Algebraic Geometry Problem Sets

Problem Set 3

D. Zack Garza
University of Georgia
dzackgarza@gmail.com

Last updated: 2022-04-01

Table of Contents

Contents

Table of Contents 2

1 Problem 1 3

2 Problem 2 5

3 Problem 3 6

4 Problem 4 8

5 Problem 5 10

1 | Problem 1

Problem 1.0.1 (Problem 1)
Let I be an index category, $\mathcal{A}$ an abelian category, and $\mathcal{A}^I$ be the category of functors $F : I \to \mathcal{A}$. Prove that the functor

$$\varprojlim_{i\in I} : \mathcal{A}^I \to \mathcal{A}, \quad F \mapsto \varprojlim_{i\in I} F_i$$

is left exact. (By duality, the functor $\varinjlim_{i\in I}$ is right exact.)
What is this functor in the case when I is a poset and F_i is a collection of stalks on the space $X = I$ with poset topology?

Solution (Part 1):
It suffices to show that $\varprojlim_{i\in I}$ is a right adjoint functor, and right adjoints are left exact by general homological algebra.

Claim: There is an adjunction

$$\begin{tikzcd} \mathsf{A} \arrow[r, shift left, "\Delta"] \arrow[r, phantom, "\perp"] & \mathsf{A}^{\mathsf{I}} \arrow[l, shift left, "\varprojlim_{i\in I}"] \end{tikzcd},$$

where Δ is the diagonal functor:

$$\begin{aligned} \Delta : \mathsf{A} &\to \mathsf{A}^{\mathsf{I}} \\ X &\mapsto \Delta_X \\ (X \xrightarrow{f} Y) &\mapsto (\Delta_X \xrightarrow{\eta_f} \Delta_Y) \end{aligned}$$

where

- The constant functor $\Delta_X : \mathsf{I} \to \mathsf{C}$ is defined on objects $i \in \mathsf{I}$ as $\Delta_X(i) := X$ and on morphisms $i \xrightarrow{\iota_{ij}} j$ as $\Delta_f(\iota_{ij}) = X \xrightarrow{\mathrm{id}_X} X$.
- η_f is a natural transformation of functors with components given by f:

$$\begin{tikzcd} \mathsf{I} & & \mathsf{C} & & & & \\ i \arrow[d, "\iota_{ij}"'] & & \Delta_X(i) \arrow[r, "\eta_f(i)"] \arrow[d, "\Delta_X(\iota_{ij})"] & \Delta_Y(i) \arrow[d, "\Delta_Y(\iota_{ij})"'] & & X \arrow[r, "f"] \arrow[d, "\mathrm{id}_X"] & Y \arrow[d, "\mathrm{id}_Y"'] \\ j \arrow[r, phantom, "\xRightarrow{\Delta:\mathsf{I}\to\mathsf{C}}"] & & \Delta_X(j) \arrow[r, "\eta_f(j)"'] & \Delta_Y(j) \arrow[r, phantom, "="] & & X \arrow[r, "f"] & Y \end{tikzcd}$$

Link to Diagram

Why this claim is true: this follows immediately from the fact that there is a natural isomorphism

$$\operatorname*{Hom}_{\mathsf{A}}(X, \lim F) \xrightarrow{\sim} \operatorname*{Hom}_{\mathsf{A}^{\mathsf{I}}}(\Delta_X, F),$$

i.e. maps from an object X into the limit of F are equivalent to natural transformations between the constant functor Δ_X and F. This follows from the fact that a morphism $X \to \lim F$ in A is the data of a family of compatible maps $\{f_i\}_{i \in \mathsf{I}}$ over the essential image of F:

$$\begin{tikzcd}
\mathsf{I} & \mathsf{A} & & \\
i & F(i) & & \\
{} \arrow[r, Rightarrow, "F"] & {} & \lim\limits_{i} F \arrow[ul] \arrow[dl] & X \arrow[l, dashed, "\exists !"'] \arrow[ull, bend right, "f_i"'] \arrow[dll, bend left, "f_j"] \\
j \arrow[uu] & F(j) \arrow[uu] & &
\end{tikzcd}$$

Link to Diagram

On the other hand, a natural transformation $\Delta_X \to F$ is precisely the same data:

$$\begin{tikzcd}
\mathsf{I} & \mathsf{A} & \\
i & F(i) & \Delta_X(i) = X \arrow[l, bend right, "f_i"'] \\
j \arrow[u, "g"] & F(j) \arrow[u, "F(g)"] & \Delta_X(j) = X \arrow[l, bend left, "f_j"] \arrow[u, equal, "F(g)=\mathrm{id}_X"']
\end{tikzcd}$$

Link to Diagram

Solution (Part 2):
If $\mathsf{I} = \mathsf{Open}(X)$ where X is given the order topology and $F : \mathsf{Open}X \to \mathsf{A}$ is a functor specified by stalks, lim sends F to the universal object $\lim F$ living over the essential image of F in A:

$$X \rightsquigarrow \mathsf{I} = \mathsf{Open}(X) \xRightarrow[F\in\mathsf{A}^{\mathsf{I}}]{} \mathsf{A}$$

$$\begin{tikzcd}
 & 3 & \\
1 \arrow[ur] & & 2 \arrow[ul]
\end{tikzcd}
\qquad
\begin{tikzcd}
 & X=\{1,2,3\} & \\
\{1,3\} \arrow[ur] & & \{2,3\} \arrow[ul] \\
 & \{3\} \arrow[ul] \arrow[ur] & \\
 & \emptyset \arrow[u] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & F(X) \arrow[dl] \arrow[dr] & & \lim F \arrow[ll, dashed] \arrow[dlll, dashed] \arrow[dl, dashed] \arrow[dddll, dashed] \\
F_1 \arrow[dr] & & F_2 \arrow[dl] & \\
 & F_3 \arrow[d] & & \\
 & 0 & &
\end{tikzcd}$$

Link to Diagram

The object corresponding to global sections $F(X) \in \mathsf{A}$ seems to also satisfies this universal property, so a conjecture would be that this construction recovers $\lim F \cong F(X) := \Gamma(X; F)$.

2 | Problem 2

Problem 2.0.1 (Problem 2)
In the category of abelian groups compute $\operatorname{Tor}_i^{\mathbb{Z}}(\mathbb{Z}_n, M)$, the left derived functors of $N \mapsto N \otimes_{\mathbb{Z}} M$.

Solution:

Claim:

$$\operatorname{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, M) \cong \ker(M \xrightarrow{\times n} M) \cong \left\{ m \in M \;\middle|\; nm = 0_M \right\},$$

which is the kernel of multiplication by n, and $\operatorname{Tor}_{\mathbb{Z}}^{i>1}(\mathbb{Z}/n\mathbb{Z}, M) = 0$.
Why this is true: in $\mathsf{R}\text{-}\mathsf{Mod}$, free implies flat, and Tor is balanced and can thus be resolved in either variable, so this can be computed by tensoring a free resolution of $\mathbb{Z}/n\mathbb{Z}$ and using the long exact sequence in Tor:

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z} \arrow[r, hook, "\times n"] \arrow[d, Rightarrow, "(-)\otimes_{\mathbb{Z}} M"] & \mathbb{Z} \arrow[r, two heads] & \mathbb{Z}/n\mathbb{Z} \arrow[r] & 0 \\
 & \mathbb{Z}\otimes_{\mathbb{Z}} M \cong M \arrow[r, "(\times n)\otimes \mathrm{id}_M"] & \mathbb{Z}\otimes_{\mathbb{Z}} M \cong M \arrow[r, two heads] & \mathbb{Z}/n\mathbb{Z}\otimes_{\mathbb{Z}} M \arrow[r] & 0 \\
 & \mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z},\mathbb{Z}) = 0 \arrow[r] & \mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z},\mathbb{Z}) = 0 \arrow[r] & \mathrm{Tor}_1^{\mathbb{Z}/n\mathbb{Z}}(\mathbb{Z}, M) \arrow[ull] & \\
 & \mathrm{Tor}_2^{\mathbb{Z}}(\mathbb{Z},\mathbb{Z}) = 0 \arrow[r] & \mathrm{Tor}_2^{\mathbb{Z}}(\mathbb{Z},\mathbb{Z}) = 0 \arrow[r] & \mathrm{Tor}_2^{\mathbb{Z}/n\mathbb{Z}}(\mathbb{Z}, M) \arrow[ull] &
\end{tikzcd}$$

Link to Diagram

In the resulting long exact sequence, since $\mathbb{Z}$ is free, thus flat, thus tor-acyclic, the first two columns vanish in degrees $d \geq 1$. As a result, in degrees $d \geq 2$, the terms $\mathrm{Tor}_d^{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, M)$ are surrounded by zeros and thus zero, meaning that only Tor_1 survives. By exactness, $\mathrm{Tor}_1(\mathbb{Z}/n\mathbb{Z}, M)$ is isomorphic to the kernel of the next map in the sequence, which is precisely $\ker(M \xrightarrow{\times n} M)$ after applying the canonical isomorphism

$$\begin{aligned} \mathbb{Z} \otimes_{\mathbb{Z}} M &\to M \\ n \otimes m &\mapsto nm. \end{aligned}$$

3 | Problem 3

Problem 3.0.1 (Problem 3)
Let k be a field and $R = k[x, y]$. In the category of R-modules compute

- $\mathrm{Ext}_R^n(R, m)$
- $\mathrm{Ext}_R^n(m, R)$, and
- $\mathrm{Tor}_n^R(m, m)$,

where $m = (x, y)$ is the maximal ideal at the origin.

Solution (Problem 3):
Note that R is a free R-module, and so $\mathrm{Ext}_R^n(R, M) = 0$ for any R-module M. This is because Ext can be computed using a free resolution of either variable. For $\mathrm{Ext}_R^n(R, m)$, compute this

as $\mathbb{R}\operatorname*{Hom}_R(-,m)$ evaluated at R. Take the free resolution

$$\cdots \to 0 \to R \xrightarrow{\mathrm{id}_R} R \to 0,$$

delete the augmentation and apply the contravariant $\operatorname*{Hom}_R(-,m)$ to obtain

$$0 \to \operatorname*{Hom}_R(R,m) \cong m \to 0 \to \cdots,$$

and take homology to obtain

$$\mathrm{Ext}^0_R(R,m) \cong m, \qquad \mathrm{Ext}^{>0}_R(R,m) = 0.$$

Compute $\mathrm{Ext}_R(m,R)$ as $\mathbb{R}\operatorname{Hom}(m,-)$ applied to R proceeds similarly: using the same resolution, applying covariant $\operatorname*{Hom}_R(m,-)$ yields

$$0 \to \operatorname*{Hom}_R(m,R) \to 0 \to \cdots,$$

and taking homology yields

$$\mathrm{Ext}^0_R(m,R) \cong \operatorname*{Hom}_R(m,R) \qquad \mathrm{Ext}^{>0}_R(m,R) = 0.$$

For the Tor calculation, we can use the Koszul resolution of m:

$$0 \to k[x,y] \xrightarrow{\cdot[x,y]} k[x,y] \oplus k[x,y] \xrightarrow{\cdot\,{}^t([-y,x])} \langle x,y\rangle \to 0,$$

so the differentials are $t \mapsto [tx,ty]$ and $[u,v] \mapsto -uy+vx$ respectively. More succinctly, this resolution is

$$0 \to R \xrightarrow{d_1} R^{\oplus 2} \xrightarrow{d_2} m \to 0,$$

so we can delete m and apply $(-)\otimes_R m$ to obtain

$$0 \to R \otimes_R m \xrightarrow{d_1\otimes \mathrm{id}_m} R^{\oplus 2} \otimes_R m \to 0$$

which simplifies to

$$C_\bullet \coloneqq 0 \to m \xrightarrow{\tilde{d}_1 \coloneqq [x,y]} m \oplus m \to 0$$

and thus we can compute Tor as the homology of this complex. We have

$$\begin{aligned}
\operatorname{Tor}_0^R(m,m) &= H^0(C_\bullet) \\
&= \operatorname{coker} \tilde{d}_1 \\
&= \frac{m \oplus m}{xm \oplus ym} \\
&\cong \frac{m}{xm} \oplus \frac{m}{ym} \\
&= \frac{\langle x,y\rangle}{\langle x^2,y\rangle} \oplus \frac{\langle x,y\rangle}{\langle x,y^2\rangle} \\
&= \left\{ f(x,y) := c_1 x \in k[x,y] \;\middle|\; c_1 \in k \right\} \oplus \left\{ g(x,y) := c_1 y \in k[x,y] \;\middle|\; c_1 \in k \right\} \\
&\cong k \oplus k
\end{aligned}$$

$$\begin{aligned}
\operatorname{Tor}_1^R(m,m) &= H^1(C_\bullet) \\
&= \ker \tilde{d}_1 \\
&= \left\{ t \in \langle x,y\rangle \;\middle|\; [tx,ty] = [0,0] \right\} \\
&= 0
\end{aligned}$$

$$\begin{aligned}
\operatorname{Tor} \geq 2^R(m,m) &= H^{\geq 2}(C_\bullet) \\
&= 0.
\end{aligned}$$

4 | Problem 4

Problem 4.0.1 (Problem 4)
Let $0 \to F' \to F \to F'' \to 0$ be a short exact triple of sheaves and assume that F' is flasque. Prove that the sequence

$$0 \to \Gamma\left(F'\right) \to \Gamma(F) \to \Gamma\left(F''\right) \to 0$$

of the spaces of global sections is exact.

Solution (Using cohomology):

Claim: Flasque sheaves are F-acyclic for the functor global sections functor $F(-) := \Gamma\left(X;-\right)$.

Proof (of claim).
Proved in class. ∎

Applying the functor $\Gamma(X; -)$ to the given short exact sequence of sheaves produces a long exact sequence of abelian groups in its right-derived functors. Using the claim above, we have $\mathbb{R}^i\Gamma(X; \mathcal{F}') = 0$ for $i \geq 1$, and thus we have the following:

$$
\begin{tikzcd}
0 \arrow[r] & \mathcal{F}' \arrow[r] & \mathcal{F} \arrow[r] \arrow[d, Rightarrow, "\Gamma(X;-)"] & \mathcal{F}'' \arrow[r] & 0 \\
0 \arrow[r] & \Gamma(X; \mathcal{F}') \arrow[r] & \Gamma(X; \mathcal{F}) \arrow[r] & \Gamma(X; \mathcal{F}'') \arrow[dll] & \\
& \mathbb{R}^1\Gamma(X; \mathcal{F}') = 0 \arrow[r] & \mathbb{R}^1\Gamma(X; \mathcal{F}) \arrow[r, "\sim"] & \mathbb{R}^1\Gamma(X; \mathcal{F}'') \arrow[dll] & \\
& \mathbb{R}^2\Gamma(X; \mathcal{F}') = 0 \arrow[r] & \mathbb{R}^2\Gamma(X; \mathcal{F}) \arrow[r, "\sim"] & \cdots &
\end{tikzcd}
$$

Link to Diagram

In particular, since $\mathbb{R}^1\Gamma(X; \mathcal{F}') = 0$, the first row forms the desired short exact sequence. As a corollary, we also obtain $\mathbb{R}^i\Gamma(X; \mathcal{F}) \cong \mathbb{R}^i\Gamma(X; \mathcal{F}'')$ for all $i \geq 1$.

Solution (Direct):
First, we'll modify the notation slightly and give names to the maps involved. We'll use the following convention for restrictions of sheaf morphisms to opens and stalks:

$$
\begin{tikzcd}
0 \arrow[r] & A \arrow[rr, "f"] \arrow[dr] \arrow[ddd, bend left] & & B \arrow[rr, "g"] \arrow[dr] \arrow[ddd, bend left] & & C \arrow[rr] \arrow[dr] \arrow[ddd, bend left] & & 0 \\
& 0 \arrow[r] & A(X) \arrow[rr, "F"] \arrow[d] & & B(X) \arrow[rr, "G"] \arrow[d] & & C(X) \arrow[d] & \\
& 0 \arrow[r] & A(U) \arrow[rr, "F|_U"] \arrow[dl] & & B(U) \arrow[rr, "G|_U"] \arrow[dl] & & C(U) \arrow[dl] & \\
0 \arrow[r] & A_x \arrow[rr, "f_x"] & & B_x \arrow[rr, "g_x"] & & C_x \arrow[rr] & & 0
\end{tikzcd}
$$

Link to Diagram

Given $c \in C(X)$, our goal is to produce a $b \in B(X)$ such that $g(b) = c$, and the strategy will be to use surjectivity at stalks to produce a maximal section of B mapping to c, and argue that

it must be a section over all of X. This will proceed by showing that if a lift is not maximal, sections over open sets that are missed can be extended using that A is flasque, contradicting maximality.
Write $c|_x$ for the image of c in the stalk C_x; by surjectivity of $g_x : B_x \twoheadrightarrow C_x$ we can find a germ b_x with $g_x(b_x) = c_x$. The germ lifts to some set $U \ni x$ and some $b \in B(U)$ with $b \mapsto c|_U$ under $F|_U : B(U) \to C(U)$. So define a poset of all such lifts:

$$P := \left\{ (U, b \in B(U)) \;\middle|\; F|_U(b) = c|_U \right\}$$
$$\text{where } (U_1, b_1) \le (U_2, b_2) \iff U_1 \subseteq U_2 \text{ and} b_2|_{U_1} = b_1.$$

As noted above, P is nonempty, and every chain $\{(U_i, b_i)\}_{i \in I}$ has an upper bound given by $(\tilde{U}, \tilde{b})$ where $\tilde{U} := \cup_{i \in I} U_i$ and $\tilde{b}$ is the unique glued section of B restricting to all of the b_i, which exists by the sheaf property for B. Thus Zorn's lemma applies, and (reusing notation) we can assume (U, b) is maximal with respect to this property.
The claim is that U must be all of X. Toward a contradiction, suppose not – then pick any $x \in X \setminus U$, and again using surjectivity on stalks at x, produce an open set $V \ni x$ and a section $b' \in B(V)$ with $G|_V(b') = c|_V$. Now on the overlap $W := U \cap V$, both b and b' map to $c|_W$, and so

$$G|_W(b|_W - b'|_W) = c|_W \, c|_W = 0 \implies b - b' \in \ker G|_W = \operatorname{im} F|_W,$$

where we've used exactness in the middle spot in the exact sequence $A(W) \to B(W) \to C(W)$. So there is some $\alpha \in A(W)$ with $F|_W(\alpha) = b|_W - b'|_W$, and since A is flasque this can be extended to a global section $\tilde{\alpha} \in A(X)$. Write $\tilde{\beta} := F(\tilde{\alpha}) \in B(X)$ with $\tilde{\beta}\Big|_W = b|_W - b'|_W$ in $B(W)$. We can now glue $\tilde{\beta}$ to a section over $U \cup V$ which extends the original section b: setting $\widehat{b} := \tilde{\beta} + b'$ yields

$$\widehat{b}\Big|_W = (b|_W - b'|_W) + b' = b|_W,$$

so this section over $U \cup V$ agrees with b on the overlap $W = U \cap V$, and thus by existence and uniqueness of gluing (using the sheaf property of B) $\widehat{b} \in B(U \cup V)$ is a section extending b over a set that strictly contains U. This contradicts the maximality of the pair (U, b).

5 | Problem 5

Problem 5.0.1 (Problem 5)
For a sheaf F on X, let

$$S(F) = \prod_{x \in X} (i_x)_* F_x, \quad i_x : x \to X$$

be the sheaf of all, possibly discontinuous section of the étale space of F. The canonical flasque resolution of F is

$$\underline{S}(F) := 0 \to F \to S(F_0) \to S(F_1) \to S(F_2) \to \dots$$

where $F_0 = F$ and F_i are defined inductively as $F_{i+1} = S(F_i)/F_i$. Some books define cohomology groups $\mathbf{H}^n(X, F)$ as the cohomology groups of the complex

$$0 \to \Gamma(S(F_0)) \to \Gamma(S(F_1)) \to \Gamma(S(F_2)) \to \dots$$

Prove that they coincide with the cohomology defined by other means by showing that this gives an exact δ-functor and that $\mathbf{H}^n$ are effaceable for $n > 0$ through the following steps:

(1) A homomorphism $F \to G$ induces a canonical homomorphism of resolutions $\underline{S}(F) \to \underline{S}(G)$.

(2) A short exact triple $0 \to F' \to F \to F'' \to 0$ induces a short exact triple of complexes $0 \to \underline{S}(F') \to \underline{S}(F) \to \underline{S}(F'') \to 0$.

(3) Applying Γ to it gives a short exact triple of complexes, i.e. $0 \to S(F'_n) \to S(F_n) \to S(F''_n) \to 0$ is exact. (You can assume the previous problem.)

(4) $(\mathbf{H}^n)$ is an exact δ-functor.

(5) For $n > 0, \mathbf{H}^n(F) \to \mathbf{H}^n(S(F))$ is the zero map.

Conclude by Grothendieck's universality theorem.

Solution (Part 1):
This follows readily from the fact that a morphism $f : F \to G$ of sheaves on X induces group morphisms $f_x : F_x \to G_x$ on stalks for every $x \in X$. Letting $y \in X$ be arbitrary, there is a morphism

$$\varphi_y : \prod_{x \in X} F_x \xrightarrow{\pi_y} F_y \xrightarrow{f_y} G_y$$

where π_y is the canonical projection out of the product. By the universal property of the product, the φ_y assemble to a morphism

$$S(f) : \prod_{x \in X} F_x \to \prod_{y \in X} G_y.$$

So there is a morphism $S(F_0) \to S(G_0)$ at the first stage of the complex. This induces a morphism on the quotient sheaves $S(F_0)/F_0 \to S(G_0)/G_0$, and thus by the same argument as above, a morphism on the second stage $S(S(F_0)/F_0) \to S(S(G_0)/G_0)$, i.e. a morphism $S(F_1) \to S(G_1)$. Continuing inductively yields levelwise morphisms $S(F_i) \to S(G_i)$. The claim is that these assemble to a chain map

$$
\begin{tikzcd}
 & & & S(F)/F \arrow[dr, hook, dashed, "S(-)"] \arrow[ddd, dashed, bend left] & & \\
0 \arrow[r] & F \arrow[r] \arrow[d, "f"] & S(F_0) = S(F) \arrow[ur, two heads, dashed] \arrow[rr] \arrow[d, "S(f)"] & & S(F_1) = S(S(F)/F) \arrow[r] \arrow[d, "S_1(F)"] & \cdots \\
0 \arrow[r] & G \arrow[r] & S(G_0) = S(G) \arrow[dr, two heads, dashed] \arrow[rr] & & S(G_1) = S(S(G)/G) \arrow[r] & \cdots \\
 & & & S(G)/G \arrow[ur, hook, dashed, "S(-)"] & &
\end{tikzcd}
$$

Link to Diagram

To see this is true, it is enough to show that the first square commutes, i.e. that applying $S(-)$ to a morphism of sheaves produces a commuting square. This is because every other square has a factorization as indicated, where the square in red naturally commutes since it involves canonically induced maps on quotients/cokernels, and the other half of the square arises by applying the S construction to some morphism of sheaves.

However, this square can be readily seen to commute using the following: first regard the sections of $\mathcal{F}$ as continuous sections of its espace étale $\text{Ét}_F \xrightarrow{\pi} X$ and regarding sections of $S(F)$ as arbitrary (potentially discontinuous) sections of π. Then $\mathcal{F} \leq S(F)$ is clearly a subsheaf and $F \to S(F)$ is an inclusion of spaces of sections.

Solution (Part 2):

By part 1, it is clear there are morphisms $\underline{S}(F') \to \underline{S}(F) \to \underline{S}(F'')$ of complexes of sheaves, yielding a double complex:

$$
\begin{tikzcd}
 & \vdots & \vdots & \vdots & \\
 & S(F'_1) \arrow[r] \arrow[u] & S(F_1) \arrow[r] \arrow[u] & S(F''_1) \arrow[u] & \\
 & S(F'_0) \arrow[r] \arrow[u] & S(F_0) \arrow[r] \arrow[u] & S(F''_0) \arrow[u] & \\
0 \arrow[r] & F' \arrow[r] \arrow[u] & F \arrow[r] \arrow[u] & F'' \arrow[r] \arrow[u] & 0 \\
 & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}
$$

Link to Diagram

It suffices to show injectivity, exactness, and surjectivity respectively along each horizontal row. Exactness is a local condition, so it suffices to show exactness on stalks.

Claim: For any open U, the following sequence at the first stage of the complex is exact:

$$0 \to S(F')(U) \to S(F)(U) \to S(F'')(U) \to 0.$$

Proof (of claim).
This follows because $S(F')(U) = \prod_{x \in U} F'_x$ and similarly for F, F'', and so if $f : F' \to F$ is injective on sheaves, then $f_x : F'_x \to F_x$ is injective on stalks. ∎

Now apply the functor $\underrightarrow{\operatorname{colim}}_{U \ni p}(-)$ to this exact sequence and use that taking stalks is exact (despite not generally being a *filtered* colimit) to conclude

$$0 \to S(F')_x \to S(F)_x \to S(F'')_x \to 0.$$

is exact for all $x \in X$, thus making the following sequence exact:

$$0 \to S(F'_0) \to S(F_0) \to S(F''_0) \to 0$$

Our double complex is now the following:

$$\begin{tikzcd}
 & \vdots & \vdots & \vdots & \\
? \arrow[r] & S(F'_1) \arrow[r] \arrow[u] & S(F_1) \arrow[r] \arrow[u] & S(F''_1) \arrow[r] \arrow[u] & ? \\
0 \arrow[r] & S(F'_0) \arrow[r] \arrow[u] & S(F_0) \arrow[r] \arrow[u] & S(F''_0) \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & F' \arrow[r] \arrow[u] & F \arrow[r] \arrow[u] & F'' \arrow[r] \arrow[u] & 0 \\
 & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

Link to Diagram

To see that

$$0 \to S(F'_k) \to S(F_k) \to S(F''_k) \to 0$$

is exact for all k, we can truncate this complex:

$$\begin{tikzcd}
 & 0 & 0 & 0 & \\
? \arrow[r] & S(F_0')/F_0' \arrow[u] \arrow[r] & S(F_0)/F_0 \arrow[u] \arrow[r] & S(F_0'')/F_0'' \arrow[u] \arrow[r] & ? \\
0 \arrow[r] & S(F_0') \arrow[u] \arrow[r] & S(F_0) \arrow[u] \arrow[r] & S(F_0'') \arrow[u] \arrow[r] & 0 \\
0 \arrow[r] & F' \arrow[u] \arrow[r] & F \arrow[u] \arrow[r] & F'' \arrow[u] \arrow[r] & 0 \\
 & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

Link to Diagram

The row highlighted in red is exact by the Nine Lemma, regarding each row as a chain complex, and since applying $S(-)$ is exact, by applying this to the top row we obtain

$$\begin{tikzcd}
 & \vdots & \vdots & \vdots & \\
0 \arrow[r] & S(F_1') \arrow[u] \arrow[r] & S(F_1) \arrow[u] \arrow[r] & S(F_1'') \arrow[u] \arrow[r] & 0 \\
0 \arrow[r] & S(F_0') \arrow[u] \arrow[r] & S(F_0) \arrow[u] \arrow[r] & S(F_0'') \arrow[u] \arrow[r] & 0 \\
0 \arrow[r] & F' \arrow[u] \arrow[r] & F \arrow[u] \arrow[r] & F'' \arrow[u] \arrow[r] & 0 \\
 & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

Link to Diagram

The remaining rows are exact by repeating this argument inductively, and regarding the columns as complexes, we obtain the desired exact sequences of complexes by deleting the first row.

Solution (Part 3):
Note: there may be a typo in the statement of this problem, so what I will show is that the following sequence of complexes is exact:

$$0 \to \Gamma\left(X; \underline{S}(F')\right) \to \Gamma\left(X; \underline{S}(F)\right) \to \Gamma\left(X; \underline{S}(F'')\right) \to 0.$$

Take the double complex from part (2) and apply the functor $\Gamma\left(X; -\right)$ to obtain the following double complex:

$$\begin{tikzcd}
 & \vdots & \vdots & \vdots & \\
0 \arrow[r] & \Gamma\left(X; S(F'_1)\right) \arrow[r] \arrow[u] & \Gamma\left(X; S(F_0)\right) \arrow[r] \arrow[u] & \Gamma\left(X; S(F''_1)\right) \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & \Gamma\left(X; S(F'_0)\right) \arrow[r] \arrow[u] & \Gamma\left(X; S(F_0)\right) \arrow[r] \arrow[u] & \Gamma\left(X; S(F''_0)\right) \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & \Gamma\left(X; F'\right) \arrow[r] \arrow[u] & \Gamma\left(X; F\right) \arrow[r] \arrow[u] & \Gamma\left(X; F''\right) \arrow[r] \arrow[u] & \mathbb{R}^1\Gamma\left(X; F'\right) \\
 & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

Link to Diagram

Here the bottom row continues in the long exact sequence for the right-derived functors of $\Gamma\left(X; -\right)$, i.e. sheaf cohomology. Since the desired sequence of complexes involved truncating this double complex by deleting the first row, consider everything from row two upward. That these levelwise maps assemble to a map of complexes is just a consequence of functoriality of $\Gamma\left(X; -\right)$, and left exactness preserves the zeros in the left-most column, so it suffices to show that the right-most column (highlighted in red) is zero as claimed.
However, this follows from the previous problem if the sheaves $S(F'_n)$ are all flasque. This is immediate since they are sheaves of discontinuous sections, and such a section on U can always be extended to a global section by simply assigning any other values on $X \setminus U$ – any choice works, since no compatibility (e.g. continuity) is required.

Solution (Part 4):
It is a general theorem in homological algebra that a short exact sequence of chain complexes induces a long exact sequence in cohomology. In this case, if we take the vertical homology of the above double complex, by the snake lemma there are connecting morphisms:

$$
\begin{tikzcd}
 & \vdots & \vdots & \vdots & \\
0 \arrow[r] & H_2(\Gamma\left(X; \underline{S}(F')\right)) \arrow[u] \arrow[r] & H_2(\Gamma\left(X; \underline{S}(F)\right)) \arrow[u] \arrow[r] & H_2(\Gamma\left(X; \underline{S}(F'')\right)) \arrow[u] \arrow[r] \arrow[ull, dashed, "\exists\delta_2"] & 0 \\
0 \arrow[r] & H_1(\Gamma\left(X; \underline{S}(F')\right)) \arrow[u] \arrow[r] & H_1(\Gamma\left(X; \underline{S}(F)\right)) \arrow[u] \arrow[r] & H_1(\Gamma\left(X; \underline{S}(F'')\right)) \arrow[u] \arrow[r] \arrow[ull, dashed, "\exists\delta_1"] & 0 \\
0 \arrow[r] & H_0(\Gamma\left(X; \underline{S}(F')\right)) \arrow[u] \arrow[r] & H_0(\Gamma\left(X; \underline{S}(F)\right)) \arrow[u] \arrow[r] & H_0(\Gamma\left(X; \underline{S}(F'')\right)) \arrow[u] \arrow[r] \arrow[ull, dashed, "\exists\delta_0"] & 0 \\
 & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}
$$

Link to Diagram

Solution (Part 5):
This holds because flasque sheaves are F-acyclic for $F(-) = \Gamma\left(X; -\right)$, so we can conclude that $\mathbf{H}^n(S(F)) = 0$ for $n > 0$ since the sheaves $S(F)$ are always flasque for any sheaf F.

Note: I realized at the last minute that this argument may not actually work, since this $\mathbf{H}^n$ a priori has nothing to do with $\mathbb{R}\Gamma(X; -)$ computed via injective resolutions.