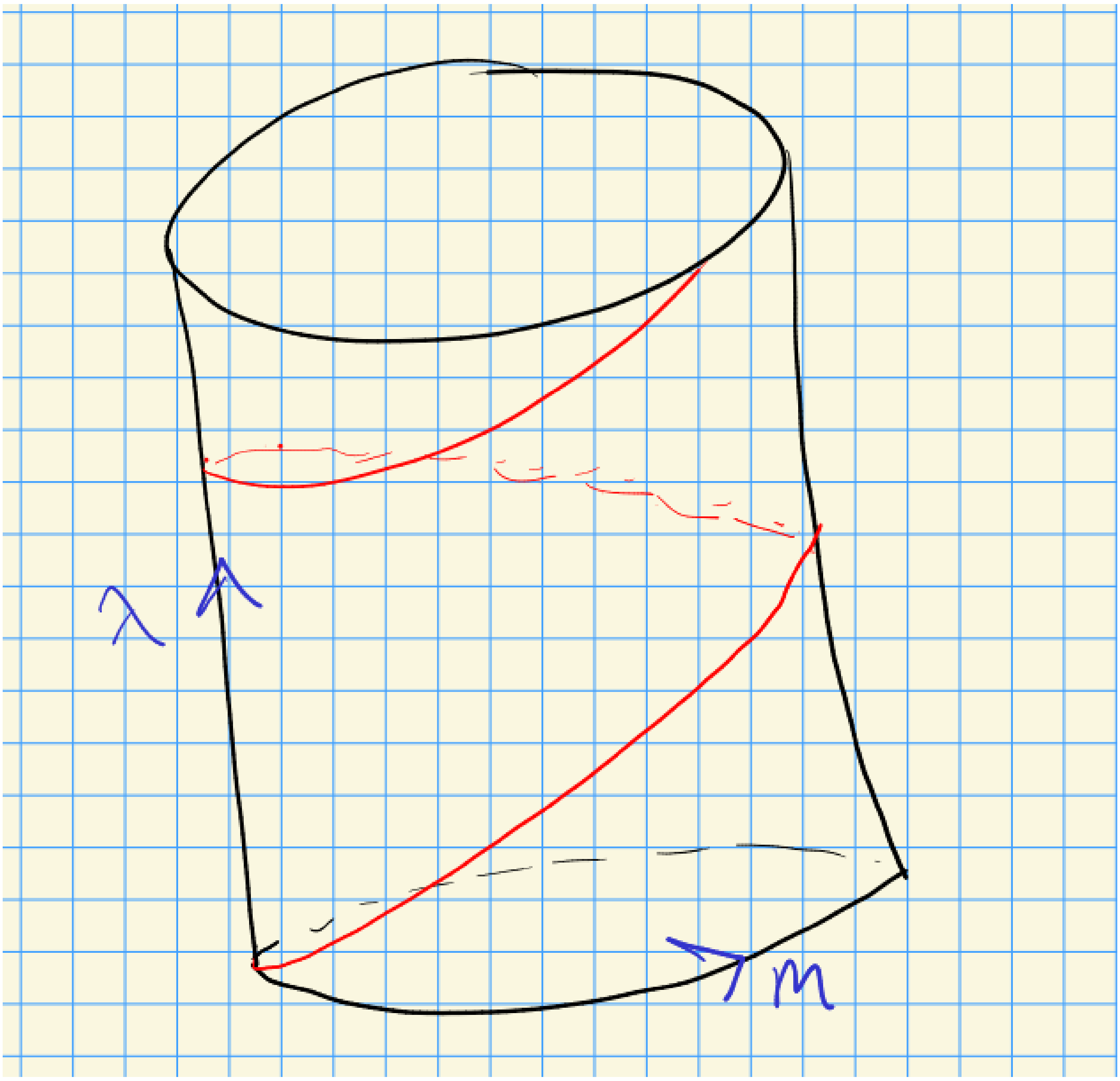
Remark 18.0.3: Let $S = S^3 \setminus \{\mathrm{pt}_1, \mathrm{pt}_2, \mathrm{pt}_3\}$ be a pair of pants and consider $X = S \times S^1$. Note $\partial X = T^2 \cup T^2 \cup T^2$:



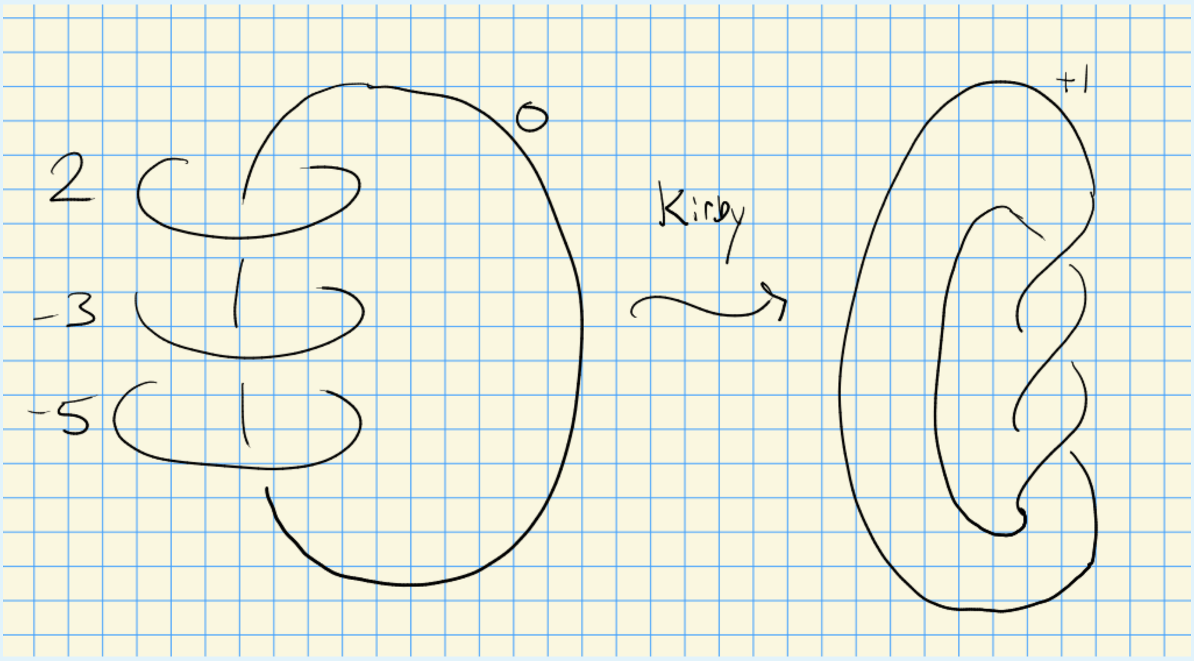
Note $-\Sigma(2, 3, 5) = \Sigma(2, -3, -5)$, since PHS^3 is -1 surgery on the trefoil. Glue in 3 solid torii by

$$A_1 = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 3 & 1 \\ -1 & 0 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 5 & 1 \\ -1 & 0 \end{bmatrix}.$$

acting on $[m, \lambda]$ in $S^1 \times \mathbb{D}^2$:



Exercise 18.0.4 (?)
Show via Kirby calculus:



Lemma 18.0.5 *(?).*
There exist Legendrian representatives F_2, F_3 with twisting numbers $m_2, m_3 = -1$.

Proof (?).
Idea: by stabilization, we can assume $m_2, m_3 < 0$, and the claim is that we can destabilize them back up to -1 simultaneously using bypass moves. Reduce to studying dividing sets on $T^2 \times I$ or $S^1 \times \mathbb{D}^2$. Check that the dividing set has slope $-1/2$, which implies that there is an overtwisted disc. Reduce to $2 \cdot 3 \cdot 5 = 30$ cases, check that an overtwisted disc can be found in each case.

■

ToDos

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Definitions

Theorems

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Figures

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