Notes: These are notes live-tex'd from a graduate course in Functional Analysis taught by Weiwei Hu at the University of Georgia in Spring 2022. As such, any errors or inaccuracies are almost certainly my own.

Functional Analysis

Lectures by Weiwei Hu. University of Georgia, Spring 2022

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Remark 1.0.1: This course: solving $Lf = g$ for L a linear operator, in analogy to solving $Ax = b$ in matrices. References:

- Hutson-Pym-Cloud, *Applications of Functional Analysis and Operator Theory*
- Reed-Simon, *Methods of Modern Mathematical Physics*
- Brezis, *Functional Analysis, Sobolev Spaces, and PDEs*

Remark 1.0.2: The issue when passing to infinite-dimensional vector spaces: the topology matters. E.g. the closure of the unit ball is closed and bounded and thus compact in finite dimensions, but this may no longer be true in $\mathbb{R}^\infty$ or $\mathbb{C}^\infty$. Recall that a Banach space is a complete normed space, and is further a Hilbert space if the norm is induced by an inner product. See the textbook for a review of vector spaces, metric spaces, norms, and inner products.

Example 1.0.3*(?)*: Our first example of infinite dimensional vector spaces: sequence spaces ℓ with elements $f := (f_1, f_2, \cdots)$ with each $f_i \in \mathbb{R}$.

Remark 1.0.4: Linear subspaces are subspaces that contain zero, as opposed to affine subspaces. An example is $C_0([0, L]; \mathbb{R}) \leq C([0, L]; \mathbb{R})$, the subspace of bounded continuous functionals on $[0, L]$ which vanish at the endpoints. For any subset $S \subseteq V$, write $[S]$ or $\operatorname{span} S$ for the linear span of S: all finite linear combinations of elements in S.

Example 1.0.5*(?)*: Let $V = C([-1, 1])$ and $x_1 \neq x_2 \in [-1, 1]$, and set $M_i := \left\{ f \in V \,\middle|\, f(x_i) = 0 \right\}$. Then $M_i \leq V$ is a linear subspace, and in fact $V = M_1 + M_2$ but $V \neq M_1 \oplus M_2$ since the zero function is in both subspaces.

Remark 1.0.6: Limits of finite operators are compact. The classical example: set $(A_N)_{i,i} = \frac{1}{i}$, so $A_N = \operatorname{diag}\left(1, \frac{1}{2}, \frac{1}{3}, \cdots, \frac{1}{N}\right)$. Then $\operatorname{Spec} A_N = \left\{ \frac{1}{n} \right\}_{n \leq N}$, but $A := \lim_N A_N$ is an operator with $0 \in \overline{\operatorname{Spec}(A)}$ as an accumulation point. Exercise: what is $\ker A$? Is it nontrivial?

> **Definition 1.0.7** (Convexity)
> A subset $S \subseteq V$ is **convex** iff
>
> $$tf + (1-t)g \in S \qquad \forall f, g \in S, \quad \forall t \in (0, 1].$$
>
> Equivalently,
>
> $$\frac{af + bg}{a + b} \in S \qquad \forall f, g \in S, \quad \forall a, b \geq 0$$
>
> where not both of a and b are zero. The **convex hull** of S is the smallest convex set containing S.

Remark 1.0.8: Recall Holder's inequality:

$$\|fg\|_1 \leq \|f\|_p \cdot \|g\|_q,$$

Schwarz's inequality

$$|\langle f,\ g\rangle| \leq \|f\|\|g\| \qquad \|f\| \coloneqq \sqrt{\langle f,\ f\rangle},$$

and Minkowski's inequality

$$\|f+g\|_p \leq \|f\|_p + \|g\|_p.$$

A nice proof of Cauchy-Schwarz:

$$\begin{aligned} 0 &\leqslant \|f-(f,h)h\|^2 \\ &= (f-(f,h)h, f-(f,h)h) \\ &= (f,f)-(f,h)(h,f)-\overline{(f,h)}(f,h)+(f,h)\overline{(f,h)} \\ &= \|f\|^2 - |(f,h)|^2. \end{aligned}$$

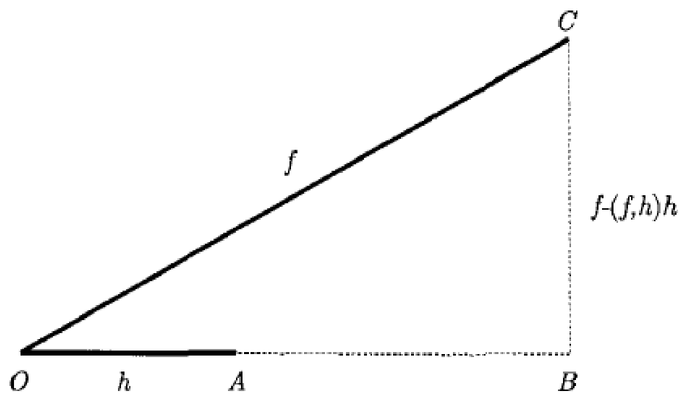


Figure 1.4: In $\mathbb{R}^2$, OA,OC represent h, f respectively, while OB is $(f,h)h$. The relation proved in 1.5.5, $\|f-(f,h)h\|^2 = \|f\|^2 - |(f,h)|^2$, is simply Pythagoras' theorem.

2 | Thursday, January 13

Remark 2.0.1: My notes:

- $K \subseteq \mathcal{H}$ is **complete** iff $K^{\perp} = 0$.
- Bessel: for $f \in \mathcal{H}$ write $f_n \coloneqq \langle f,\ e_n \rangle e_n$, then $\|(f_n)\|_{\ell^2(\mathbb{C})} \leq \|f\|_{\mathcal{H}}^2$.
- Best estimate: for any other sequence $(c_n) \in \ell^2(\mathbb{C})$, $\left\| f - \sum c_n e_n \right\| \geq \left\| f - \sum f_n e_n \right\|$.
- For $\{e_n\}$ orthonormal, $(c_n) \in \ell^2(\mathbb{C}) \iff \sum c_n e_n$ converges. If the series converges, it can be rearranged.
- Differentiating through an integral:

2.4.16 Theorem. *Assume that $f(\cdot, y)$ is differentiable for each $y \in Y$, and that $f(x, \cdot)$ is integrable for each $x \in X$. Suppose that there is an integrable function $g\colon Y \to \bar{\mathbb{R}}$ such that $|\partial f / \partial x(x, y)| \leqslant g(y)$ for all $x \in X$, $y \in Y$. Then F defined by (2.4.1) is differentiable, and*

$$\frac{dF}{dx}(x) = \int_Y \frac{\partial f}{\partial x}(x, y)\, d\mu(y).$$

- Parseval, Plancherel, and Fourier inversion:

version of the well-known Fourier Integral Theorem is given here.

2.6.1 Theorem. *Suppose that* $f \in \mathcal{L}_2(\mathbb{R}^n)$. *Let* Ω *be a bounded cube centre the origin, and set*

$$\hat{f}_\Omega(\xi) = (2\pi)^{-n/2} \int_\Omega e^{i\xi\cdot x} f(x)\, dx, \tag{2.6.1}$$

where $\xi \cdot x$ *denotes the scalar product in* $\mathbb{R}^n$. *Then as* $\Omega \to \mathbb{R}^n$, $\hat{f}_\Omega$ *converges in* $\mathcal{L}_2(\mathbb{R}^n)$ *to a function* $\hat{f}$ *called the Fourier transform of* f; *also*

$$\|\hat{f}\|_2 = \|f\|_2 \qquad \text{(Parseval's formula)},$$

and

$$(\hat{f}, \hat{g}) = (f, g) \qquad \text{(Plancherel's formula)}.$$

With the limit being taken in the same sense, the inversion formula is

$$f(x) = \lim_{\Omega \to \mathbb{R}^n} (2\pi)^{-n/2} \int_\Omega e^{-i\xi\cdot x} \hat{f}(\xi)\, d\xi.$$

Remark 2.0.2: Last time: any norm yields a metric: $d(f,g) := \|f - g\|$.

- Open/closed balls: $B_r(f) := \left\{ x \;\middle|\; \|f - x\| < r \right\}$.
- Bounded subsets: contained in some ball of finite radius.
- $\operatorname{diam} S = \inf_{r,f} \operatorname{diam} B_r(f)$ is the diameter of the smallest ball containing S.
- $d(f, S) := \inf_{x \in S} \|f - x\|$.
- $V = \mathbb{R}^n$ with $\|f\|_2^2 := \sum_{k \le n} f_i^2$, $\overline{B}_0(1)$ is a metric space but not a vector space.
- For L^2, there are unique least squares projections, but uniqueness may fail for L^1.
  - Counterexample: take a line $M = \{[\alpha, \alpha]\}$ in $\mathbb{R}^2$ of angle $\pi/4$ with respect to the x-axis and consider $f := [[]0, 1]$. Then for $g := (\alpha, \alpha)$, $\|f - g\| = |1 - \alpha| + |\alpha| \ge |1 - \alpha + \alpha| = 1$, and the minimizer occurs for *any* $\alpha \in [0, 1]$.
  - Similar issues may happen for L^∞ – but L^1, L^∞ have sharper tails than L^2, so this can be useful e.g. in image problems.
- If limits of sequences (f_n) exist, i.e. $\|f_n - f_m\| \to 0$, then the limiting function $f_n \to f$ is unique by the triangle inequality.
- Example from last time: $\operatorname{diag}\left(1, \frac{1}{2}, \cdots, \frac{1}{n}\right) \to A$ a compact self-adjoint operator with $\operatorname{Spec} A = \left\{ \frac{1}{n} \right\}_{n \ge 0}$.
  - What is $\ker A$? Note that $0 \in \sigma(A)$, where $\sigma(A)$ is the set where $(A - I\lambda)^{-1}$ is not defined. It turns out $\ker A = \{0\}$.

- Defining closures of subsets: for $S \subseteq V$, say $f \in \overline{S}$ iff there exists a sequence of not necessarily distinct points $f_n \in S$ with $f_n \to f$.
  - Say $S_1 \subseteq S_2 \subseteq V$ is closed in S_2 iff $S_1 = C \cap S_2$ for some C closed in V. The closure of S_1 in S_2 is $\mathrm{cl}_V(S_1) \cap S_2$.
- A set that is neither open nor closed: $X := [a,b] \cap \mathbb{Q}$, and $\partial X = [a,b] \supseteq X$ is actually larger.
- Recall the little ℓ^p norms: $\|(f_n)\|_p := \left(\sum |f_n|^p\right)^{\frac{1}{p}}$ and $\|(f_n)\|_\infty := \sup_n |f_n|$.

Exercise 2.0.3 (?)
- Prove Jensen's inequality for concave functions.
- Prove Young's inequality.
- Prove Holder's inequality.
  - Idea: consider $a = \widehat{f_n} := |f_n| / \|f_n\|^p, b = \widehat{g_n} := |g_n| / \|g_n\|_q$ and apply Young's after summing over n.
- Prove Minkowski's inequality.
  - Idea: use that $(p-1)q = p$ and apply the triangle inequality and then Holder to $\sum |f_n + g_n|^p$. Also use that $q^{-1} = 1 - p^{-1}$, and divide through this inequality at the end. Be sure to check the cases $\|f+g\|_p = 0, \infty$.

3 | Tuesday, January 18

Remark 3.0.1: Last time:

- $\|f\|_{\ell^p} = \left(\sum |f_n|^p\right)^{\frac{1}{p}}$
- $\|f\|_{\ell^\infty} = \sup_n |f_n|$.

Today:

- $\ell_p = \left\{ f := (f_n) \;\middle|\; \|f\|_{\ell^p} < \infty \right\}$.
- Example: set $f^k := (0, 0, \cdots, 1, 1, \cdots)$ with zeros for the first $k-1$ entires and ones for all remaining entries. Then $f_i^k \overset{k\to\infty}{\longrightarrow} 0$ for each fixed component at index i. So $f^k \to (0)$ component-wise, but $\|f_k\|_{\ell^\infty} = 1$ for every k, so this doesn't converge in ℓ_∞.
- Recall the ε-δ and limit definitions of continuity.
- Recall the definition of uniform continuity.
- For $\Omega \subseteq \mathbb{R}^n$, write $C(\Omega)$ for the $\mathbb{R}$-vector space of continuous bounded functionals $f : \Omega \to \mathbb{C}$ with the norm $\|f\|_{L^\infty} = \sup_{x\in\Omega} |f(x)|$.

- Define $C^k(\Omega, \mathbb{C}^m)$ to be functions with k continuous partial derivatives which are bounded, and set $C^\infty(\Omega, \mathbb{C}^m) = \cap_{k\geq 0} C^k(\Omega, \mathbb{C}^m)$. Define a norm $\|f\|_{C^k} = \sum_{j\leq k} \sup_{x\in\Omega} \left| f^{(j)}(x) \right|$.

- Take $g(x) = 2 - x^2$ and consider $\mathbb{B}_{\frac{1}{2}}(g)$ in $C[0,1]$ with $\|-\|_{L^\infty}$.

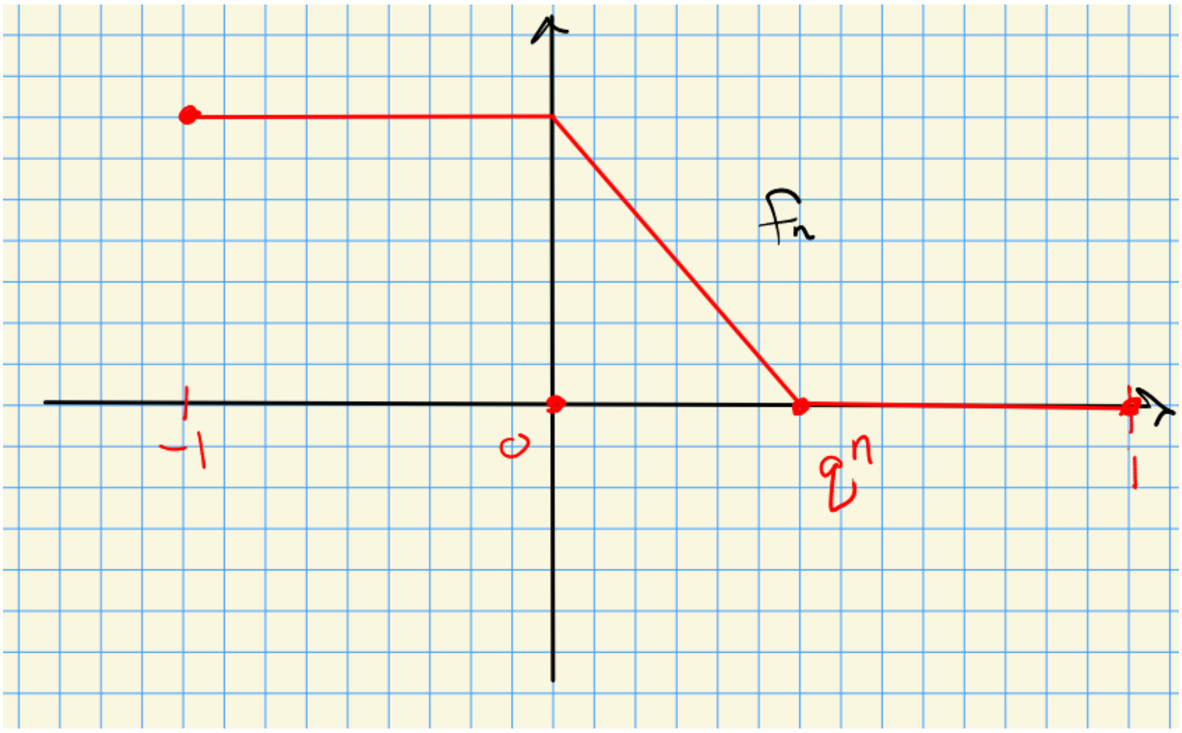
- Show that convergent implies Cauchy-convergent using the triangle inequality.

- Lipschitz with $|c| < 1$ implies Cauchy.

- Lemma 1.4.2: $\|f_{n+k} - f_n\| \leq q^n (1-q)^{-1} \|f_1 - f_0\|$ for all $k \geq 0$. Use

$$\|f_{n+k} - f_n\| = \left\| \sum_{j=1}^{k} (f_{n+j} - f_{n+j-1}) \right\| \leq \|f_1 - f_0\| \sum_{j=1}^{k} q^{n+j-1} \overset{n\to\infty}{\longrightarrow} 0.$$

- Counterexample, not all normed spaces are complete: take $V = C[-1,1]$ with $\|f\|_{L^1} := \int_{-1}^{1} |f(x)|\, dx$. Define a sequence of functions (f_n):



Check that $\|f_{n+1} - f_n\|_{L^1} \leq q \|f_n - f_{n-1}\|_{L^1}$, and pointwise $f_n \to \chi_{[-1,0]}$ which is discontinuous and not in $C[-1,1]$.

- Banach spaces: complete normed vector spaces.

- Series:

  - Convergent: $f := \lim_N \sum_{n \leq N} f_n \in V$.
  - Absolute convergence: $\sum \|f_n\| < \infty$.
  - In a Banach space, absolutely convergent series can be rearranged.

- Theorem: A normed space is complete iff absolute convergence $\implies$ convergence. Proof:

  - Step 1: show that every Cauchy sequence has a convergent subsequence.
  - Set $a_n := \sup_{m>n} \|f_n - f_m\|$, then Cauchy implies $a_n \to 0$ in $\mathbb{R}$.
  - Get a convergent subsequence $a_{n_j} \leq j^{-2}$.
  - Set $g_j := f_{n_j} - f_{n_{j+1}}$, then $g := \sum g_j$ absolutely converges, say to g.
  - Note $f_{n_i} - f_{n_{i+1}} = \sum_{j=1}^{i} g_j$, so the subsequence (f_{n_j}) is convergent.
  - Step 2: use this to show that the original sequence (f_n) converges.
  - Set $f = \lim f_n$, then $\|f_n - f\| \leq \|f_n - f_{n_i}\| + \|f_{n_i} - f\|$, using Cauchy for the first ε and the convergent subsequence for the second.

Remark 3.0.2 ***(Some random notes):*** Some theorems that hold in Hilbert spaces but not necessarily Banach spaces:

allowed.

1.5.18 Theorem. *Suppose $\mathcal{H}$ is a separable Hilbert space, and assume that $\mathcal{K} = \{\phi_n\}$ is an orthonormal set in $\mathcal{H}$. Then the following conditions are equivalent:*

(i) $\mathcal{K}$ *is complete, that is* $\mathcal{K}^\perp = 0$;

(ii) $\overline{[\mathcal{K}]} = \mathcal{H}$;

(iii) $\mathcal{K}$ *is an orthonormal basis;*

(iv) for any $f \in \mathcal{H}$

$$\|f\|^2 = \sum |(f, \phi_n)|^2 \qquad (\textit{Parseval's formula}). \tag{1.5.9}$$

the sum in (1.5.9) is zero, whence $f = 0$. □

1.5.19 Theorem. *A separable Hilbert space has an orthonormal basis.*

Proof. Let $\{f_n\}$ be a countable dense set, and apply the Gram Schmidt

2.4.12 Example. Consider the integrals

$$I_n = \int_0^\infty e^{-nx} x^{-1/2}\, dx.$$

We want to show that $\lim I_n = 0$. With $f_n(x) = e^{-nx}x^{-1/2}$, certainly $\lim_n f_n(x) = 0$ if $x \neq 0$, but the convergence is not uniform (because $\lim_{x\to 0} f_n(x) = \infty$), and using the Riemann integral it would be necessary to invent a delicate argument based on a subdivision of the range of

Absolute continuity:

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2.4.13 Definition. Let S be a finite interval. Then the real valued function f is said to be **absolutely continuous** on S iff for each $\varepsilon > 0$ there is a $\delta > 0$ such that

$$\sum_1^n |f(b_j) - f(a_j)| < \varepsilon$$

for any finite set $\{[a_j, b_j]\}$ of disjoint intervals with total length less than δ. f is absolutely continuous on $\mathbb{R}$ iff it is absolutely continuous on every finite subinterval.

2.4.14 Theorem. *A real-valued function f defined on an interval in $\mathbb{R}$ is an indefinite integral of a locally Lebesgue integrable function, g say, iff f is absolutely continuous, in which case f is differentiable almost everywhere and $f' = g$ (a.e.).*

L_p^{loc}:

2.5.2 Definition. $\mathcal{L}_p^{\mathrm{loc}}$ is the set of functions which lie in $\mathcal{L}_p(S)$ for every $S \subset X$ closed in $\mathbb{R}^n$ and bounded.

$\ker L = 0$ may not be sufficient to guarantee bijectivity in infinite dimensions:

3.3.13 Example. Let $L\colon \mathcal{C}([0,1]) \to \mathcal{C}([0,1])$ be the operator defined by

$$Lf(x) = \int_0^x f(t)\,dt \qquad (f \in \mathcal{C}([0,1])).$$

Obviously $Lf(0) = 0$ for every $f \in \mathcal{C}([0,1])$. Therefore $R(L)$ is a proper subset of $\mathcal{C}([0,1])$ and L is not surjective. However, $N(L) = 0$, for differentiation shows that the only continuous solution of $Lf = 0$ is $f = 0$. Thus condition (ii) of Theorem 3.3.12 does not imply (iii) or (i).

Boundedness:

3.4.2 Definition. Suppose that L is a linear operator from $\mathcal{B}$ into $\mathcal{C}$. L is said to be **bounded on** $D(L)$ iff there is a finite number m such that

$$\|Lf\| \leqslant m\,\|f\| \qquad (f \in D(L)). \tag{3.4.1}$$

Bounded iff continuous:

maximum gradient.

3.4.3 Theorem. *Suppose that L is a linear operator from $\mathcal{B}$ into $\mathcal{C}$. Then L is bounded on $D(L)$ if and only if it is continuous.*

Proof. If L is bounded, from (3.4.1) $f_n \to 0 \implies Lf_n \to 0$. Thus L is continuous at zero, and so by Lemma 3.4.1, L is continuous. On the other hand, if L is not bounded there is a sequence (g_n) such that $a_n = \|Lg_n\| / \|g_n\| \to \infty$. But if $f_n = g_n/(a_n\,\|g_n\|)$, then $\|f_n\| = a_n^{-1} \to 0$ and $\|Lf_n\| = 1$. Since $L0 = 0$, L is not continuous at zero, and therefore is not continuous. $\square$

4 More Banach Spaces (Thursday, January 20)

Remark 4.0.1: Last time: complete iff absolutely convergent implies convergent. Today: wrapping up some results on Banach and Hilbert spaces, skipping a review of L^p spaces, and starting on operators next week.

Remark 4.0.2: Note that $S \coloneqq (0,1]$ is open and not complete, but $\mathrm{cl}_{\mathbb{R}}(S) = [0,1]$ is both closed and complete. Generalizing:

Lemma 4.0.3 *(?)*.
A subset $S \subseteq B$ of a Banach space is complete iff S is closed in B.

Proof (?).
$\Longleftarrow$: If S is closed, a Cauchy sequence (f_n) in S converges to some $f \in B$. Since S is closed in B, in fact $f \in S$.
$\Longrightarrow$: Suppose that for $f \in \mathrm{cl}_B(S)$, there is a Cauchy sequence $f_n \to f$ with $f_n \in S$ and $f \in B$. Since S is complete, $f \in S$, so $\mathrm{cl}_B(S) \subseteq S$ making S closed. ■

Theorem 4.0.4 *(?)*.
For $\Omega \subseteq \mathbb{R}^n$, the space $X = (C(\Omega; \mathbb{C}), \|-\|_\infty)$ is a Banach space.

⚠ **Warning 4.0.5**
This is *not* complete with respect to any other L^p norm with $p < \infty$!

Proof (of theorem).
Use the lemma – write B for the space of all bounded (not necessarily continuous) functions on Ω, which is clearly a normed vector space, so it suffice to show

- $X \subseteq B$ is closed,
- B is a Banach space (i.e. complete).

Step 1: we'll show $\mathrm{cl}_B(X) \subseteq X$. Take f to be a limit point of X, then for every $\varepsilon > 0$ there is a $g \in X$ with $\|f - g\| < \varepsilon$. Apply the triangle inequality:

$$\|f\| \leq \|f - g + g\| \leq \|f - g\| + \|g\| = \varepsilon + C < \infty,$$

so $f \in \mathrm{cl}_B(X) \subseteq B$ since it is bounded. It remains to show f is continuous. Use that $\|f - g\|\infty < \varepsilon$ and continuity of g to get $|g(x) - g(x_0)| < \varepsilon$ for $|x - x_0| < \varepsilon$. Now

$$\begin{aligned} |f(x) - f(x_0)| &= |f(x) - g(x) + g(x) - g(x_0) + g(x_0) - f(x_0)| \\ &\leq |f(x) - g(x)| + |g(x) - g(x_0)| + |f(x_0) - g(x_0)| \\ &\leq 3\varepsilon. \end{aligned}$$

So X is closed in B.
Step 2: Let f_n be Cauchy in B, and note that we have a pointwise bound $|f_n(x) - f_n(x_0)| \leq \|f_n - f_m\| \to 0$. So pointwise, $f_n(x)$ is a Cauchy sequence in $\mathbb{C}$ which is complete, so $f_n(x) \to f(x)$ for some $f : \Omega \to \mathbb{C}$. We now want to show $f_n \to f$ in X. Using that f_n is Cauchy in X,

produce an N_0 such that $n, m \geq N_0 \implies \|f_n - f_m\| < \varepsilon$. Now

$$\begin{aligned}\|f - f_n\| &= \sup_{x\in\Omega} |f(x) - f_n(x)| \\ &\leq \sup_{x\in\Omega} \sup_{m\geq N_0} |f_m(x) - f_n(x)| \\ &= \sup_{m\geq N_0} \sup_{x\in\Omega} |f_m(x) - f_n(x)| \\ &= \sup_{m\geq N_0} \|f_m - f_n\| \\ &\leq \varepsilon.\end{aligned}$$

Now use the reverse triangle inequality to show f_n is bounded

$$\|f\| - \|f_n\| \leq \|f - f_n\| < \varepsilon \implies \|f\| < \infty.$$

Now by problem 1.13, every Cauchy sequence is bounded, so $f_n \to f \in B$.

■

Remark 4.0.6: Extending to vector-valued functions: for $\Omega \subseteq \mathbb{R}^n$, take $\mathbf{x} = [x_1, \cdots, x_n]$ and $F = [f_1, \cdots, f_m] : \mathbb{C}^n \to \mathbb{C}^m$. Then there is a Banach space

$$X = C^(\Omega, \mathbb{C}^m), \qquad \|f\|_{C_1(\Omega)} := \sum_{i\leq m} \sup_{x\in\Omega} |f(x)| + \sum_{i\leq m, j\leq n} \sup_{x\in\Omega} \left| \frac{\partial f_i}{\partial x_j}(x) \right|.$$

Similarly define $L^p(\Omega)$, noting that $\|f\|_{L^\infty(\Omega)}$ is the *essential supremum.*

Theorem 4.0.7 *(?).*
For $p \in (1, \infty)$, the sequence space $X = (\ell^p, \|-\|_{\ell^p})$ is a Banach space.

Definition 4.0.8 (Closed subspaces)
A **closed subspace** of a Banach space is a linear subspace $M \leq B$ which is norm-closed in B.

Example 4.0.9 *(?):*

$$M := \ker \nabla\cdot = \left\{ f \in C(\Omega) \;\middle|\; \nabla\cdot f = 0 \right\} \leq C(\Omega)$$

is closed, where $\nabla\cdot f$ is the divergence of a function f.

For any $S \subseteq B$, one can also take the corresponding closed subspace $\overline{[S]} := \text{cl}_B \operatorname{span}_{\mathbb{C}} \{s \in S\}$, i.e. all linear combinations of elements in S and their limits. This is called the **closed linear span** of S.

Exercise 4.0.10 (?)
Let $B = (C[a, b], \|-\|_{L^\infty}$ and for $x_0 \in [a, b]$ define

$$M := \left\{ f \in C[a, b] \;\middle|\; f(x_0) = 0 \right\}, \qquad N := \left\{ f \in C[a, b] \;\middle|\; f(x_0) \leq c \right\}.$$

Show that these are closed subspaces with no nontrivial open subsets of B, since any $f \in M$ can be perturbed to be nonzero at x_0 with an arbitrarily small norm difference.

Remark 4.0.11: Recall that for $S_1 \subseteq S_2 \subseteq B$, S_1 is **dense** in S_2 iff $\operatorname{cl}_{S_2}(S_1) = S_2$. Recall Weierstrass' theorem: for $\Omega \subseteq \mathbb{R}^n$ is closed and bounded and write $\mathcal{O} := \mathbb{R}[x_1, \cdots, x_n]$ for the polynomials in n variables. Then $\mathcal{O} \subseteq C(\Omega)$, and $\operatorname{cl}_{C(\Omega)}\mathcal{O} = C(\Omega)$, i.e. $\mathcal{O}$ is a dense subspace in the L^∞ norms. In fact, piecewise linear functions are dense.

Remark 4.0.12: Norms are equivalent iff $c_1 \|f\|_a \leq \|f\|_b \leq c_2 \|f\|_a$ for some constants c_i. All norms on $\mathbb{R}^n$ (resp. $\mathbb{C}^n$) are equivalent.

Example 4.0.13 ***(?):*** For $a > 0, f \in C[0,1]$, define

$$\|f\| := \sup_{x \in I} |e^{-ax} f(x)|,$$

which can be thought of as a weighting on the uniform norm which de-emphasizes the tails of functions near the endpoints. This is equivalent to $\|-\|_\infty$, since

$$e^a \sup |f| \leq \sup |e^{-ax} f| \leq 1 \cdot \sup |f|.$$

Remark 4.0.14: Note that a basis for a norm can be used as a basis with respect to an equivalent norm in finite dimensions. In infinite dimensions this may not hold – e.g. for Fourier series, $\{e_k(x)\}_{k \in \mathbb{Z}}$ is not a basis for $C[0, 2\pi]$ with the sup norm.

Definition 4.0.15 (Separable Banach spaces)
An $B \in \mathcal{B}$ is **separable** iff X contains a countable dense subset $S = \{f_k\}_{k \geq 0}$ such that for each $f \in B$ and $\varepsilon > 0$, there is an $f_k \in S$ with $\|f - f_k\| < \varepsilon$.

Example 4.0.16 ***(?):*** Show that

- $\Omega \subseteq \mathbb{R}^n$ a bounded subset, $C(\Omega), \|-\|_{\sup}$ is separable.
- ℓ^p is separable for $p \in (1, \infty)$.
- ℓ^∞ is *not* separable.

4.1 Random Notes

Remark 4.1.1:

Exercise 1.12. Recall (Example 3.12 in Prerequisite Material) that continuous function f on a X loca compact and Hausdorff space X is invertible if $1/f$ is continuous on X. What is the spectrum of $f(z) =$ in $C(\mathbb{T})$?

Definition 2.2. A nonzero homomorphism into the base field of an algebra is called a *character*. The *spectrum* of a commutative Banach algebra A, denoted $\hat{A}$, is the set of all nonzero characters from A into $\mathbb{C}$. Hence $\hat{A}$ is often called the *character space* for $\hat{A}$.

5 | Tuesday, January 25

Remark 5.0.1:

- Inner products:
  - $\langle f,\ f\rangle \geq 0$ and $\langle f,\ f\rangle = 0 \iff f = 0$
  - $\langle f,\ g+h\rangle = \langle f,\ g\rangle + \langle f,\ h\rangle$
  - $\langle f,\ g\rangle = \overline{\langle g,\ f\rangle}$
  - $\langle \alpha f,\ g\rangle = \alpha\langle f,\ g\rangle$
- A pre-Hilbert space is an inner-product space.
- Example: $\langle f,\ g\rangle := \int_\Omega w(x)f(x)\overline{g(x)}\,dx$ for $f, g \in C(\Omega)$, where w is any weighting function.
- Example: $\langle f,\ g\rangle = \sum f_i \overline{g_i}$ for $f, g \in \mathbb{C}^n$.
- Inner products induce norms: $\|f\| := \sqrt{\langle f,\ f\rangle}$
  - Orthogonality: write $f \perp g$ iff $\langle f,\ g\rangle = 0$ and $S^\perp = \left\{ f \in \mathcal{H} \;\middle|\; f \perp s \,\forall s \in S \right\}$.
- Definition of a Hilbert space: a pre-Hilbert space complete with respect to the norm induced by its inner product.
- Recall $C(\Omega)$ with $\langle f,\ g\rangle := \int_\Omega f\bar{g}$ is not complete, thus not a Hilbert space.
- Example: a common optimization problem, $\operatorname{argmin}\|x\|$ such that $Ax = 0$.

Theorem 5.0.2 ***(Cauchy-Schwarz).***

$$|\langle f,\ g\rangle| \leq \|f\|\|g\|.$$

Proof (?).
Use $\langle f,\ g\rangle = |f||g|\cos\theta_{fg}$ where $|\cos\theta| \leq 1$. - Assume $g \neq 0$, then STS $\left\langle f,\ \frac{g}{\|g\|}\right\rangle \leq \|f\|$. - Use

$$\begin{aligned} 0 &\leq \|f - \langle f,\ g\rangle g\|^2 \\ &= \langle f - \langle f,\ g\rangle g,\ f - \langle f,\ g\rangle g\rangle \\ &= \langle f,\ f\rangle - \langle f,\ g\rangle\langle g,\ f\rangle - \overline{\langle f,\ g\rangle}\langle f,\ g\rangle + \langle f,\ g\rangle\overline{\langle f,\ g\rangle} \\ &= \|f\|^2 - |\langle f,\ g\rangle|^2. \end{aligned}$$

■

Theorem 5.0.3 ***(Closest approximations).***
Let $f \in M \leq \mathcal{H}$ be a closed (and thus complete) subspace of a Hilbert space $\mathcal{H}$. Then there is a unique element g in $\mathcal{H}$ closest to M in the norm.

Proof (?).
Let $d := \operatorname{dist}(f, M)$, choose a sequence $g_n \in M$ such that $\|f - g_n\| \to d$, which is possible since $d = \inf_{g \in M} \|f - g\|$. Apply the parallelogram law to write

$$\begin{aligned}\|g_n - g_m\|^2 &= \|(g_n - f) - (g_m - f)\|^2 \\ &= 2\|g_n - f\|^2 + 2\|g_m - f\|^2 - 4\left\|\frac{1}{2}(g_n + g_m) - f\right\|^2 \\ &\leq 2\|g_n - f\|^2 + 2\|g_m - f\|^2 - 4d^2 \\ &\leq 2d^2 + 2d^2 - 4d^2 \\ &= 0,\end{aligned}$$

so the g_n are Cauchy. Here we've used that $\frac{1}{2}(g_n + g_m) = m \in M$ since M is a subspace, and $\|m - f\| \geq d$. Since M is complete, $g_n \to g \in M$ and moreover $\|f - g\| = d$. For uniqueness, if $\|f - g'\| = d$ then

$$\left\|f - \frac{1}{2}(g + g')\right\|^2 = d^2 - \|g - g'\|^2 < d^2 \qquad \lightning.$$

■

Theorem 5.0.4 ***(Projection theorem).***
Let $M \leq \mathcal{H}$ be a closed subspace of a Hilbert space. Then $M^\perp \leq \mathcal{H}$ is closed, and $\mathcal{H} = M \oplus M^\perp$. In the decomposition $f = g + h$, in fact $g \in M$ is the closest approximation to f in M, making this decomposition unique.

Proof (?).
STS $\mathcal{H} = M + M^\perp$ by the exercises. If $f \in M$, take $f = g + h$ where $g = f$ and $h = 0$, so suppose $f \notin M$. Let $g = \operatorname{argmin} \operatorname{dist}(f, M) \in M$, and we claim $f - g \in M^\perp$, so $\langle f - g,\ h\rangle = 0$ for any $h \in M$. For all $h \in M$ and $\alpha > 0$, we have $g + \alpha h \in M$, so

$$\begin{aligned}\|f - g\|^2 &\leq \|f - (g + \alpha h)\|^2 \\ &= \|f - g\|^2 - 2\Re\alpha\langle h,\ f - g\rangle + \alpha^2\|h\|^2,\end{aligned}$$

so

$$2\Re\alpha\langle h,\ f - g\rangle \leq \alpha^2\|h\|^2 \implies 2\Re\langle h,\ f - g\rangle \leq \alpha\|h\|^2 \overset{\alpha \to 0}{\longrightarrow} 0,$$

so $\Re\langle h,\ f - g\rangle = 0$. Similarly $\Im\langle h,\ f - g\rangle = 0$.

■

Exercise 5.0.5 (?)
Show $S^\perp$ is closed for any $S \in \mathcal{H}$, and in fact $S^\perp = \operatorname{span}_{\mathbb{C}}(\operatorname{cl}_{\mathcal{H}}(S))^\perp$, and if $f \in S \cap S^\perp$ then $f = 0$.

Exercise 5.0.6 (?)
Prove the parallelogram law

$$\|f - g\|^2 + \|f + g\|^2 = 2\|f\|^2 + 2\|g\|^2.$$

6 | Thursday, January 27

Remark 6.0.1: Notes:

- Bessel:

$$\sum_n |\langle f,\ \varphi_n\rangle| \leq \|f\|^2, \qquad \|f\| := \sqrt{\langle f,\ f\rangle}.$$

  - Prove using the fact that

$$0 \leq \left\| f - \sum_{n\leq m} \langle f,\ \varphi_n\rangle \varphi_n \right\|^2 = \|f\|^2 - \sum_{n\leq m} |\langle f,\ \varphi_n\rangle|^2.$$

- Best fit:

$$\left\| f - \sum_{n\leq m} c_n\varphi_n \right\| \geq \left\| f - \sum_{n\leq m} \langle f,\ \varphi_n\rangle \varphi_n \right\|,$$

  so define projections $P_M(f) := \sum \langle f,\ \varphi_n\rangle\varphi_n$ for span $\{\varphi_n\} = M$.

  - Prove using

$$\begin{aligned}\left\| f - \sum_{n\leq m} c_n\varphi_n \right\| &= \|f\|^2 - \sum_{n\leq m} |\langle f,\ \varphi_n\rangle|^2 + \sum_{n\leq m} |\langle f,\ \varphi_n\rangle - c_n|^2 \\ &\geq \|f\|^2 - \sum_{n\leq m} |\langle f,\ \varphi_n\rangle|^2.\end{aligned}$$

- Hilbert spaces are separable: have countable dense subsets.

  - $\ell^\infty(\mathbb{Z})$ is not separable.

- For $\{\varphi_n\}$ orthonormal and $\{c_n\}$ scalars, $\sum c_n\varphi_n$ is convergent iff $\{a_n\} \in \ell^2(\mathbb{Z})$, so $\sum |c_n|^2 < \infty$. If it converges, it can be rearranged, and

$$\left\|\sum c_n\varphi_n\right\|^2 = \|\{c_n\}\|^2_{\ell^2(\mathbb{Z})} = \sum |c_n|^2.$$

  - To prove, use that $\left\| \sum_{i\leq n\leq j} c_n\varphi_n \right\|^2 = \sum_{i\leq n\leq j} |c_n|^2$, so the sequence $\{S_m\}_{m\geq 0}$ where $S_m := \sum_{n\leq m} c_n\varphi_n$ is Cauchy since $\sum |c_n|^2$ converges.

- If $g = \sum c_n \varphi_n$ and $f = \sum c_{m_n} \varphi_{m_n}$ is a rearrangement, if $\{c_n\} \in \ell^2$ then $\|f\|^2 = \|g\|^2 = \sum |c_n|^2$. Then $\|f - g\|^2 = \|f\|^2 + \|g\|^2 - 2\Re\langle f,\ g\rangle$, but $\langle f,\ g\rangle = \sum |c_n|^2$, so $\|f - g\| = 0$.

- If $K = \{\varphi_n\} \subseteq \mathcal{H}$ is a proper subset, so $\operatorname{span}\{\varphi_n\} \neq \mathcal{H}$, write $P_K(f) = \sum \langle f,\ \varphi_n\rangle \varphi_n$. Then $P_K(f) = 0$ if $f \in \operatorname{cl}(K)^\perp$, $P_K(f) = f$ if $f \in \operatorname{cl}(K)$, and if $f \notin K$, since $\operatorname{cl}(K) \leq \mathcal{H}$ is closed then there exists a $g \in \operatorname{cl}(K)$ where $\|f - g\| = \operatorname{dist}(f, K)$. Write $f = h + (f - g) \in \operatorname{cl}(K) \oplus \operatorname{cl}(K)^\perp$, so $f = P_K f + (I - P_K) f$.
- Recall complete subspaces of Banach spaces are closed.
- Next theorem: every separable Hilbert space admits an orthonormal basis.

Theorem 6.0.2 *(?).*
If $\mathcal{H}$ is a separable Hilbert spaces and $K = \{\varphi_n\}$ is an orthonormal set, then TFAE

- K is complete, i.e. $K^\perp = 0$ (taking the closure is not needed),
- $\operatorname{clspan} K = \mathcal{H}$,
- K is an orthonormal *basis*,
- For all $f \in \mathcal{H}$, $\|f\| = \sum_{n \geq 0} |\langle f,\ \varphi_n\rangle|^2$ (Parseval).

Remark 6.0.3: Note $\langle f,\ \varphi_n\rangle$ is the nth Fourier coefficient $\widehat{f}(\xi) = \sum \langle f,\ \varphi_n\rangle \varphi_n(\xi)$, and Parseval says $\|f\|^2 = \left\|\widehat{f}\right\|^2$.

Proof (?).
$1 \implies 2$: Let $f \in \mathcal{H} \setminus \operatorname{cl}\operatorname{span}(K)$ and project, so $f = g + h$ with $g, h \neq 0$. But $g \in \operatorname{cl}\operatorname{span}(K)$ and $h \in \operatorname{cl}\operatorname{span}(K)^\perp = 0$, forcing $K^\perp = \operatorname{cl}(K)^\perp \neq 0$. $\lightning$
$2 \implies 3$: Follows directly from previous lemma that $f = P_K f + (I - P_K) f$.
$3 \implies 4$: Write $f \in \mathcal{H}$ as $f = \sum \langle f,\ \varphi_n\rangle \varphi_n$ by sending $m \to \infty$ in the previous lemma.
$4 \implies 1$: Toward a contradiction, suppose $f \neq 0 \in K^\perp$. Then $\|f\| \neq 0$ but $\langle f,\ \varphi_n\rangle = 0$ for all n, contradicting Parseval. $\lightning$ ∎

7 | Tuesday, February 01

Remark 7.0.1: Notes:

- Assume $\mathcal{H}$ is a **separable** Hilbert space: there exists a countable set of vectors $\{v_i\}_{i \in \mathbb{Z}}$ which span a subspace that is dense in $\mathcal{H}$, so $\operatorname{cl}(\operatorname{span}\{v_i\}) = \mathcal{H}$.
- Complete subspaces: $M \leq \mathcal{H}$ with $M^\perp = 0$.
- Show that for any $S \subseteq \mathcal{H}$, $S^\perp$ is closed in $\mathcal{H}$, $S^\perp = (\operatorname{cl}\operatorname{span} S)^\perp$, and $f \in S \cap S^\perp \implies f = 0$.
- If $K = \{\varphi_k\}_{k \in \mathbb{Z}}$ is an orthonormal set in $\mathcal{H}$, then TFAE
  - $K \leq \mathcal{H}$ is a complete subspace, so $K^\perp = 0$, i.e. $\langle f,\ \varphi_k\rangle = 0$ for all k implies $f = 0$.

- $\operatorname{cl}\operatorname{span} K = \mathcal{H}$, so every $f \in \mathcal{H}$ is the limit of a sequence of vectors from $\operatorname{span} K$.
- K is an orthonormal *basis*
- Parseval: equality in Bessel, i.e. $\|f\| = \sum |\langle f,\ \varphi_k\rangle|^2$

- Lemma: if $M, N \leq \mathcal{H}$ with $\dim M < \dim N$, then $M^\perp \cap N \neq 0$.
  - Without loss of generality assume $\dim N = n + 1$ where $n \coloneqq \dim M$, take a basis $\{\psi_k\}_{k \leq n+1}$ for N.
  - Try to find $f \in N$ with $f \perp M$, i.e. coefficients $\{b_i\}_{i \leq n}$ with $\sum b_i \psi_i \perp \varphi_k$ for every φ_k basis elements of M.
  - This is a linear system of n equations in $n + 1$ unknowns, so it has a nontrivial solution.

- Theorem, orthonormal bases are stable: if $\{\varphi_k\}$ is an orthonormal basis and $\{\psi_k\}$ is an orthonormal *system*, if $\sum \|\varphi_k - \psi_k\|^2 < \infty$ then $\{\psi_k\}$ is a basis.
  - Assume note, then find a $\psi_0 \in \mathcal{H}$ with $\|\psi_0\| = 1$ and $\langle \psi_0,\ \psi_j\rangle = 0$ for all j.
  - Choose $N \gg 1$ so that $\displaystyle\sum_{k \geq N} \|\psi_k - \varphi_k\| < 1$.
  - Use previous lemma to produce $w \in \operatorname{span}\{\psi_0, \psi_1, \cdots, \psi_N\}$ with $w \neq 0$ and $w \perp \varphi_j$ for all $j \leq N$.
  - Note $w \perp \operatorname{span}\{\psi_n\}_{n > N}$.
  - Apply Parseval:

$$
\begin{aligned}
0 &< \|w\|^2 \\
&= \sum_{n \geq 1} |\langle w,\ \varphi_n\rangle|^2 \\
&= \sum_{n \geq N+1} |\langle w,\ \varphi_n\rangle|^2 \\
&= \sum_{n \geq N+1} |\langle w,\ \varphi_n - \psi_n\rangle|^2 \\
&\leq \|w\|^2 \sum_{n \geq N+1} \|\varphi_n - \psi_n\|^2 \\
&< \|w\|^2 \cdot 1,
\end{aligned}
$$

    where we've used that $\langle w,\ \psi_n\rangle = 0$ for $n \geq N$. ↯

- $\mathcal{H}$ admits a countable orthonormal basis iff $\mathcal{H}$ is separable.
  - $\implies$: clear, since the basis is countable, and every element is a limit of partial sums against the basis.
  - $\impliedby$: Gram-Schmidt.
    - ◇ $h_1 = \psi_1$ and $\varphi_1 = h_1/\|h_1\|$
    - ◇ $h_n = \psi_n - \displaystyle\sum_{1 \leq k \leq n-1} \langle \psi_k,\ \varphi_k\rangle \varphi_k$ and normalize.

- Exercise: a closed subspace of a separable Hilbert space is separable.
- Linearly isometric inner product spaces: $E \sim F$ iff there is a map $A : E \twoheadrightarrow F$ with
  - $A(au + bv) = aAu + bAv$
  - $\|Au\|_F = \|u\|_E$

- Theorem: if $\mathcal{H}_1, \mathcal{H}_2$ are infinite dimensional separable Hilbert spaces, then $\mathcal{H}_1 \sim \mathcal{H}_2$. Thus for any Hilbert space $\mathcal{H}$ over $\mathbb{C}$, $\mathcal{H} \sim \ell^2(\mathbb{C})$.
  - Pick orthonormal bases $\{\varphi_k\} \subseteq \mathcal{H}_1, \{\psi_k\} \subseteq \mathcal{H}_2$.
  - For $u \in \mathcal{H}_1$, define $Au := \sum \langle u,\ \varphi_k\rangle \psi_k$, which converges – this will be the linear isometry, and satisfies condition (i).
  - Check $\|Au\|^2_{\mathcal{H}_2} = \sum_{k\geq 1} |\langle u,\ \varphi_k\rangle|^2 = \|u\|_{\mathcal{H}_2}$, which is condition (ii).
  - Check A is surjective: for $y \in \mathcal{H}_2$, write $y = \sum_k \langle y,\ \psi_k\rangle \psi_k = Av$ for $v := \sum_k \langle y,\ \psi_k\rangle \varphi_k \in \mathcal{H}_1$.
- Non-separable spaces: look at *almost-periodic functions*.
  - E.g. $\sum_{k\leq n} c_k \exp(i\lambda_k t)$ for $\lambda_k \in \mathbb{R}$.

8 | Tuesday, February 08

Remark 8.0.1: Motivating question: when is an operator equation solvable? Today: relation between boundedness and continuity for linear operators. Nonlinear operators next week.

- A map of vector spaces $V \to W$ is a linear map defined on some domain $D(A) \subseteq V$, where $D(A)$ need not equal V.
  - Notation: $A(f) = Af = g$.
  - $Af \subseteq W$ is the image of f, and $R(A) := \left\{ Af \;\middle|\; f \in D(A) \right\} \subseteq W$ is the range. Preimages of $S \subseteq W$: $A^{-1}(S) = \left\{ f \;\middle|\; f \in D(A) \text{ and } Af \in S \right\}$.
- We distinguish operators with different domains, e.g. $Af := f'$ can be the formula for distinct operators A_1, A_2 where $D(A_1) = C^\infty[0,1] \subseteq C^0[0,1]$ or $D(A_2) = C^1[0,1] \subseteq C^0[0,1]$, so $A_1 \neq A_2$.
  - I.e. $A_1 = A_2 \iff D(A_1) = D(A_2)$ and $A_1 f = A_2 f$ for all $f \in D(A_1) = D(A_2)$.
  - If $A_1 f = A_2 f$ with $D(A_1) \subseteq D(A_2)$, say A_2 is an extension of A_1. The extension is proper iff $D(A_1) \neq D(A_2)$.
- Example operator: the Laplace equation $\Delta f = g$ where $\Delta = \partial_{xx} + \partial_{yy}$. We can take domains $g \in C[0,1], L^2[0,1], H^2[0,1] = \left\{ f \in L^2(0,1) \;\middle|\; \partial_{xx} f, \partial_{yy} f \in L^2(0,1) \right\}$.
  - Why domains matter: boundary conditions affect what eigenfunctions you get. Examples where $A_1 \neq A_2$:
  - Dirichlet boundary conditions: $\Delta f = g,\ f|_{\partial\Omega} = 0$. The relevant solution spaces is $D(A_1) = \left\{ \varphi \in C^2[0,1]^2 \;\middle|\; \varphi|_{\partial\Omega} = 0 \right\}$ for $A_1\varphi := \Delta\varphi$.

- Neumann boundary conditions: $\Delta f = g$, $\left.\frac{\partial f}{\partial \mathbf{n}}\right|_{\partial\Omega} = 0$, i.e. there is no flux across the boundary. The relevant solution space is $D(A_2) = \left\{ \psi \in C^2[0,1]^2 \;\middle|\; \left.\frac{\partial \psi}{\partial \mathbf{n}}\right|_{\partial\Omega} = 0 \right\}$ for $A_2\psi \coloneqq \Delta\psi$.

- Injectivity: for $A : V \to W$, for every $g \in R(A)$ there is exactly one $f \in D(A)$ with $Af = g$. Equivalently for linear operators, $Af = 0 \implies f = 0$.
- Surjectivity: $R(A) = W$.
- Example: $A \coloneqq x \mapsto \sin(x)$ regarded as a function $A : \mathbb{R} \to \mathbb{R}$ is neither surjective nor injective: $R(A) = [-1, 1] \subsetneq \mathbb{R}$, and $\sin(\pi\mathbb{Z}) = 0$.
- Linearity: for $Lf = g$, L is linear if $L(af + bg) = aLf + bLg$.

Exercise 8.0.2 (?)
Show that the following are equivalent conditions for continuity of $A : V \to W$ at $f_0 \in D(A)$:

- $\|Af - Af_0\|_W < \varepsilon$ for all $f \in D(A)$ with $\|f - f_0\|_V < \delta$
- For every sequence $\{f_k\} \to f_0$, $Af_k \to Af_0$.
- Preimages of open sets in W are again open in V.

9 | Tuesday, February 15

Remark 9.0.1: Last time:

- Continuous operators are bounded:
  - If $\|Lf_n\| = 1$ and $\|f_n\| \to 0$, check $\lim(Lf_n) = L(\lim f_n) = L0 = 0$.
  - Take norms to contract $\|Lf_n\| = 1$.

Theorem 9.0.2 *(3.4.4).*
If $L : B \to C$ with dense image (so $\text{cl}_B(D(L)) = B$), if L is continuous on $D(L)$ then it has a unique extension $\tilde{L}$ to all of B, so $D(\tilde{L}) = B$, with $\|L\| = \left\|\tilde{L}\right\|$.

Proof (of theorem).
In steps:

- Defining the extension:
  - For $f \in B$, pick $f_n \to f$ with $f_n \in D(L)$ using density.
  - Convergent implies Cauchy, so estimate:

$$\|Lf_n - Lf_m\| = \|L(f_n - f_m)\| \le \|L\|\|f_n - f_m\| \to 0.$$

  - Thus Lf_n is Cauchy, by completeness $Lf_n \to g$ for some g.

- Preservation of norm:
  - Define the extension as $\tilde{L}f := g$, by continuity it is independent of the sequence $\{f_k\} \to f$.
  - Check that $\tilde{L}$ is a bounded operator:

$$\left\|\tilde{L}f\right\| := \|g\| = \|\lim Lf_n\| = \lim \|Lf_n\| \le \lim \|L\|\|f_n\| = \|L\|\|f\| \implies \left\|\tilde{L}\right\| \le \|L\|.$$

  - Since $\|A\|_{\text{op}}$ is defined in terms of sups over test functions in $D(A)$ for any operator A and here $D(\tilde{L}) \supseteq D(L)$ is a larger set, we have $\left\|\tilde{L}\right\| \ge \|L\|$ by definition, yielding $\left\|\tilde{L}\right\| = \|L\|$.
- Uniqueness of the extension:
  - Take $\tilde{L}_1, \tilde{L}_2$ extending L, then

$$\tilde{L}_1 f = \lim \tilde{L}_1 f_n = \lim Lf_n = \lim \tilde{L}_2 f_n = \tilde{L}_2 f.$$

  - Use linearity:

$$\tilde{L}_1 f - \tilde{L}_2 f = (\tilde{L}_1 - \tilde{L}_2) f = 0 \implies \tilde{L}_1 - \tilde{L}_2 = 0.$$

■

Example 9.0.3 ***(?):*** Let $\mathcal{L} \in L(\mathbb{C}^n, \mathbb{C}^n)$ be defined in coordinates by $(\mathcal{L}f)_i := \sum_{1 \le j \le n} \alpha_{ij} f_j$ for $1 \le i \le n$. Take $\|-\|_{\ell^\infty}$ and check

$$\begin{aligned} \|Lf\|_\infty &:= \sup_i \left|\sum \alpha_{ij} f_j\right| \\ &\le \left(\sup_i \sum |\alpha_{ij}|\right) \sup_j |f_j| \\ &:= m\|f\|_{\ell^\infty}. \end{aligned}$$

So $\|L\| \le m$, where m is the largest row sum. Is there an f for which equality holds? In this case, we'd need

$$\|Lf\|_{\ell^\infty} \ge m\|f\|_{\ell^\infty}.$$

Identify the row k so that $m = \sum_{1 \le j \le n} |\alpha_{kj}|$. Set f to be a unit vector with coefficients $(f)_j =$

$\overline{\alpha_{kj}}/|\alpha_{kj}|$. Then

$$\begin{aligned}
\|Lf\|_\infty &= \sup_i \left|\sum_j \alpha_{ij} f_j\right| \\
&\geq \left|\sum_j \alpha_{kj} f_j\right| \\
&= \left|\sum_j \alpha_{kj} \bar{\alpha}_{kj}/|\alpha_{kj}|\right| \\
&= \sum_j |\alpha_{kj}| \\
&= m\|f\|_{\ell^\infty}.
\end{aligned}$$

So the answer is yes in this case. Does this also work for $\|-\|_{\ell^p}$ with $p \in (1, \infty)$? Recall Holder's inequality:

$$\begin{aligned}
\left|\sum \alpha_{ij} f_j\right| &\leq \left(\sum |\alpha_{ij}|^q\right)^{\frac{1}{q}} \left(\sum |f_j|^p\right)^{\frac{1}{p}} \\
&= \left(\sum |\alpha_{ij}|^q\right)^{\frac{1}{q}} \|f\|_{\ell^p}.
\end{aligned}$$

Check that

$$\begin{aligned}
\|Lf\|_{\ell^p}^p &= \sum_i |(Lf)_i|^p \\
&= \sum_i \left|\sum_j \alpha_{ij} f_j\right|^p \\
&\leq \sum_i \left(\sum_j |\alpha_{ij}|\right)^{\frac{p}{q}} \|f\|_{\ell^p}^p,
\end{aligned}$$

where we've applied Holder in the last line. Thus

$$\|L\| \leq \left(\sum_i \left(\sum_j |\alpha_{ij}|\right)^{\frac{p}{q}}\right)^{\frac{1}{p}}.$$

Exercise 9.0.4 *(?)*
Is there an f that attains this bound in the ℓ^p case?

Remark 9.0.5: For $\mathcal{L} \in L(\mathbb{C}^\infty, \mathbb{C}^\infty)$ defined by $(Lf)_i = \sum_{j\geq 11} \alpha_{ij} f_j$ for $j \geq 1$, one needs a notion of convergence of the coordinates α_{ij} in order for $\mathcal{L}$ to be bounded. A sufficient condition is $m := \sup_i \sum_{j\geq 1} |\alpha_{ij}| < \infty$.

Definition 9.0.6 (?)
Some notation:

$$
\begin{aligned}
|||\alpha|||_1 &\coloneqq \sup_j \sum_i |\alpha_{ij}| \\
|||\alpha|||_p &\coloneqq \left(\sum_i \left(\sum_j |\alpha_{ij}|^q \right)^{\frac{p}{q}} \right)^{\frac{1}{p}} \\
|||\alpha|||_\infty &\coloneqq \sup_i \sum_j |\alpha_{ij}|.
\end{aligned}
$$

Remark 9.0.7: Note that if $\mathcal{L} : \ell^p \to \ell^p$, then $\|L\| \leq |||\alpha|||_p$ for $p \in [1, \infty)$ and for $p = \infty$ this is an equality.

Example 9.0.8 ***(Kernels):*** Consider $C[a, b]$ with $\|-\|_\infty$ and $k \in C^0([a, b]^{\times 2}, \mathbb{C})$. Define

$$
\begin{aligned}
K : C[a, b] &\to C[a, b] \\
f &\mapsto \int_a^b k(x, y) f(y) \, dy.
\end{aligned}
$$

What is $\|K\|$? Estimate

$$
\begin{aligned}
\|Kf\| &\leq \sup_{y \in [a,b]} |f(y)| \sup_{x \in [a,b]} \int_a^b |k(x, y)| \, dy \\
&\leq \|f\|_\infty \|k\|_\infty,
\end{aligned}
$$

so $\|K\| \leq \|k\|_\infty$.

Define

$$
\begin{aligned}
|||k|||_1 &\coloneqq \sup_y \int |k| \, dx \\
|||k|||_p &\coloneqq \left(\int \left(\int |k|^q \, dy \right)^{\frac{p}{q}} \, dx \right)^{\frac{1}{p}} \\
|||k|||_\infty &\coloneqq \sup_x \int |k| \, dy.
\end{aligned}
$$

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