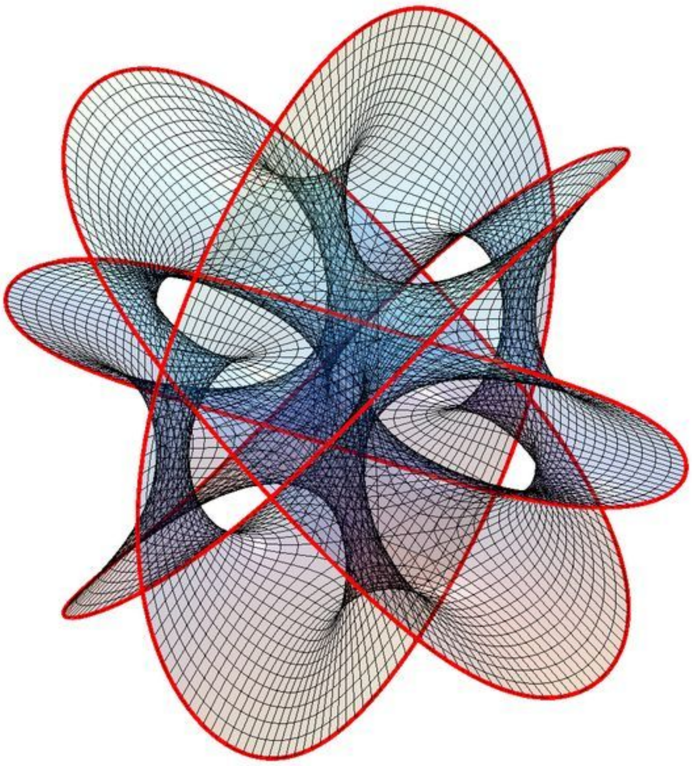

Notes: These are notes live-tex'd from a graduate course in Sheaf Cohomology taught by Valery Alexeev at the University of Georgia in Spring 2022. As such, any errors or inaccuracies are almost certainly my own.

Sheaf Cohomology

Lectures by Valery Alexeev. University of Georgia, Spring 2022

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1 | Intro, Motivations (Monday, January 10)

Remark 1.0.1: Topic: cohomology of sheaves and derived categories. The plan:

- Sheaves (see ELC notes)
- Derived functors and coherent sheaves (see ELC notes)
- Derived categories (Gelfand-Manin, Tohoku)

References:

- Valery's notes (see ELC)
- Gelfand-Manin, *Methods of Homological Algebra*.

Remark 1.0.2: Compare (genus g) Riemann surfaces in the classical topology to (genus g, projective) algebraic curves over $\mathbb{C}$ in the Zariski topology. Recall that

$$H^*(\Sigma_g; \mathbb{Z}) = \begin{cases} \mathbb{Z} & * = 0, 2 \\ \mathbb{Z}^{2g} & * = 1 \\ 0 & \text{else.} \end{cases}$$

Note that this is a linear invariant in the sense that the constituents are free abelian groups, and we can extract a numerical invariant. For surfaces up to homeomorphism, this distinguishes them completely.

For algebraic curves, note that the topology is very different: the only closed sets are finite. In this topology,

$$H^*(X; \mathbb{Z}) = \begin{cases} \mathbb{Z} & * = 0 \\ 0 & \text{else,} \end{cases}$$

which doesn't see the genus at all. In fact all such curves are homeomorphic in this topology, witnessed by picking any bijection and noting that it sends closed sets to closed sets. The linear replacement: $H^*(X; \mathcal{O}_X)$ for $\mathcal{O}_X$ the structure sheaf, which yields

$$H^*(X; \mathcal{O}_X) = \begin{cases} \mathbb{C} & * = 0 \\ \mathbb{C}^g & * = 1 \\ 0 & \text{else.} \end{cases}$$

These surfaces can be parameterized by the moduli space $\mathcal{M}_g$, which is dimension $3g - 3$ for $g \geq 2$.

Remark 1.0.3: The POV in classical topology is to fix the coefficients: $\mathbb{Z}, \mathbb{R}, \mathbb{C}, \mathbb{Z}/n$, or R a general ring. A minor variation is to consider a local system $\mathcal{L}$, which are locally constant but may have nontrivial monodromy around loops. For example, one might have $\mathbb{R}$ locally, but traversing a loop induces an automorphism $f \in \operatorname{Aut}(\mathbb{R}) = \mathbb{R}^\times$. In this setting, we have a functor $F(-) = H(-; R)$.

For sheaf cohomology, instead fix X and take $G(-) = H(X; -)$. In general, one can take sheaves of abelian groups, $\mathcal{O}_X$-modules, quasicoherent sheaves, or coherent sheaves:

$$\mathsf{Sh}(X, \mathsf{AbGrp}) \hookrightarrow \mathcal{O}_{\mathsf{X}}\text{-}\mathsf{Mod} \hookrightarrow \mathsf{QCoh}(X) \hookrightarrow \mathsf{Coh}(X).$$

Remark 1.0.4: We'll be looking at three kinds of topologies:

- The order topology: start with a poset and define the open sets to be the *decreasing/lower sets*, i.e. subsets U_{x_0} that contain every element below a point x_0. In other words, if $x \in U$ and $y \leq x$, then $y \in U$.
- The Zariski topology: let R be a DVR, so $\operatorname{Spec} R = \{\langle 0 \rangle, \mathfrak{m}\}$. E.g. for $R \coloneqq \mathbb{C}[t]$, $\mathfrak{m} = \langle t \rangle$, and the open sets are $\{\langle 0 \rangle\}, \operatorname{Spec} R$, corresponding to the poset $\mathrm{pt} \to \mathrm{pt}$.
- The classical topology, usually paracompact and Hausdorff.

One can define sheaves in all three cases, which have different properties. For posets, e.g. one can take $C^0(-, R)$ for $R = \mathbb{R}, \mathbb{C}, \mathbb{Z}_{\widehat{p}}$.

Remark 1.0.5: Some computational tools:

- Vanishing theorems
- Riemann-Roch

2 | Topological Notions (Wednesday, January 12)

Remark 2.0.1: Some topological notions to recall:

- T_0, Kolmogorov spaces: distinct points don't have the exact same neighborhoods, i.e. there exists a neighborhood of x not containing y **or** a neighborhood of y not containing x.
- T_1, Frechet spaces: points are separated, so replace "or" with "and" above.
- T_2, Hausdorff spaces: points are separated by disjoint neighborhoods.
- Alexandrov spaces: arbitrary intersections of opens are open.
- Metrizability
- Paracompactness

Remark 2.0.2: Recall that a topology τ on X satisfies

- $\emptyset, X \in \tau$
- $A, B \in \tau \implies A \cap B \in \tau$
- $\bigcup_{j \in J} A_j \in \tau$ if $A_j \in \tau$ for all j.

Equivalently one can specify the closed sets and require closure under finite unions and arbitrary intersections.

Example 2.0.3 ***(of topologies):*** Running examples:

- Any subset $S \subseteq \mathbb{R}^n$ is Hausdorff and paracompact.
- Order topologies on posets
- Zariski topologies on varieties over $k = \bar{k}$, e.g. mSpec A for $A \in \mathsf{Alg}_{/k}^{\mathrm{fg}}$ or affine schemes Spec A.
- The discrete/initial topology $\tau = 2^X$.
- The indiscrete topology $\tau = \{\emptyset, X\}$.

Remark 2.0.4: Recall the separation axioms:

- T_0: points can be topologically distinguished. Note that the indiscrete topology s not T_0 if $\sharp X \geq 2$.
- T_1: points can be separated by (not necessarily disjoint) neighborhoods. Equivalently, points are closed.
- T_2/Hausdorff: points can be separated by disjoint neighborhoods.
- $T_{3.5}$/Tychonoff:?
- T_6:?

Exercise 2.0.5 (?)
Show that points are closed in X iff X is T_1.

Definition 2.0.6 (Paracompactness)
A space X is **paracompact** iff every open cover $\mathcal{U} \rightrightarrows X$ admits a *locally* finite refinement $\mathcal{V} \rightrightarrows X$, i.e. any $x \in X$ is in only finitely many V_i.

Exercise 2.0.7 (Euclidean space is paracompact)
Show that any $S \subseteq \mathbb{R}^n$ is paracompact, and indeed any metric space is paracompact.

Solution:
Let $\mathcal{U} \rightrightarrows X := \mathbb{R}^d$ be an open cover and define a proposed locally open refinement in the following way:

- Write $\mathcal{U} := \left\{ U_\alpha \;\middle|\; \alpha \in A \right\}$ for some index set.
- Use that $W_n := \mathrm{cl}_X(\mathbb{B}_n(\mathbf{0}))$ is compact, and since $\mathcal{U} \rightrightarrows W_n$ there is a finite subcover $\mathcal{V}_n := \{U_{n,1}, \cdots, U_{n,m}\} \rightrightarrows \mathrm{cl}_X(\mathbb{B}^n(\mathbf{0}))^c$.
- Show that $\mathcal{V} := \{\mathcal{V}\}_{n \in \mathbb{Z}_{\geq 0}}$ is an open refinement of $\mathcal{U}$.

- Why: it is a subcollection, and every $x \in X$ is in a ball of radius $R \approx N := \lceil \|x\| \rceil$. So $x \in \mathbb{B}_N(0)$, thus $x \in U_{N,k}$ for some k.

- Show that $\mathcal{V}$ is locally finite.
  - Why: each $\mathcal{V}_n$ misses the $\mathbb{B}_{k<n}(0)$, so each $x \notin \bigcup_{k \geq N} \mathcal{V}_n$ if N is defined as above. So x is in only finitely many $\mathcal{V}_n$.

Fact 2.0.8
Paracompact spaces admit a POU – for $\mathcal{U} \rightrightarrows X$, a collection A of function $f_\alpha : X \to \mathbb{R}$ where for all $\alpha \in \operatorname{supp} f_\alpha = \operatorname{cl}(\{f \neq 0\})$, for all $x \in X$, there exists a $V \ni x$ such that for only finitely many α, $f_\alpha|_V \not\equiv 0$, and $\sum_{\alpha \in A} f_\alpha(x) = 1$.

Remark 2.0.9: Recall the order topology: for $(P, \leq)$ a poset, so

- $x \leq y, y \leq x \implies x = y$,
- $x \leq y \leq z \implies x \leq z$
- $x \leq x$

Define

- Open sets to be increasing sets, so $x \in U, x \leq y \implies y \in U$,
- Closed sets to be decreasing sets, so $x \in U, x \geq y \implies y \in U$

Note that this is a free choice!

Exercise 2.0.10 (?)
Show that the order topology is closed under arbitrary unions *and* intersections of opens.

Exercise 2.0.11 (?)
Show that the order topology is not T_1 by showing $\operatorname{cl}_P(\{x\}) = Z_{\leq}(x) := \left\{ y \in P \;\middle|\; y \leq x \right\}$.

Fact 2.0.12
For k an infinite field, $\mathbb{A}^1_{/k}$ is the cofinite topology and thus not Hausdorff.

3 | Friday, January 14

3.1 Posets, Zariski Topologies

Remark 3.1.1: Recall the definition of a **poset**.

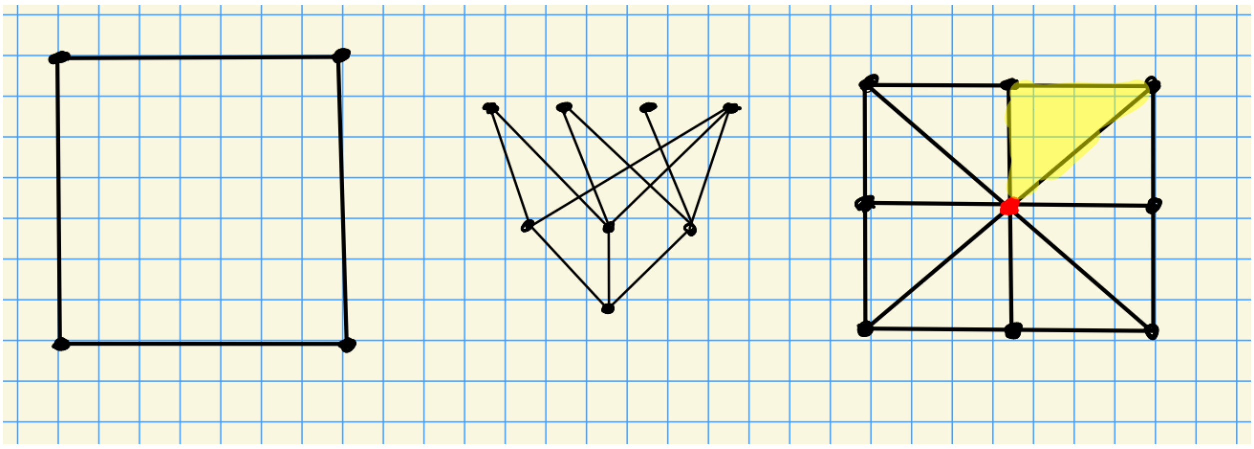
Example 3.1.2 ***(?):*** Given a polytope, one can take its face poset $\mathrm{FP}(P) = \{F \leq P\}$ where $F_1 \leq F_2$ iff $F_1 \subseteq F_2$ for the faces F_i. More generally, one can take a complex of polytopes, i.e. a collection of polytopes that only intersect at faces. An example of a complex is the fan of a toric variety.

Similarly, one can take **cones** $\sum c_i \mathbf{v}_i \subseteq \mathbb{R}^d$ for some positive coefficients.

Remark 3.1.3: Conversely, given a poset I, one can associate a simplicial complex $|I|$, the geometric realization. Any chain $i_{n_1} < \cdots i_{n_k}$ is sent to a face and glued.

Example 3.1.4 ***(?):*** Consider a polytope P, taking the face poset $\mathrm{FP}(P)$, and its geometric realization $|\mathrm{FP}(P)|$. A square has

- $\sharp P_2 = 1$
- $\sharp P_1 = 4$
- $\sharp P_0 = 1$



Note that one can take the geometric realization of a category by using the nerve to first produce a poset.

Remark 3.1.5: With the right choices, there exists a continuous map $|I| \to I$ where I is given the order topology. Pulling back sheaves on the latter yields constructible sheaves on convex objects, which are locally constant on the interior components.

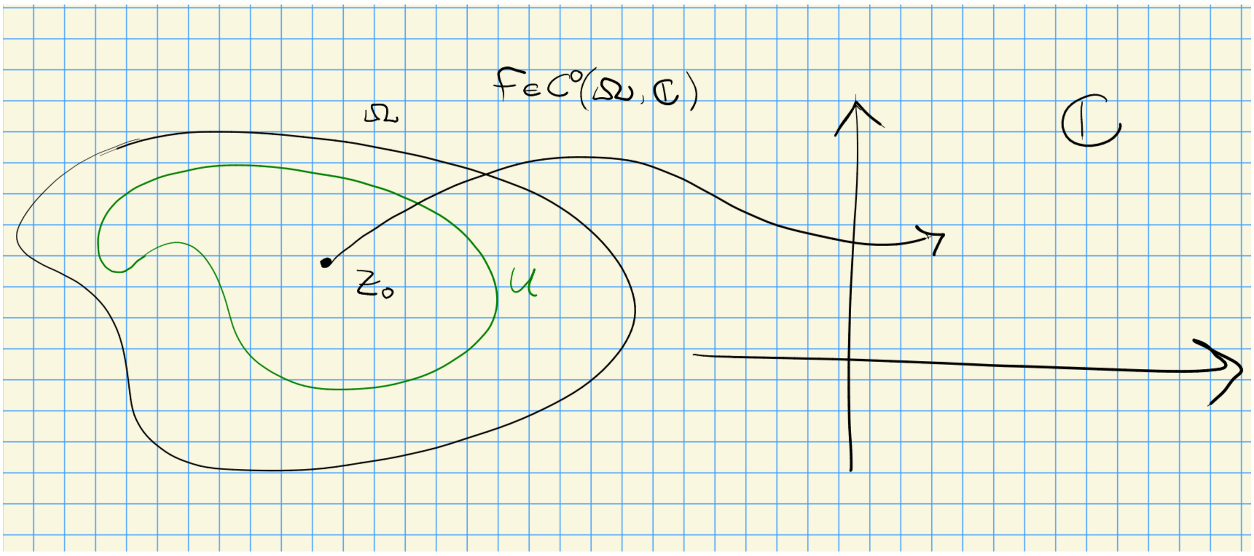
Remark 3.1.6: A first version of the Zariski topology: let $k = \bar{k} \in \mathsf{Field}$ and let $R \in \mathsf{Alg}^{\mathrm{fg}}_{/k}$ be of the form $R = k[x_1, \cdots, x_n]/\left\langle f_a \right\rangle$. We can consider $X \coloneqq \mathrm{mSpec}\, R \subseteq \mathbb{A}^n_{/k}$ as the points $\mathbf{x} \in k^{\times^n}$ such that $f_a(\mathbf{x}) = 0$. Recall Noether's theorem – the f_a can be replaced with a finite collection. The closed subsets are of the form $V(g_b)$. Note that this is T_1 since points are closed: given $\mathbf{p} = [p_0, \cdots, p_n]$, take $f(\mathbf{p}) = \prod_{i \leq n} (x - p_i)$ so that $V(f) = \{\mathbf{p}\}$. These points biject with maximal ideals in R.

Remark 3.1.7: An improved version of the Zariski topology: $X = \operatorname{Spec} R$, including prime ideals. The points are as before, and additionally for every irreducible subvariety $Z \subseteq X$, there is a generic point η_Z. This adds new points which can't be described in coordinates.

Remark 3.1.8: Note that this generalizes to arbitrary (associative, commutative) rings. For rings that aren't finitely generated, one loses the coordinate interpretation. These generally won't embed into $\mathbb{A}^n_{/k}$ for any n, but can be embedded into (say) $\mathbb{A}^1_{/R}$. Use that a closed embedding $X \hookrightarrow Y$ corresponds precisely to a surjection of associated rings $R_Y \twoheadrightarrow R_X$.

3.2 Sheaves

Example 3.2.1 ***(?):*** Let $U \subseteq \Omega \subseteq \mathbb{C}$ and consider $C^0(\Omega, \mathbb{C}) := \operatorname{Hom}_{\mathsf{Top}}(\Omega, \mathbb{C})$ – this forms a sheaf of abelian groups, $\mathbb{C}$-algebras, rings, sets, etc.



We'll refer to this as $\mathcal{O}_X^{\text{cts}}$.

Remark 3.2.2: Some properties:

- For every $\iota : V \subseteq U \implies$ there is a restriction ma

$$\begin{aligned} \mathcal{F}(\iota) : \mathcal{F}(U) &\to \mathcal{F}(V) \\ f &\mapsto f|_V. \end{aligned}$$

- $\mathcal{F}(\emptyset^{\downarrow}) = {}_{\uparrow}$, so e.g. for rings ${}_{\uparrow} = \{0\}$ is the zero ring.

- (Sheaf 1, uniqueness): if $\mathcal{U} \rightrightarrows U$ and $s_1, s_2 \in \mathcal{F}(\mathcal{U})$, then $s_1|_{U_i} = s_2|_{U_i} \implies s_1 = s_2$.
- (Sheaf 2, existence): if $s_i \in \mathcal{F}(U_{ij})$ and $s_1|_{U_{ij}} = s_2|_{U_{ij}}$, then there is a global section $s \in \mathcal{F}(U_1 \cup U_2)$.

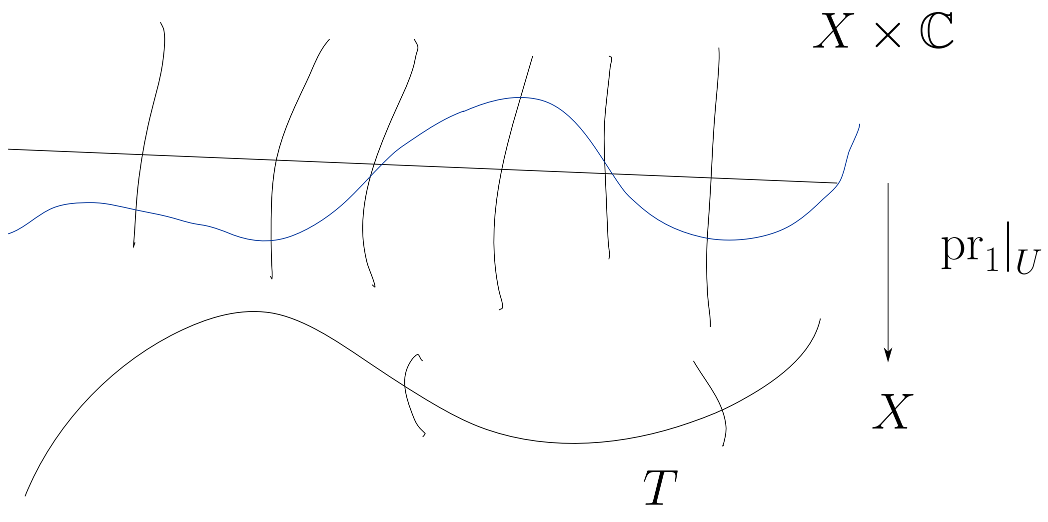
Example 3.2.3 ***(?):*** Other examples of sheaves:

- $\mathcal{O}_X^{\text{cts}}$. One can check the sheaf properties directly.
- $\mathcal{O}_X^{\text{hol}} = \mathcal{O}_X^{\text{an}}$ the holomorphic (complex differentiable) and thus analytic (locally equal to a convergent power series) functions on X.
- Given a fixed continuous map $f : Y \to X$, setting $\mathcal{F}(U) = \{s : U \to Y\}$ the set of continuous sections of f.

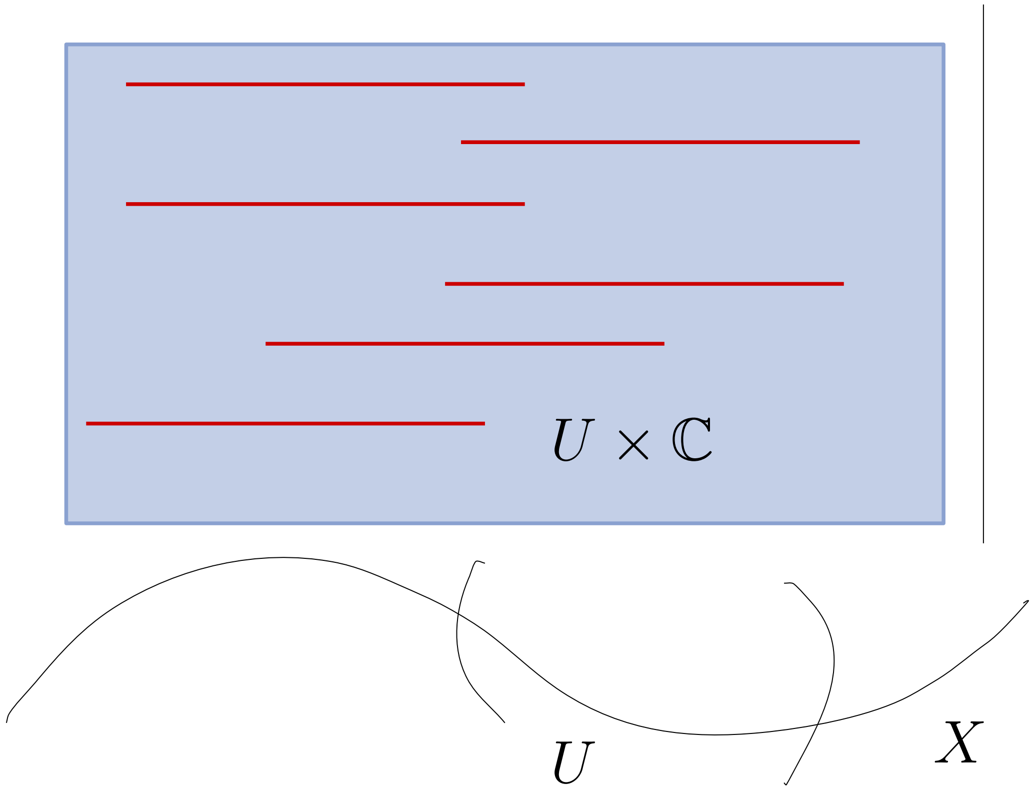
4 | Wednesday, January 19

Example 4.0.1 ***(of sheaves):*** Some examples of sheaves:

- For $X \subseteq \mathbb{C}$ open, consider $\text{pr}_1 : X \times \mathbb{C} \to X$ and consider the space of continuous sections $\mathcal{O}_X^{\text{cts}}(U) := \underset{\text{Top}}{\text{Hom}}(U, U \times \mathbb{C})$.



- Analytic functions $\mathcal{O}_X^{\text{an}}$
- $\mathcal{O}_X^{\text{cts}}$ where $\mathbb{C}$ is given the discrete topology instead of the Euclidean topology. The opens in $U \times \mathbb{C}$ are of the form $U \times V$ for $V \subseteq \mathbb{C}$ any set at all:



- Constant sheaves $\underline{\mathbb{C}}(U)$ defined as the locally constant continuous $\mathbb{C}$-valued functions on U.

Remark 4.0.2: Recall the sheaf properties:

- $U \to F(U)$ and $\iota_{U,V} \mapsto \operatorname{Res} F(V), F(U)$.
- $\emptyset^{\downarrow} \mapsto F(\emptyset^{\downarrow}) = {}_{\uparrow}$.
- Sheaf conditions:
  - Unique gluing: $\mathcal{U} \rightrightarrows X$ with $\underset{X,U_i}{\operatorname{Res}}\, s = \underset{X,U_i}{\operatorname{Res}}\, t \implies s = t \in F(X)$
  - Existence of gluing: $\{s_i \in F(U_i)\}$ with $\underset{U_i,U_{ij}}{\operatorname{Res}}\, s_i = \underset{U_j,U_{ij}}{\operatorname{Res}}\, s_j$ implies $\exists! s \in F(X)$ with $\underset{X,U_i}{\operatorname{Res}}\, s = s_i$ for all i.

Example 4.0.3 ***(?):*** Recall that a **basis** of a topology is a collection B_i where every $U \in \tau_X$ can be written as $\bigcup_{i \in I} B_i$ for some index set $I = I(X)$. Some examples:

- For $X \in \mathsf{AlgVar}_{/k}$, the distinguished opens $D(f) = \{f \neq 0\}$ and $Z(f) = \{f = 0\}$.
- For $X = \operatorname{Spec} R \in \mathsf{AffSch}_{/k}$, take $D(f) = \left\{\mathfrak{p} \in \operatorname{Spec} R \;\middle|\; f \neq 0 \in R/\mathfrak{p}\right\} = \left\{\mathfrak{p} \in \operatorname{Spec} R \;\middle|\; f \notin \mathfrak{p}\right\}$
  - Note that $\mathcal{O}_{\operatorname{Spec} R}(D(f)) = R\left[f^{-1}\right]$.

Exercise 4.0.4 (?)
Formulate the sheaf condition with a basis instead of arbitrary opens.

Hint: keep all of the same conditions, but since intersections may not be basic opens, write $B_\alpha \cap B_\beta = \cup_k B_k$.

Remark 4.0.5: Some upcoming standard notions:

- Stalks F_x
- Sheafification $F \mapsto F^+$

A less standard topic:

- The espace etale or "flat space" of F.

Definition 4.0.6 (Stalks)
Recall that

$$F_x = \underset{\substack{\longrightarrow \\ U \ni x}}{\operatorname{colim}} F(U) = \{(U, s \in F(U))\} / \sim \quad (U, s) \sim (V, t) \iff \exists W \supseteq U, V, \operatorname*{Res}_{U,W} s = \operatorname*{Res}_{V,W} t.$$

Example: $\mathcal{O}_{X,p}^{\text{an}} = \left\{ f(z) := \sum c_k (z-p)^k \;\middle|\; f \text{ has a positive radius of convergence} \right\}$. Note that $\mathcal{O}_{X,p}^{\text{cts}}$ doesn't have such a nice description, since continuous functions can be distinct while agreeing on a small neighborhood. Similarly, $\underline{\mathbb{C}}_p = \mathbb{C}$, since locally constant is actually constant on a small enough neighborhood.

Remark 4.0.7: Recall that morphisms of (pre)sheaves are natural transformations of functors. There is a forgetful functor Forget : $\mathsf{Sh}(X) \to \underset{\text{pre}}{\mathsf{Sh}}(X)$, which has a left adjoint $(-)^+ : \underset{\text{pre}}{\mathsf{Sh}}(X) \to \mathsf{Sh}(X)$. There is a description of $F^+(U)$ as collections of local compatible sections of F modulo equivalence – compatibility here means that if $\mathcal{U} \rightrightarrows X$, then writing $U_{ij} = \cup V_k$ we have $\operatorname*{Res}_{X,V_k} s_i = \operatorname*{Res}_{X,V_k} s_j$ for all i, j.

5 | Friday, January 21

Remark 5.0.1: Last time: definitions of presheaves and sheaves. There is an adjunction

$$\underset{\text{pre}}{\mathsf{Sh}}(X) \underset{\text{Forget}}{\overset{(-)^+}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathsf{Sh}(X).$$

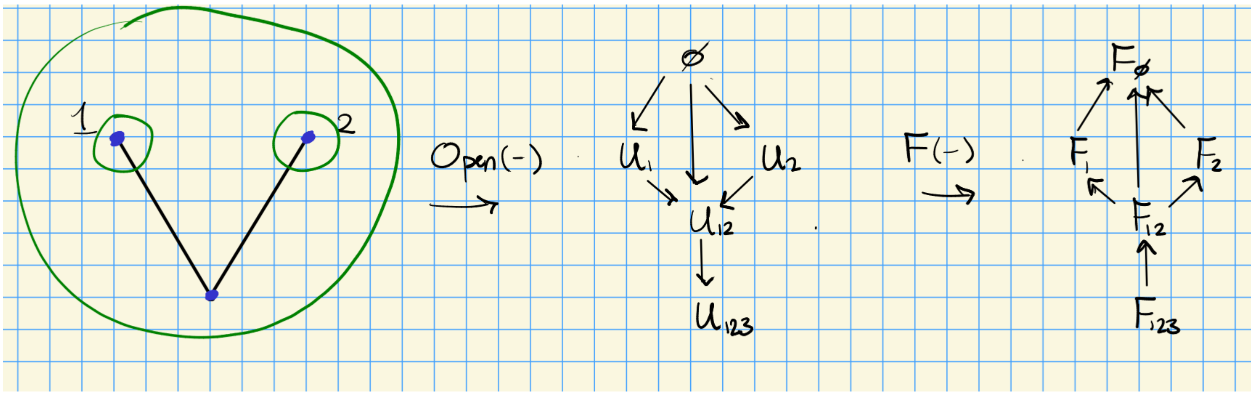
Recall that constant sheaves for $A \in \mathsf{D}$ are defined as $\underline{A}(-) := \underset{\mathsf{Top}}{\operatorname{Hom}}(-, A)$ where A is equipped with the discrete topology.

Exercise 5.0.2 (?)
What is $\Gamma(\underline{A}, X)$ for $X \coloneqq \{1/n\}_{n \in \mathbb{Z}_{\geq 0}} \subseteq \mathbb{R}$? So $A(U) \neq A^{\sharp \pi_0 U}$ in general, since there may not be a notion of connected components for an arbitrary topological space.

Exercise 5.0.3 (?)
Is it true that for any $X \in \mathsf{Top}$ there is a unique decomposition $X = \coprod_{i \in I} U_i$ into connected components?

Hint: form a poset of such decompositions ordered by refinement and apply Zorn's lemma.

Example 5.0.4*(?):* Consider the following poset with a prescribed topology, and applying some functor F:



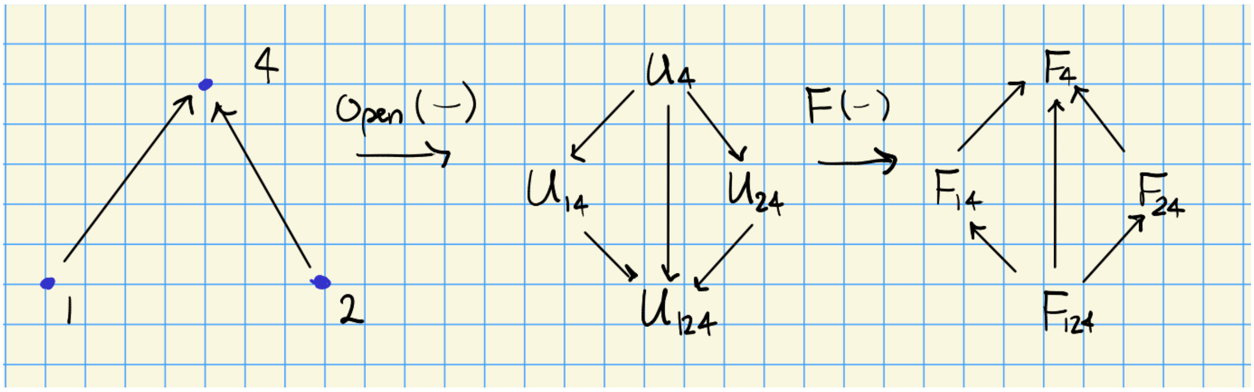
For this to be a sheaf, this forces

- $F(\emptyset) = {}_{\uparrow}$
- $F_{12} \cong F_1 \oplus F_2$ by the universal property of $\oplus$ if this is to be a sheaf.
- F_3 can be anything mapping to F_{12}.

What are the stalks?

- $F_x = F(X)$ for $x = 3$, since X is the smallest open set containing 3.
- $F_{x_i} = F_i$ for $x_i = 1, 2$.

Example 5.0.5*(?):* Consider now a poset in the order topology:

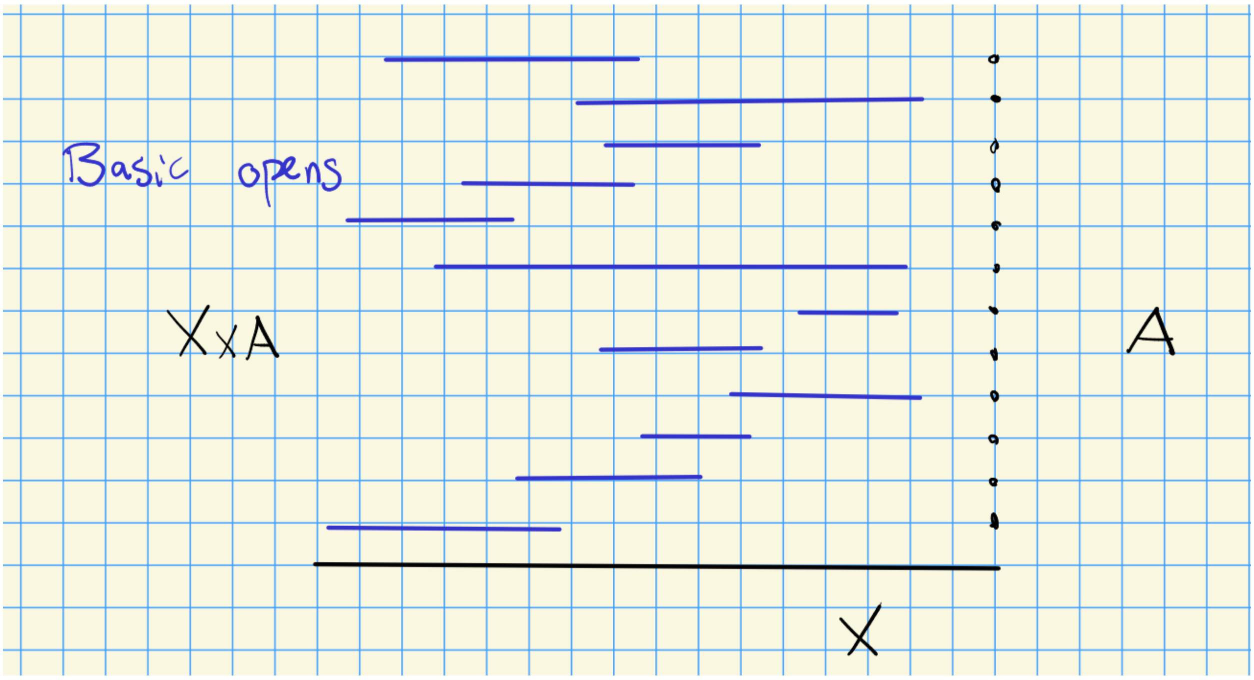


Now F is a sheaf iff $F_{124} \cong F_1 \underset{F_4}{\times} F_2$ is the fiber product.

Definition 5.0.6 (Sheaf space)
A map $\pi : Y \to X \in \mathsf{Top}$ is a **sheaf space** if it is a local homeomorphism, so every $y \in Y$ admits a neighborhood $U_y \ni y$ where $\pi|_{U_y} : U_y \to \pi(U_y)$ is a homeomorphism onto its image.

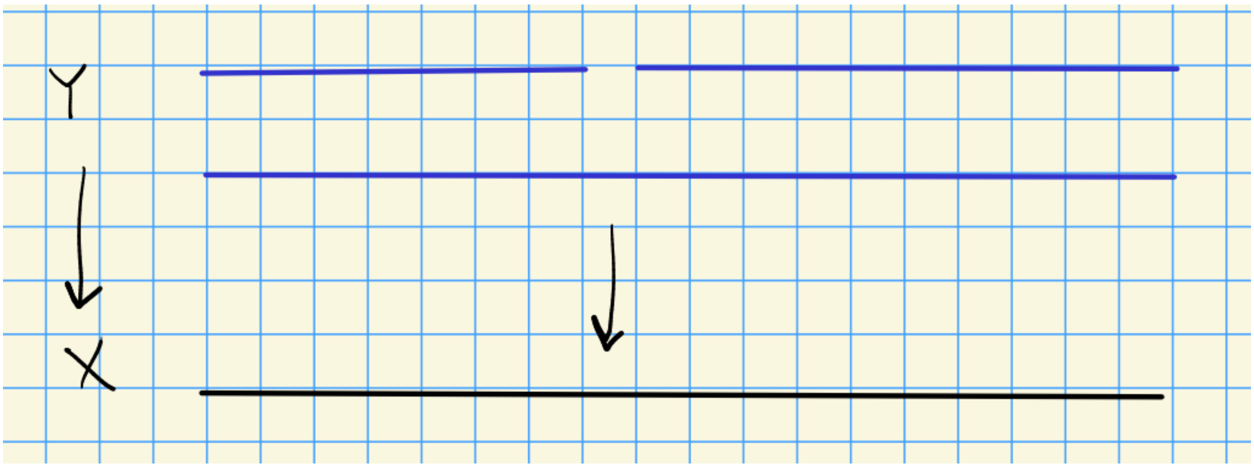
Example 5.0.7 *(?):* Some examples:

- $X \times A \to X$ for A discrete.



Example 5.0.8 *(?):* One possibility: "jumping". Take $Y := X \coprod_{X\setminus\{0\}} X$ for $X \subseteq \mathbb{R}$, which is a version of the line with two zeros. Then $Y \to X$ is a sheaf space, since it is a local homeomorphism.

The other possibility is "skipping":



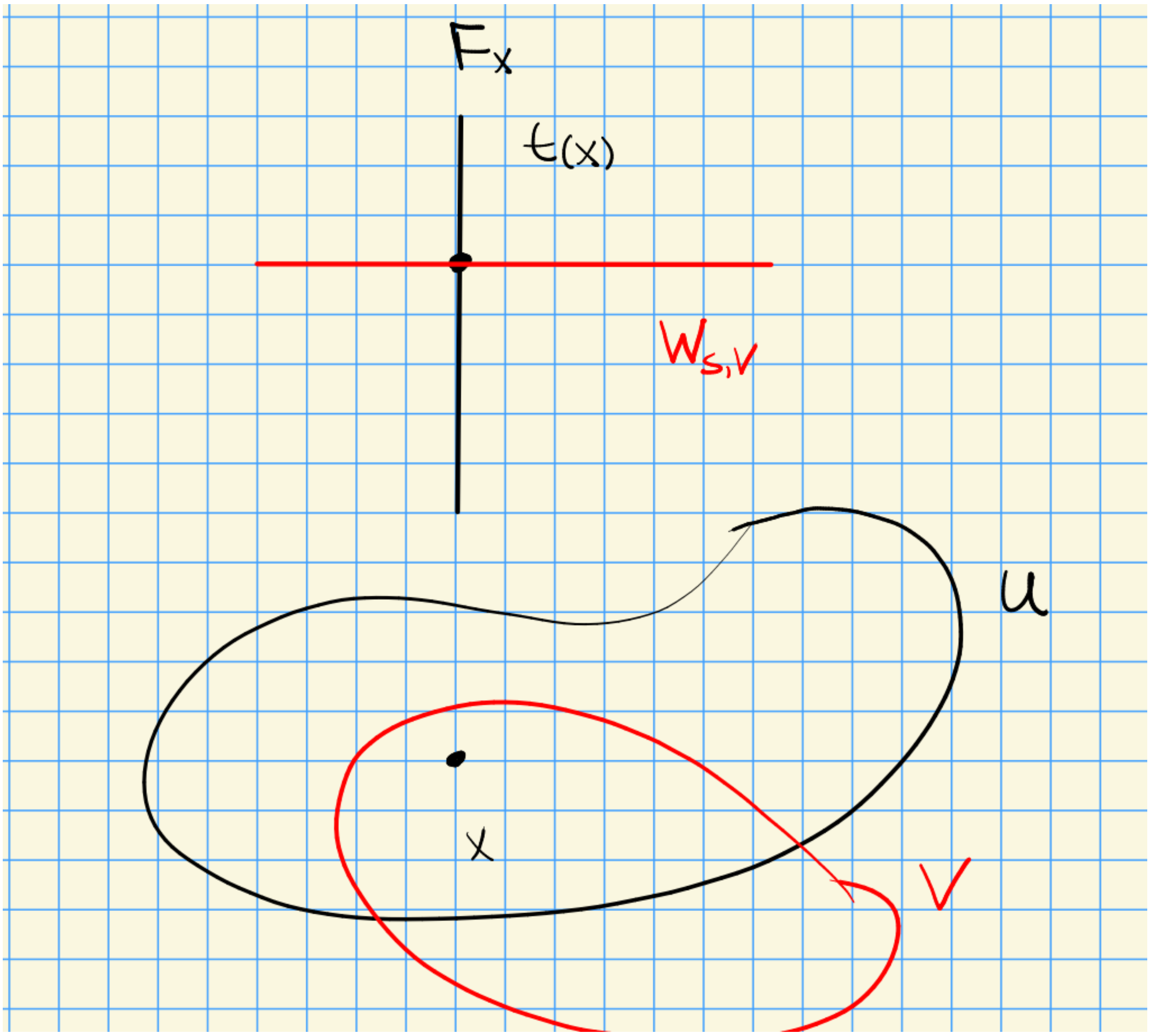
Remark 5.0.9: These two definitions of sheaf coincide: for new to old, given $Y \xrightarrow{\pi} X$ apply $\text{ContSec}_\pi \subseteq \underset{\mathsf{Top}}{\text{Hom}}(X, Y)$. In the other direction, define $Y := \coprod_{x \in X} F_x$ and prove it is a local homeomorphism.

Remark 5.0.10: Next time: direct/inverse image, shriek functors, sheaves of modules.

6 | Monday, January 24

Remark 6.0.1: Recall the definitions of presheaves and sheaves, and sheafification as an adjoint to Forget : $\mathsf{Sh}(X) \to \underset{\mathsf{pre}}{\mathsf{Sh}}(X)$. For $F \in \underset{\mathsf{pre}}{\mathsf{Sh}}(X)$ we concretely construct its sheafification F^+ using the sheaf space $\pi : Y := \coprod_{x \in X} F_x \to X$.

What are the sections of π? For a basic open $U \subseteq X \ni x$, the fiber is $\pi^{-1}(x) = F_x := \underset{\overrightarrow{V \ni x}}{\text{colim}}\, F(V)$, which receives a map $\underset{U,x}{\text{Res}} : F(U) \to F_x$. Writing $s \in F(U)$, define $s_x := \underset{U,x}{\text{Res}}(s)$, and set $W_{s,U} := \left\{ s_x \;\middle|\; x \in U \right\}$ to be $\pi^{-1}(U)$. Then define F^+ to be the continuous sections of $Y \xrightarrow{\pi} X$. What does such a section look like? For $t : U \to \pi^{-1}(U)$ and $x \in U$, the vertical fiber is F_x. For a basic open $V \ni X$ in the base, there is a basic open $W_{s,V}$ in Y for $s \in F(V)$:



There are maps $s_{ij} : U_{ij} \to \pi^{-1}(U_{ij})$, but note that $\mathrm{Res}(U_i, U_{ij})s_i$ does not necessarily equal $\mathrm{Res}(U_j, U_{ij})s_j$ in $F(U_{ij})$ – instead, there is an open cover $U_{ij} = \bigcup V_\alpha$ with $\mathrm{Res}(U_i, V_\alpha)s_i = \mathrm{Res}(U_j, V_\alpha)s_j$ for each α.

Todo

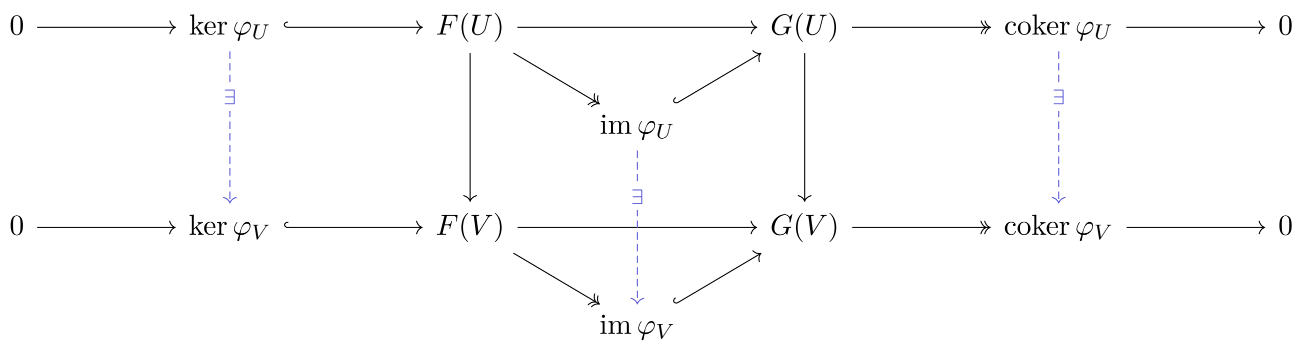
Remark 6.0.2: For $f \in \mathsf{Top}(X, Y)$ we have the following constructions:

- The direct image $f_* : \mathsf{Sh}(X) \to \mathsf{Sh}(Y)$, which is easy with the sheaf definition, and
- The inverse image $f^{-1} : \mathsf{Sh}(Y) \to \mathsf{Sh}(X)$ which is easier with the sheaf space definition.

Recall the definition of a morphism of sheaves as a natural transformation.

For sheaves of abelian groups and $\varphi : F \to G$ a morphism of sheaves, there are notions of $\ker \varphi$, $\operatorname{coker} \varphi$, $\operatorname{im} \varphi$, and extension of a sheaf by zero.

To show these exist as presheaves, one only has to show existence of the following blue morphisms of abelian groups:



Link to Diagram

Write $(\operatorname{coker}\varphi)^-$ and $(\operatorname{im}\varphi)^-$ for these presheaves.

Proposition 6.0.3 ***(?)*.**
$\ker\varphi$ is a sheaf.

Proof (?).
Axiom 1: use that F is a sheaf and $\ker\varphi_U \subseteq F(U)$ can be viewed as an inclusion. Axiom 2: write $s_i \in \ker\left(F(U_i) \xrightarrow{\varphi_{U_i}} F(U_j)\right)$, then there exists a unique $s \in F(U)$. Then check that $s \in \ker(F(U) \to G(U))$ by noting that if $s \mapsto t$ then $t|_{U_i} = 0$ for all i, making $t \equiv 0$ by the sheaf property of G. ■

Definition 6.0.4 (Cokernel and image sheaves)
Define

$$\operatorname{coker}\varphi := ((\operatorname{coker}\varphi)^-)^+$$
$$\operatorname{im}\varphi := ((\operatorname{im}\varphi)^-)^+.$$

Example 6.0.5 ***(of necessity of sheafifying):*** Take $X = \mathbb{C}$ and consider $\exp : \operatorname{Hol}(X) \to G$ the sheaf of nowhere zero holomorphic functions. Then on $U_i \in \mathbb{C} \setminus \{0\}$, take $z \in G$. Then $z = \exp(f_i)$ in each U_i with $f_i \in \operatorname{Hol}(X)$, so $f_i = \log(z)$ locally and $z = \exp(\log z)$, but there is no global $f \in \operatorname{Hol}(X)$ with $\exp(f) = z$. So $z \in \ker\varphi_i(\operatorname{Hol}(U_i) \to G(U_i))$ but $z \notin \ker\exp$. For the same reason, $z = 0$ in $\operatorname{coker}\varphi_i$ since it's locally in the image. but $z \neq 0 \in \operatorname{coker}\exp$ since it's not globally in the image.

7 | Wednesday, January 26

Remark 7.0.1: Recall last time: presheaf vs sheaf properties, images, kernel, cokernel. We can state the uniqueness sheaf axiom as the following: if $s \in F(U)$ with $s|_{U_i} = 0$ for $\mathcal{U} \rightrightarrows U$, then $s = 0$ in $F(U)$.

- $\mathcal{F} := (\operatorname{im}\varphi)^{-}$ satisfies uniqueness.
- $\mathcal{G} := (\operatorname{coker}\varphi)^{-}$ satisfies existence.
- $\mathcal{F}$ fails existence $\iff$ $\mathcal{G}$ fails uniqueness
- $\mathcal{F}$ fails uniqueness iff $\mathcal{G}$ fails existence.

The presheaf image and cokernel can sometimes fail to be a sheaf: use $\operatorname{Hol}(X) \xrightarrow{\exp} \operatorname{Hol}(X)^{\times}$. The kernel presheaf $(\ker\varphi)^{-}$ is already a sheaf.

Exercise 7.0.2 (?)
Show the following:

- A sheaf $\mathcal{F}$ is the zero sheaf iff $\mathcal{F}_p = 0$ for all p.
- $\ker(\varphi)_p = \ker(\varphi_p)$, which is $\ker(\mathcal{F}_p \xrightarrow{\varphi_p} \mathcal{G}_p)$ the kernel of the induced map.
- $\operatorname{coker}(\varphi)_p = \operatorname{coker}(\varphi_p) := \operatorname{coker}(\mathcal{F}_p \xrightarrow{\varphi_p} \mathcal{G}_p)$.
- $\varphi : \mathcal{F} \to \mathcal{G}$ is injective iff $\varphi_p : \mathcal{F}_p \to \mathcal{G}_p$ is injective for all p.
- $\varphi : \mathcal{F} \to \mathcal{G}$ is surjective iff $\varphi_p : \mathcal{F}_p \to \mathcal{G}_p$ is surjective for all p.

Remark 7.0.3: Hints:

- Suppose $s \neq 0$ in $F(U)$, does there exist a p with $s_p = 0$?
- Use that $s_p \in (\ker\varphi)_p$ can be regarded as $s \in \ker(F(V) \to G(V))$ mod equivalence.

Definition 7.0.4 (?)
If there exists an injective morphism $\varphi : \mathcal{F} \to \mathcal{G}$, we regard $\mathcal{F} \leq \mathcal{G}$ as a **subsheaf** and define the **quotient sheaf** $\mathcal{F}/\mathcal{G} := \operatorname{coker}(\mathcal{F} \xrightarrow{\varphi} \mathcal{G})$.

Exercise 7.0.5 (?)
Show by example that $(\mathcal{F}/\mathcal{G})^{-}$ need not be a sheaf.

Remark 7.0.6: Note that for $\varphi : \mathcal{F} \to \mathcal{G}$, the image $\operatorname{im}\varphi$ is a secondary notion in additive categories, and can instead be defined as either

- $\operatorname{coker}(\ker\varphi \to \mathcal{F})$
- $\ker(\mathcal{G} \to \operatorname{coker}\varphi)$

These need not coincide in general.

Remark 7.0.7: Defining the direct image: easier using the sheaf axioms. For $f \in \mathsf{Top}(X, Y)$, define $f_* : \mathsf{Sh}(X) \to \mathsf{Sh}(Y)$ by

$$f_*\mathcal{F}(U) \coloneqq \mathcal{F}(f^{-1}(U)), \qquad \in \mathsf{Sh}(Y) \text{ for } \mathcal{F} \in \mathsf{Sh}(X).$$

Remark 7.0.8: For the preimage: easier to use the espace étalé. As a special case, consider $\iota : S \hookrightarrow Y$ where S is a subspace of Y (with the subspace topology). Then for $\mathcal{F} \in \mathsf{Sh}(Y)$, we can now define sections not only on open subsets U but arbitrary subsets S as

$$\mathcal{F}(S) \coloneqq (\iota^{-1}\mathcal{F})(S).$$

8 | Friday, January 28

Remark 8.0.1: Last time:

- Morphisms of sheaves φ,
- $\ker \varphi$ (already a sheaf),
- $(\operatorname{im} \varphi)^-, (\operatorname{coker} \varphi)^-$ (need to sheafify),
  - All defined to commute with taking stalks: $(\ker \varphi)_p = \ker(\varphi_p)$, etc
- $(\operatorname{im} \varphi)^-$ may fail the existence axioms for sheaves, using $\exp : \mathcal{O}^{\mathrm{an}} \to (\mathcal{O}^{\mathrm{an}})^\times$ for X a complex analytic space,
- $(\operatorname{coker} \varphi)^-$ may fail the uniqueness axioms for sheaves,
- $(\operatorname{im} \varphi)^-$ satisfies existence $\iff$ $(\operatorname{coker} \varphi)^-$ satisfies uniqueness,
- For $\mathcal{F} \hookrightarrow \mathcal{G}$ injective, the presheaf quotient $(\mathcal{G}/\mathcal{F})^-$ may fail to be a sheaf.

Example 8.0.2 ***(of the last claim):*** For $X \in \mathsf{AlgVar}_{/k}$ for $k = \bar{k}$, let $\mathcal{O}_X$ be its regular algebraic functions. Take $X = \mathbb{P}^1$ and $U \coloneqq \mathbb{A}^1 \setminus \{\mathrm{pt}\} \subseteq \mathbb{A}^1 \subseteq \mathbb{P}^1 \setminus \{a_1, \cdots, a_k\}$. Then $\mathcal{O}_X(U) = k[x]\left[f^{-1}\right]$ for $f(x) \coloneqq \prod(x - a_k)$, $\mathcal{O}_X(X) = k$, $K_X(U) = k(x)$, and $K_X^\times(U) = k(x) \setminus \{0\}$ if $U \neq \emptyset$. Define **Cartier divisors** as global sections of the sheaf $\operatorname{Cart}\operatorname{Div} \coloneqq K_X^\times / \mathcal{O}_X^\times$. Recall that Weil divisors are finite sums of codimension 1 subvarieties, and these notions coincide for nonsingular varieties.

For $p \in \mathbb{A}^1 \subseteq \mathbb{P}^1$, we have

$$(K_X^\times / \mathcal{O}_X^\times)_p = \frac{K_{X,p}}{\mathcal{O}_{X,p}} = \frac{k(x)}{\left\{ f/g \;\middle|\; f(p) \neq 0, g(p) \neq 0 \right\}} \cong \mathbb{Z},$$

using that any element in the quotient can be written as $h(x) = (x-p)^n g(x)$ for some $g \in \mathcal{O}_{X,p}^\times$. Here $\operatorname{Cart}\operatorname{Div}(X) = \sum n_p P$ are all finite sums with $n_p \in \mathbb{Z}$. The claim is that sheaf existence fails for this quotient – there are local sections that do not glue. Here

- $K^\times(\mathbb{P}^1) = k(x)^\times$

- $\mathcal{O}^\times(\mathbb{P}^1) = k^\times$
- $K^\times(\mathbb{P}^1)/\mathcal{O}^\times(\mathbb{P}^1) = \dfrac{k(x)^\times}{k^\times}$

For any s in the quotient, we can associated $(s)_0 - (s)_\infty = \sum n_p P$, but not every Cartier divisor is of this form – these are the *principal* divisors. This form a group $\text{Pic}(X) = \text{Cart Div}(X)/\text{Prin Cart Div}(X)$, which may not be trivial. This proof generalizes to locally Noetherian schemes, not necessarily reducible, with no embedded components.

Remark 8.0.3: Note that $\text{Pic}(X)$ is also the group of invertible sheaves on X, and for irreducible algebraic varieties these coincide. Use the SES $0 \to \mathcal{O}^\times \to K^\times \to K^\times/\mathcal{O}^\times \to 0$ to obtain

$$1 \to H^0(\mathcal{O}^\times) \to H^0(K^\times) \to \text{Prin Cart Div}(X) \to H^1(\mathcal{O}^\times) \cong \text{invertible sheaves}/\sim \to 0,,$$

where $H^1(K^\times)$ vanishes since it's a constant sheaf on an irreducible scheme in the Zariski topology.

Proposition 8.0.4 *(?).*
$(\operatorname{im} \varphi)^-$ satisfies existence $\iff$ $(\operatorname{coker} \varphi)^-$ satisfies uniqueness.

Proof (?).
$\implies$: Let $s \in \operatorname{coker}(F(U) \to G(U))$ and write $U = \cup U_i$. We want to show that s_{U_i} implies $s \in \operatorname{coker}(F(U_i) \to G(U_i))$ for all i. Note that $s = 0$ in $\operatorname{coker}(F(U) \to G(U))$ iff $s \in \operatorname{im}(F(U) \to G(U))$ ■

9 | Monday, January 31

Remark 9.0.1: Direct image sheaf: for $\mathcal{F} \in \mathsf{Sh}(X), \mathcal{G} \in \mathsf{Sh}(Y), f \in \mathsf{Top}(X,Y)$, the $f_* \in [\mathsf{Sh}(X), \mathsf{Sh}(Y)]$ is defined by $f_*\mathcal{F}(U) := \mathcal{F}(f^{-1}U)$. The inverse image functor $f^{-1} \in [\mathsf{Sh}(Y), \mathsf{Sh}(X)]$ is slightly more complicated. An easy case: if $\iota : S \hookrightarrow Y$ is a subspace, then it is just restriction: $(\iota^{-1}G)(S) := G(S)$.

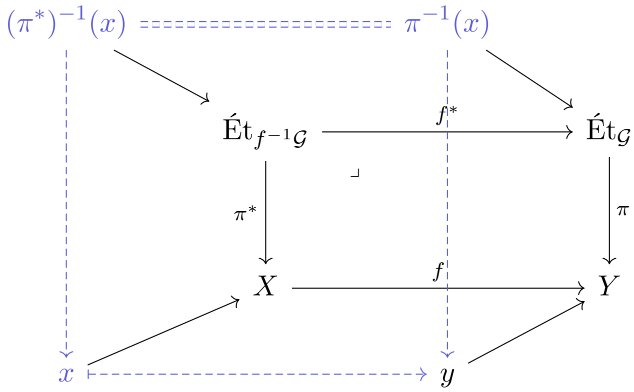
Idea for sheaf space: there are strictly horizontal neighborhoods as the homeomorphic preimages of small opens in the base. So for $\text{Ét}_\mathcal{G} \xrightarrow{\pi} Y$ the sheaf space of $\mathcal{G}$, define the inverse image as

$$\text{Ét}_{\iota^{-1}\mathcal{G}} := \pi^{-1}(S) \subseteq \text{Ét}_\mathcal{G},$$

and define a basis of sections in the following way: for $s \in \mathcal{G}(U)$, set $t(U) := s(U) \cap \pi^{-1}(S) \in \text{Ét}_\mathcal{G}$ to be sections of $\text{Ét}_{\iota^{-1}\mathcal{G}}$. Declare these to be a basis of opens, i.e. take the subspace topology for $\pi^{-1}(S) \subseteq \text{Ét}_\mathcal{G}$ in the sheaf topology on the total space. More generally, for $f \in \mathsf{Top}(X,Y)$, set

$$\text{Ét}_{f^{-1}\mathcal{G}} := \text{Ét}_\mathcal{G} \underset{Y}{\times} X.$$

The fibers are identical:



Link to Diagram

The topology on $\text{Ét}_{f^{-1}\mathcal{G}}$ is the coarsest topology for which π^* and f^* are continuous. This is generated by $\left(f^{-1}(s)(f^{-1}U)\right) \cap (\pi^*)^{-1}(W)$ for $W \subseteq X$ open. Define $f^{-1}(s) \in f^{-1}\mathcal{G}(f^{-1}(U)) := (f^{-1}U) \underset{U}{\times} s(U)$. This makes the pullback continuous both vertically and horizontally.

Corollary 9.0.2 *(?)*.

$$(f^{-1}\mathcal{G})_y = \mathcal{G}_{f(y)}.$$

Definition 9.0.3 (Inverse image sheaf)

$$f^{-1}\mathcal{G} := \left(V \mapsto \underset{\overrightarrow{U, V\subseteq f^{-1}(U)}}{\operatorname{colim}} \mathcal{G}(U)\right)^+.$$

Remark 9.0.4: How to prove this coincides with the previous definition:

- Show the stalks are isomorphic,
- Show that there is a map of presheaves $(f^{-1}\mathcal{G}) \to f^{-1}\mathcal{G}$,
- Show that the map induces an isomorphism on stalks, and lift using the universal property of sheafification.

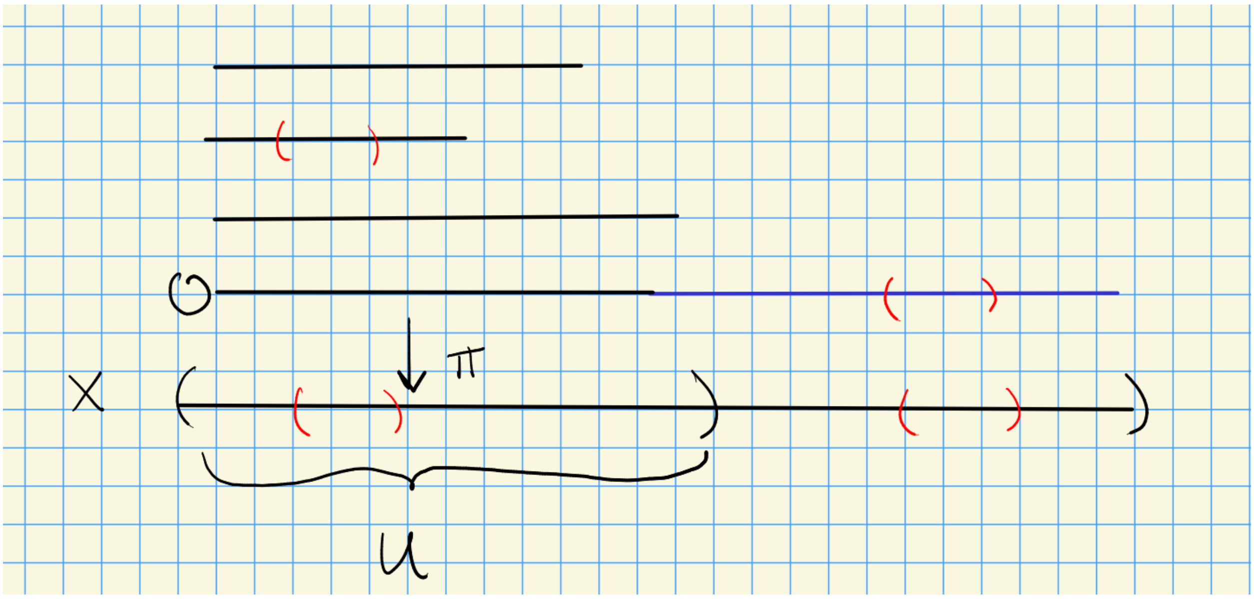
Exercise 9.0.5 (?)
Try to prove this by commuting limits.

Remark 9.0.6: Recall that $K^\times/\mathcal{O}^\times \cong \bigoplus_{x\in X} (\iota_*)_* \iota_*^{-1}\mathbb{Z}$ which had stalks $\mathbb{Z}$ but was not constant – check that the local sections differ.

Question 9.0.7
For $S \hookrightarrow Y$, does every section of $\mathcal{G}$ over S extend to Y?

10 | Wednesday, February 02

Remark 10.0.1: Extending by zero: for $i : U \hookrightarrow X$ an open subspace and $\mathcal{F} \in \mathsf{Sh}(U)$, define $i_!\mathcal{F} \in \mathsf{Sh}(X)$. If the target category has a zero object, define this in the sheaf space by extending the zero section:



Thus $\text{Ét}_{i_!\mathcal{F}} = \text{Ét}_{\mathcal{F}} \coprod \{s_0\}$ for s_0 the zero section.

Proposition 10.0.2 *(?)*.
Define a presheaf are given by

$$(i_!\mathcal{F})^-(V) = \begin{cases} F(V) & V \subseteq U \\ 0 & \text{else.} \end{cases}.$$

Sheafifying produces an equivalent sheaf, i.e. $(i_!\mathcal{F})^{-+} \cong i_!\mathcal{F}$.

Proof (?).
Idea: produce a map $(i_!\mathcal{F})^- \to i_!\mathcal{F}$ and show it is an isomorphism on stalks. What are the stalks? By the sheaf space definition,

$$(i_!\mathcal{F})_p = \begin{cases} \mathcal{F}_p & p \in U \\ 0 & \text{else.} \end{cases}.$$

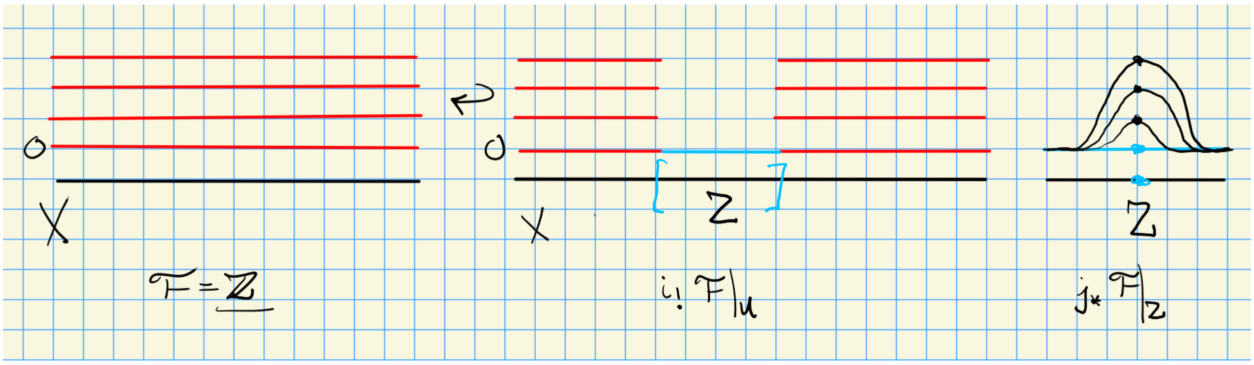
On the other hand, $(i_! \mathcal{F})_p^- = \operatorname*{colim}_{\overrightarrow{V \ni p}} \mathcal{F}(V)$, but this limit can be taken over the system of open sets $V \subseteq U$, so it yields $\mathcal{F}_p$.

■

Remark 10.0.3: Consider $X = U \coprod Z$ with U open and Z closed. Let $U \overset{i}{\hookrightarrow} X$ and $Z \overset{j}{\hookrightarrow} X$, and consider $i_* \mathcal{F}|_U$ and $j_* \mathcal{F}|_U$. There is a SES

$$0 \to i_! \mathcal{F}|_U \to \mathcal{F} \to j_* \mathcal{F}|_Z \to 0.$$

Example 10.0.4*(?)*: The sheaf $i_! \mathcal{F}|_U$ is a subsheaf of $\mathcal{F}$, and $j_* \mathcal{F}|_Z$ is a quotient.



Here $\text{Ét}_{\underline{\mathbb{Z}}} = \coprod_{n \in \mathbb{Z}} X$, and $\text{Ét}_{j_* \underline{\mathbb{Z}}|_Z} X$ glued along $X \setminus Z$. So $i_! \mathcal{F}|_U \hookrightarrow \mathcal{F}$. It's important that Z is closed here to get a surjection, since then any point in its complement has a neighborhood V missing Z entirely and $(i_! \mathcal{F})^-(V) = 0$. Checking the stalks:

	$\mathcal{F}$	$i_! \mathcal{F}\|_U$	$j_* \mathcal{F}\|_V$
$p \in U$	$\mathcal{F}_p$	$\mathcal{F}_p$	0
$p \in Z$	$\mathcal{F}_p$	0	$\mathcal{F}_p$

Example 10.0.5*(?)*: Let $X \in \mathsf{AlgVar}_{/k}$, e.g. $X = \mathbb{P}^1$, let $Z \subseteq X$ be closed, and let $\mathcal{F} := \mathcal{O}_X$. There is a SES $0 \to I_Z \to \mathcal{O}_X \to \mathcal{O}_Z \to 0$.

Remark 10.0.6: Note that we have adjunctions

$$\mathsf{Sh}X \underset{f_*}{\overset{f^{-1}}{\rightleftarrows}} \mathsf{Sh}Y$$

$$\mathsf{Sh}? \underset{-|_U}{\overset{i_!}{\rightleftarrows}} \mathsf{Sh}?$$

$$\mathsf{Sh}? \underset{-|_Z}{\overset{j_*}{\rightleftarrows}} \mathsf{Sh}?.$$

11 | Friday, February 04

Remark 11.0.1: Last time: extension by zero, inverse image, pushforward on closed sets and adjunctions.

$$f \in \underset{\mathsf{Top}}{\mathrm{Hom}}(X,Y) \rightsquigarrow \underset{\mathsf{Sh}(X)}{\mathrm{Hom}}(f^{-1}\mathcal{G}, \mathcal{F}) \cong \underset{\mathsf{Sh}(Y)}{\mathrm{Hom}}(\mathcal{G}, f_*\mathcal{F}).$$

Warning 11.0.2

Pushing forward open sets is not generally a good idea! Take $X = \mathbb{R}^{\mathrm{zar}}$, $Z = \{\mathrm{pt}\}, U = X \setminus Z$. Then $(i_*\underline{\mathbb{Z}_U})_p = \mathbb{Z}^{\oplus^2}$ if $p = \mathrm{pt}$, since any neighborhood of p pulls back to two connected components.

Remark 11.0.3: Consider $U \overset{i}{\hookrightarrow} X$ with U open and $Z \overset{j}{\hookrightarrow} X$ with Z closed, then for $\mathcal{F} \in \mathsf{Sh}(X), \mathcal{H} \in \mathsf{Sh}(U), \mathcal{G} \in \mathsf{Sh}(Z)$,

$$\underset{\mathsf{Sh}(Z)}{\mathrm{Hom}}(\mathcal{F}|_Z, \mathcal{G}) \xrightarrow{\sim} \underset{\mathsf{Sh}(X)}{\mathrm{Hom}}(\mathcal{F}, j_*\mathcal{G})$$
$$\underset{\mathsf{Sh}(U)}{\mathrm{Hom}}(\mathcal{H}, \mathcal{F}|_U) \xrightarrow{\sim} \underset{\mathsf{Sh}(X)}{\mathrm{Hom}}(i_!\mathcal{H}, \mathcal{F}).$$

Remark 11.0.4: We'll consider $(X, \mathcal{O}_X) \in \mathsf{LocRingSp}_{/\mathsf{CRing}}$ with sheaves of reduced commutative rings – note that noncommutative rings are also important, e.g. GL_n or $\mathfrak{gl}_n$.

Example 11.0.5 *(?)*: Common examples of locally ringed spaces:

- $(X, \underline{R})$ any space with a constant sheaf.
- $(X, \mathcal{F})$ for $\mathcal{F} := \mathcal{O}_X^{\mathrm{cts}} := \underset{\mathsf{Top}}{\mathrm{Hom}}(-, \mathbb{R})$.
- $(X, \mathcal{O}_X^{\mathrm{zar}})$ for $X \in \mathsf{AffAlgVar}_{/k}$ and $\mathcal{O}_X^{\mathrm{zar}}$ the sheaf of Zariski-regular functions. In this case, for $k = \bar{k}$, these are of the form $\mathrm{mSpec}\, R \subseteq \mathbb{A}^n_{/k}$ for $R := k[x_1, \cdots, x_n]/\langle f \rangle$. Recall distinguished opens are $D(g) = \{g \neq 0\}$ for $g \in k[x_1, \cdots, x_n]$, and sections are $\mathcal{O}_X(D(g)) = R[g^{-1}]$ are functions $\rho : X \to k$ of the form $\rho = h/g^k$ for some regular function h. It's a theorem that these assemble to a sheaf.

Remark 11.0.6: Define algebraic varieties as locally ringed spaces $(X, \mathcal{O}_X)$ that

1. X is covered by finitely affine algebraic varieties, so $X = \cup U_i$ with $(U_i, \mathcal{O}_{U_i})$ affine algebraic, and
2. X is separated, i.e. $X \xrightarrow{\Delta_X} X^{\times_X^2}$ is closed.

Note that affine and even quasiprojective schemes are automatically separated. We require the separated condition here to rule out things like $\mathbb{A}^1$ with two origins, i.e. $X := \mathbb{A}^1 \coprod_{\mathbb{A}^1 \setminus \{0\}} \mathbb{A}^1$.

Example 11.0.7 *(?)*: Affine schemes: for $R \in \mathsf{CRing}$, take $X := \mathrm{Spec}\, R$ with a basis $D(g)$ and define a presheaf by $\mathcal{O}_X(D(g)) = R[g^{-1}]$. It's a theorem that this yields a sheaf.

Definition 11.0.8 ($\mathcal{O}_X$-modules)
For $(X, \mathcal{O}_X) \in \mathsf{LocRingSp}$, $\mathcal{F}$ is a **sheaf of $\mathcal{O}_X$-modules** iff every section $F(U)$ is an $\mathcal{O}_X(U)$-module and restriction is compatible with the module structures in the sense that $(rm)|_V = r|_V m|_V$:

$$\begin{array}{ccccc} m \in & \mathcal{F}(U) & \longrightarrow & \mathcal{O}_X(U) & \ni r \\ & \downarrow & & \downarrow & \\ & \mathcal{F}(V) & \longrightarrow & \mathcal{O}_X(V) & \end{array}$$

Link to Diagram

Example 11.0.9 *(?):* Any constant sheaf $\underline{M}$ for $M \in \mathsf{R\text{-}Mod}$.

Definition 11.0.10 (Quasicoherent and coherent sheaves)
An $\mathcal{O}_X$-module is

- **Quasicoherent** if locally $\mathcal{F} \cong \underline{M}$ (there exists an open cover $X = \cup U_i$ with $\mathcal{F}|_{U_i} \cong \underline{M_{U_i}}$),
- **Coherent** iff $\mathcal{F}$ is quasicoherent and $M \in \mathsf{R\text{-}Mod}^{\mathrm{fg}}$ and X is locally Noetherian.

Example 11.0.11 *(?):* Of an $\mathcal{O}_X$-module for a constant sheaf: $M = R/p$ for $\mathcal{O}_X = \underline{R}$.

Example 11.0.12 *(?):* For complex analytic varieties, take $(X, \mathcal{O}_X^{\mathrm{an}})$ so $\mathcal{O}_X(U)$ are locally meromorphic functions regular on U, i.e. whose denominator does not vanish on U. This is the setting where Cartan, Serre, etc defined original notions of coherence, and e.g. Serre vanishing, and scheme theory is developed by analogy to this situation. Here, $\mathcal{F}$ is a **coherent** sheaf iff $\mathcal{F}$ is a sheaf of $\mathcal{O}_X^{\mathrm{an}}$-modules and admits a presentation

$$\mathcal{O}_X^{\mathrm{an}\oplus^m} \to \mathcal{O}_X^{\mathrm{an}\oplus^n} \to \mathcal{F} \to 0.$$

Remark 11.0.13: Next time: locally free, invertible, tensor, and hom.

12 | Monday, February 07

Remark 12.0.1: Examples of sheaves:

- $\mathcal{O}_X^{\mathrm{cts}}$ for $X \in \mathsf{Top}$, where $\mathcal{O}_X^{\mathrm{cts}}(-) = \mathsf{Top}(-, \mathbb{R})$
- $\mathcal{O}_X^{\mathsf{sm}}(-) = C^\infty(-, \mathbb{R})$.
- $\mathcal{O}_X^{\mathrm{hol}}(-) = \mathrm{Hol}(-, \mathbb{C})$
- $\mathcal{O}_X^{\mathrm{an}}(-) \subseteq \mathsf{Top}(-, \mathbb{R})$ the sheaf of analytic functions, those locally equal to power series.

- For $X \in \mathsf{AlgVar}_{/k}$, $\mathcal{O}_X(-) = \mathsf{Top}((-)^{\mathrm{zar}}, k)$ the Zariski-regular k-valued functions.

In all cases, $\mathcal{O}_X$ can be regarded as sheaves of *regular* sections to $X \times \mathbb{A}^1_{/k} \xrightarrow{\pi} X$. Note that this doesn't necessarily coincide with sections of the espace etale, since e.g. the fibers are $\mathbb{A}^1$ and not necessarily the stalks. For $\mathcal{O}^{\oplus^d}$, one instead takes $X \times \mathbb{A}^d_{/k} \to X$.

Definition 12.0.2 (Locally free and invertible sheaves)
A sheaf $\mathcal{F} \in \mathsf{Sh}(X)$ is **locally free** iff there exists an open cover $\mathcal{U} \rightrightarrows X$ with $\mathcal{F}|_{U_i} \cong {\mathcal{O}_{U_i}}^{\oplus^n}$.
The quantity n is the **rank** of $\mathcal{F}$. If $\operatorname{rank} \mathcal{F} = 1$, then $\mathcal{F}$ is **invertible**.
A **vector bundle** over X is $V \xrightarrow{\pi} X$ with $\pi^{-1}(U_i) \cong U_i \times \mathbb{A}^r$. For $r = 1$, this is a **line bundle**.

Remark 12.0.3: Maps between bundles are linear in the second coordinate.

Note that there is a correspondence between vector bundles and locally free sheaves. Consider the rank 1 case, matching invertible sheaves and line bundles. The necessary data:

- An open cover $\mathcal{U} \rightrightarrows X$, where $\mathcal{U} = \{U_i\}_{i \in I}$
- For all $i, j \in I$, transition functions $\varphi_{ij} \in \mathcal{O}^\times(U_{ij}) = \operatorname*{Aut}_{\mathcal{O}_X\text{-}\mathsf{Mod}}(U_{ij})$.
- A cocycle condition: $\varphi_{ii} = \mathrm{id}, \varphi_{ij} = \varphi_{ji}^{-1}$, and $\varphi_{ij}\varphi_{jk}\varphi_{ki} = \mathrm{id} \in \mathcal{O}^\times(U_{ijk})$

Note that any morphism of sheaves $\mathcal{O}_V \to \mathcal{O}_V$ induces a morphism of $\mathcal{O}_V$-modules on global sections

$$\begin{aligned} \mathcal{O}_V(V) &\xrightarrow{\sim} \mathcal{O}_V(V) \in \mathcal{O}_{\mathsf{V}}\text{-}\mathsf{Mod} \\ 1 &\mapsto \varphi, \end{aligned}$$

and this being an isomorphism manes φ is invertible. Note that these are not isomorphic as rings.

Write $Z_1(\mathcal{U}; \mathcal{O}^\times) = \left\{ \varphi_{ij} \in \mathcal{O}^\times(U_{ij}) \;\middle|\; \cdots \right\}$ for those φ_{ij} satisfying the conditions above, and $B_1(\mathcal{U}; \mathcal{O}^\times) = \left\{ \varphi_{ij} \in \mathcal{O}^\times(U_{ij}) \;\middle|\; \varphi_{ij} \sim \varphi_{ij} \frac{\psi_j}{\psi_i} \right\}$ for any $\frac{\psi_i}{\psi_j} \in \mathrm{GL}_1(\mathcal{O}) \cong \mathcal{O}^\times$. More generally, we let $\varphi_{ij} = \psi_j \varphi_{ij} \psi_i^{-1}$ for $\psi_i, \psi_j \in \mathrm{GL}_n(\mathcal{O})$.

Remark 12.0.4: Recall that for a given space X, the open covers of X form a poset under refinement, where $\mathcal{U} \geq \mathcal{V}$ iff for every $U_i \in \mathcal{U}$ there is some $V_j \in \mathcal{V}$ with $U_i \supseteq V_j$. This yields a system of maps $Z^1(\mathcal{U}; \mathcal{O}^\times) \to Z^1(\mathcal{V}; \mathcal{O}^\times)$ compatible with transition maps, so we define

$$\check{H}^1(X; \mathcal{O}_X^\times) \coloneqq \operatorname*{colim}_{\overrightarrow{\mathcal{U} \rightrightarrows X}} \check{H}^1(\mathcal{U}; \mathcal{O}^\times).$$

Exercise 12.0.5 (?)
Compute $\check{H}^1(\mathbb{P}^1; \mathcal{O}_{\mathbb{P}^1}^\times)$ using an open cover by two sets.

13 | Wednesday, February 09

Remark 13.0.1: Plan:

- $\underset{\mathsf{Sh}(X)}{\operatorname{Hom}}(-,-)$
- $\underset{\mathcal{O}_X\text{-}\mathsf{Mod}}{\operatorname{Hom}}(-,-)$
- $(-) \otimes_{\mathcal{O}_X} (-)$

Remark 13.0.2: For $(X, \mathcal{O}_X) \in \mathsf{LocRingSp}$ and $\mathcal{F}, \mathcal{G} \in \mathsf{Sh}(X; \mathsf{AbGrp})$, define $\underset{\mathcal{O}_X}{\operatorname{Hom}}(\mathcal{F}, \mathcal{G})$ to be natural transformations which are $\mathcal{O}_X$-linear. This forms an abelian group under pointwise operations, and more generally an $\mathcal{O}_X$-module since one can act on morphisms by global sections. There is a sheaf version, the local hom $\underline{\operatorname{Hom}}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})(U) \coloneqq \underset{\mathcal{O}_U}{\operatorname{Hom}}(\mathcal{F}_U, \mathcal{G}_U)$ where we write $\mathcal{F}_U \coloneqq \mathcal{F}|_U$.

Proposition 13.0.3 *(?).*
This forms a sheaf of $\mathcal{O}_X$-modules.

Proof (?).
Let

$$f_i \in \underset{\mathcal{O}_{U_i}}{\operatorname{Hom}}(\mathcal{F}_{U_i}, \mathcal{G}_{U_i})$$
$$f_j \in \underset{\mathcal{O}_{U_j}}{\operatorname{Hom}}(\mathcal{F}_{U_j}, \mathcal{G}_{U_j}).$$

If $f_i|_{U_{ij}} = f_j|_{U_{ij}}$, then the claim is that there exists a unique $F \in \underset{\mathcal{O}_{U_{ij}}}{\operatorname{Hom}}(\mathcal{F}_{U_{ij}}, \mathcal{G}_{U_{ij}})$. For $V \subseteq X$, decompose as $V = \bigcup_i U_i$. ■

Proposition 13.0.4 *(?).*
If $\mathcal{F} \in \mathcal{O}_X\text{-}\mathsf{Mod}^{\mathrm{lf,rank}=r}$ and $\mathcal{G} \in \mathcal{O}_X\text{-}\mathsf{Mod}^{\mathrm{lf,rank}=s}$ then $\underline{\operatorname{Hom}}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \in \mathcal{O}_X\text{-}\mathsf{Mod}^{\mathrm{lf,rank}=rs}$.

Proof (?).
Choose trivializations $\mathcal{F}_{U_i} \xrightarrow{\sim} {\mathcal{O}_{U_i}}^{\oplus r}$ and $\mathcal{G}_{U_i} \xrightarrow{\sim} {\mathcal{O}_{U_i}}^{\oplus s}$. The claim is that $\underline{\operatorname{Hom}}_{\mathcal{O}_U}(\mathcal{O}_U, \mathcal{O}_U) = \mathcal{O}_U$ for any $\mathcal{O}_U$. Given this, $\underset{\mathcal{O}_X}{\operatorname{Hom}}({\mathcal{O}_X}^{\oplus r}, {\mathcal{O}_X}^{\oplus s}) \cong \operatorname{Mat}_{r\times s}(\mathcal{O}_X)$ split out as matrices. To prove this, just check on global sections that $\underline{\operatorname{Hom}}_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{O}_X) \cong \underset{\mathsf{R}\text{-}\mathsf{Mod}}{\operatorname{Hom}}(R, R) \cong R$ for $R \coloneqq \Gamma(R)$. ■

Remark 13.0.5: Recall that $\mathsf{Sh}(X)^{\mathrm{lf,rank}=1} \cong \mathsf{Bun}_{\mathrm{GL}}^{\mathrm{rank}=1}$, i.e. we identify rank 1 locally free sheaves

with line bundles. We can write $\underset{\mathcal{O}}{\mathrm{Hom}}(\mathcal{F},\mathcal{G}) = \left\{ \frac{\varphi_{ij}}{\psi_{ij}} \;\middle|\; \varphi_{ij} \in \mathcal{O}_X^\times(U_{ij}) \text{ satisfies the cocycle condition} \right\}$.
What are the transition functions?

We also define $\underset{\mathcal{O}}{\mathrm{Hom}}(\mathcal{F},\mathcal{O}) := \mathcal{F}^\vee$, and there is a relation to $\mathrm{Pic}(X)$.

Remark 13.0.6: Note also that $\underline{\mathrm{Hom}}_{\mathcal{O}}(\mathcal{O},\mathcal{F}) \cong \mathcal{F}$, so global sections coincide with homs. This will be useful later when defining H^* in terms of derived functors.

Definition 13.0.7 (Tensor product)
Define the tensor product of $\mathcal{F},\mathcal{G} \in \mathcal{O}_{\mathsf{X}}\text{-}\mathsf{Mod}$ as the sheafification of

$$(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G})^- := U \mapsto \mathcal{F}(U) \otimes_{\mathcal{O}_U} \mathcal{G}(U).$$

Note that there is a formula for stalks:

$$(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G})_x = \mathcal{F}_x \otimes_{\mathcal{O}_x} \mathcal{G}_x.$$

Moreover $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G} \in \mathcal{O}_{\mathsf{X}}\text{-}\mathsf{Mod}^{\mathrm{lf},\mathrm{rank}=rs}$. This endows $\mathcal{O}_{\mathsf{X}}\text{-}\mathsf{Mod}$ with a symmetric monoidal structure with duals, so

- $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{F}^\vee \cong \mathcal{O}_X$
- $\mathcal{O}_X \otimes_{\mathcal{O}_X} \mathcal{F} \cong \mathcal{F}$

Remark 13.0.8: Recall that $f \in \mathsf{Top}(X,Y)$ for $X,Y \in \mathsf{AffSch}$ induces $f^{-1} \in \mathsf{Sh}(X)(f^{-1}\mathcal{O}_Y, \mathcal{O}_X)$. For varieties, this is just given by pullback of regular functions. More generally, for $X,Y \in \mathsf{LocRingSp}$, define the **full pullback** f^* as

$$f^*\mathcal{F} = f^{-1}\mathcal{F} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X.$$

Lemma 13.0.9 *(?)*.
For the full pullback,

$$f^*\mathcal{O}_Y \cong \mathcal{O}_X,$$

which is not true for f^{-1}. This essentially follows from $R \otimes_R S \cong S$.

Remark 13.0.10: Consider $f \in \mathsf{Alg}_{/k}(S,R)$ for $k = \bar{k}$ where we only consider reduced algebra (no nonzero nilpotents). This induces maps $\tilde{f} : \mathrm{Spec}\, R \to \mathrm{Spec}\, S$ and $\tilde{f}' : \mathrm{mSpec}\, R \to \mathrm{mSpec}\, S$. If $\mathcal{A} \in \mathsf{Sh}(X;\mathcal{O}_X\text{-}\mathsf{Alg})$, there are induced maps $\mathcal{O}_X(U) \to \mathcal{A}(U)$ and thus affine morphism $\pi : \mathrm{Spec}\,\mathcal{A}(U) \to U$ covering the affine open U.

Example 13.0.11 *(?)*:

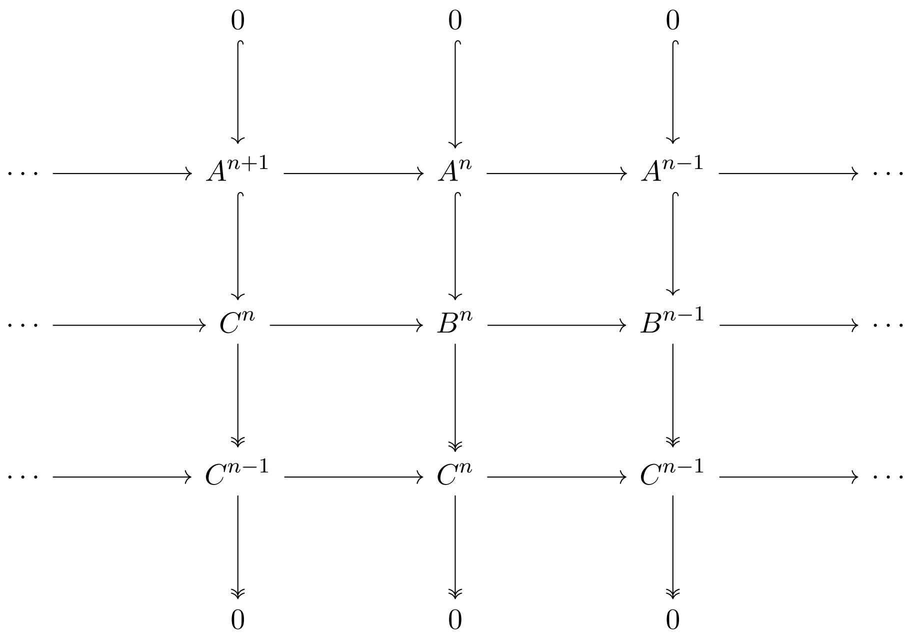
- $\mathcal{A} = \mathcal{O}_X[x_1, \cdots, x_n]$ yields a trivial vector bundle $\mathrm{Spec}\,\mathcal{A} = X \times \mathbb{A}^n \to X$.

- For $\mathcal{F} \in \mathsf{Sh}(X, \mathcal{O}_{\mathsf{X}}\text{-}\mathsf{Mod}^{\mathrm{lf,rank}=n})$, set
$$\mathcal{A} = \mathrm{Sym}^*_{\mathcal{O}}(\mathcal{F}) \coloneqq \mathcal{O}_X \oplus \mathcal{F} \oplus \mathrm{Sym}^2(\mathcal{F}) \oplus \cdots,$$
which yields a nontrivial vector bundle $\mathrm{Spec}\,\mathcal{A} \to X$.
- For $\mathcal{F}$ rank 1, $\mathcal{F}^{\otimes^n_{\mathcal{O}_X}} \xrightarrow{\sim} \mathcal{O}_X(-D)$, set
$$\mathcal{A} \coloneqq T_{\mathcal{O}}(\mathcal{A}) \coloneqq \mathcal{O} \oplus \mathcal{F} \oplus \mathcal{F}^{\otimes^2_{\mathcal{O}_X}} \oplus \cdots,$$
then $\mathrm{Spec}\,\mathcal{A} \to X$ is a cyclic Galois cover for $G = \mu_n$.

14 | Friday, February 11

Remark 14.0.1: Recall the definitions of:

- Cochain complexes,
- Boundaries,
- Cycles,
- Homology as cycles mod boundaries,
- Morphisms of chain complexes
- Chain homotopies
- Nullhomotopic morphisms
- Homotopic morphisms of chain complexes
- Short exact sequences of complexes:



Link to Diagram

- Small categories
  - Sets of objects, sets of morphisms, a pairing $\mathrm{Mor}(A, B) \times \mathrm{Mor}(B, C) \to \mathrm{Mor}(A, C)$.
- Universes

Exercise 14.0.2 (?)
Show that a morphism of chain complexes induces a morphism on homology.

Exercise 14.0.3 (?)
Show that $f \simeq g \implies H_\bullet(f) = H_\bullet(g)$, i.e. homotopic chain morphisms induce equal maps on homology.

Hint: reduce to showing that f nullhomotopic implies $H_\bullet(f) = 0$.

Exercise 14.0.4 (Show a SES induces a LES in homology)
Show that a SES of complexes induces a LES in homology. Write a formula for the connecting morphism, and do the check that everything is well-defined! Use the grid diagram from above.

Example 14.0.5 *(?):* Examples of categories:

- Set
- R-Mod
- Mod-R
- Top
- CRing, assumed to be unital
- $\mathsf{Sch}_{/k}$
- $\mathsf{AlgVar}_{/k}$ for $k = \bar{k}$
- $\mathsf{Sh}(X; \mathbb{Z}\text{-}\mathsf{Mod})$
- $\mathcal{O}_X$-Mod
- TopAbGrp
- $G \curvearrowright$ R-Mod

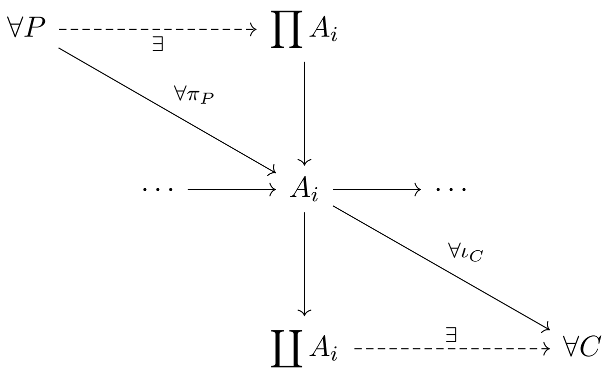
Note that many of these are not abelian, since they are not even additive, or e.g. are not closed under kernels.

15 | Monday, February 14

Remark 15.0.1: Recall the definitions of:

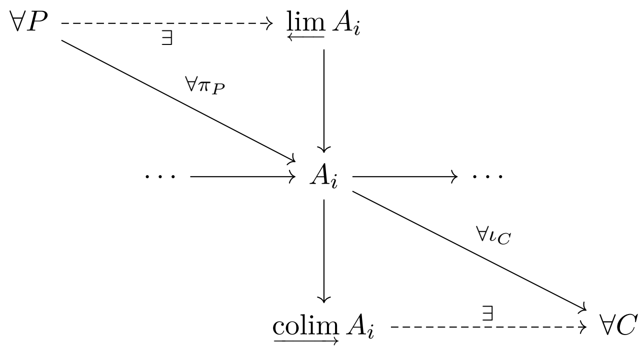
- Categories
- Functors

- Diagram/index categories
  - $\bullet \to \bullet \leftarrow \bullet$
  - $\bullet \leftarrow \bullet \to \bullet$
  - $\bullet \rightleftharpoons \bullet$
  - $\mathbb{N} : \bullet \to \bullet \to \cdots$
  - $\bullet \rightrightarrows \bullet$
- Sets and posets as categories
- Collections of objects C as functors $F \in [\mathsf{I}, \mathsf{C}]$ for I an index category
- Products and coproducts (via their universal properties). Useful mnemonic diagram:



Link to Diagram

- Algebraic cats over sets (concrete categories) will be closed under products, i.e. $\prod A_i$ will admit the same algebraic structure by taking pointwise operations.
- Examples of (co)products in common categories:
  - Set: direct cartesian product and disjoint union.
  - AbGrp: direct cartesian product and direct sum $\oplus$.
  - Ring: $\prod$ and $\otimes_{\mathbb{Z}}$
  - Top: $\prod$ whose underlying set is the cartesian product with the product topology and $\coprod$ as the disjoint union with the union of topologies Note the difference between the box and product topologies.
- A diagram in C defined as a functor.
- (co)filtered diagram categories I: for any pair i, j, $\sharp \mathrm{Mor}_{\mathsf{I}}(i, j) \leq 1$ and there exists a k with $i, j \to k$. Reverse arrows for cofiltered.
  - This allows for distinct but isomorphic objects, useful e.g. in $\mathsf{Vect}_{/k}$ where abstractly $V \cong V^\vee$ but it's useful to distinguish.
- Limits (injective, cones that live above) and colimits (projective, cocones that live below):



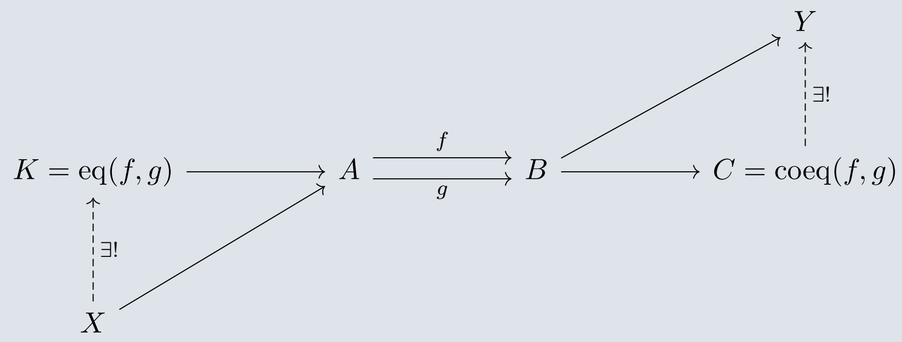
Link to Diagram

- Fiber products/pullbacks and pushouts
- Equalizers/difference kernels K and coequalizers/difference cokernels C fitting into $K \to A_1 \rightrightarrows A_2 \to C$.
- Computing cofiltered colimits in AbGrp: for the cofiltered set $\{A_i, \varphi_{ij} : A_i \to A_j\}_{i,j}$, can construct as $\underset{\longrightarrow}{\operatorname{colim}} A_i = \coprod A_i / \sim$ here $a_i \sim \varphi_{ik}(a_i)$ for any k with $i \to k$.
  - For filtered limits, one generally gets $\varprojlim A_i = \bigoplus A_i / \sim$ where $a_i \sim \varphi_{ik}(a_i)$
- Example: $\coprod A_i \in \mathsf{AbGrp}$ is not a cofiltered colimit, since the diagram category is discrete.
  - Claim: the underlying set is not $\coprod A_i$.
- For fixed objects $A \in \mathsf{C}$, the functors $\underset{\mathsf{C}}{\operatorname{Mor}}(A, -) : \mathsf{C} \to \mathsf{Set}$ and $\underset{\mathsf{C}}{\operatorname{Mor}}(-, A) : \mathsf{C} \to \mathsf{Set}^{\mathrm{op}}$.
  - More generally the target can be $\mathsf{AbGrp}, \mathsf{CRing}$, etc.

Remark 15.0.2: Next time: additive and abelian categories, why $\mathsf{Sh}(X; \mathsf{AbGrp})$ is an abelian category.

16 | Wednesday, February 16

Definition 16.0.1 (Equalizer and coequalizer)
The definition of equalizers and coequalizers:



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Remark 16.0.2: Notes:

- $\ker f \to A \rightrightarrows_0^f B \to \operatorname{coker} f$.
- $B \overset{h}{\hookrightarrow} X$ is injective iff $A \rightrightarrows_g^f B \to X$
- $X \overset{h}{\twoheadrightarrow} A$ is surjective iff $X \overset{h}{\to} A \rightrightarrows_g^f B$
- Iso = mono and epi

Exercise 16.0.3 (?)
Show that if $\operatorname{eq}(f,g) \to A$ exists then $\operatorname{eq}(f,g) \hookrightarrow A$ is mono.

Warning 16.0.4
Iso implies bijective on underlying sets, but not conversely.

Take the subcategory of $\mathsf{TopAbGrp}$ whose objects are $\mathbb{R}$ with various topologies, then take $\mathrm{id} : \mathbb{R}^{\mathrm{disc}} \to \mathbb{R}^{\mathrm{Euc}}$. Note that $\ker \mathrm{id} = \operatorname{coker} \mathrm{id} = 0$ but this is not an isomorphism. The issue: this is an additive category that isn't abelian.

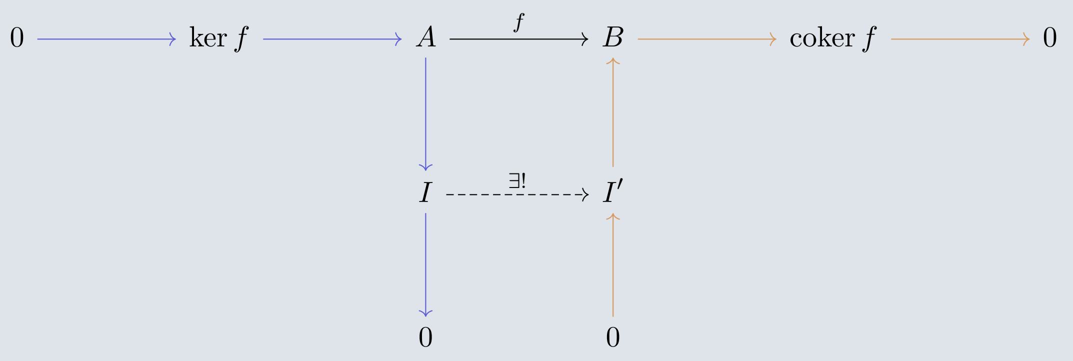
Definition 16.0.5 (Additive categories)
For $\mathsf{C} \in \mathsf{Cat}$,

- $\operatorname{Hom}_{\mathsf{C}}(A, B) \in \mathsf{AbGrp}$
- Composition is distributive, so $f(g+h) = fg + gh$ and $(g+h)f = gf + hf$.

Definition 16.0.6 (Abelian categories)
For $\mathsf{C} \in \mathsf{Cat}$,

- Closed under all kernels and cokernels
- Closed under products $\prod A_i$
  - Equivalently, closed under coproducts $\bigoplus A_i$, and in fact $A \times B = A \oplus B$ in C.
- There exists a zero object $0 = \emptyset^{\downarrow} = {}_{\uparrow}$ with $\operatorname{Hom}(0, X) = \operatorname{Hom}(X, 0) = 0$.

- Images are uniquely isomorphic to coimages:



Link to Diagram

Remark 16.0.7: For $\mathsf{C} = \mathsf{AbGrp}$, $\underset{\mathsf{C}}{\mathrm{Hom}}$ form abelian groups under pointwise operations. For morphisms $\mathsf{C} = \mathsf{Sh}(X; \mathsf{AbGrp})$ and $f, g \in \mathsf{C}(\mathcal{F}, \mathcal{G})$, writing $f = \{f_U\}, g = \{g_U\}$ in components, one can set $f + g = \{f_U + g_U\}$ to make $\underset{\mathsf{C}}{\mathrm{Hom}}$ an abelian group. Images will be isomorphic to coimages in C since the induced maps will be isomorphisms on stalks, using that AbGrp is abelian.

Remark 16.0.8: If $\mathcal{A} \in \mathsf{AbCat}$, then $\mathsf{Sh}(X; \mathcal{A}) \in \mathsf{AbCat}$.

Exercise 16.0.9 (?)
Show that $A_1 \times A_2 = A_1 \oplus A_2$ in an abelian category using the universal properties.

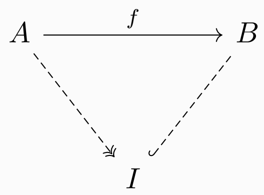
Solution:
See course notes.

17 | Friday, February 18

Remark 17.0.1: Last time: abelian categories C.

1. Existence of kernels, cokernels, and biproducts: $\exists A \times B \iff \exists A \oplus B$.
2. Existence of isomorphisms $\operatorname{coim} \varphi \to \operatorname{im} \varphi$ for all $\varphi \in \mathsf{C}(A, B)$

Corollary 17.0.2 *(?).*
For $A \in \mathsf{AbCat}$, every morphism has a mono-epi factorization:



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Remark 17.0.3: The main technical tool: every SES induces a LES in cohomology. The proof used for $\mathsf{C} = \mathsf{AbGrp}$ works nearly identically in an arbitrary abelian category using either

- *generalized elements*, c/o MacLane, or
- the full Freyd-Mitchell embedding.

MacLane's idea: define a functor

$$\begin{aligned} F : \mathsf{A} &\to \mathsf{Set}_{\text{pt}} \\ A &\mapsto \left\{ X \in \mathsf{A} \;\middle|\; X \hookrightarrow A \right\} / \sim, \end{aligned}$$

sending A to the set of its subobjects (equivalence classes of monomorphisms), and on morphisms $A \xrightarrow{f} B$ sending $X \hookrightarrow A$ to its image $f(X) \hookrightarrow B$, so $F(f)(X) = \text{im}_B(X)$. The point in the pointed set is the subobject $0_A \to A$. One then proves

- $f = 0 \iff F(f) = 0$,
- f is mono/epi $\iff F(f)$ is mono/epi,
- Thus F is exact.

So one can reduce checking exactness of f (where A may not have sets of elements) to checking exactness of $F(f)$, where the source/target are sets.

Theorem 17.0.4 ***(Freyd-Mitchell).***
For $\mathsf{A} \in \mathsf{AbCat}$, there is a fully faithful embedding $\mathsf{C} \xhookrightarrow{F} R\text{-}\mathsf{Mod}$ for some ring R. Here *full* means that $\hom_{\mathsf{A}}(A, B) \cong \hom_{R\text{-}\mathsf{Mod}}(FA, FB)$.

Proof (Idea).

- Use the Yoneda/functor of points embedding, which is fully faithful:

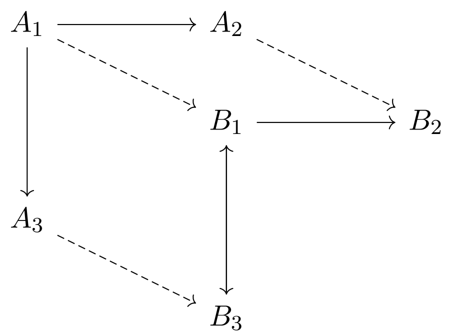
$$\begin{aligned} \mathsf{A} &\to [\mathsf{A}, \mathsf{Set}] \\ X &\mapsto h^X(-) := \underset{\mathsf{A}}{\text{Mor}}(X, -). \end{aligned}$$

- Identify $[A, \mathsf{Set}] \simeq R\text{-}\mathsf{Mod}$ where $R = \underset{\mathsf{A}}{\text{Mor}}(I, I)$ for I an injective generator of this category, so every object comes from a subobject or quotient of I. Then every $M = \underset{\mathsf{A}}{\text{Mor}}(I, M)$ becomes an R-module.

■

Observation 17.0.5
Some observations about abelian categories:

- AbCat is closed under $(-)^{\text{op}}$, i.e. $\mathsf{A} \in \mathsf{AbCat} \iff \mathsf{A}^{\text{op}} \in \mathsf{AbCat}$
- $\mathsf{A} \in \mathsf{AbCat} \implies \mathsf{ChA} \in \mathsf{AbCat}$.
- If I is any index category, $\mathsf{A}^{\mathsf{I}} = [\mathsf{I}, \mathsf{A}] \in \mathsf{AbCat}$.
  - E.g. $\mathbb{Z}$ with $i \to j \iff i \leq j$ yields $\mathsf{A}^{\mathbb{Z}}$ the category of sequences of elements of A, i.e. $\cdots \to A_{-1} \to A_0 \to A_1 \to \cdots$.
  - E.g. for $I = \bullet \to \bullet \leftarrow \bullet$, A^{I} is the category of pushouts in A whose morphisms are commuting diagrams:



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Remark 17.0.6: Some additional axioms that hold in AbGrp which we could ask $\mathsf{A} \in \mathsf{AbCat}$ to have:

- AB3: existence of arbitrary sums $\bigoplus_i A_i$.
- AB4: AB3 and if $A_i \hookrightarrow B_i$ for all i, then $\bigoplus_i A_i \hookrightarrow \bigoplus_i B_i$ is again injective.
- The dual of AB4, with products replaced by coproducts and injectives replaced by surjections.
- AB5: AB3 and for all filtered system of subobjects $A_i \subseteq A$ and a subobject $B \subseteq A$,

$$\left(\sum A_i\right) \cap B \cong \sum (A_i \cap B).$$

- AB6: AB3 and for all filtered systems $B_i^j \subseteq B^j \subseteq A$,

$$\cap_{j\in J}\left(\sum_{i\in I_j} B_i^j\right) = \sum_{i\in\coprod I_j}\left(\cap_{j\in J} B_i^j\right).$$

- AB: AB6 and AB4$^\vee$, the dual conditions for AB4.

The categories $\mathsf{Sh}_X(\mathsf{AbGrp})$ and $\mathsf{Sh}_X(\mathcal{O}_X\text{-}\mathsf{Mod})$ satisfy AB5 and AB3$^\vee$

18 | Monday, February 21

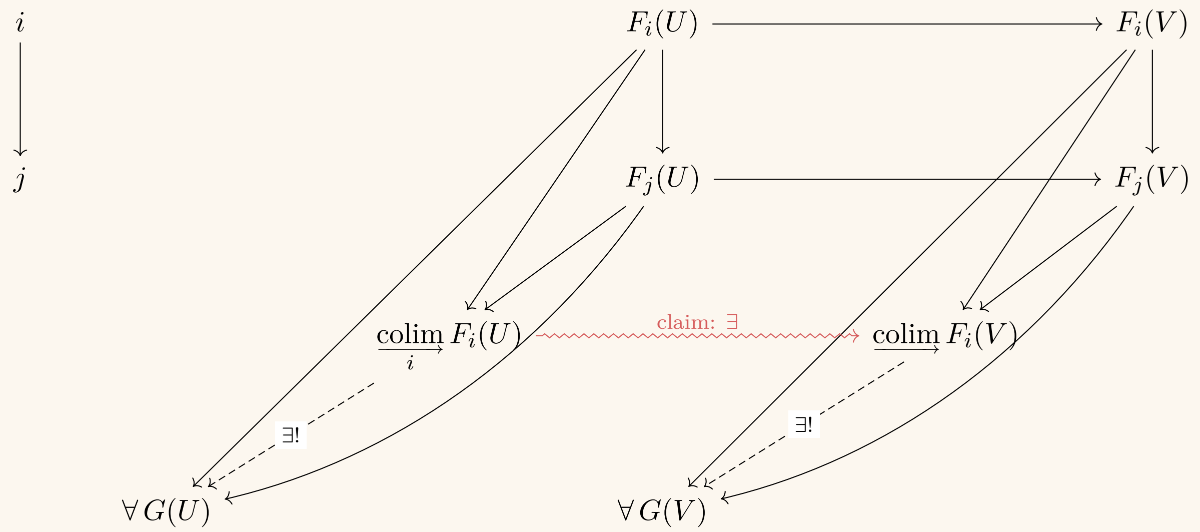
Remark 18.0.1: Recall the definitions of $\varprojlim F$ and $\varinjlim F$ for $F \in [\mathsf{I}, \mathsf{C}] = \mathsf{C}^{\mathsf{I}}$ with I a small index category. Note that if $\mathsf{N} := \mathsf{Open}(X)^{\mathrm{op}}$, the functor category $\mathsf{C}^{\mathsf{N}} = \mathsf{Sh}_{\mathrm{pre}}(X; \mathsf{C})$ consists of presheaves on X.

Lemma 18.0.2 ***(?).***
If any of the following exist in C:

- $\prod A_i$
- $\coprod A_i$
- $\varprojlim F$
- $\varinjlim F$

Then the same is true in C^{N}.

Proof (?).



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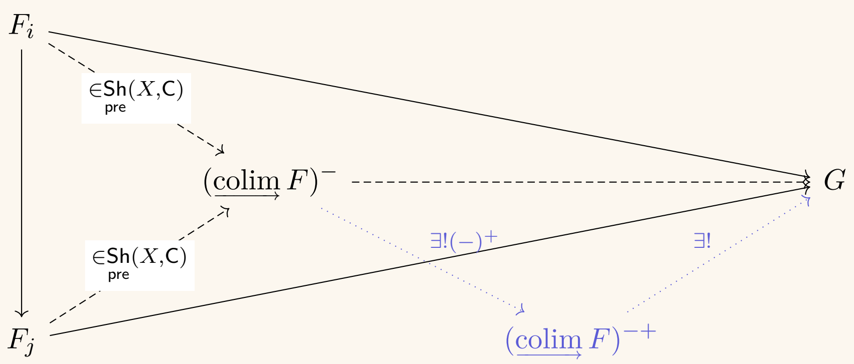
Lemma 18.0.3 *(?).*
If C has coproducts or colimits, then so does $\mathsf{Sh}(X;\mathsf{C})$.

Proof (?).
Factor through the sheafification:



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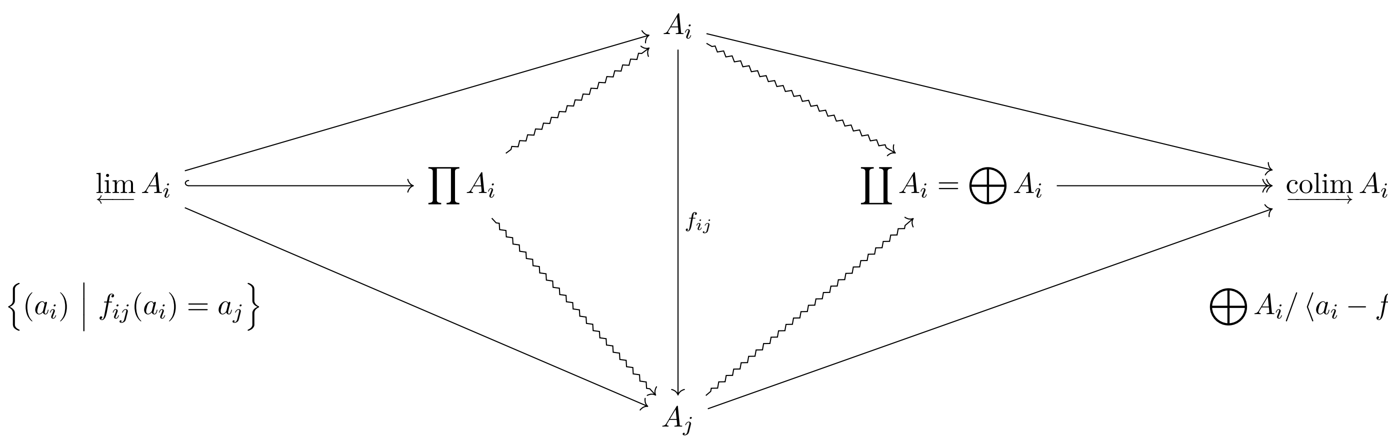
Remark 18.0.4: In AbGrp, we have $\prod, \coprod = \bigoplus, \underrightarrow{\operatorname{colim}}, \underleftarrow{\lim}$.



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Note that the inner diamond doesn't necessarily commute. The same diagram holds in $\mathsf{R}\text{-}\mathsf{Mod}$.

Corollary 18.0.5 *(?).*
In $\mathsf{Sh}(X, \mathsf{AbGrp})$ and $\mathsf{Sh}(X, \mathcal{O}_X\text{-}\mathsf{Mod})$, both $\oplus$ and $\varinjlim$ exist.

Lemma 18.0.6 *(?).*
In $\mathsf{Sh}(X, \mathsf{AbGrp})$ and $\mathsf{Sh}(X, \mathcal{O}_X\text{-}\mathsf{Mod})$, both $\prod$ and $\varprojlim$ exist.

Proof (?).
In $\underset{\mathsf{pre}}{\mathsf{Sh}}(X, \mathsf{AbGrp})$, there exist $\prod, \varprojlim$ where $(\prod F_i)(U) = \prod F_i(U)$, but this already forms a sheaf. Check that if $U = \bigcup U_\alpha$, then a collection of sections $F_i(U_\alpha)$ agreeing on intersections is the same as an element of the product. ■

Warning 18.0.7
Luckily we don't need to sheafify here, since the arrow for sheafification goes the wrong way. However, the presheaf $U \mapsto \oplus_i F_i(U)$ is not necessarily a sheaf. Take $X = \mathbb{Z}$ with the discrete topology, then any global section has infinitely many nonzero components. Note that $(\oplus F_i)^{-+} \subseteq \prod F_i$ is the subsheaf of the product where every local section has all but finitely many entries zero.

Question 18.0.8

$$\left(\bigoplus F_i\right)_p^{-+} =_? \oplus (F_i)_p,$$

i.e. is the stalk given as $\left\{(a_i) \in (F_i)_p \;\middle|\; \text{all but finitely many entries are zero}\right\}$. Idea: each a_n might only lift to a disc of radius $1/n$, which intersect to $\{p\}$. For example, take $\mathcal{F} = C^\infty$ and take smooth compactly supported functions on $[-1/n, 1/n]$ converging to $\chi_{x=0}$.

19 | Wednesday, February 23

Remark 19.0.1: Recall the definition of an additive category:

- $\underset{\mathsf{C}}{\mathrm{Mor}}(-,-)$ are abelian groups,
- Compositions distribute
- A zero object
- Finite products $A \times B \iff$ finite coproducts $A \oplus B \iff$ finite biproducts:

$$A \underset{p_1}{\overset{i_1}{\rightleftarrows}} A \oplus B \underset{p_2}{\overset{i_2}{\leftrightarrows}} B$$

Link to Diagram

where we require

- $p_j i_j = \mathrm{id}$
- $p_j i_k = 0$ for $i \neq j$
- $i_1 p_1 + i_2 p_2 = \mathrm{id}_{A\oplus B}$.
- Abelian cats: additive, plus existence of kernels, cokernels, images.

Definition 19.0.2 (Additive Functors)
A functor $F \in [\mathsf{A}, \mathsf{B}]$ is **additive** iff the induced map $F_* : \underset{\mathsf{A}}{\mathrm{Mor}}(A, B) \to \underset{\mathsf{B}}{\mathrm{Mor}}(FA, FB) \in \mathsf{AbGrp}$ is a morphism of groups.

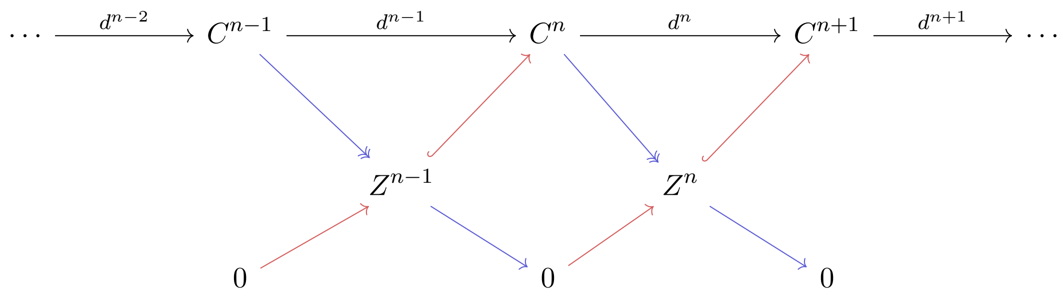
Slogan 19.0.3
Additive functors preserve

- polynomial identities in morphisms,
- biproducts, so $F(A \oplus B) \cong FA \oplus FB$,
- complexes, so $d_{n+1} d_n = 0$,
- chain homotopy equivalences of complexes, which is a polynomial identity of the form $ds + sd = h$.

Example 19.0.4 ***(of additive functors):***

- For $A \in \mathsf{A} \in \mathsf{AddCat}$, the functors $\underset{\mathsf{A}}{\mathrm{Mor}}(A, -) : \mathsf{A} \to \mathsf{AbGrp}$ and $\underset{\mathsf{A}}{\mathrm{Mor}}(-, A) : \mathsf{A} \to \mathsf{AbGrp}^{\mathrm{op}}$.
- For $A \in \mathsf{R}\text{-}\mathsf{Mod}$, $F_A(-) \coloneqq A \otimes_R (-) : \mathsf{R}\text{-}\mathsf{Mod} \to \mathsf{AbGrp}$.
    - If $R \in \mathsf{CRing}$, is commutative $F_A(-) : \mathsf{R}\text{-}\mathsf{Mod} \to \mathsf{R}\text{-}\mathsf{Mod}$.
- For I and index category, recalling $\mathsf{A}^{\mathsf{I}} = [\mathsf{I}, \mathsf{A}]$, the functors $\varprojlim \mathsf{A}^{\mathsf{I}} \to \mathsf{A}$ and $\varinjlim : \mathsf{A}^{\mathsf{I}} \to \mathsf{A}$ when they exist.
- For $\mathsf{Sh}(X; \mathsf{AbGrp})$, the global sections functor $\Gamma(X; -) : \mathsf{Sh}(X, \mathsf{AbGrp}) \to \mathsf{AbGrp}$.
    - For $f \in \mathsf{Top}(X, Y)$, pushforward $f_* : \mathsf{Sh}(X) \to \mathsf{Sh}(Y)$ (which includes inclusion of a point, i.e. taking stalks at a point) and $f^{-1} : \mathsf{Sh}(Y) \to \mathsf{Sh}(X)$ (which includes restriction).
- Local homs $\mathrm{Hom}(\mathcal{F}, -) : \mathsf{Sh}(X; \mathcal{O}_{\mathsf{X}}\text{-}\mathsf{Mod}) \to \mathsf{Sh}(X; \mathcal{O}_{\mathsf{X}}\text{-}\mathsf{Mod})$.
- $\Big|_{x} : \mathsf{Sh}(X; \mathsf{AbGrp}) \to \mathsf{AbGrp}$ where $\mathcal{F} \mapsto \mathcal{F}_x$.

Remark 19.0.5: Recall the definition of exactness for chain complexes over abelian categories: $\mathrm{im}\, d^{n-1} = \ker d^n$. Note that one can use epi-mono factorization to **splice**:



Link to Diagram

This yields collections of SESs,

$$0 \to Z^{n-1} \to C^n \to Z^n \to 0.$$

Recall the definition of right/left/middle exactness: for $0 \to A \to B \to C \to 0$ and covariant functors F:

- Right exact: $FA \to FB \to FC \to 0$,
- Long exact: $0 \to FA \to FB \to FC$,
- Middle exact: $FA \to FB \to FC$.

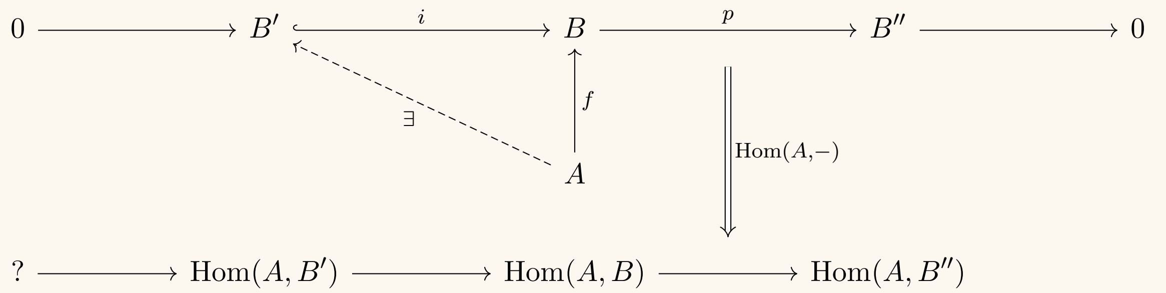
For contravariant functors, e.g. left exactness means $0 \to FC \to FB \to FA$, so injectivity is preserved. Equivalently, $F : \mathsf{A} \to \mathsf{B}$ is left exact iff the covariant $F : \mathsf{A}^{\mathrm{op}} \to \mathsf{B}$ is left-exact.

Example 19.0.6 ***(?):*** Of exactness:

- $\Gamma(-)$ is left-exact,
- $\Big|_x$ is fully exact,
- f_* is left exact,
- f^{-1} is fully exact, since this preserves stalks,
- $A \otimes_R (-)$ is right exact,
- $\underset{\mathsf{R\text{-}Mod}}{\mathrm{Hom}}(A, -)$, $\underset{\mathsf{R\text{-}Mod}}{\mathrm{Hom}}(-, A)$ are both left exact, which we'll prove.

Proposition 19.0.7 ***(?).***
$\underset{\mathsf{R\text{-}Mod}}{\mathrm{Hom}}(A, -)$, $\underset{\mathsf{R\text{-}Mod}}{\mathrm{Hom}}(-, A)$ are both left exact.

Proof (?).
Use that kernels are monomorphisms:



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Then show $if = 0 \implies f = 0$, using that $B' \to B$ is mono. Similarly $pf = 0 \implies f = ig$ for some g.

■

Remark 19.0.8: A nice proof that $\Gamma(-)$ is left-exact: realize $\Gamma(X; -) \cong \underset{\mathsf{Sh}(X)}{\mathrm{Hom}}(\underline{\mathbb{Z}}, -)$, which is left-exact for free. Use that the map $\underline{\mathbb{Z}} \to \mathcal{F}(X)$ is determined by $1 \mapsto s$ and extend using $n = n \cdot 1$.

20 | Friday, February 25

20.1 Adjoint Functors, Exactness

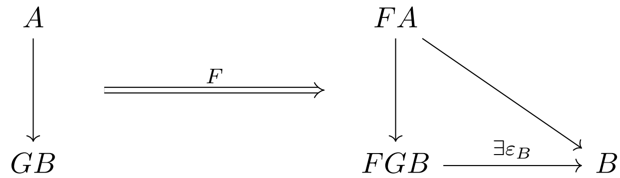
Remark 20.1.1: Consider the setup:

$$\mathsf{A} \underset{G}{\overset{F}{\rightleftarrows}} \mathsf{B}.$$

We say F is a **left adjoint** and G is a **right adjoint**, so F *has* a right adjoint and G *has* a left adjoint, if there are natural isomorphisms

$$[FA, B]_{\mathsf{B}} \xrightarrow{\sim} [A, GB]_{\mathsf{A}},$$

i.e. there is a natural isomorphism of functors $[A, G(-)] \xrightarrow{\sim} [FA, (-)]$. For a fixed object B, there is a natural transformation $\varepsilon_B : FG \to \mathrm{id}_B$ which we call the **counit** and $\eta_A : \mathrm{id}_A \to GF$ called the **unit**:

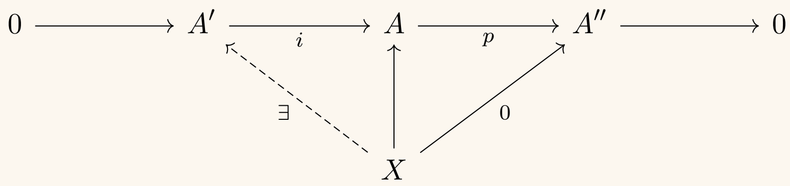


Link to Diagram

Theorem 20.1.2 *(?)*.
If $\mathsf{A}, \mathsf{B} \in \mathsf{AbCat}$, then

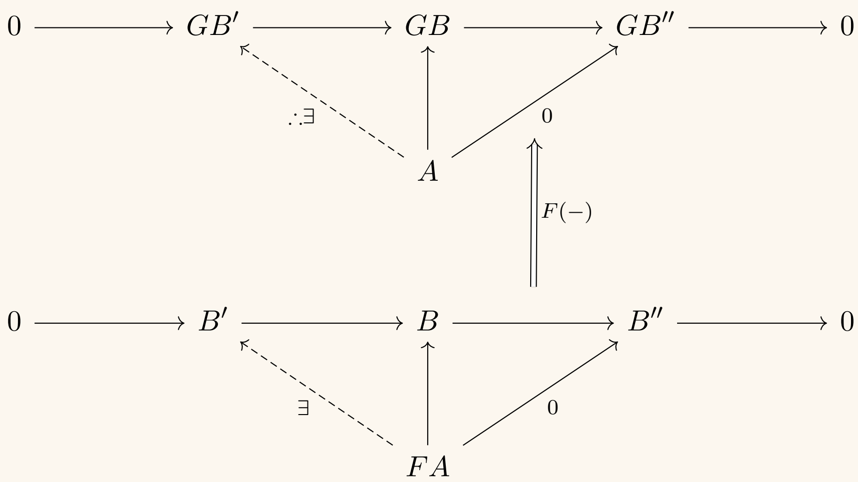
- If F is a right adjoint, F is left exact.
- If G is a left adjoint, G is right exact.

Proof (?).
Note that the following lift exists iff $\ker(A \to A'') = (A' \to A)$:



Link to Diagram

Given $0 \to B' \to B \to B''$, we want to show $0 \to GB' \to GB \to GB''$ is exact. Given $A \to B''$ factoring through zero, we can use adjointness to flip diagrams:



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Example 20.1.3 *(?)***:** There is an adjunction between global sections and constant sheaves:

$$\mathsf{Sh}(X; \mathsf{AbGrp}) \underset{\underline{(-)}}{\overset{\Gamma(X;-)}{\rightleftarrows}} \mathsf{AbGrp}.$$

One can define the map explicitly:

$$\begin{aligned} [A, \Gamma(X; \mathcal{F})]_{\mathsf{AbGrp}} &\to [\underline{A}, \mathcal{F}]_{\mathsf{Sh}(X;\mathsf{AbGrp})} \\ (a \mapsto s_a) &\mapsto (a_U \mapsto s_a|_U). \end{aligned}$$

It suffices to check this locally. Use that $\Gamma(X; \underline{A})$ contains a copy of A to define the reverse map, and check they are mutually inverse.

Example 20.1.4 ***(?):*** For $f \in [X, Y]_{\mathsf{Top}}$, there is an induced adjunction

$$\mathsf{Sh}(X; \mathsf{AbGrp}) \underset{f^{-1}}{\overset{f_*}{\rightleftarrows}} \mathsf{Sh}(Y; \mathsf{AbGrp}).$$

Thus f_* is left exact.

Exercise 20.1.5 (?)
Define the map

$$[\mathcal{G}, f_*\mathcal{F}]_{\mathsf{Sh}_Y} \to [f^{-1}\mathcal{G}, \mathcal{F}]_{\mathsf{Sh}_X}.$$

Remark 20.1.6: Note that f_* is fully exact, as we knew before by checking on stalks. Also note that $\big|_x$ for $\mathcal{F} \in \mathsf{Sh}(X)$ is $f^{-1}\mathcal{F}$ for $f : \{x\} \hookrightarrow X$.

Example 20.1.7 ***(?):***

$$\underset{\mathsf{pre}}{\mathsf{Sh}}(X; \mathsf{AbGrp}) \underset{\text{Forget}}{\overset{(-)^+}{\rightleftarrows}} \mathsf{Sh}(X; \mathsf{AbGrp}),$$

so sheafification is right exact and the forgetful functor is left exact. In fact, $(-)^+$ is fully exact since it preserves stalks.

Example 20.1.8 ***(?):*** For $j \in [U, X]_{\mathsf{Top}}$ with U open in X,

$$\mathsf{Sh}(U; \mathsf{AbGrp}) \underset{j^{-1}}{\overset{j_!}{\rightleftarrows}} \mathsf{Sh}(X; \mathsf{AbGrp}).$$

In general there is a SES

$$0 \to j_! \mathcal{F}|_U \to \mathcal{F} \to i_* \mathcal{F}|_{X \setminus U} \to 0.$$

Example 20.1.9 ***(from algebra):***

$$\mathsf{R\text{-}Mod} \underset{[-,-]_{\mathsf{R\text{-}Mod}}}{\overset{(-)\otimes_R(-)}{\rightleftarrows}} \mathsf{R\text{-}Mod},$$

so tensoring is right exact when an object is fixed. Note the isomorphism

$$[A \otimes_R B]_{\mathsf{R\text{-}Mod}} \xrightarrow{\sim} [A, [B, C]_{\mathsf{R\text{-}Mod}}]_{\mathsf{R\text{-}Mod}}.$$

21 | Monday, February 28

21.1 Tensors

Remark 21.1.1: Recall $\mathsf{R\text{-}Mod} = \mathsf{Mod\text{-}R} = (R, R)\text{-}\mathsf{biMod}$ for $R \in \mathsf{CRing}$ associative, but for noncommutative rings these may differ.

- The tensor product is a bifunctor

  $(-) \otimes_R (-) : \mathsf{Mod\text{-}R} \times \mathsf{R\text{-}Mod} \to \mathsf{AbGrp}$

  $$M_R \times_R N \mapsto M_R \otimes_R {}_RN = \frac{F(M \times N)}{(m_1 + m_2) \otimes n - m_1 \otimes n - m_2 \otimes n, ma \otimes n - m \otimes an, \cdots}$$

  satisfying the usual universal property.

- This generalizes:

  $$(-) \otimes_R (-) : (R, R)\text{-}\mathsf{biMod} \times \mathsf{R\text{-}Mod} \to \mathsf{R\text{-}Mod}.$$

- If $\varphi \in \mathsf{CRing}(R, S)$, then

  $$(-) \otimes_R S : \mathsf{Mod\text{-}R} \to \mathsf{S\text{-}Mod}.$$

- This extends to algebras:

  $$(-) \otimes_R (-) : \mathsf{Alg}R \times \mathsf{Alg}R \to \mathsf{Alg}R,$$

  with multiplication given by $(s_1 \otimes s_2) \cdot (t_1 \otimes t_2) := (s_1 t_1) \otimes (s_2 t_2)$.

- There is an adjunction:

  $$[A \otimes_R C, B]_{\mathsf{R\text{-}Mod}} \xrightarrow{\sim} [A, B^C]_{\mathsf{R\text{-}Mod}}.$$

Corollary 21.1.2 *(?)***.**
Since $A \otimes_R (-)$ is a left adjoint, it is right exact. Thus presentations $R^J \to R^I \to M \to 0$ yield presentations $M^J \to M^I \to M \otimes_R N \to 0$.

Example 21.1.3 *(?)***:**

$$\mathbb{C} \otimes_R R\mathbb{C} \cong \mathbb{C} \oplus \mathbb{C},$$

writing $\mathbb{C} = \mathbb{R}[x] / \left\langle x^2 + 1 \right\rangle$, so

$$\mathbb{C} \otimes_{\mathbb{R}} \frac{\mathbb{R}[x]}{\langle x^2 + 1 \rangle} \cong \frac{\mathbb{C}[x]}{\langle x^2 + 1 \rangle} \cong \frac{\mathbb{C}[x]}{\langle x - i \rangle} \oplus \frac{\mathbb{C}[x]}{\langle x + i \rangle}.$$

Geometrically, this corresponds to $\underrightarrow{\operatorname{colim}}(\operatorname{Spec}\mathbb{C} \to \operatorname{Spec}\mathbb{R} \leftarrow \operatorname{Spec}\mathbb{C}) \cong X := \operatorname{Spec}\mathbb{C}\otimes_{\mathbb{R}}\mathbb{C}$, where point $\left\langle x^2+1 \right\rangle$ splits geometrically and $X \to \operatorname{Spec}\mathbb{R}$ is a 2-to-1 cover over this point.

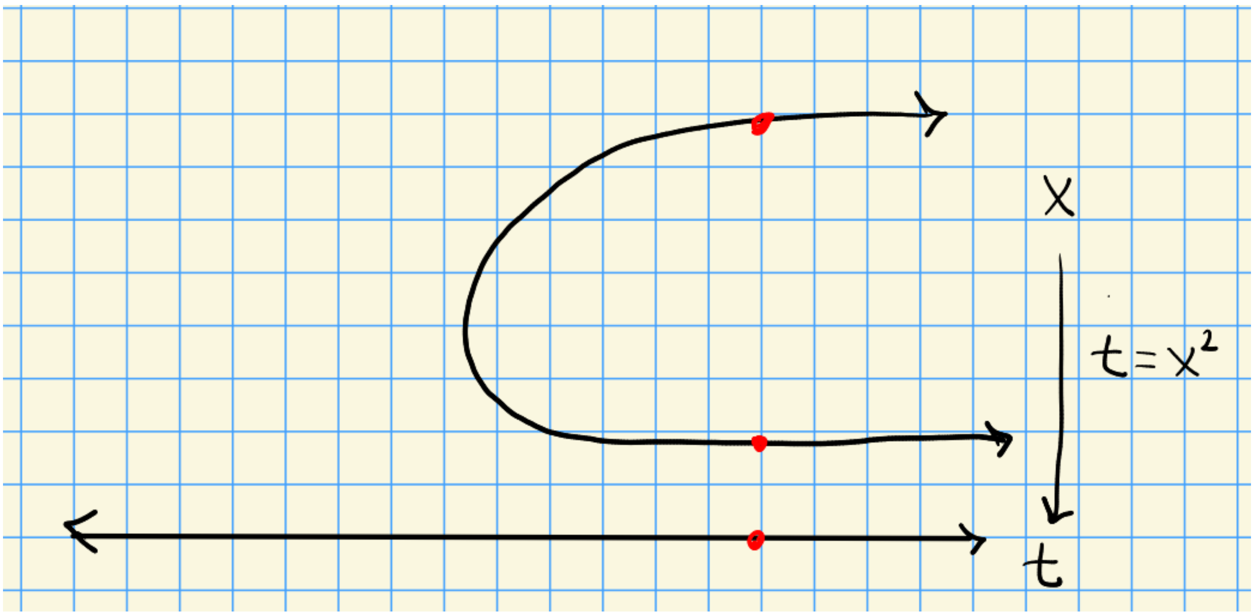
$$\begin{array}{ccccc} X \underset{Y}{\times} \operatorname{Spec} k \cong \operatorname{Spec}(S\otimes_R k) & \longrightarrow & X & \hookrightarrow & \mathbb{A}^{m+n} \\ \downarrow & \lrcorner & \downarrow & & \downarrow \\ \operatorname{Spec} k & \longrightarrow & Y & \hookrightarrow & \mathbb{A}^n \ni [a_1,\cdots,a_n] \\ & & \uparrow & \nearrow & \\ & & \operatorname{Spec} k & & \end{array} \qquad S = R$$

Link to Diagram

Conclusion:

$$S\otimes_R k = \frac{k[x_1,\cdots,x_m]}{\langle q_j(a,x)\rangle}.$$

In the previous example, the fiber over a is $\operatorname{Spec} k[x]/\left\langle x^2-a \right\rangle$ and the covering map looks like the following:

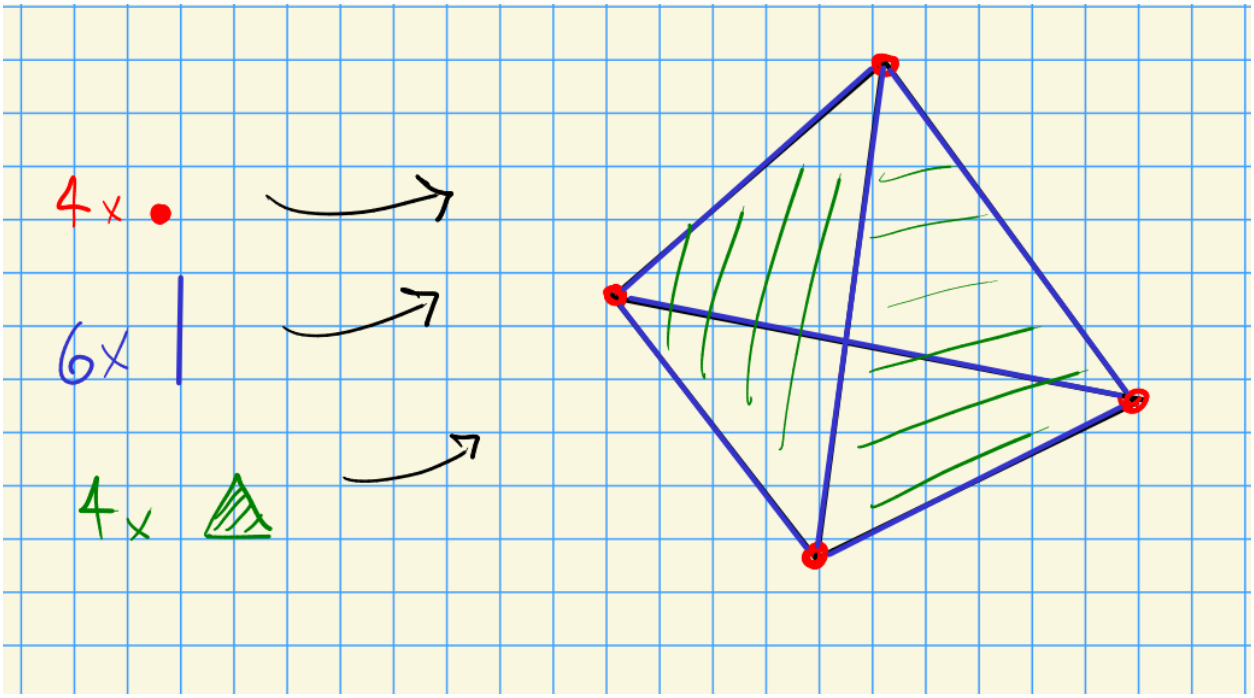


Question 21.1.4
Is direct sum exact as a functor $\mathsf{A}^{\times 2} \to \mathsf{A}$? Regard $\mathsf{A}^{\times 2} = \mathsf{A}^{\mathsf{I}}$ where $I = \{\bullet,\bullet\}$ is the discrete 2-object diagram category. The map $(A_1, A_2) \to A_1 \oplus A_2$ is exact by just summing SESs.

21.2 Cohomology

Remark 21.2.1: Recall that one can compute $H_*(S^2;\mathbb{Z})$ in several ways.

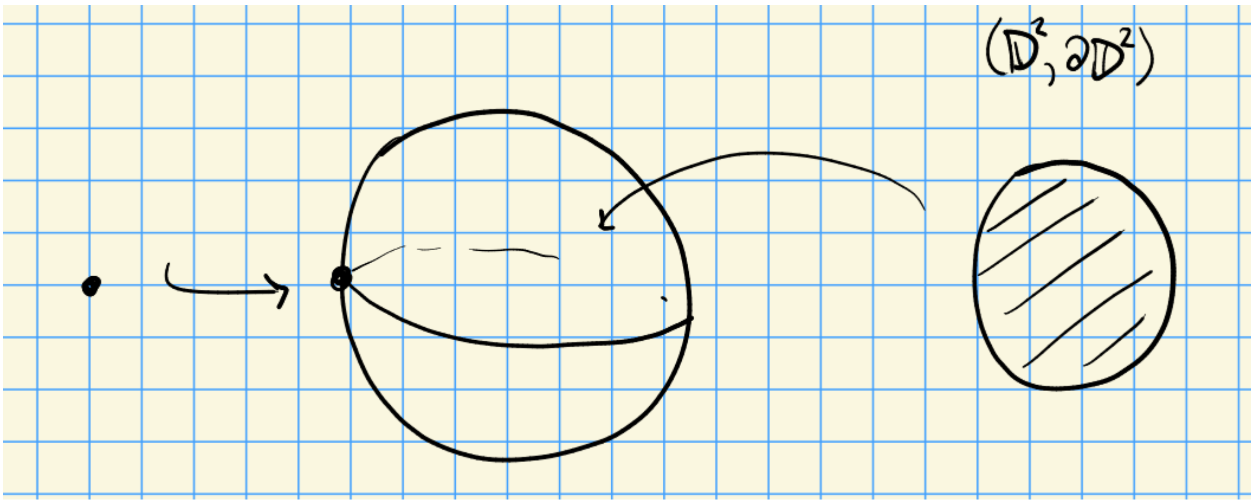
Method 1: triangulation.



This yields

$$0 \leftarrow \mathbb{Z}^{\times 4} \leftarrow \mathbb{Z}^{\times 6} \leftarrow \mathbb{Z}^{\times 4} \leftarrow 0 \rightsquigarrow 0 \leftarrow \mathbb{Z} \leftarrow 0 \leftarrow \mathbb{Z} \leftarrow 0.$$

Method 2: cell complexes.



This directly yields

$$0 \leftarrow \mathbb{Z} \leftarrow 0 \leftarrow \mathbb{Z} \leftarrow 0.$$

Question 21.2.2
Why are simplices Δ_n or discs D^n the right things?

Answer 21.2.3
They are contractible, but more importantly do not themselves have higher homology and are thus *acyclic*.

Remark 21.2.4: More generally, for $F \in \mathsf{AbCat}(\mathsf{A}, \mathsf{B})$, we'll want to resolve by acyclic objects. Injectives and projectives will be universal such objects, but are often hard to work with, so we'll work on finding more economical acyclic resolutions. Next time: injectives/projectives and derived functors.

22 | Wednesday, March 02

Remark 22.0.1: For $F \in \mathsf{AbCat}(\mathsf{A}, \mathsf{B})$ left exact, assuming A has enough injectives, there is a right derived functor $\mathbb{R}F$ so that a SES $0 \to A \to B \to C \to 0$ admits a LES with a connecting morphism δ:

$$0 \to \mathbb{R}FA \to \mathbb{R}FB \to \mathbb{R}FC \xrightarrow{\delta} \Sigma^1 \mathbb{R}FA \to \cdots .$$

Note that δ depends on the triple appearing in the SES.

Theorem 22.0.2 ***(Grothendieck).***
$\mathbb{R}F$ and δ are universal among δ-functors.

Remark 22.0.3: Injectives will be acyclic and homology will measure how things are glued. Analogy: simplicial or cellular homology uses contractible objects (with trivial homology) to measure how spaces are glued from simplices or spheres.

Remark 22.0.4: Recall the definitions of projective and injective objects, which require existence (but not uniqueness) of certain lifts. In $R\text{-}\mathsf{Mod}$, free implies projective, so free resolutions usually suffice and one can study generators, relations, syzygies, etc.

We'll show that $\mathsf{A} := \mathsf{Sh}(X; \mathsf{AbGrp})$ has enough injectives, but usually won't have enough projectives. Recall that this means that every $A \in \mathsf{A}$ admits a monomorphism $A \hookrightarrow I$ for I an injective object. If there are enough injectives, every object admits an injective resolution, and any two such resolutions are homotopy equivalent.

Remark 22.0.5: Recall that

$$\mathbb{R}^{\bullet}F(X) = H_{\bullet}(F(X \leftleftarrows I^{\bullet}))$$

and $\mathbb{R}^{i\geq 1}F(I) = 0$ if I is itself injective.

Remark 22.0.6: Recall the Horseshoe lemma:

$$\begin{array}{ccccccc}
0 & & 0 & & 0 & & \\
\downarrow & & \downarrow & & \downarrow & & \\
A & \longleftarrow & I_A^1 & \longleftarrow & I_A^2 & \dashleftarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
B & \longleftarrow & \exists I_B^1 := I_A^1 \oplus I_C^1 & \longleftarrow & \exists I_B^2 := I_A^2 \oplus I_C^2 & \dashleftarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
C & \longleftarrow & I_C^1 & \longleftarrow & I_C^2 & \dashleftarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & & 0 & & 0 & &
\end{array}$$

Link to Diagram

Note that the complex in the middle is not the direct sum of the two outer complexes, just the terms – the differential d_B on $I^{\bullet}{}_B$ will be of the form

$$d_B = \begin{bmatrix} d_A & * \\ 0 & d_C \end{bmatrix}.$$

Exercise 22.0.7 (?)
Prove this, using that additive functors preserve direct sums. Conclude using that this construction yields a SES of complexes $0 \to FI^{\bullet}{}_A \to FI^{\bullet}{}_B \to FI^{\bullet}{}_C \to 0$.

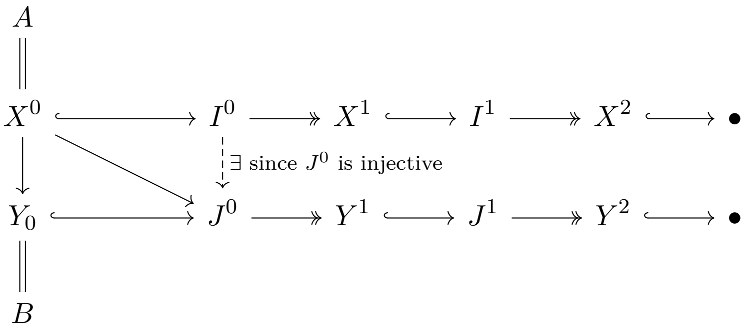
Exercise 22.0.8 (?)
Prove that if I is injective then $0 \to I \to B \to C \to 0$ splits by explicitly constructing a left and right splitting to show that B satisfies the universal property of the biproduct. Show also that the same conclusion holds for $0 \to A \to B \to P \to 0$ with P projective.

23 | Friday, March 04

Remark 23.0.1: Idea: regard A as a chain complex supported in degree zero and $A \underset{\leftleftarrows}{\eta} I^{\bullet}$ an injective resolution, then the induced map $\eta^* : H^*(A) \to H^*(I^{\bullet})$ is an isomorphism, so A and $I^{\bullet}$ are quasi-isomorphic.

> **Exercise 23.0.2** (?)
> Show that if $A \leftleftarrows I^\bullet, J^\bullet$, then there exists a chain homotopy $f : I \simeq J$.

Remark 23.0.3: Hints:



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Given

$$f^{n-1} - g^{n-1} = h^n d^{n-1} - d^{n-2} h^{n-1},$$

construct $h^{n+1} d_I^n$ such that

$$(f^n - g^n) d_I^n = (h^{n+1} d_I^n + d_J^{n-1} h^n) d_I^n$$

and extend arbitrarily to $h^{n+1} : I^{n+1} \to J^n$.

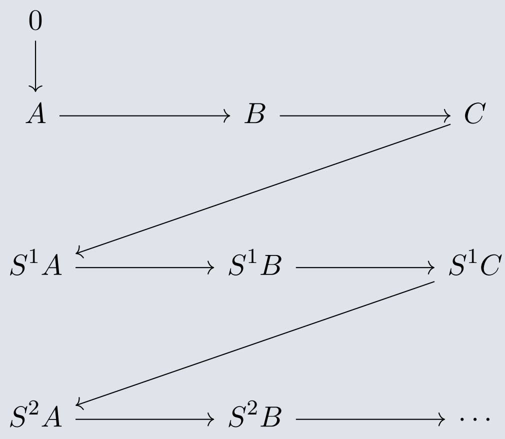
> **Exercise 23.0.4** (?)
> Prove the Horseshoe lemma.

24 | Monday, March 14

Remark 24.0.1: Call the definition of the derived functor $\mathbb{R}F$ for a left-exact functor $F \in \mathsf{AbCat}(\mathsf{C}, \mathsf{D})$ where A has enough injectives. These satisfy $\mathbb{R}^0 F = F$, and for a SES $0 \to A \to B \to C \to 0$ there is an induced LES $\mathbb{R}FA \to \mathbb{R}FB \to \mathbb{R}FC \to \mathbb{R}FA[1]$ which is functorial in the triple (A, B, C). Next: Grothendieck's universality theorem.

> **Definition 24.0.2** (δ-functor)
> A δ-functor is a sequence of functors $\left\{ S^i : \mathsf{A} \to \mathsf{B} \right\}_{i \geq 0}$ such that for all SESs $0 \to A \to B \to$

$C \to 0$ there is a (not necessarily exact) complex:



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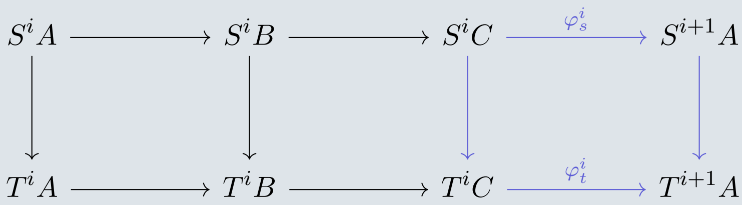
A **morphism** of δ-functors is a collection $\left\{f^i : S^i \to T^i\right\}_{i\geq 0}$ such that for all such SESs, there is a commutative diagram:



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Note that the first 2 square are commutative by functoriality, and the content here is that the map commutes with the connecting morphisms.

Definition 24.0.3 (Effaceable functors)

An additive functor $G : \mathsf{A} \to \mathsf{B}$ is **effaceable** iff for all $A \in \mathsf{A}$ there is a monomorphism $A \xhookrightarrow{f} M$ such that $GA \xrightarrow{Gf} GM$ is the zero map.

Slogan 24.0.4

Effaceable functors are those which erase some monomorphism.

Definition 24.0.5 (Universal delta functors)

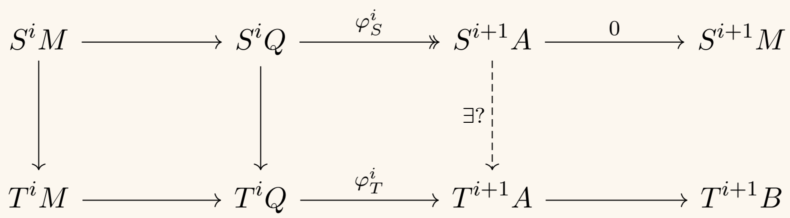
A delta functor (S_i, φ_S) is *exact* iff the induced complex is a LES, and is **universal** iff for any other delta functor (T_i, φ_T) and any natural transformation $\eta : S^0 \to T^0$, there is a unique morphism $(S_i, \varphi_S) \to (T_i, \varphi_T)$ extending η.

Theorem 24.0.6 *(Grothendieck, Tohoku: exact fully effaceable functors are universal).*
Suppose $\left\{S^i F, \varphi\right\}_{i\geq 0}$ is an exact delta functor and that the S^i are effaceable for all i. Then it is a universal δ functor.

Corollary 24.0.7 *(?).*
When $F \in \mathsf{AbCat}(\mathsf{A}, \mathsf{B})$ where A has enough injectives, $(\mathbb{R}^i F, \varphi)$ is universal and there is a unique such delta functor with $\mathbb{R}^0 F = F$.

Proof (of corollary).
Embed $A \hookrightarrow I$ into an injective object, which is F-acyclic, and thus $\mathbb{R}^i FA \xrightarrow{0} \mathbb{R}^i FI = 0$. ■

Proof (of theorem).
Proceed by induction. Let $0 \to A \to M \to Q \to 0$ be arbitrary, and use a diagram chase to define a map $f^i(\iota)$:



Link to Diagram

One needs to show:

1. $f^i(\iota)$ does not depend on ι
2. It is a ? for all $A \to B$
3. This map commutes with φ_S, φ_T.

■

25 | Wednesday, March 16

25.1 Grothendieck's Universal Theorem

Remark 25.1.1: Setup from last time: $F \in \mathsf{AddCat}(\mathsf{A}, \mathsf{B})$ left-exact, $\{(S^n, \varphi_S^n)\}_{n\geq 0}$ exact δ-functors where for $n > 0$ the S^n are effaceable. Then it is universal: for all δ-functors $\{(T^n, \varphi_T^n)\}_{n\geq 0}$ with a natural transformation $S^0 \to T^0$ there exist unique morphisms $(S^n, \varphi_S^n) \to (T^n, \varphi_T^n)$, i.e. natural transformations $S^n \to T^n$ commuting with the φ^n.

25.1.1 Proof of Universality

Remark 25.1.2: Take an effacement $0 \to A \overset{i}{\hookrightarrow} M$ for S^{n+1} and extend to a SES $0 \to A \to M \to Q \to 0$. We'll define the ladder of morphisms inductively using the following commutative diagram:

$$\begin{array}{ccccc} S^nQ & \xrightarrow{\varphi_S} & S^{n+1}A & \longrightarrow & S^{n+1}M \\ {\scriptstyle f^n}\downarrow & & \downarrow{\scriptstyle \exists f^{n+1}=f^{n+1}(A,i)} & & \\ T^nQ & \xrightarrow{\varphi_T} & T^{n+1}A & & \end{array}$$

Link to Diagram

We need to show

- $f^{n+1}(A,i)$ only depends on A
- f^{n+1} is functorial in A
- f^{n+1} commutes with φ_S, φ_T.

Lemma 25.1.3 *(?).*
Assume that given two effacements of two delta functors, there exist morphisms:

$$\begin{array}{ccccc} 0 & \xrightarrow{g} & A_1 & \xrightarrow{i_1} & M_1 \\ & & \downarrow & & \downarrow \\ 0 & \longrightarrow & A_2 & \xrightarrow{i_2} & M_2 \end{array}$$

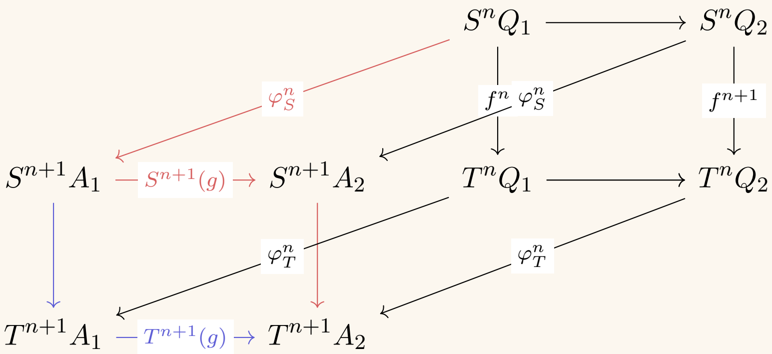
Link to Diagram

Then there is a commuting square

$$\begin{array}{ccc} S^{n+1}A_1 & \xrightarrow{S^{n+1}(g)} & S^{n+1}A_2 \\ \downarrow & & \downarrow \\ T^{n+1}A_1 & \xrightarrow{T^{n+1}(g)} & T^{n+1}A_2 \end{array}$$

Link to Diagram

Proof (?).
There is a cube:



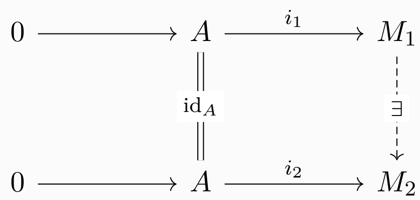
Link to Diagram

Here all faces but the front form commuting squares.

Exercise *(?)*
Show that one can move the red path to the blue through the other commuting faces.

■

Corollary 25.1.5 ***(?).***
$f^{n+1}(A, i)$ only depends on A. Take two effacements, and assume there is a commuting diagram:



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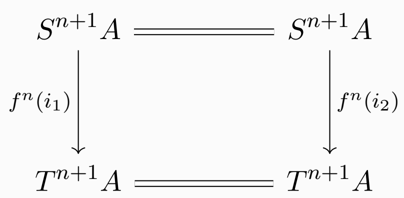
By the lemma:



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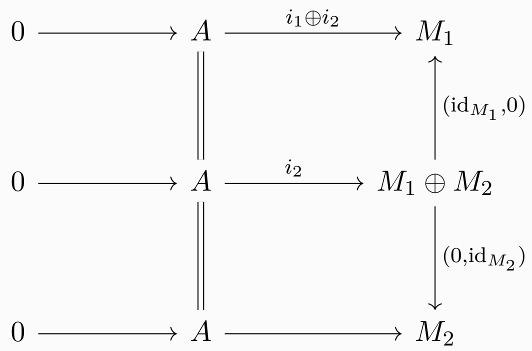

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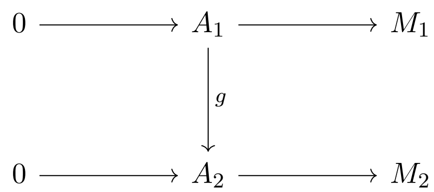
See notes for finished proof.

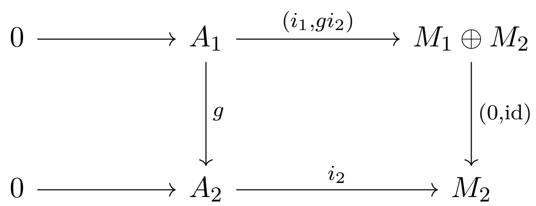
26 | Friday, March 18

Remark 26.0.1: Given effacements:



Link to Diagram

There exists an effacement extending g. Use



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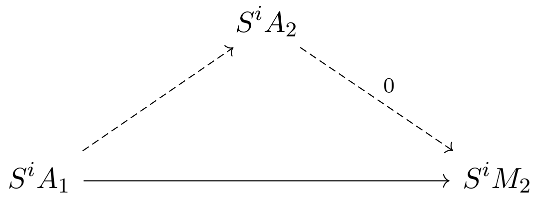
There is a factorization:



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?? Concludes theorem from last time.z

Remark 26.0.2: Recall that $\operatorname{Hom}(C, -)$ is left exact covariant and $\operatorname{Hom}(-, C)$ is left exact contravariant. For left exact functors,

- Right derived functors are computed with injective resolutions.
- C needs enough injectives

For right exact functors,

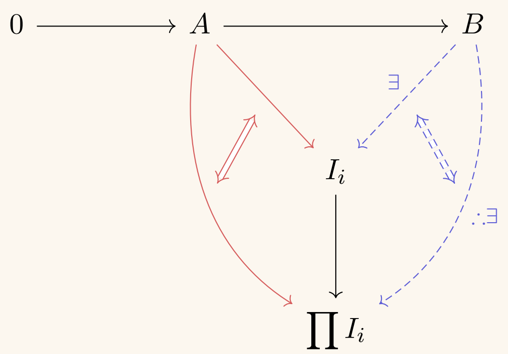
- Left derived functors are computed with projective resolutions.
- C needs enough projectives

Remark 26.0.3: Projective sheaves are locally free.

Exercise 26.0.4 (?)
Show:

- Injectives are closed under $\prod$,
- Projectives are closed under $\bigoplus$.

Proof (?).



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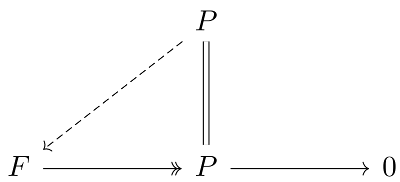
■

Exercise 26.0.5 (?)
Show that in R-Mod, M is projective $\iff$ M is a direct summand of a free module iff M is locally free.

Solution:
Some hints:



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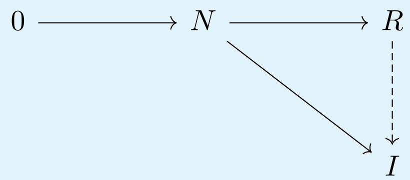
Exercise 26.0.6 (?)
Show

- $\mathbb{R}\underset{\mathbb{Z}\text{-Mod}}{\operatorname{Hom}}(C_n, M) = M[n] \oplus \Sigma^1(M/nM)$ using $0 \to \mathbb{Z} \xrightarrow{\times n} \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \to 0$.
  - Conclude that divisible module has vanishing $\operatorname{Ext}^1(C_n, -)$.
- If R is a PID, then $M \in$ R-Mod is injective $\iff$ M is divisible.
- For all rings R, R is injective iff

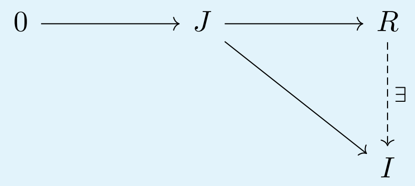


Link to Diagram

27 | Monday, March 21

Remark 27.0.1: Recall free $\implies$ projective and R-Mod has enough projectives and enough injectives.

Exercise 27.0.2 (?)
Show I is injective iff



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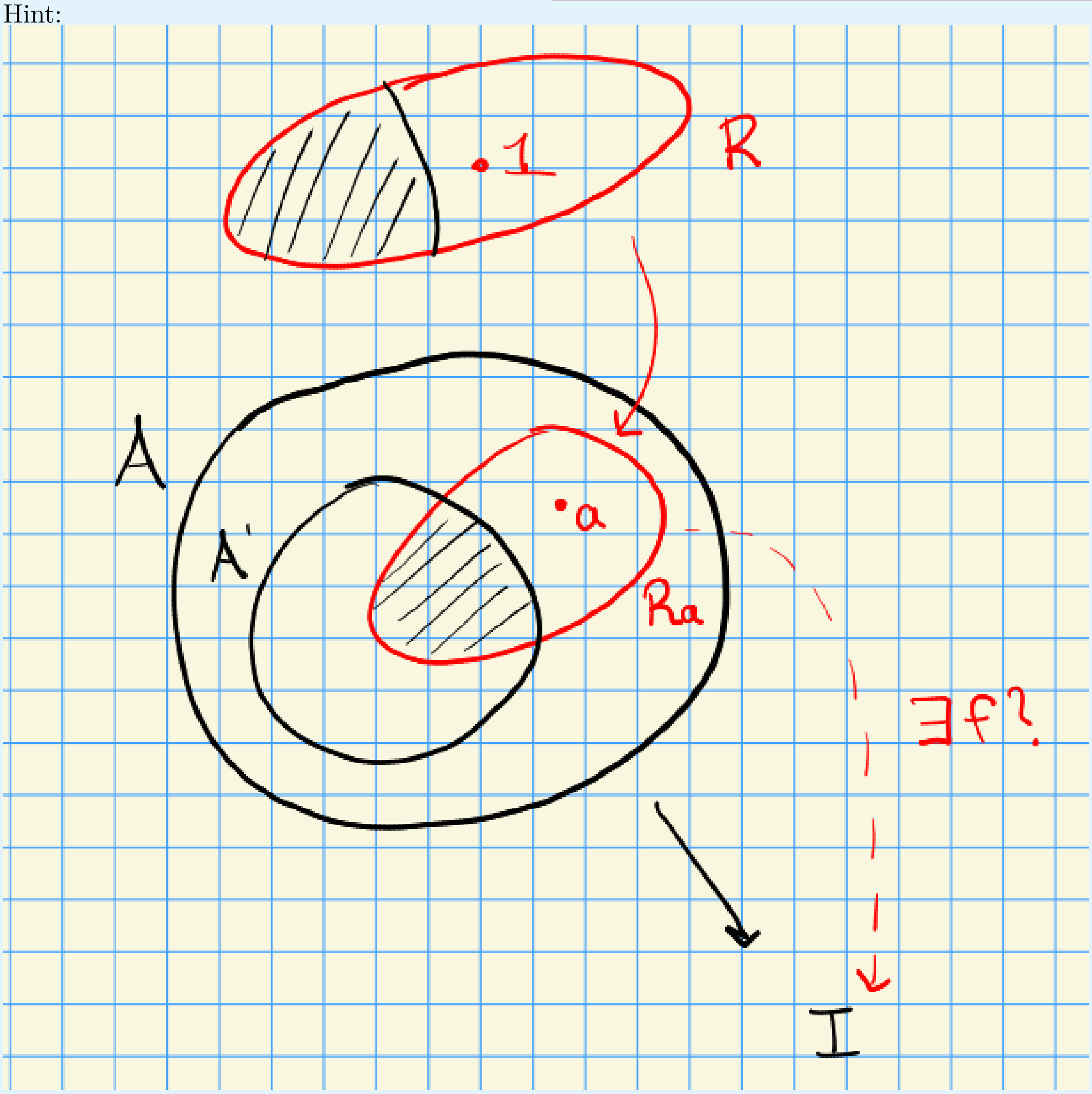
Hint:



Extend to $A' + Ra$ using $1 \mapsto a \mapsto i \in I$ under $R \to Ra \to I$. Take a poset of all $B \subseteq A$ with $g : B \to I$ extending $A' \to I$ and apply Zorn's lemma.

Exercise 27.0.3 (?)
Show that for R a PID, $M \in$ R-Mod is injective iff divisible.

Exercise 27.0.4 (?)
Show that $\mathbb{Z}$-Mod has enough innjectives.

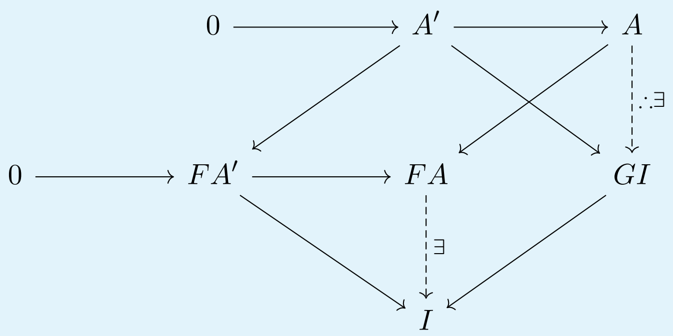
Hint: write $A = \bigoplus \mathbb{Z}/K \hookrightarrow \bigoplus \mathbb{Q}/K$.

Remark 27.0.5: On adjoint functors:

$$\mathsf{A} \underset{G}{\overset{F}{\underset{\perp}{\rightleftarrows}}} \mathsf{B} \implies \mathsf{B}(FX, Y) \xrightarrow{\sim} \mathsf{A}(X, GY).$$

Here F is a left adjoint hence right exact, and G is a right adjoint and is left exact.

Exercise 27.0.6 (?)
Show that if F is left exact then G preserves in injectives, and if F is right exact then G preserves projectives.
Hint:



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Remark 27.0.7: For $f \in \mathsf{CRing}(S \to R)$, there is an adjunction

$$\mathsf{R\text{-}Mod} \underset{\mathsf{S\text{-}Mod}(R,-)}{\overset{M_R \mapsto M_S}{\underset{\perp}{\rightleftarrows}}} \mathsf{S\text{-}Mod}$$

where $\mathsf{S\text{-}Mod}(R, N) \in \mathsf{R\text{-}Mod}$ via the action $(rf)(x) := f(rx)$, sometimes called the *induced R-module*. Note that $\mathsf{R\text{-}Mod}(R, N) \xrightarrow{\sim} N$ by $1_R \mapsto n$, and there is an iso

$$\begin{aligned} \mathsf{S\text{-}Mod}(M_S, N) &\rightleftharpoons \mathsf{R\text{-}Mod}(M_R, \mathsf{S\text{-}Mod}(R, N)) \\ (m \mapsto \psi(m)(1)) &\leftarrow\!\shortmid \psi \\ \varphi &\mapsto (m \mapsto \psi(m)(i) := \psi(im) := \varphi(im)) . \end{aligned}$$

Remark 27.0.8: Proving $\mathsf{R\text{-}Mod}$ has enough injectives if $\mathsf{S\text{-}Mod}$ has enough injectives: use $M_R \cong \mathsf{R\text{-}Mod}(R, M) \hookrightarrow \mathsf{S\text{-}Mod}(R, M_S) \hookrightarrow \mathsf{S\text{-}Mod}(R, I)$ where $M_S \hookrightarrow I$ embeds into some injective. Take R arbitrary and $S = \mathbb{Z}$ to conclude any $\mathsf{R\text{-}Mod}$ has enough injectives.

Exercise 27.0.9 (?)
This is a theoretical tool and not particularly practical. Consider $S \to R := \mathbb{Q} \to \mathbb{C}$ and $M = \mathbb{C}$. Then $\mathbb{Q}\text{-}\mathsf{Mod}(\mathbb{C}, \mathbb{C}_{\mathbb{Q}}) = G\mathbb{C}_{\mathbb{Q}}$.

Remark 27.0.10: Any $M \in \mathsf{R\text{-}Mod}$ admits a minimal injective hull $M \hookrightarrow I$.

Theorem 27.0.11 *(?).*
$\mathsf{Sh}(X \to \mathsf{AbGrp})$ and $\mathcal{O}_{\mathsf{X}}\text{-}\mathsf{Mod}$ have enough injectives.

Proof (?).
Take

$$\mathcal{F} \hookrightarrow \prod_{x \in X} (\iota_x)_* \mathcal{F}_x \hookrightarrow \prod_{x \in X} I_x.$$

The claim is that the last term is an injective sheaf. Using that products of injective are injective, it STS I_x is injective. For $\iota_x : \{x\} \hookrightarrow X$, use that modules on a point are $\mathbb{Z}\text{-}\mathsf{Mod}$ and obtain an adjunction

$$\mathbb{Z}\text{-}\mathsf{Mod} \underset{(\iota_x)^{-1}}{\overset{(\iota_x)_*}{\rightleftarrows}} \mathsf{Sh}(X \to \mathsf{AbGrp}).$$

Finally use that $\mathbb{Z}\text{-}\mathsf{Mod}$ has enough injectives. ■

28 | Wednesday, March 23

Remark 28.0.1: Induced and coinduced modules:

$$\begin{array}{ccccc} S & & \mathsf{R\text{-}Mod} & \xrightarrow{\text{Forget}} & \mathsf{S\text{-}Mod} \\ \downarrow & & & & \\ R & & R \otimes_S N & \xleftarrow[\text{Ind}]{} & N \\ & & \operatorname{Hom}_S(R, N) & \xleftarrow[\text{coInd}]{} & N \end{array}$$

Link to Diagram

Note that coinduction sends injective to injectives, and induction sends projectives to projectives. Recall that $\mathsf{Sh}(X;\mathsf{AbGrp})$ and $\mathcal{O}_X\text{-}\mathsf{Mod}$ have enough injectives, so left exact covariant functors F admit right-derived functors $\mathbb{R}F$, and similarly right exact contravariant functors F admit left-derived functors $\mathbb{L}F$.

Example 28.0.2 ***(?):*** Important functors:

- Global sections $\Gamma(-) : \mathsf{Sh}(X;\mathsf{C}) \to \mathsf{C}$ where $\mathcal{F} \mapsto \mathcal{F}(X)$, e.g. for $\mathsf{C} = \mathsf{AbGrp}$. $\mathbb{R}\Gamma(\mathcal{F}) = H^i(X;\mathcal{F})$ is sheaf cohomology.
- For $f \in \mathsf{Top}(X,Y)$, the pushforwards $f_* : \mathsf{Sh}(X;\mathsf{C}) \to \mathsf{Sh}(Y;\mathsf{C})$ where $\mathcal{F} \mapsto (U \mapsto \mathcal{F}(f^{-1}U))$. $\mathbb{R}f_*\mathcal{F}$ are *derived pushforwards.*
- Inverse image, which is exact.
- $(-) \otimes_{\mathcal{O}_X} \mathcal{F}$
- $\underset{\mathcal{O}_X}{\mathrm{Hom}}(-,\mathcal{F})$.

Theorem 28.0.3 ***(?).***
If $F \in [\mathsf{A},\mathsf{B}]$ is left exact covariant and A has enough injectives, then for every $A \in \mathsf{A}$ there exists an acyclic resolution $0 \to A \leftleftarrows J^\bullet$ whose homology computes $\mathbb{R}R$.

Proof (Sketch).
Why this homology computes the derived functors: let $A = A^0$ and take an injective resolution $A \leftleftarrows J^\bullet$. Break this into SESs, letting Z_i denote images:

- $0 \to Z^0 \to J^0 \to Z^1 \to 0$
- $0 \to Z^1 \to J^1 \to Z^2 \to 0$
- $\cdots$

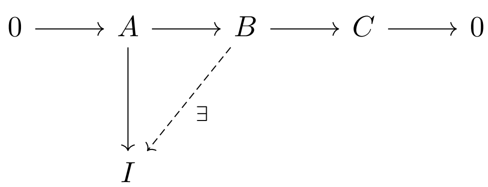
Note that $Z^n \leftleftarrows \Sigma^n J^\bullet = (J^n \to J^{n+1} \to \cdots)$ is an injective resolution. Splice to obtain

$$0 \to FA \to FJ^0 \to FZ^1 \to \mathbb{R}^1 FA \to 0, \qquad \mathbb{R}^n FZ^1 \xrightarrow{\sim} \mathbb{R}^{n+1} FA$$
$$0 \to \ker(FJ^0 \to FJ^1) \to FJ^0 \to \ker(FJ^1 \to FJ^2) \to \mathbb{R}^1 FA \to 0.$$

Proceed by induction.

■

Remark 28.0.4: Consider $F = \mathsf{A}(A,-)$ (covariant) or $\mathsf{A}(-,A)$ (contravariant), so $F \in \mathsf{Cat}(\mathsf{A},\mathsf{AbGrp})$. Note that acyclic objects for F are exactly injectives: take $0 \to A \to B \to C \to 0$ to obtain $0 \to [C,I] \to [B,I] \to [A,I] \to \mathrm{Ext}^1(C,I) = 0$ by acyclicity of I, meaning that $[B,I] \twoheadrightarrow [A,I]$ and thus there exist lifts:



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Definition 28.0.5 (Flasque and soft sheaves)
A sheaf $\mathcal{F} \in \mathsf{Sh}(X; \mathbb{Z}\text{-}\mathsf{Mod})$ is **flasque** iff for all $U \subseteq X$ open, $F(X) \twoheadrightarrow F(U)$. It is **soft** iff the same holds for all *closed* sets instead, and **fine** if $\mathcal{F}$ has a partition of unity property.

Remark 28.0.6: Note that fine $\implies$ soft and flasque $\implies$ soft. Fine sheaves are best for paracompact Hausdorff spaces, and flasque are better for e.g. the order topology.

29 | Friday, March 25

29.1 Flasque Sheaves

Remark 29.1.1: Important classes of sheaves:

- Universal: flasque or flabby.
- Classical topologies (Hausdorff, paracompact): fine $\implies$ soft.
- AG: quasicoherent sheaves on affine sets and covers.

Theorem 29.1.2 ***(Sufficient conditions for acyclicity).***
Suppose $\mathsf{A} \in \mathsf{AbCat}$ has enough injectives and $\mathcal{F} \in \mathsf{Cat}(\mathsf{A}, \mathsf{B})$ is left exact. Suppose $\mathcal{C} \subseteq \operatorname{Ob}\mathsf{A}$ satisfies

- Any $A \in \mathsf{A}$ admits an embedding $A \hookrightarrow C$ for some $C \in \mathcal{C}$.
- If $A_1 \bigoplus A_2 \in \mathcal{C}$ then $A_1, A_2 \in \mathcal{C}$.
- Given a SES $0 \to A \to B \to C \to 0$ with $A, B \in \mathcal{C}$, $C \in \mathcal{C}$ and $0 \to FA \to FB \to FC \to 0$ is exact.

Then every $C \in \mathcal{C}$ is F-acyclic.

Exercise 29.1.3 (?)
Use this to show that flasque implies F-acyclic for $F(-) := \Gamma(-)$.

Solution:
Recall $U \subseteq X$ open $\implies F(X) \twoheadrightarrow F(U)$.

- Take an embedding $0 \to F \to \prod_{x \in X} (\iota_x)_* F_x$ where $\iota_x : \{x\} \hookrightarrow X$. Use that for any group A, $\mathcal{G} := (\iota_x)_* A$ satisfies $\mathcal{G}(X) \twoheadrightarrow \mathcal{G}(S)$ for any $S \subseteq X$ since $\mathcal{G}$ is flasque and soft and this is preserved under products.

- Apply the lifting property to direct sums.
- Use that restrictions of flasque sheaves to opens are again flasque to prove that there is a surjection:

$$\begin{array}{ccc} B(X) & \twoheadrightarrow & C(X) \\ \downarrow & & \vdots\ \therefore \\ B(U) & \twoheadrightarrow & C(U) \end{array}$$

Link to Diagram

Proof (of theorem).
Any injective is in $\mathcal{C}$ by assumption: since $J \hookrightarrow C$ splits for any injective J, one has $C \cong J \oplus J'$, making J a direct summand and thus in $\mathcal{C}$ by the 2nd property.
Since there are enough injectives, form $0 \to C \to I \to C'' \to 0$. Take the LES, using that $\mathbb{R}^{>0} FI = 0$ to obtain

$$\begin{array}{l} 0 \to FC \to FI \to FC'' \to 0 \\ 0 \to \mathbb{R}^1 FC \to \mathbb{R}^1 FI = 0 \to \mathbb{R}^1 FC'' \xrightarrow{\cong} \\ \mathbb{R}^2 FC \to \mathbb{R}^2 FI = 0 \to \mathbb{R}^2 FC'' \xrightarrow{\cong} \\ \cdots \end{array}$$

Link to Diagram

■

Remark 29.1.4: There is a canonical flasque resolution:

$$\begin{array}{ccccc} 0 & \to & \mathcal{F} & \to & S(F) \coloneqq \prod_{x \in X} (\iota_x)_* \mathcal{F}_X \\ & & \downarrow & & \vdots \\ 0 & \to & \mathcal{G} & \to & S(\mathcal{G}) \end{array}$$

Link to Diagram

This is useful e.g. for finite sets with the order topology, but less useful if $|X|$ is infinite and there are non-closed points.

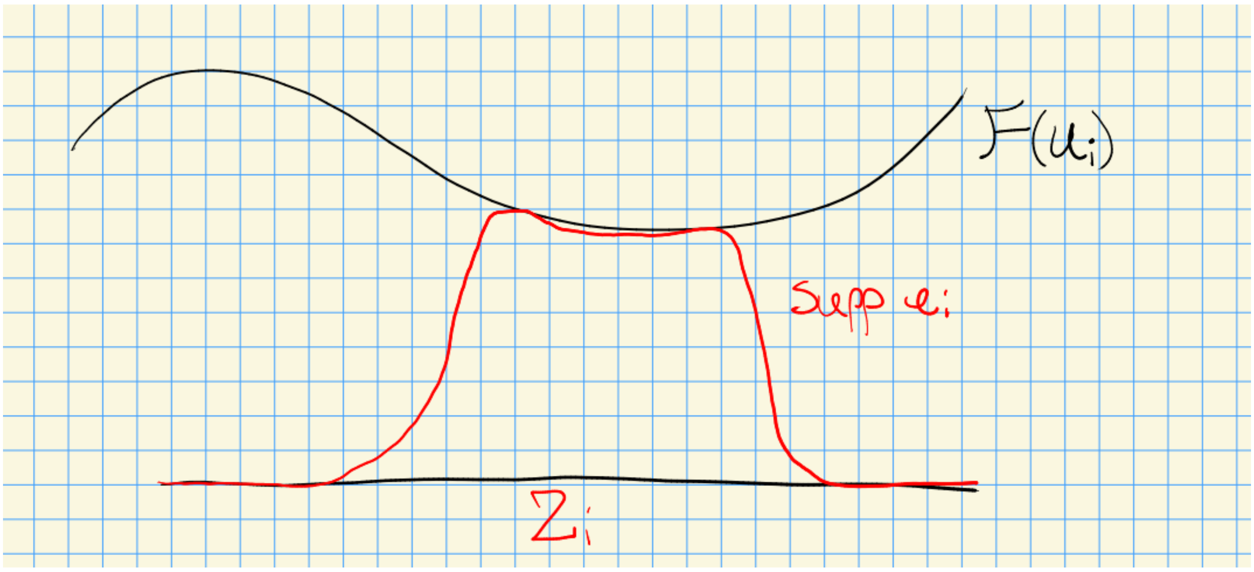
Exercise 29.1.5 (?)
Show that if X is Hausdorff paracompact, flasque implies soft. As a corollary, soft sheaves are acyclic for such spaces.

Solution:
See notes.

29.2 Fine Sheaves

Remark 29.2.1: Recall that a sheaf is fine iff it satisfies the POU property.

- Classically: there is an open cover $\mathcal{U} \rightrightarrows X$ and $\varphi_i : U_i \to \mathbb{R}$ with $\operatorname{supp} \varphi_i \subseteq U_i$ where $\sum \varphi_i = 1$ and locally there are only finitely many nonzero φ.
- For sheaves: there is an open cover $\mathcal{U} \rightrightarrows X$ and $\varphi_i : \mathcal{F} \to \mathcal{F}$ with $\operatorname{supp} \varphi$ a closed set Z_i where $\sum \varphi_i = \operatorname{id}_{\mathcal{F}}$ and locally there are only finitely many i with $\varphi(\mathcal{F}) \neq 0$.



Example 29.2.2 *(?)*: Suppose X is Hausdorff paracompact, set $\mathcal{F} := \mathcal{O}_X^{\text{cts}}$. Thus $\mathcal{O}_X$ has a POU property, as does any $\mathcal{O}_X$-module. Take a usual POU $\{f_i\}$ and define

$$\begin{aligned} \varphi : \mathcal{F} &\to \mathcal{F} \\ s &\mapsto f_i s. \end{aligned}$$

So any $\mathcal{F} \in \mathcal{O}_X^{\text{cts}}\text{-Mod}$ is soft.

Remark 29.2.3: In this case, fine implies soft.

29.3 de Rham and Dolbeault cohomology

Remark 29.3.1: Let X be a smooth manifold over $\mathbb{R}$. Note that $\underline{\mathbb{R}}$ is not fine and not soft, and not even an $\mathcal{O}_X$-module. However it admits a resolution $0 \leftarrow \underline{\mathbb{R}} \leftleftarrows \Omega_X{}^\bullet$ where $\Omega_X^0 := \mathcal{O}_X^{\mathsf{sm}}$, and this resolution computes the sheaf cohomology $H^\bullet(X; \underline{\mathbb{R}})$.

Similarly, $0 \to \underline{\mathbb{C}} \leftleftarrows_{\bar\partial} \Omega^{0,\bullet}$ where $\bar\partial = \sum \frac{\partial}{\partial \bar z_i} \, dz_i$.

30 | Computing Cohomology (Monday, March 28)

Remark 30.0.1: Upcoming topics related to $H^\bullet(X; \mathcal{F})$:

- General vanishing theorems
- Čech cohomology
- Riemann-Roch

30.1 Vanishing Theorems

Theorem 30.1.1 *(Grothendieck)*.
If X is a Noetherian space, then $\tau_{\geq n+1} H^\bullet(X; \mathcal{F}) = 0$ for $n := \dim X$.

Remark 30.1.2:

- Note that the theorem statement uses the Zariski topology, and so doesn't contradict that $H^{2d}_{\text{sing}}(X; \mathbb{Z}) \neq 0$ for (say) X a compact complex manifold.
  - The theorem uses algebraic dimension $d := \dim_{\mathbb{C}} X$, which is generally twice the real dimension.
- Recall that X is Noetherian iff X satisfies the DCC on closed sets.
- Algebraic varieties with the Zariski topology are Noetherian, since dimension strictly decreases on proper closed subsets.
- Affine schemes over Noetherian rings are Noetherian, since closed subsets corresponds to radical ideals, which satisfy the ACC.
- $\dim X$ is defined as $\sup \left\{ d \;\middle|\; Z_0 \subsetneq Z_1 \subsetneq \cdots \subsetneq Z_d \right\}$.
- Noetherian spaces can have infinite dimension (see examples by Nagata)

- Schemes are nonsingular if the completions of local rings are formal power series.
- Smallest class of nice rings in AG: referred to as "Japanese rings" in the literature, finitely generated rings over DVRs, plus localizations, completions, direct sums, etc.

Definition 30.1.3 (Quasicoherent sheaves)
A sheaf $\mathcal{F} \in \mathsf{Sh}(X, \mathcal{O}_X\text{-}\mathsf{Mod})$ is **quasicoherent** if for all $U = \operatorname{Spec} R \subseteq X$, the restrictions $\mathcal{F}|_U \cong \tilde{M}$ for $M \in \mathsf{R\text{-}Mod}$. Recall that $\mathcal{O}_X(D(f)) = R\left[\frac{1}{f}\right]$, and we define $\tilde{M}(D(f)) := M\left[\frac{1}{f}\right]$, so e.g. $\tilde{R} = \mathcal{O}_X$.

Theorem 30.1.4 ***(Serre).***
A sheaf $\mathcal{F} \in \mathsf{Sh}(X, \mathcal{O}_X\text{-}\mathsf{Mod})$ is quasicoherent iff

$$\mathcal{O}_X{}^{\oplus J} \to \mathcal{O}_X{}^{\oplus I} \to \mathcal{F} \to 0.$$

Remark 30.1.5: Analogy:

- Quasicoherent: arbitrary modules M
- Coherent: finitely presented modules M.

Example 30.1.6 ***(Coherent sheaves):*** Examples of coherent sheaves

- For $X \subseteq \mathbb{P}^N$ projective (or quasiprojective, i.e. open in a projective), the **twisting sheaves** $\mathcal{O}_X(d)$ whose local sections are $p(\mathbf{x})/q(\mathbf{x})$ for p, q homogeneous where $\deg p - \deg q = d$.
- For any $Z \subseteq X$ as above, the **ideal sheaf** $\mathcal{I}_Z \subseteq \mathcal{O}_Z$ and their twists $\mathcal{I}_Z(d) := \mathcal{I}_Z \otimes_{\mathcal{O}_X} \mathcal{O}(d)$.
- Tangent sheaves $\mathbf{T}_X$ and cotangent sheaves $\mathbf{T}_X^\vee$, and their tensor powers, e.g. Ω_X^n.

Theorem 30.1.7 ***(Serre Vanishing 1).***

$$\mathcal{F} \in \mathsf{QCoh}(X),\ X \in \mathsf{AffSch}_{/k} \implies \tau_{\geq 1} H^\bullet(X; \mathcal{F}) = 0.$$

Theorem 30.1.8 ***(Serre Vanishing 2).***

$$\mathcal{F} \in \mathsf{Coh}(X),\ X \in \operatorname{Proj} \mathsf{Sch}_{/k} \implies \tau_{\geq 1} H^\bullet(X; \mathcal{F}(n)) = 0 \text{ for some } n \gg 0.$$

Remark 30.1.9: Affine schemes correspond to general rings, and projective schemes correspond to graded rings. In the second statement, coherence is used as a kind of finiteness.

30.2 Čech Cohomology

Definition 30.2.1 (The Cech complex an differential)
For open covers, write $\mathcal{U} \rightrightarrows X$ iff $X = \cup_i U_i$. Define $U_{i_0, i_1, \cdots, i_p} := U_{i_1} \cap U_{i_1} \cap \cdots \cap U_{i_p}$. Define a complex

$$0 \to \check{C}^0(\mathcal{U}; \mathcal{F}) = \bigoplus_{i_0 \in I} \Gamma(\mathcal{F}; U_{i_0}) \xrightarrow{\partial_1} \bigoplus_{i_1 < i_2} \Gamma(\mathcal{F}; U_{i_0, i_1}) \xrightarrow{\partial_2} \cdots .$$

where we specify where elements land componentwise:

$$\partial_i|_{i_1 < \cdots < i_{p+1}} : \bigoplus_{i_0 < \cdots < i_p} \mathcal{F}(U_{i_0, \cdots, i_p})$$

$$f \mapsto \sum_{0 \le k \le p+1} (-1)^k f|_{i_0 < \cdots \widehat{k} < i_{p+1}} \Big|_{U_{i_0, \cdots, i_{p+1}}} .$$

Remark 30.2.2: Why $\partial^2 = 0$: if $k < \ell$, forget ℓ first and then k to get a sign $(-1)^\ell (-1)^k$, or forget k first then ℓ to get $(-1)^k (-1)^{\ell - 1}$ due to the shift. So these contributions cancel.

Theorem 30.2.3 ***(?).***
Suppose that for all inclusions $j_{i_0, \cdots, i_p} : U_{i_0, \cdots, i_p} \to X$, the pushforwards of $\mathcal{F}$

$$(j_{i_0, \cdots, i_p})_* \mathcal{F}|_{U_{i_0, \cdots, i_p}}$$

have vanishing cohomology in degrees $p \ge 1$. Then

$$H^\bullet(X; \mathcal{F}) \xrightarrow{\sim} \check{H}^\bullet(\mathcal{U}; \mathcal{F}).$$

This is true for all affine schemes if $\mathcal{F} \in \mathsf{QCoh}(X)$, e.g. for algebraic varieties or separated schemes.

31 | Wednesday, March 30

Remark 31.0.1: Topics:

- General vanishing (Serre 1 and 2)
- Čech cohomology
- Riemann-Roch and Serre duality
- Advanced vanishing (e.g. Kodaira vanishing)

31.1 Čech Cohomology

Remark 31.1.1: Setup: X and $\mathcal{F} \in \mathsf{Sh}(X; \mathsf{AbGrp})$, an open cover $\mathcal{U} \rightrightarrows X$. We defined the Čech complex:

$$\check{C}^p(\mathcal{U}; \mathcal{F}) = \bigoplus_{i_1 < \cdots < i_p} \mathcal{F}(U_{i_1, \cdots, i_p}),$$

which had certain differentials.

> **Theorem 31.1.2** *(?).*
> Suppose $X \in \mathsf{AlgVar}$ or $X \in \mathsf{Sch}$ is separated (e.g. a quasiprojective scheme), $F \in \mathsf{QCoh}(X)$ an $\mathcal{O}_X$-module, and let $\mathcal{U} \rightrightarrows X$ be an affine open cover. Then
>
> $$\check{H}(\mathcal{U}; F) = \mathbb{R}\Gamma(X; F).$$

Remark 31.1.3: More generally, we can just assume that all intersections of affines are affine, and instead there is a spectral sequence. This can fail if X is not separated, e.g. $X := \mathbb{A}^2 \coprod_{\mathbb{A}^2 \setminus \{0\}} \mathbb{A}^2$ where the intersection $\mathbb{A}^2 \setminus \{0\}$ is not affine. Recall that X is separated iff $X \xhookrightarrow{\Delta_X} X^{\times^2}_X$ is closed.

Example 31.1.4 *(?):* Consider $X = \mathbb{P}^1$ and $F = \bigoplus_{d \in \mathbb{Z}} \mathcal{O}_X(d)$, we can compute $\check{H}(X; \mathcal{O}(d))$ for all d. Take a cover $U_i = \{x_i \neq 0\}$ where U_0 has coordinate $x := x_1/x_0$ and U_1 has coordinate $y = x_0/x_1$ which intersect at $U_{01} = \{x, y \neq 0\}$ and are glued by $y = 1/x$. The Čech resolution is

$$0 \to F(U_0) \oplus F(U_1) \xrightarrow{f} F(U_{01}) \to 0,$$

so $H^0 = \ker f$ and $H^1 = \operatorname{coker} f$. Recall that sections of $\mathcal{O}(d)$ are locally ratios of polynomials with valuation d. We have $\mathcal{O}_{\mathbb{P}^1}(d)|_{\mathbb{A}_1} = x_0^d \mathcal{O}_{\mathbb{P}^1}$ by rewriting $p/q = x_0^d p'/q'$. We can thus write this sequence as

$$0 \to \bigoplus_{d \in \mathbb{Z}} x_0^d k\left[x = \tfrac{x_1}{x_0}\right] = \bigoplus_d \left\langle \text{degree } d \text{ monomials in } x_0^{\pm 1}, x_1 \right\rangle \oplus \bigoplus_d \left\langle \text{degree } d \text{ monomials in } x_0, x_1^{\pm 1} \right\rangle \to \bigoplus_d \left\langle \text{degr} \right.$$

Claim:

$$H^0(X; F) = k[x_0, x_1], \qquad H^1 = \frac{1}{x_0 x_1} k\left[\tfrac{1}{x_0}, \tfrac{1}{x_1}\right].$$

Being in the kernel means $v_{x_0}(f) > 0$ and $v_{x_1}(f) > 0$, which yields monomials $x_0^n x_1^m$ where $d = n + m$. For the cokernel, note $(p, 1) \mapsto p - q$, what's missing? Monomials where both powers are negative.

Example 31.1.5 *(?):* Similar computations work for $X = \mathbb{P}^n$ and yield

$$H^0\left(X; \bigoplus_{d \in \mathbb{Z}} \mathcal{O}_{\mathbb{P}^n}(d)\right) = k[x_1, \cdots, x_n], \quad H^n\left(X; \bigoplus_{d \in \mathbb{Z}} \mathcal{O}_{\mathbb{P}^n}(d)\right) = \frac{1}{\prod x_i} k\left[\tfrac{1}{x_0}, \cdots, \tfrac{1}{x_n}\right].$$

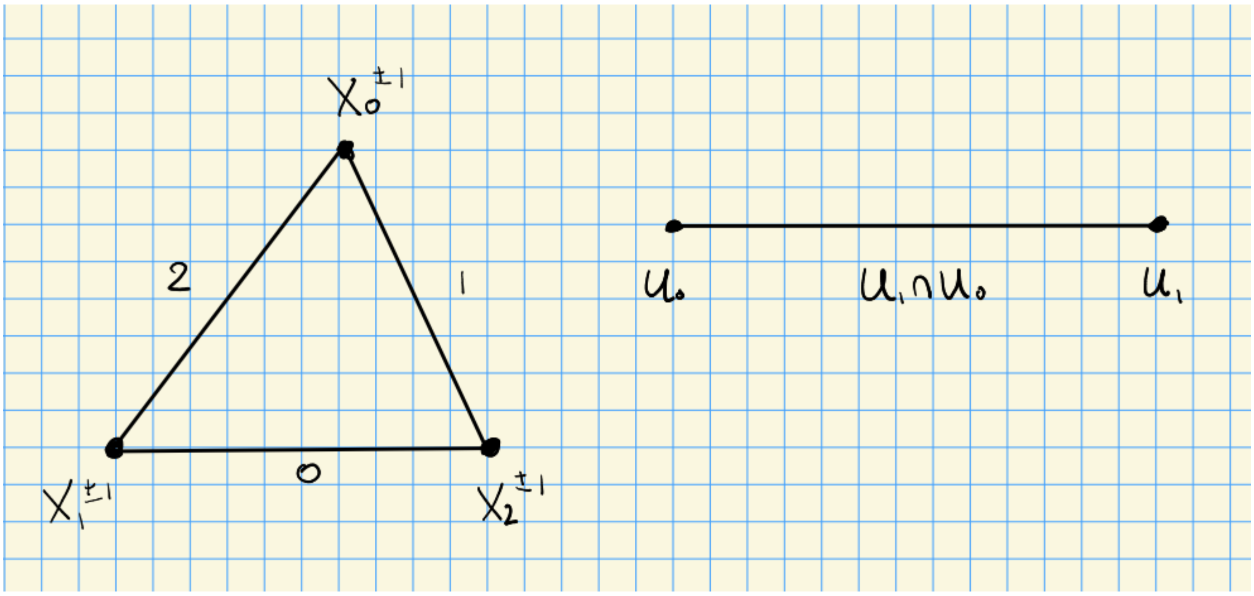
Note that both sides are graded by degree. This can be done in affine opens $U_i = \{x_i \neq 0\} \cong \mathbb{A}^n$, $\mathcal{O}_X(d)|_= x_i^d \mathcal{O}_X$, and similarly

$$0 \to \bigoplus_d \left\langle \text{degree } d \text{ monomials in } x_0^{\pm 1}, x_1, \cdots, x_n \right\rangle \oplus \bigoplus_d \left\langle \text{degree } d \text{ monomials in} x_0, x_1^{\pm 1}, \cdots, x_n \right\rangle \oplus \cdots \to \cdots \to$$

The kernel is again spanned by monomials f with $v_{x_i}(f) \geq 0$ for all i. Which monomials don't come from the middle step? Those where $v_{x_i}(f) < 0$ for all i.

Remark 31.1.6: A combinatorial device to keep track of monomials: let $X = \mathbb{P}^2$, and build simplices which track which monomials are allowed to be negative.

See Hartshorne for a description of how to encode this as a simplicial set:



Remark 31.1.7: As a result, we can compute

$$\dim H^0(\mathbb{P}^n; \mathcal{O}_{\mathbb{P}^n}(d)) = \binom{n+d}{n} = \binom{n+d}{d}$$

by counting monomials using a stars and bars argument. Moreover

$$\dim H^n(\mathbb{P}^n; \mathcal{O}(d)) = \dim H^0(\mathbb{P}^n; \mathcal{O}(n-1-d)) = \dim H^0(\mathbb{P}^n; \mathcal{O}(K) \otimes \mathcal{O}(d)^{-1})$$

where the canonical class of $\mathbb{P}^n$ is given by $\mathcal{O}(K_{\mathbb{P}^n}) = \mathcal{O}(-n-1)$.

32 | Friday, April 01

Reference for toric geometry: Fulton's Toric Varieties, Oda's Convex bodies in algebraic geometry.

Proposition 32.0.1 *(?).*
Claim from last time:

$$H^\bullet(\mathbb{P}^n; \mathcal{O}(d)) \coloneqq \mathbb{R}\Gamma(\mathbb{P}^n; \mathcal{O}_{\mathbb{P}^n}(d)) \cong \check{H}(\mathcal{U}; \mathcal{O}_{\mathbb{P}^n}(d)),$$

where this isomorphism is of graded vector spaces. We also saw

$$\bigoplus_{d\in\mathbb{Z}} H^0(\mathbb{P}^n; \mathcal{O}(d)) \cong k[x_1, \cdots, x_n] = \bigoplus_{\mathbf{d}\geq 0} k \prod x_i^{d_i},$$

and in top degree,

$$\bigoplus_{d\in\mathbb{Z}} H^n(\mathbb{P}^n; \mathcal{O}(d)) \cong \prod x_i^{-1} k[x_0^{-1}, \cdots, x_n^{-1}],$$

with all intermediate degrees vanishing. There is a nondegenerate pairing

$$H^0(\mathbb{P}^n; \mathcal{O}(d)) \times H^n(\mathbb{P}^n; \mathcal{O}(-n-1-d)) \to k \cdot \prod x_i^{-1} \cong k$$

which is concretely realized by multiplying monomials and projecting onto the span of $\prod x_i^{-1}$ (so setting all other monomials to zero). This is an instance of Serre duality, but this example is in fact used in the proof.

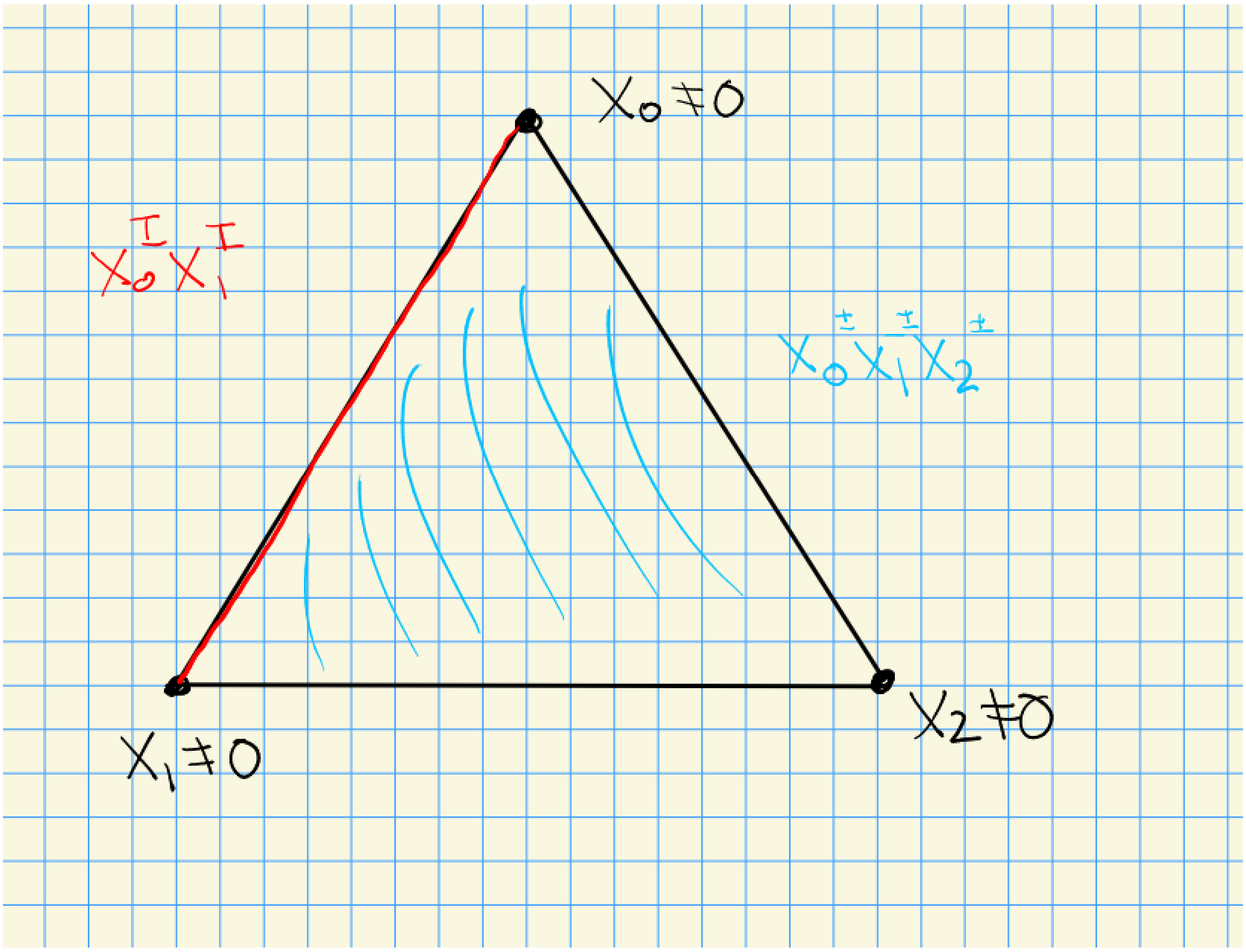
Proof (?).
Compute $\oplus_d \check{H}(\mathcal{U}; \mathcal{O}(d))$ by first writing $\mathbb{P}^n = \mathbb{A}^n_{x_0 \neq 0} \cup \mathbb{A}^n_{x_1 \neq 0}$ and look at global sections:

$$0 \to k[x_0^{\pm 1}, x_1, \cdots, x_n] \oplus k[x_0, x_1^{\pm 1}, x_2, \cdots, x_n] \oplus \cdots \to k[x_0^{\pm 1}, x_1^{\pm 1}, x_2, \cdots] \oplus \cdots \to \cdots \to\to k[x_0^{\pm 1}, x_1^{\pm 1}, \cdots, x$$

where we choose 1 coordinate to invert at the 1st stage, 2 coordinate to invert at the 2nd stage, and so on. Note that this is not only $\mathbb{Z}$-graded, but $\mathbb{Z}^{\times^{n+1}}$- graded by monomials. The claim is that the contribution of a monomial $\prod x_i^{d_i}$ to cohomology will only depend on the pattern of signs, i.e. $I \coloneqq \left\{ k \;\middle|\; d_k < 0 \right\} \subseteq [n]$.

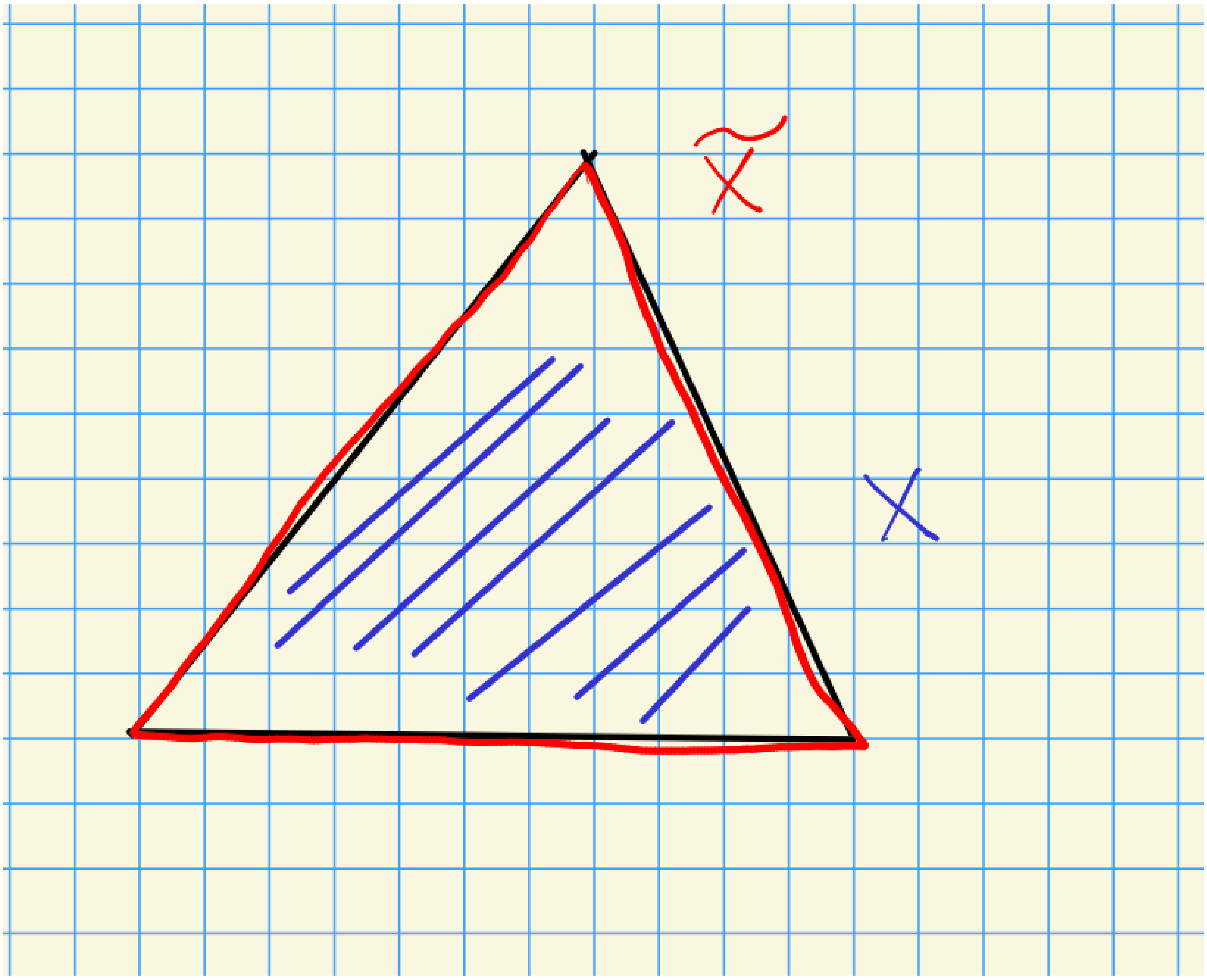
■

Example 32.0.2 *(?):* Consider $I = \emptyset$, and the contribution of $\prod x_i^{d_i}$ with $d_i \geq 0$ for all i. Form a simplicial complex X:



The cohomology computes $H^{\bullet}{}_{\triangle}(X;\mathbb{Z}) \cong \mathbb{Z}$ since X is contractible.

Example 32.0.3 ***(?):*** For $I = [n]$, so all $d_i < 0$, one obtains just the faces of the complex with the boundaries deleted.

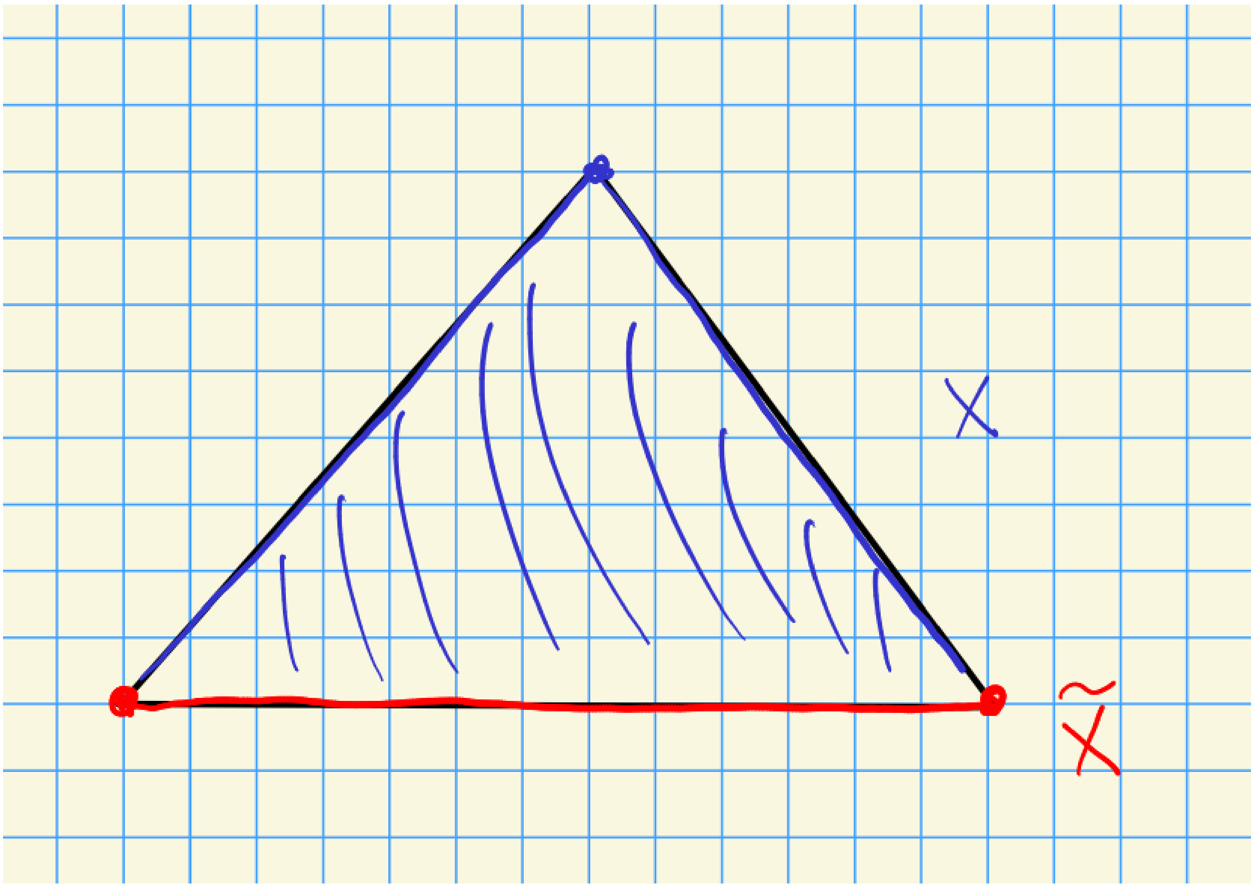


This computes $H^{\bullet}{}_{\Delta}(X, \tilde{X}; \mathbb{Z}) \cong \tilde{H}^{\bullet}{}_{\Delta}(\tilde{X})$ by the LES of a pair:

Remark 32.0.4: Recall that this LES arises from

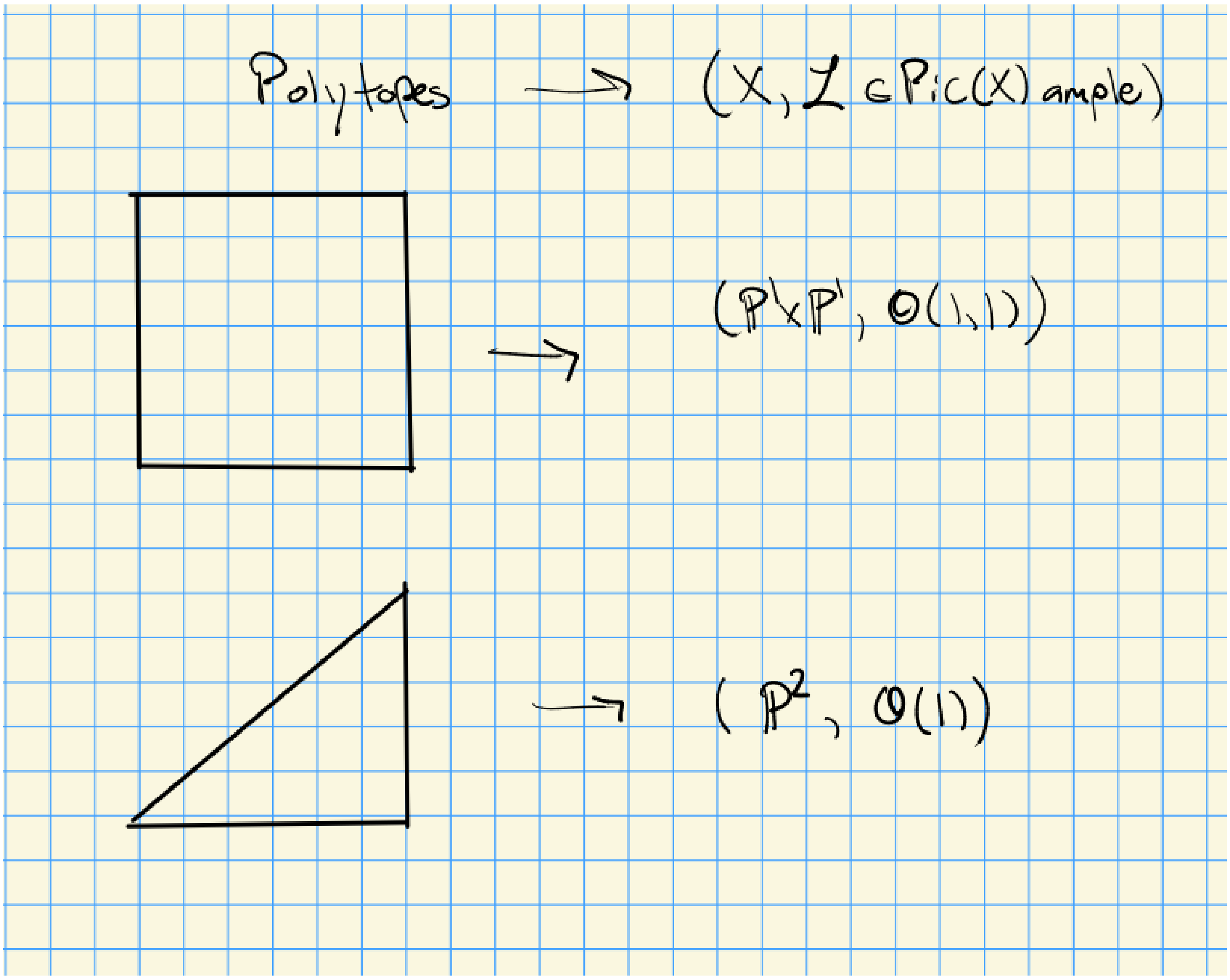
$$0 \to C^n(\tilde{X}) \to C^n(X) \to C^n(X, \tilde{X}) \to 0.$$

Example 32.0.5 ***(?):*** For $I = \{0\}$, so $I = \{0\}$ with $d_0 < 0$ and $d_i \geq 0$ for $i \geq 1$.



This computes $H^{\bullet}{}_{\Delta}(X, \tilde{X}; \mathbb{Z}) \cong \tilde{H^{\bullet}}{}_{\Delta}(\tilde{X}) = 0$.

Remark 32.0.6: When does this trick work? For any pair (X, L) with $L \in \mathrm{Pic} X$ where the sections are $\mathbb{Z}^{n+1}$-graded where each graded piece is dimension at most 1. These are referred to as **multiplicity-free**. Examples: toric varieties:



33 | Monday, April 04

33.1 Riemann-Roch and Serre Duality

Remark 33.1.1: Let $X \in \operatorname{Proj}\mathsf{Var}_{/k}$ and $F \in \mathsf{Coh}(\mathcal{O}_X\text{-}\mathsf{Mod})$. By Grothendieck, $H^\bullet(X;F)$ is supported in degrees $0 \leq d \leq \dim X$ and $h^i = \dim_k H^d(X;F) < \infty$ for all d.

Proposition 33.1.2 ***(Riemann-Roch).***
If $X \in \mathsf{sm}\operatorname{Proj}\mathsf{Var}_{/k}$,

$$\chi(X;F) := \sum_{0 \leq i \leq \dim X} (-1)^i h^i(F) = \int_X \operatorname{ch}(F)\operatorname{Td}(\mathbf{T}_X).$$

Remark 33.1.3: What this formula means: for X smooth projective, there is a Chow ring $A^*(X) = \bigoplus_{0 \leq i \leq \dim X} A^i(X)$ where A^i is analogous to $H^{2i}_{\text{sing}}(X; \mathbb{C})$. These are often different, but sometimes coincide (which can only happen if odd cohomology vanishes). For curves, these differ, and $A^1(X) \cong \operatorname{Pic}(X)$ which breaks up as a discrete part (degree) and continuous part (Jacobian). Define $A^i(X) := \mathbb{Z}[C_i] \sim$ where C_i are codimension i algebraic cycles (subvarieties) and we quotient by linear equivalence. Recall that for divisors, $D_1 \sim D_2$ if $D_1 - D_2$ is the divisor of zeros/poles of a rational functions. More generally, for Z of codimension i and $Z \xrightarrow{f} X$, consider $f_* D_1 \sim f_* D_2$ in order to define linear equivalence.

Example 33.1.4*(?):* Consider $X_4 \subseteq \mathbb{P}^3$ a quartic, the easiest example of a K3 surface. Then $A^0[X] = \mathbb{Z}[X]$, $A^1(X) = \operatorname{Pic}(X)$, so what is $A^2(X)$? These are linear equivalence classes of points, and any two points are equivalent if they are equivalent in the image of a curve. It's a fact that K3s are not covered by rational curves – instead these form a countable discrete set, with finitely many in each degree. There is a formula which says that the generating function of curve counts is modular, and

$$\sum n_d x^d = \frac{1}{x} \frac{1}{\prod_{1 \leq n \leq \infty} (1 - x^n)^{24}},$$

where n_d is the number of rational curves of degree $2d$. So $A^2(X)$ is not obvious! A theorem of Mumford says that it's torsionfree and infinitely generated. Note that $n_d = p_{24}(d+1)$ where $p_\ell(-)$ is the numbered of *colored* integer partitions

Remark 33.1.5: The integration map:

$$\begin{aligned} \int_X : A^{\dim X}(X) &\to \mathbb{Z} \\ \sum n_i p_i &\to \sum n_i. \end{aligned}$$

There are two non-homogeneous polynomials $\operatorname{ch}(F)$ and $\operatorname{Td}(\mathbf{T}_X)$ in $A^*(X) \otimes_{\mathbb{Z}} \mathbb{Q}$, and the formula for Riemann-Roch says to multiply and extract only the top-dimensional component, i.e. take $\deg(\operatorname{ch}(F) \operatorname{Td}(\mathbf{T}_X))_{\dim X}$. This is very computable!

Example 33.1.6*(?):* A Chern class: if $F = \mathcal{O}_X(D)$, then

$$\operatorname{ch}(F) = e^D = \sum_{1 \leq i \leq n} D^i / i!$$

where

$$\mathcal{O}_X(D)(U) = \left\{ f \in \mathcal{O}_X(U) \;\middle|\; (f) + D \geq 0 \right\}$$

and $D^n = D \smile D \smile \cdots \smile D$ is the n-fold self-intersection of D. Note that $c_1(F) = D$.

Remark 33.1.7: The Chern character of F is additive on SESs, i.e. $0 \to A \to B \to C \to 0$ yields $\operatorname{ch}(B) = \operatorname{ch}(A) + \operatorname{ch}(C)$.

Proposition 33.1.8 ***(RR for curves).***
If X is a smooth projective curve,

$$h^0(X) - h^1(X) = \deg D - g(X) + 1.$$

In this case, $\mathrm{ch}(F) = 1 + D$ and $\mathrm{Td}(\mathbf{T}_X) = 1 + (1 - g)[\mathrm{pt}]$ where $[\mathrm{pt}]$ is a certain well-defined divisor in $A^1(X)$. One can rewrite this as $\mathrm{Td}_X = 1 + \frac{1}{2}c_1 = 1 - \frac{1}{2}K_X$ (the canonical class, where $\deg K_X = 2g - 2$). This uses that

$$c_1 = c_1(\mathbf{T}_X) = -c_1(\Omega_X) = -K_X.$$

Example 33.1.9 ***(?):*** For X a smooth surface,

- $\mathrm{ch}(F) = 1 + D + \frac{D}{2}$
- $\mathrm{Td}(\mathbf{T}_X) = 1 - \frac{1}{2}c_1 + \frac{1}{12}(c_1^2 + c_2)$,

thus

$$\chi(X; \mathcal{O}_X(D)) = \frac{D(D-2)}{K} + \chi(X; \mathcal{O}_X).$$

Example 33.1.10: If X is a K3 surface, then $K_X = 0$ and $h^0(\mathcal{O}_X) = h^1(\mathcal{O}_X) = 0$, so $\chi(X; \mathcal{O}_X) = 2$ and

$$\chi(X; \mathcal{O}_X(D)) = \frac{D^2}{2} + 2.$$

Example 33.1.11 ***(?):*** For $X = \mathbb{P}^2$ with $F = \mathcal{O}(d)$, note

- $K_X = \mathcal{O}(-3)$
- $h^0(\mathcal{O}_X) = 1, h^1(\mathcal{O}_X) = h^2(\mathcal{O}_X) = 0$

So

$$\chi(X; \mathcal{O}(d)) = \frac{d(d+3)}{2} + 1 = \binom{d+2}{2}.$$

As a corollary, for $d \geq 0$,

$$h^0(\mathcal{O}_{\mathbb{P}^n}(d)) = \binom{d+n}{n}.$$

34 | Friday, April 08

34.1 Vanishing theorems

Remark 34.1.1: Setup: $X \in \operatorname{Proj}\mathsf{Var}_{/k}, \mathcal{F} \in \mathsf{Sh}(X; \mathsf{AbGrp})$. What is $H^0(X; \mathcal{F})$? Note that if

$$\chi(X; \mathcal{F}) \coloneqq \sum_k (-1)^k h^k(X; \mathcal{F}),$$

if $\tau_{\geq 1} H^\bullet(X; \mathcal{F}) = 0$ then this $\chi(X; \mathcal{F}) = h^0(X; \mathcal{F})$. By Serre duality, $h^n(X; \mathcal{F}) = h^0(\omega_X \otimes \mathcal{F}^{-1})$ which holds if X is Gorenstein, e.g. a locally complete intersection.

Recall that $\mathcal{O}_X(D)(U) = \left\{ \varphi \,\middle|\, (\varphi) + D \geq 0 \right\}$. Note that if $\mathcal{F} = \mathcal{O}(D)$ then $h^0(X; \mathcal{F}) \neq 0 \iff D \sim D'$ where $D' > 0$ is effective.

Remark 34.1.2: If $D \sim D'$ where $-D' > 0$ is effective, then $h^0(X; \mathcal{O}(D)) = 0$. Note that if $D \subseteq X \subseteq \mathbb{P}^N$ is projective, take $H \subseteq \mathbb{P}^N$ and $\mathcal{O}_{\mathbb{P}^N}(1) = \mathcal{O}_{\mathbb{P}^N}(H)$ and intersect to obtain $D \cdot H^{n-1} = \deg D$.

Example 34.1.3 ***(?):*** If X is a smooth projective curve and $\mathcal{F} = \mathcal{O}_X(D)$ is a line bundle. Riemann-Roch yields

$$h^0(X; \mathcal{F}) - h^1(X; \mathcal{F}) = \deg D - g + 1$$

and

$$\deg D = h^0(D) - h^0(K_X - D) \implies \deg(K_X - D) = 2g - 2 - \deg D.$$

Example 34.1.4 ***(?):*** If X is a smooth projective curve,

- $\mathcal{O}_X(D)$ is ample $\iff$ $D > 0$ (some large multiple is a hyperplane section).
- $\mathcal{O}_X(D)$ is very ample $\impliedby$ $\deg D \geq 2g - 2 + 3$ (very ample: some multiple is ample).

There exists an embedding $X \hookrightarrow \mathbb{P}^N$, and $\mathcal{O}_X(D) = \mathcal{O}_X(1) = \mathcal{O}_{\mathbb{P}^N}(1)\big|_X$. One can show $h^0(D - \mathrm{pt}) < h^0(D)$.

Example 34.1.5 ***(?):*** An effective but not ample divisor: take two lines in $\mathbb{P}^1 \times \mathbb{P}^1$ which do not intersect.

Theorem 34.1.6 ***(Kodaira).***
Suppose $X \in \mathsf{sm}\operatorname{Proj}\mathsf{Var}_{/k}$ where $k = \mathbb{C}$ or $\operatorname{ch} k = 0$ with $k = \bar{k}$ and let $\mathcal{F} = \omega_X(L)$ with L

ample. Then

$$\tau_{\geq 1} h^\bullet(X; \mathcal{F}) = 0.$$

Remark 34.1.7: A note on the proof: uses Deligne-Illusie and liftability from Witt vectors. This liftability holds for all curves, all K3s, and some Calabi-Yau threefolds.

Remark 34.1.8: For curves, $h^1(X; \omega_X(L)) = h^0(-L)$.

Theorem 34.1.9 ***(Kawamata-Viehweg vanishing (generalized Kodaira vanishing)).*** Let $X \in \mathsf{sm\,Proj\,Var}_{/\mathbb{C}}$ with $D = \cup_k D_K$ normal crossing union of smooth divisors and write its formal boundary as $\Delta := \sum a_i D_i$ with $0 < a_i < 1$ and $a_i \in \mathbb{Q}$. Suppose $\mathcal{F} \equiv K_X + \Delta + A$ for A ample, then

$$\tau_{\geq 1} h^\bullet(X; \mathcal{F}) = 0.$$

Remark 34.1.10: Say X has **klt singularities** (Kawamata log terminal) iff there exists a projective morphism $Y \xrightarrow{f} X$ with $Y \supseteq \cup_i D_i$ with each D_i snc, and $f^* K_X = K_Y + \Delta$. Generally Y is smooth and f is a resolution.

Remark 34.1.11: A note on the MMP: take X_0 a variety, produce a variety X with K_X nice, e.g. $-K_X > 0$ or $K_X \geq 0$ numerically. At each stage, contract a curve (the result is a -1 curve) are perform a **flip**. So if $C \in X_0$, produce $X_0 \to X_1$ with $CK_X < 0$.

35 | Monday, April 11

35.1 Spectral sequences

Proposition 35.1.1 ***(Leray spectral sequence).*** If $f \in \mathsf{Top}(X, Y)$ and $\mathcal{F} \in \mathsf{Sh}(X; \mathsf{AbGrp})$, there is a spectral sequence

$$E_2^{p,q} = H^p(X; \mathbb{R}^q f_* \mathcal{F}) \Rightarrow H^{p+q}(X; \mathcal{F}).$$

Example 35.1.2 ***(?):*** If $0 \to A \leftleftarrows J^\bullet$ is an injective resolution of a sheaf A, then $E_1^{p,q} = H^p(J^q) \Rightarrow H^{p+q}(A)$. More generally, for any functor $F \in \mathsf{Cat}(\mathsf{A}, \mathsf{B})$,

$$E_1^{p,q} = \mathbb{R}^p F(J^q) \Rightarrow \mathbb{R}^{p+q} F(A).$$

So if J^q are F-acyclic, then $\tau_{\geq 1} \mathbb{R}^\bullet F(J^q) = 0$ and thus $\mathbb{R}^n F(A)$ is the homology of the complex $FJ^\bullet$.

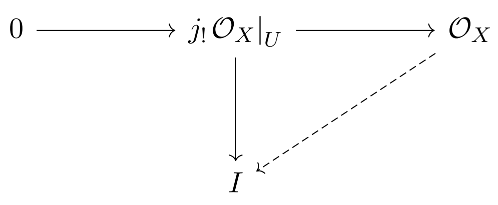
Proposition 35.1.3 *(Grothendieck).*
If

- $\mathsf{A} \xrightarrow{F} \mathsf{B} \xrightarrow{G} \mathsf{C}$ are left-exact functors between abelian categories
- A, B have enough injectives, and
- $F(I)$ for I injective in A yields a G-acyclic object in B,

then there is a first-quadrant spectral sequence

$$E_2^{p,q} = \mathbb{R}^p G(\mathbb{R}^q G(A)) \Rightarrow \mathbb{R}^{p+q}(F \circ G)(A).$$

Remark 35.1.4: This recovers the Leray spectral sequence via $\mathsf{Sh}(X; \mathsf{AbGrp}) \xrightarrow{f_*} \mathsf{Sh}(Y; \mathsf{AbGrp}) \xrightarrow{\Gamma(Y;-)} \mathsf{AbGrp}$, where the composition is $\Gamma(X; -)$. Note that injective sheaves are flasque, and pushforwards of flasque sheaves are again flasque. Why flasque implies injective:



Link to Diagram

Remark 35.1.5: Recall that cohomology vanishes above the dimension of a Noetherian space. The analog for pushforward involves the relative dimension.

Remark 35.1.6: General setup:

- $d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}$ (down and to the right) moves between diagonals.
  - For a fixed p, q, all differentials out of $E_{p,q}$ land on the same diagonal.
- $E_{r+1} = H(E_r, d_r)$.
- Letting $E_n = \bigoplus_{p+q=n} E_\infty^{p,q}$, there is a descending filtration $\mathrm{Fil}_\bullet E_n$ such that $\mathsf{gr}_p \mathrm{Fil}_\bullet E_n := \mathrm{Fil}_p E_n / \mathrm{Fil}_{p+1} E_n = E_\infty^{p,n-p}$.
- Extension problem: $\mathbb{Z} \supseteq 2\mathbb{Z} \supseteq 0$ where $\mathsf{gr}_1 = C_2$ and $\mathsf{gr}_1 \cong \mathbb{Z}$, but another group and filtration may have the same associated graded, e.g. $\mathbb{Z} \oplus C_2 \supseteq \mathbb{Z} \supseteq 0$.
- Double complexes naturally arise by taking an injective resolution $A \leftleftarrows J^\bullet$ and individually resolving the pieces by $J^n \leftleftarrows C^{n,\bullet}$. Writing $\mathrm{Tot}(C^{p,q})_n := \oplus_{p+q=n} C^{p,q}$, there are maps

$A \to C^{0,0} \to (C^{0,1} \oplus C^{1,0}) \to \cdots$ by summing horizontal and vertical differentials. Using the sign trick makes this a differential (multiply the vertical differentials in every even column by -1).

- There are spectral sequences

$$E_1 p, q = H^p(C^{\bullet,q}, d_h) \Rightarrow H^{p+q}(\mathrm{Tot}(C^{\bullet,\bullet}))$$
$$E_1 p, q = H^q(C^{p,\bullet}, d_v) \Rightarrow H^{p+q}(\mathrm{Tot}(C^{\bullet,\bullet})).$$

  - Why this is useful: resolve A by J which are not necessarily injective, and resolve each J^n by injectives, then Tot is now an injective resolution.

-

36 | Wednesday, April 13

36.1 Spectral sequences continued

Remark 36.1.1: Recall that for spectral sequences, the diagonal entries $p+q=n$ are the successive quotients in a filtration on $E^n \coloneqq \mathrm{Tot}(E_\infty^{\bullet,\bullet})_n$. Kodaira vanishing: for the original argument, go to characteristic p and look at liftability.

Example 36.1.2 ***(Deligne-Illusie's proof of Kodaira vanishing):*** We'll have some spectral sequence which we'll want to degenerate at E_2. It STS that $d_r = 0$ for $r \geq 1$, which in fact forces (E, d) to degenerate at E_1. Strategy: find another spectral sequence (E', d') with the same $E_1' \cong E_1$ and a differential $d \neq d'$ which converges to the same thing and more patently stabilizes at E_1'. It then follows that E stabilizes at E_1. Note the $\dim_k E_r^{p,q} \leq \dim_k E_{r-1}^{p,q}$ since we're taking kernels mod images.

Lemma 36.1.3 ***(A 5-term sequence).***
Suppose $E_2^{p,q} \Rightarrow E^n$ for $n = p+q$ is first quadrant. Then

- $E_2^{0,0} = E_{0,0}^{\infty}$ and $E_2^{1,0} = E_\infty^{1,0}$.
- $E_3^{0,1} = E_\infty^{0,1}$ and $E_3^{2,0} = E_\infty^{2,0}$
- There is a 5-term exact sequence

$$0 \to E_2^{1,0} \to E^1 \to E_2^{0,1} \to E_2^{2,0} \to E^2.$$

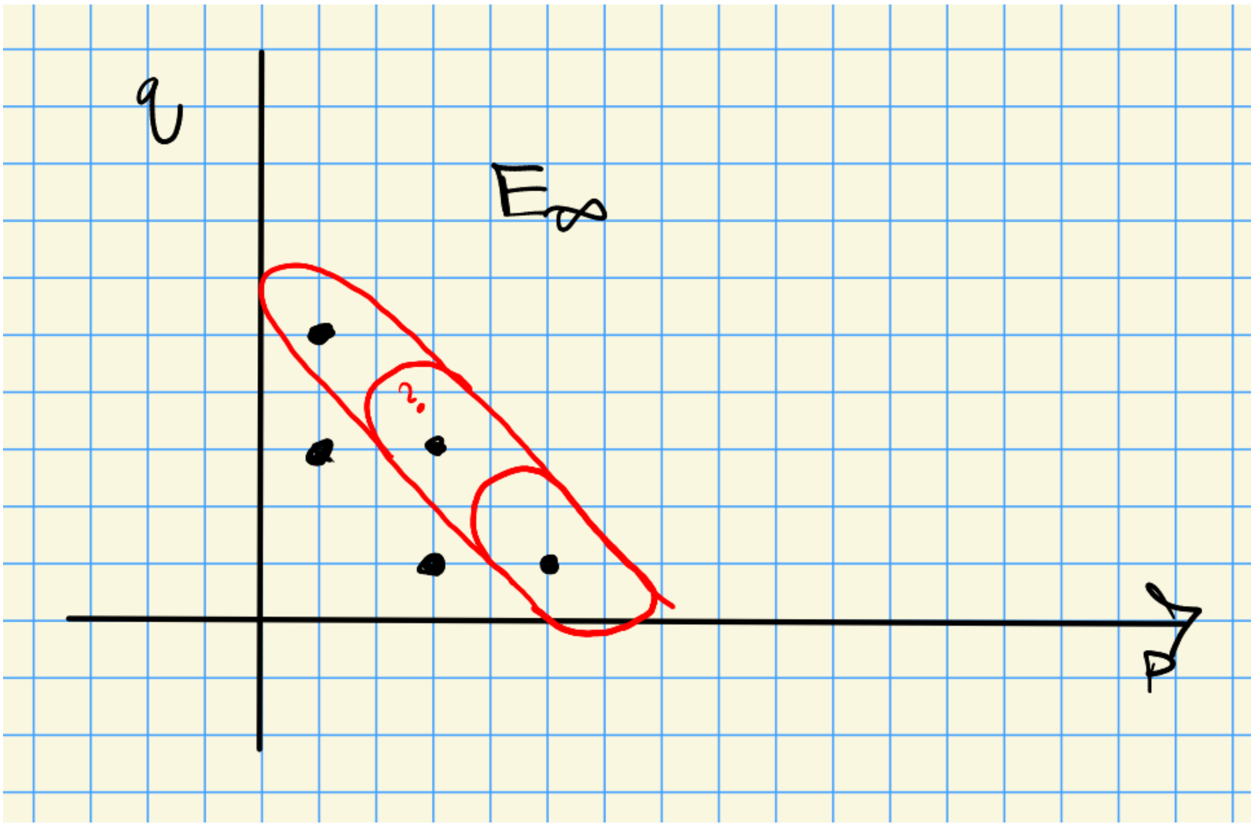
Example 36.1.4 ***(?):*** The Leray spectral sequence: for $f \in \mathsf{Top}(X, Y)$ and $\mathcal{F} \in \mathsf{Sh}(X; \mathsf{Vect}_{/k})$,

$$E_2^{p,q} = H^p(Y; \mathbb{R}^q f_* \mathcal{F}) \Rightarrow H^{p+q}(X; \mathcal{F}).$$

This yields

$$0 \to H^1(X; f_*\mathcal{F}) \to H^1(X;\mathcal{F}) \to H^0(X;\mathbb{R}^1 f_*\mathcal{F}) \to H^2(X; f_*\mathcal{F}) \to H^2(F).$$

Consider the filtration on E_∞:



This yields exact sequences

- $0 \to E_\infty^{1,0} \to E^1 \to E_\infty^{0,1} \to 0$
- $0 \to E_\infty^{2,0} \to ? \to E_\infty^{1,1} \to 0$
- $0 \to ? \to E^2 \to E_\infty^{0,2} \to 0.$

Remark 36.1.5: Recall the definition of a double complex: $(C^{\bullet,\bullet}, d_h, d_v)$ where each row is a complex for d_h and each column for d_v, and each square skew-commutes. Note that the sign trick does not change the cohomology. The totalized complex is is $(\mathrm{Tot}(C), \partial)$ where $C^n := \bigoplus_{p+q=n} C^{p,q} \xrightarrow{\partial} C^{n+1} := \bigoplus_{p+q=n+1} C^{p,q}$ and the differential is constructed from $C^{p,q} \xrightarrow{d_h \oplus d_v} C^{p+1,q} \oplus C^{p,q+1}$. There is a descending filtration $\mathrm{Fil}_\bullet \mathrm{Tot}(C)$ where $\mathrm{Fil}_n \mathrm{Tot}(C) = \tau_{\geq n,\bullet} \mathrm{Tot}(C) = \bigoplus_{p \geq n} C^{p,q}$, which is the double complex obtained by truncating all columns to the left of column n.

37 | Friday, April 15

37.1 Filtrations and Gradings

Remark 37.1.1: Given $\mathrm{Fil}A$ a descending filtration, define $\mathsf{gr}_i A \coloneqq \mathrm{Fil}_i A/\mathrm{Fil}_{i+1}A$. Convention: everywhere we'll set $p+q \coloneqq n, p = n-q$, etc.

This results in a collection of short exact sequences:

$$0 \to \mathrm{Fil}_{i+1}A \to \mathrm{Fil}_i A \to \mathsf{gr}_i A \to 0.$$

Remark 37.1.2: Our main example: a double complex $C^{\bullet,\bullet}$ with $A^\bullet \coloneqq \mathrm{Tot}^\bullet C^{\bullet,\bullet}$ with $A^n \coloneqq \oplus_{p+q=n} C^{p,q}$ and differentials $\partial = (d_v, d_h)$ producing skew-commuting squares. The main question is computing $H^*(A)$.

Each A^n is a filtration $\mathrm{Fil}A^n$ where $\partial \mathrm{Fil}^i A^n \subseteq \mathrm{Fil}^{i+1}A^n$. The filtration is defined by

$$\mathrm{Fil}^{p_0} A^n = \bigoplus_{p+q=n, p\geq p_0} C^{p,q},$$

taking everything to the right of column p_0. The claim is that this induces a filtrations on $Z^n(A), B^n(A), H^n(A)$ (cycles, boundaries, and homology). One can restrict the differential on $A^\bullet$ to $\mathrm{Fil}A^\bullet$; note that cycles $Z_n \mapsto 0$ and boundaries are the image and we're taking cycles mod boundaries. Writing $\mathrm{Fil}^p Z^n \coloneqq \mathrm{Fil}^p A^n \cap Z_n$ and similarly for B^n, H^n, one gets a filtration $\mathrm{Fil}H(\mathrm{Fil}^p A)$ on $H(\mathrm{Fil}^p A)$. This yields

$$E_\infty^{p,q} = \mathsf{gr}_p H^n = \mathrm{Fil}^p H^n / \mathrm{Fil}^{p+1} H^n.$$

If all of the SESs split, then $H^n = \bigoplus_{p+q=n} E_\infty^{p,q}$.

Remark 37.1.3: Set

- $E_0^{p,q} \coloneqq C^{p,q}$
- $E_1^{p,q} = H^n(C^{p,\bullet}, d_v)$.
- $E_2^{p,q} = H^n(E_1^{p,q}, d_v) = H^*(\cdots \to H^{n-1}C^{p,\bullet} \to H^n(C^{p,\bullet}) \to H^{n+1}C^{p+1,\bullet} \to \cdots)$.

What are the cycles in E_0? To map to zero under the total differential ∂, things emanating from column p must go to zero, and for the columns $p+k$, images under $d_h^{p+k,\ell}$ must cancel with images under $d_h^{p+k+1,\ell-1}$. Define the *approximate homology*

$$\mathrm{Fil}^p H_{p\pm r}^{\approx} = \frac{\partial^{-1}(\mathrm{Fil}^{p+r} A^{n+1})}{\partial(\mathrm{Fil}^{p-r+1} A^{n-1})}.$$

Note that this *increases* the number of allowed cycles and *decreases* the number of allowed boundaries. Then $E_r^{p,q} = \mathsf{gr}_p H_{p\pm r}^n$.

Remark 37.1.4: Note that the statement is not the E_r is computed as $H^*(E_{r-1})$; instead there is a formula for $E_r^{p,q}$ for all r, p, q a priori, and it is a property that taking homology of pages computes this.

Remark 37.1.5: Claim: $\mathsf{gr}_p H^n_{p\pm 0} = C^{p,q}$. Check that

$$\mathrm{Fil}^{p_0} H^n_{p\pm 0} = \frac{\bigoplus_{p+q=n, p\geq p_0} C^{p,q}}{d\left(\bigoplus_{p+q=n-1, p\geq p_0+1} C^{p,q}\right)}.$$

38 | Monday, April 18

38.1 Spectral Sequences

Remark 38.1.1: A filtered complex:

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & A^{n-1} & \xrightarrow{d^{n-1}} & A^n & \xrightarrow{d^n} & A^{n+1} & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
\cdots & \longrightarrow & \mathrm{Fil}^p A^{n-1} & \xrightarrow{d^{n-1}} & \mathrm{Fil}^p A^n & \xrightarrow{d^n} & \mathrm{Fil}^p A^{n+1} & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
\cdots & \longrightarrow & \mathrm{Fil}^{p+1} A^{n-1} & \xrightarrow{d^{n-1}} & \mathrm{Fil}^{p+1} A^n & \xrightarrow{d^n} & \mathrm{Fil}^{p+1} A^{n+1} & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

Link to Diagram

This yields

$$H^n_{p\pm r} = \frac{A^n \cap d^{-1}(\mathrm{Fil}^{p+r} A^{n+1})}{A^n \cap d(\mathrm{Fil}^{p-r+1} A^{n+1})}$$

$$H^n_{p\pm\infty} = \frac{A^n \cap d^{-1}(0)}{A^n \cap d(A^{n+1})}.$$

Notation: write

$$ {}^{n}E_r^p := E_r^{p,q} = \operatorname{gr}^p H^n_{p\pm r} = \frac{\mathrm{Fil}^p A^n \cap d^{-1}\left(\mathrm{Fil}^{p+r}A^{n+1}\right)}{\mathrm{Fil}^{p+1}A^n \cap d^{-1}(\mathrm{Fil}^{p+r}A^{n+1}) + \mathrm{Fil}^p A^n \cap d^{-1}\left(\mathrm{Fil}^{p-r+1}A^{n-1}\right)}. $$

The main properties:

- $d_r : E_r^{p,q} \to E_r^{p+r, q-r+1}$
- $H(E_r{}^{\bullet,\bullet}, d_r) = E_{r+1}{}^{\bullet,\bullet}$.

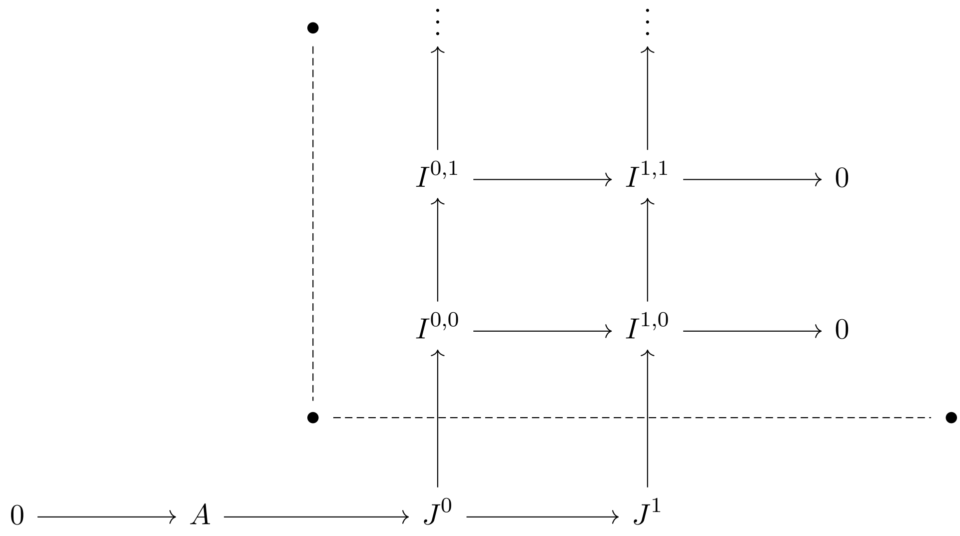
Note that ${}^{n}E_r^p \xrightarrow{d_r} {}^{n+1}E_r^{p+r}$, so

$$ \frac{\mathrm{Fil}^p A^n \cap d^{-1}\left(\mathrm{Fil}^{p+r}A^{n+1}\right)}{\mathrm{Fil}^{p+1}A^n \cap d^{-1}(\mathrm{Fil}^{p+r}A^{n+1}) + \mathrm{Fil}^p A^n \cap d^{-1}\left(\mathrm{Fil}^{p-r+1}A^{n-1}\right)} \xrightarrow{d_r} \frac{\mathrm{Fil}^{p+r} A^{n+1} \cap d^{-1}\left(\mathrm{Fil}^{p+2r}A^{n+2}\right)}{\mathrm{Fil}^{p+r+1}A^{n+1} \cap d^{-1}(\mathrm{Fil}^{p+2r}A^{n+2}) + \cdots}, $$

and $d_r^2 = 0$ since the first denominator above appears as the next numerator.

38.2 Applications

Remark 38.2.1: An application: consider a 2-step resolution $0 \to A \to J^0 \to J^1$, and take injective resolutions of each J^i to form an E_0:



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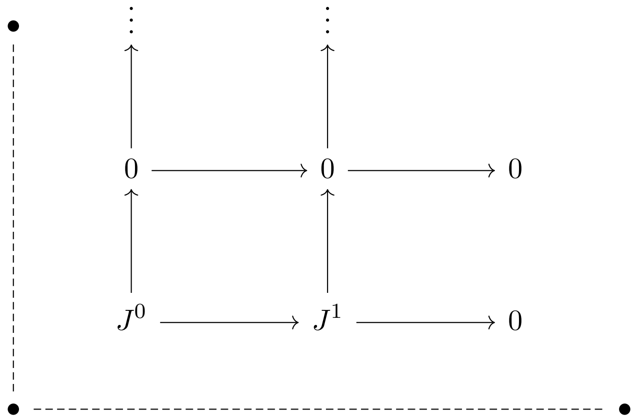
Then $0 \to A \to \mathrm{Tot}^\bullet(A^{\bullet,\bullet})$ is exact, i.e. this is an injective resolution of A. Take vertical cohomology to get E_1:



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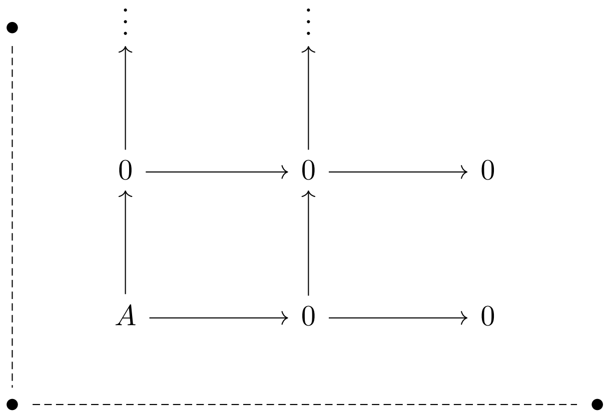
Since no functor has been applied, we obtain the follow E_2 after taking horizontal cohomology:



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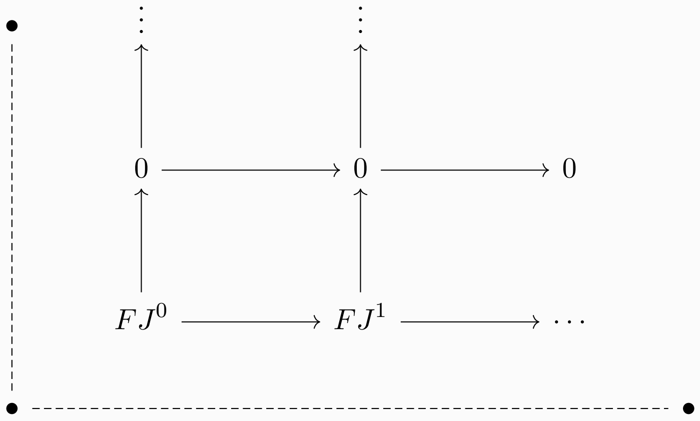
So $H^n(\mathrm{Tot} I) = A[0]$.

Remark 38.2.2: Let $F \in \mathsf{Cat}(A, B)$ be additive left-exact, then $\mathbb{R}^n FA = H^n(F\mathrm{Tot} I^{\bullet,\bullet})$ for $0 \to A \to I$ a biresolution as above. Define $E_0 = FI$, then $E_1^{p,q} = \mathbb{R}^q F J^p$.

Corollary 38.2.3 *(?).*
If J^p are F-acyclic, then E_1 has the form



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So $E_2^{p,q} = H^q(FJ^p)$, i.e. $\mathbb{R}FA$ can be compute using the resolution $0 \to A \to J^{\bullet,\bullet} \to \cdots$. For example, for $F(-) = \Gamma(X; -)$, we can resolve by flasque, soft, or fine sheaves.

Remark 38.2.4: Using two spectral sequences for a single bicomplex: given $C_{\bullet,\bullet}$,

$$E_2^{p,q} = H_h^p H_v^q C^{p,\bullet} \Rightarrow H^n(\mathrm{Tot}_\bullet C_{\bullet,\bullet})$$
$$E_2^{p,q} = H_v^q H_h^p C^{\bullet,q} \Rightarrow H^n(\mathrm{Tot}_\bullet C_{\bullet,\bullet}).$$

Remark 38.2.5: Grothendieck spectral sequences: for $\mathsf{A} \xrightarrow{F} \mathsf{B} \xrightarrow{G} \mathsf{C}$, form the composite $\mathsf{A} \xrightarrow{GF} \mathsf{C}$ to obtain

$$E_2^{p,q} = \mathbb{R}^p G \mathbb{R}^q FA \Rightarrow \mathbb{R}^{p+q} GFA,$$

provided F sends injectives to G-acyclics. This comes from running the two spectral sequences above, where one collapses onto a single row.

39 | Wednesday, April 20

39.1 Derived Categories

Remark 39.1.1: Recall how to construct derived functors. It is advantageous to embed $\mathsf{C} \hookrightarrow \mathsf{ChC}$ and resolve by nicer objects. A complex contains strictly more information than homology: e.g. $0 \to \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \to 0$ and $0 \to \mathbb{Z} \hookrightarrow \mathbb{Z} \oplus \dfrac{\mathbb{Z}}{2\mathbb{Z}} \to 0$ have isomorphic homology but aren't isomorphic as complexes.

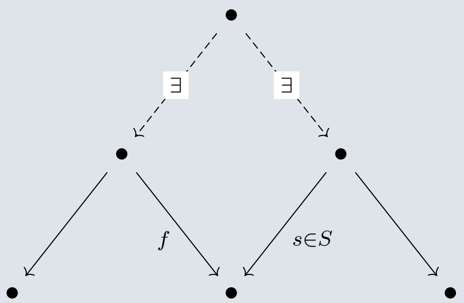
> **Definition 39.1.2** (Quasi-isomorphism)
> A morphism $f \in \mathsf{ChC}(A, B)$ is a **quasi-isomorphism** iff the induced map $f^* \in \mathsf{ChC}(H^\bullet A, H^\bullet B)$ is an isomorphism.

> **Definition 39.1.3** (The derived category)
> There is a category $\mathbb{D}\mathsf{C}$ and a functor $\mathsf{ChC} \to \mathbb{D}\mathsf{C}$ with the following universal property: if $\mathsf{ChC} \to \mathsf{B}$ is any functor sending quasi-isomorphisms to isomorphisms, there is a unique functor $\mathbb{D}\mathsf{C} \to B$ factoring it. We call $\mathbb{D}\mathsf{C}$ the **derived category** of C.

Remark 39.1.4: The basic morphisms in $\mathbb{D}\mathsf{C}$ are given by usual chain maps $f : A \to B$, and if f is a quasi-isomorphism we formally add inverses $X_f : B \to A$. A general morphism is a sequence of morphisms $\bullet \to \bullet \to \cdots \to \bullet$ where we quotient by

- $\bullet \xrightarrow{f} \bullet \xrightarrow{g} \bullet \sim \bullet \xrightarrow{gf} \bullet$
- $A \xrightarrow{f} B \xrightarrow{X_f} A \sim A \xrightarrow{\text{id}} A$
- $B \xrightarrow{X_f} A \xrightarrow{f} B \sim B \xrightarrow{\text{id}} B.$

One would like a calculus of fractions, so define:

> **Definition 39.1.5** (Localizing morphisms)
> Given $\mathsf{C} \in \mathsf{Cat}$, and subset $S \subseteq \text{Mor}(\mathsf{C})$ of morphisms is **localizing** iff
>
> - $\text{id}_A \in S$ for all objects A
> - S is closed under compositions
> - For every roof with f arbitrary and $s \in S$, there exist arrows:
>
> 
>
>
> *Link to Diagram*
>
> As a corollary, arrows in $\mathsf{C}[\frac{1}{S}]$ are roofs modulo equivalence.

Remark 39.1.6: The set S of quasi-isomorphisms in ChA is localizing.

Note that we can take

- ChC : all complexes,
- $\mathsf{Ch}^+\mathsf{C}$: complexes bounded from below,

- $\mathsf{Ch}^-\mathsf{C}$: complexes from above,
- $\mathsf{Ch}^b\mathsf{C}$: complexes from above and below.

These yield derived categories $\mathbb{D}\mathsf{C}, \mathbb{D}^+\mathsf{C}, \mathbb{D}^-\mathsf{C}, \mathbb{D}^b\mathsf{C}$. Note: frequently $\mathbb{D}\mathsf{C}$ actually means $\mathbb{D}^+\mathsf{C}$ in the literature. When $\mathbb{D}^b\mathsf{C}$ is used: if $\mathcal{F} \in \mathsf{Coh}(X)$ and X is projective, which corresponds to a graded module which (by Hilbert) has a finite resolution.

One can similarly define homotopy categories $\mathsf{hoChC}, \mathsf{hoCh}^+\mathsf{C}, \mathsf{hoCh}^-\mathsf{C}, \mathsf{hoCh}^b\mathsf{C}$ with $\mathrm{Ob}(\mathsf{hoChC}) := \mathrm{Ob}(\mathsf{ChC})$ and $\mathrm{Mor}(\mathsf{hoChC}) := \mathrm{Mor}(\mathsf{hoCat}C)/\sim$ where $\sim$ denotes chain homotopy equivalence.

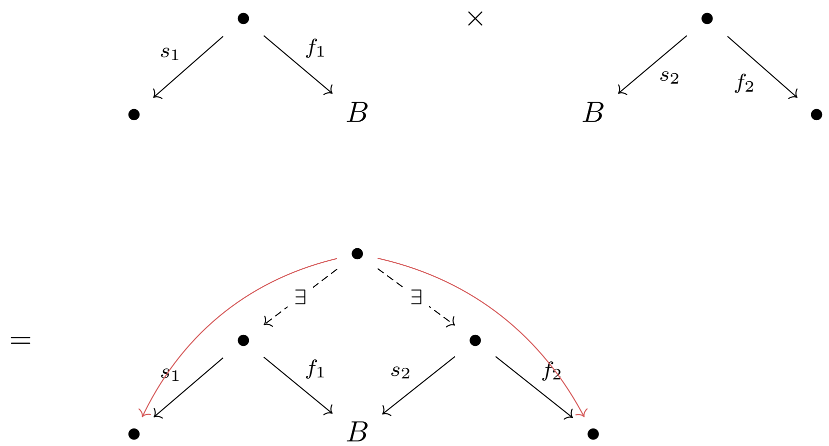
Theorem 39.1.7 *(?).*
$\mathbb{D}^+\mathsf{A} \cong \mathsf{hoCh}^+\mathsf{I}_\mathsf{A}$ where I_A is the homotopy category of complexes of injective objects in ChA.

Remark 39.1.8: Generally there is a functor $\mathsf{hoChA} \hookrightarrow \mathbb{D}\mathsf{A}$ since chain homotopy equivalences induce isomorphisms on homology (where we apply the universal property of $\mathbb{D}\mathsf{A}$) There is also a functor $\mathbb{D}\mathsf{A} \to \mathsf{hoChA}$ where $A \mapsto \mathrm{Tot}(I^{\bullet,\bullet})$ is a quasi-isomorphism.

40 | Friday, April 22

Remark 40.0.1: Recall that for $S \subseteq \mathrm{Mor}(\mathsf{C})$, there is a localized category $\mathsf{C}\left[S^{-1}\right]$ whose morphisms are chains $s_0^{-1} \circ f_0 \circ s_1^{-1} \circ \cdots$ modulo an equivalence, and if S is **localizing** then

- Morphisms are single roofs (i.e. we can collect the product fraction involving s_i, f_i into a single fraction).
  - Note that roofs can be multiplied, and roofs are equivalent when they admit a common roof:



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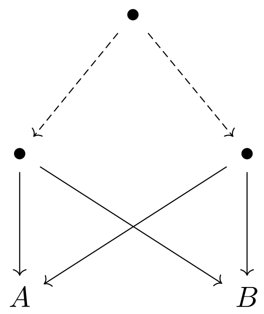
- Morphisms are equivalent when they admit a common roof:



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- If $\mathsf{C} \in \mathsf{AddCat}$ then $\mathsf{C}\left[S^{-1}\right] \in \mathsf{AddCat}$, where the calculus of fractions behaves as in ring localization: $\frac{f_1}{s} + \frac{f_2}{s} = \frac{f_1 + f_2}{s}$.
- If $I \leq \mathsf{C}$ is a full subcategorya nd S is *compatible*, i.e. $S \cap \mathrm{Mor}(I)$ is localizing, then $I\left[S^{-1}\right] \leq \mathsf{C}\left[S^{-1}\right]$ is a full subcategory.

Remark 40.0.2: ChC with S quasi-isomorphisms yields $\mathbb{D}\mathsf{A} := \mathsf{C}\left[S^{-1}\right]$.

Theorem 40.0.3 ***(?).***
The collection S of quasi-isomorphisms is localizing.

Corollary 40.0.4 ***(?).***
$\mathbb{D}\mathsf{A}$ is additive and morphisms are roofs in ChA.

Theorem 40.0.5 ***(?).***
I defined as $\mathsf{hoChC}^{\mathrm{inj}}$, the homotopy category of complexes of injective objects, is compatible with S.

Theorem 40.0.6 ***(?).***
$I\left[S^{-1}\right] \leq \mathsf{A}\left[S^{-1}\right] = \mathbb{D}\mathsf{A}$, with an equivalence if A has enough injectives.

⚠ **Warning 40.0.7**
These last two theorems do *not* hold just for $I = \mathsf{ChC}^{\mathrm{inj}}$.

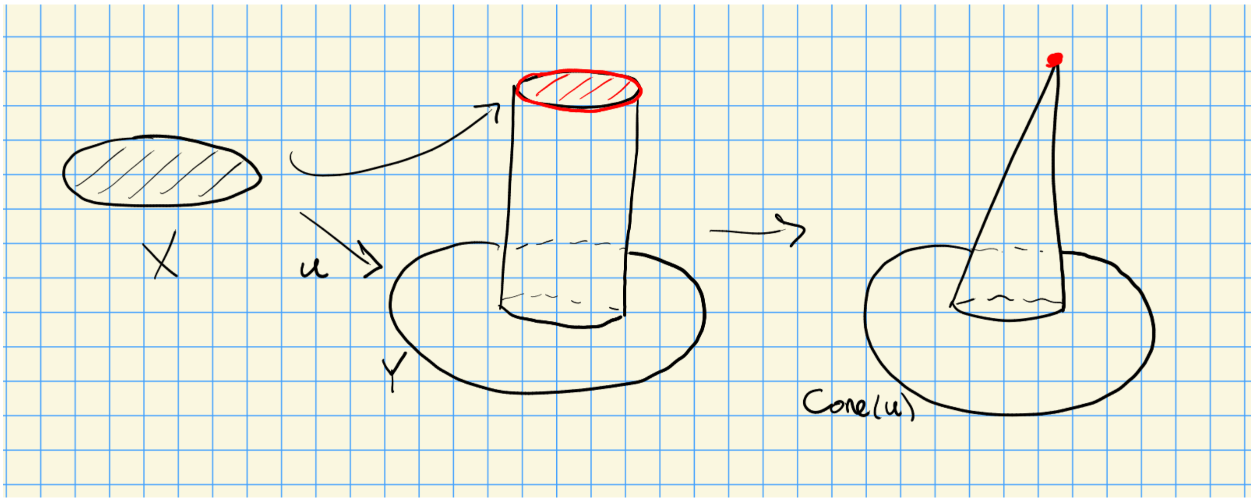
Remark 40.0.8: An application: for $F \in \mathsf{AbCat}(\mathsf{A}, \mathsf{B})$ additive (with no left/right exactness conditions), there is a derived functor $\mathbb{D}F \in \mathbb{D}^{+}\mathsf{A}, \mathbb{D}^{+}\mathsf{B}$ if A has enough injectives. Note that $\mathbb{D}\mathsf{A}$ is never abelian but admits a triangulated structure.

Example 40.0.9 ***(?):*** For $X \in \mathsf{sm}\,\mathrm{Proj}\,\mathsf{Var}_{/k}$, the usual notation is $\mathbb{D}(X) := \mathbb{D}^{b}\mathsf{Coh}(X)$. Global sections $\Gamma \in \mathsf{Cat}(\mathsf{Coh}X \to \mathsf{AbGrp})$ induce a derived functor $\mathbb{R}\Gamma \in \mathsf{Cat}(\mathbb{D}X \to \mathbb{D}^b\mathsf{AbGrp})$. Note that $\mathsf{Coh}X \hookrightarrow \mathbb{D}(X)$ by $\mathcal{F} \mapsto \mathcal{F}[0]$.

Remark 40.0.10: For X Proj $\mathsf{Var}_{/k}$, recall $\mathsf{K}_0X := \mathsf{K}_0\mathsf{Coh}X$ where $[b] = [a] + [c]$ for $0 \to a \to b \to c$, and $\mathsf{K}^0X := \mathsf{K}^0\mathsf{Sh}^{\mathrm{locfree}}(X)$. If X is smooth, these are isomorphic, but generally they are not if X is singular. In general, $\mathbb{D}X := \mathbb{D}^+\mathsf{Coh}X$ replaces $\mathsf{K}_0(X)$, and $\mathbb{D}^+\mathsf{Sh}^{\mathrm{locfree}}(X)$ replaces K^0X.

Theorem 40.0.11 *(?)*.
$\mathbb{D}\mathsf{A} \in \mathsf{triangCat}$.

Remark 40.0.12: Although these do not have SESs, there are distinguished triangles for which any morphism $X \to Y$ can be completed to $X \to Y \to Z \to \Sigma^{[1]}X$. This can be accomplished using mapping cylinders/cones:



Remark 40.0.13: See tilting of complexes, exceptional sequences.

41 | Monday, April 25

41.1 Triangulated categories

Definition 41.1.1 (Triangulated categories)
A **triangulated category** is an additive category $\mathsf{C} \in \mathsf{AddCat}$ with an additive autoequivalence $T : \mathsf{C} \to \mathsf{C}$ and a set of distinguished triangles $X \to Y \to Z \to TX$ satisfying

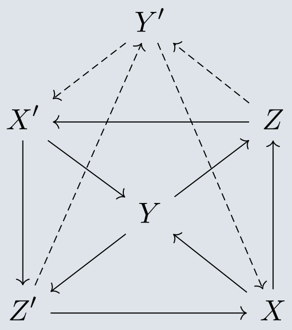
- **TR1**:
  - $X \xrightarrow{\mathrm{id}} X \to 0 \to TX$ is distinguished,

- Any triangle isomorphic to a distinguished triangle is again distinguished,
- For every $X \xrightarrow{u} Y$ there is a distinguished triangle $X \xrightarrow{u} Y \to Z \to X[1]$. Idea: $Z \approx Y/X$.

- **TR2**:
  - For every $X \to Y \to Z \to X[1]$, there is a triangle $Y \to Z \to X[1] \xrightarrow{Tu} Y[1]$.
- **TR3**:
  - Given 3 triangles

$$X \to Y \to Z' \to Y \qquad \to Z \to X' \to X \to Z \to Y' \to$$

there is a triangle $Z' \to Y' \to X'$ making the relevant octahedral diagram commute.



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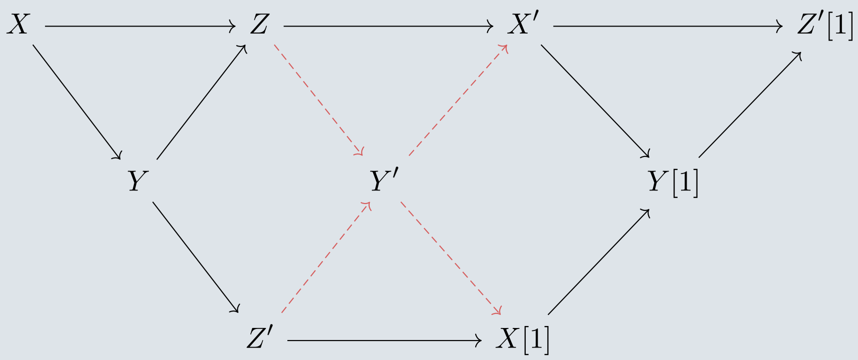
This can equivalently be expressed as a braid lemma:



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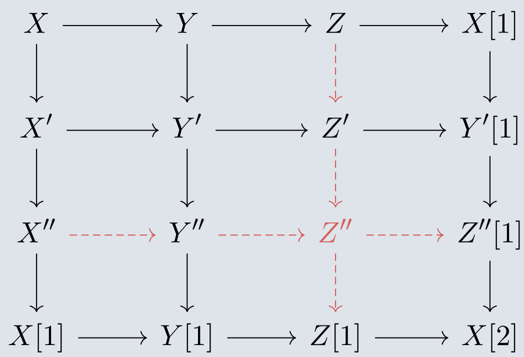
Equivalently, a 3x3 lemma holds:



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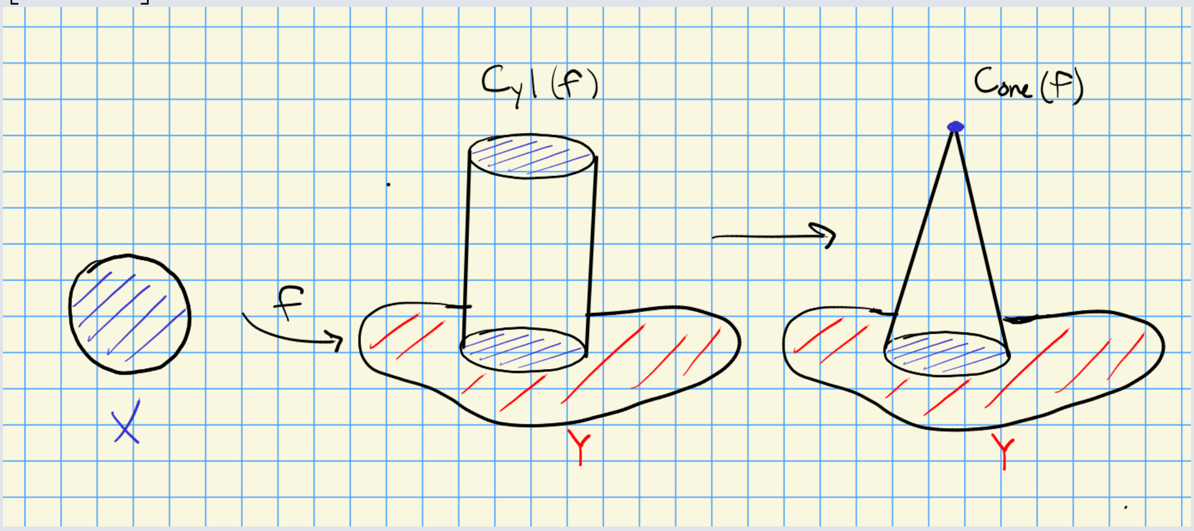
Theorem 41.1.2 *(?).*
For $\mathsf{A} \in \mathsf{AbCat}, \mathbf{D}\mathsf{A} \in \mathsf{triangCat}$.

Definition 41.1.3 (?)
For $f \in \mathsf{ChA}(X, Y)$, there is a *cone* complex $\mathrm{Cone}(f) = TX \oplus Y$ with differential $d_{\mathrm{Cone}(f)} = \begin{bmatrix} d_{X[1]} & 0 \\ f[1] & d_Y \end{bmatrix}$ and a *cylinder* complex $\mathrm{Cyl}(f)$:



Note that $d_{\mathrm{Cone}(f)}[x_{i+1}, y_i] = [-d_X x_{i+1}, f(x_{i+1}) + d_Y(y_i)]$, and one can check $d^2 = 0$.

Remark 41.1.4: Any distinguished triangle $X \xrightarrow{f} Y \to Z \to X[1]$ in $\mathbf{D}\mathsf{A}$ is isomorphic to a triangle of the form $X \to \mathrm{Cyl}(f) \to \mathrm{Cone}(f) \to X[1]$. For ChA, define $T^n A := A[n]$, so $(T^n A)_k = A[n]_k = A_{n+k}$, and $\partial_{TA} := (-1)^n \partial_A$.

42 | Wednesday, April 27

42.1 Cohomological Functors

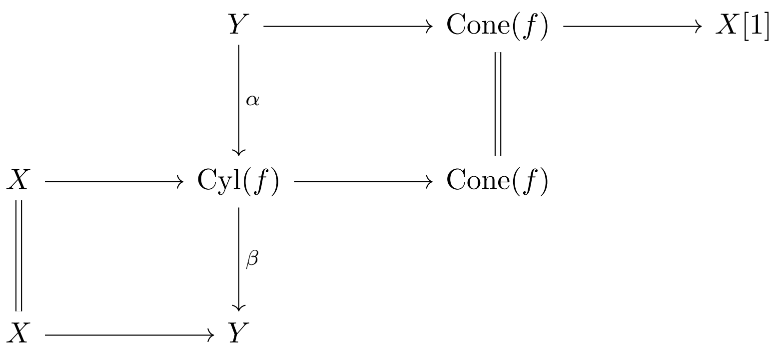
Remark 42.1.1: Recall that for $X \xrightarrow{f} Y$, $\mathrm{cone}(f) \approx X[1] \oplus Y$ and $\mathrm{Cyl}(f) \approx X \oplus X[1] \oplus Y$ with differential

$$d_{\mathrm{Cyl}(f)} := \begin{bmatrix} d_X & -1 & \\ & d_X[1] & \\ & f[1] & d_Y \end{bmatrix} \curvearrowright [x_i, x_{i+1}, y_i] \in \mathrm{Cyl}(f)^i.$$

42 Wednesday, April 27

Note: I use ≈ above because these formulas hold levelwise, but the SESs they fit into may not be split exact, so $\mathrm{cone}(f), \mathrm{Cyl}(f)$ *may not such direct sums.*

There are related exact triples, here the first and second rows:



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Here $\beta\alpha = \mathrm{id}_Y$ and $\alpha\beta \simeq \mathrm{id}_{\mathrm{Cyl}(f)}$.

Definition 42.1.2 (Cohomological functors)
A functor $H \in [\mathsf{C}, \mathsf{A}]$ with $\mathsf{C} \in \mathsf{triangCat}, \mathsf{A} \in \mathsf{AbCat}$ (where A is not necessarily related to C) is a **cohomological functor** iff every distinguished triangle $A \to B \to C \in \mathsf{C}$ is sent to an exact sequence $HA \to HB \to HC \in \mathsf{A}$.

Corollary 42.1.3 *(?).*
If H is cohomological, there is an associated LES

$$\cdots \to HA \to HB \to HC \to H(A[1]) \to H(B[1]) \to \cdots .$$

Lemma 42.1.4 *(?).*
The functor $H : \mathbf{D}A \to A$ where $X \mapsto H^0(X)$ is cohomological, noting that $H^i(X)$ can be written as $H^0(X[i])$.

Definition 42.1.5 (Ext for triangulated categories)

$$\mathrm{Ext}^i(X, Y) := \mathop{\mathrm{Hom}}_{\mathsf{C}}(X, Y[1]).$$

Lemma 42.1.6 *(?).*

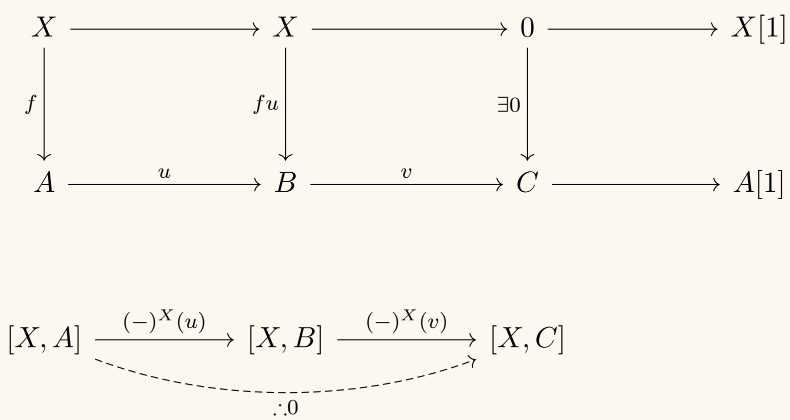
$$\mathrm{Ext}^i_{\mathsf{A}}(X, Y) \cong \mathrm{Ext}_{\mathbf{D}\mathsf{A}}(\iota X, \iota Y)$$

where $\iota : \mathsf{A} \to \mathsf{ChA}$ is given by $\iota(A) = \cdots \to 0 \to A \to 0 \to \cdots$ supported in degree zero.

Theorem 42.1.7 ***(?).***
For all $\mathsf{C} \in \mathsf{triangCat}$, for all $X, Y \in \mathsf{C}$ the (co)representable hom functors are cohomological:

$$h_Y := \mathop{\mathrm{Hom}}_{\mathsf{C}}(-, Y) \qquad \text{covariant}$$
$$(-)^X := \mathop{\mathrm{Hom}}_{\mathsf{C}}(X, -) \qquad \text{contravariant.}$$

Proof (?).
The proof uses the octahedral axiom TR3. To show that applying homs yields a complex, show that the maps on homs square to zero using the following:



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42.2 Exceptional Collections

Definition 42.2.1 (Exceptional collections)
For $\mathsf{C} \in \mathsf{triangCat}$, an **exceptional collection/sequence** is a chain of morphisms

$$\mathcal{E}_1 \to \mathcal{E}_2 \to \cdots \to \mathcal{E}_n \in \mathsf{C}$$

such that

1. Self-Exts are supported only in degree zero, i.e. $\operatorname{Hom}(\mathcal{E}_i, \mathcal{E}_i[k]) = 0$ for $k \neq 0$.
2. There are no homs in the opposite direction, i.e. $\operatorname{Hom}(\mathcal{E}_j, \mathcal{E}_i[m]) = 0$ for $j > i$ and for any m.

Example 42.2.2 ***(?):*** From a paper of Valery's: let X be a smooth projective surface with $H^1(\mathcal{O}_X) = H^2(\mathcal{O}_X) = 0$, which cohomologically look like rational surfaces. Examples: X rational with $|nK_X| = \emptyset$ (so "negative" canonical class), or X of general type with $q = p_g = 0$ and $|nK_X|$ big for $n \gg 0$. In these cases, there are line bundles $\mathcal{E}$ with $\operatorname{Ext}^i(\mathcal{E}, \mathcal{E}) = H^1(\mathcal{O}, \mathcal{O}) = H^i(\mathcal{O}_X)$ and one can use that $\operatorname{Ext}(\mathcal{E}_i, \mathcal{E}_j) = H^i(\mathcal{E}_i \otimes \mathcal{E}_j^{-1})$.

Theorem 42.2.3 ***(Beilinson, Bondal, Kapranov).***
If $\mathsf{C}' \leq \mathsf{C}$ is the full subcategory generated by $\mathcal{E}_1, \cdots, \mathcal{E}_n$, then $\mathsf{C}' \simeq \mathbf{D}^b(Q)$ for Q a quiver. In particular, if $\{\mathcal{E}_i\}$ is a full exceptional collection, $\mathsf{C} = \mathsf{C}'$.

43 | Friday, April 29

43.1 Applications of derived categories

Remark 43.1.1: Some major work in this area:

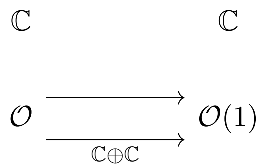
- Beilinson-Gelfand-Gelfand (BGG)
- Bendel-Kapranov
- Mukai
- Bendal-Orlov
- Orlov
- Kutznatsov
- Kontsevich, Fukaya (homological mirror symmetry)
- Beilinson-Bernstein-Gabber-Deligne on perverse sheaves
- Bridgeland

Remark 43.1.2: Results:

- BGG: $\mathbf{D}^b(\mathsf{Coh}\mathbb{P}^n) \cong \mathbf{D}^b(\mathsf{R\text{-}Mod})$ for R a certain ring.
- BK, K: same for quadrics and grassmannians.

Recall that given $\mathsf{T} \in \mathsf{triangCat}$ with an exceptional collection $\{\mathcal{E}_i\}$, they generate a triangulated subcategory $\langle \mathcal{E}_i \rangle \leq \mathsf{T}$. It turns out that $\langle \mathcal{E}_i \rangle \cong \mathsf{R\text{-}Mod}$ for $R = \bigoplus \operatorname{End} \mathcal{E}_i$. Beilinson produces a collection $\left\{\mathcal{O}, \Omega^1, \cdots, \Omega^{n-1}\right\}$, but an easier alternative is $\{\mathcal{O}, \mathcal{O}(1), \cdots, \mathcal{O}(n-1)\}$. If the collection is full, then $\mathsf{T} \cong \langle \mathcal{E}_i \rangle$. As an alternative to R, one can take the corresponding quiver: make a directed graph $\mathcal{E}_1 \to \mathcal{E}_2 \to \cdots$ where each node has $\operatorname{End} \mathcal{E}_i$ attached and each edge $\mathcal{E}_i \to \mathcal{E}_j$ is assigned $\oplus_n \operatorname{Hom}(\mathcal{E}_i, \mathcal{E}_j[n])$. So the derived category corresponds to representations of this quiver.

Example: for $\mathbb{P}^1$, one obtains the following quiver:



Link to Diagram

Proposition 43.1.3 *(?).*
If $X \in \mathsf{AlgVar}_{/k}$ admits a full exceptional collection, then the following also admit a full exceptional collection:

- Any $\mathbb{P}^n$-bundle $Y = \mathbb{P}(V) \to X$, and
- Any blowup $Y = \operatorname{Bl}_Z X$ for Z a smooth subvariety.

Corollary 43.1.4 *(?).*
Any rational smooth projective surface admits a full exceptional collection, by running the MMP.

Conjecture 43.1.5.
Given a smooth surface admitting a full exceptional collection, is it rational? For a threefold, is it a blowup of something rational?

Definition 43.1.6 (Semiorthogonal decompositions)
Given $\mathsf{T} \in \mathsf{triangCat}$ and $A \leq \mathsf{T}$ a full triangulated subcategory, one can define two subcategories ${}^\perp A$ and $A^\perp$:

$$A^\perp = \left\{ F \;\middle|\; \operatorname{Hom}(F, A) = 1 \right\}.$$

Remark 43.1.7: For $\mathsf{C} \in \mathsf{triangCat}$, one can take HHC. For $\mathsf{C} = \mathbf{D}(X)$, the $\mathrm{HH}_0 D(X) \cong \mathbb{Z}^n \oplus A$ as a group, for A some finite torsion group. If one has a full exceptional collection, then $A = \mathrm{HH}_0(\langle \mathcal{E}_1, \cdots, \mathcal{E}_n \rangle^\perp)$. As a corollary, the length m of an exceptional collection satisfies $m \leq \operatorname{rank}_{\mathbb{Z}} \mathrm{HH}_0 \mathbf{D}(X)$.

Conjecture 43.1.8 ***(Kaznutsov).***
If $\{\mathcal{E}_1, \cdots, \mathcal{E}_n\}$ is an exceptional collection and $n = \operatorname{rank}_{\mathbb{Z}} \mathrm{HH}_0 \mathbf{D}(X)$, then this is a *full* exceptional collection.

Remark 43.1.9: For surfaces of general type, a special Godeaux surface produces a counterexample. There is a much easier counterexample coming from a Burnist (?) surface – generally fake $\mathbb{P}^2$, fake Fanos, etc. See A-Orlov, Orlov-Gorheaise, Katgerov-?, ??

Remark 43.1.10: Phantoms: categories with zero HH, so no full exceptional collections.

43.2 Well-known classical results

Theorem 43.2.1 ***(Bondal-Orlov (very important!)).***
If $X \in \mathsf{sm\,proj\,Var}$ where either K_X or $-K_X$ is ample, then X can be recovered from $\mathbf{D}(X)$.

Remark 43.2.2: Having $-K_X$ ample yields **Fano** varieties, and K_X ample yields **general-type** surfaces.

Theorem 43.2.3 ***(Mukai).***
If $A \in \mathsf{AbVar}$, then $\mathbf{D}(A) \cong \mathbf{D}(A^\vee)$. Such pairs are referred to as **Mukai partners**.

Remark 43.2.4: How to construct the equivalence $\mathbf{D}(A) \to \mathbf{D}(A^\vee)$: take the Fourier-Mukai transform. Use the Poincare bundle $P_A \to A \times A^\vee$, and construct the functor as a push-pull over the span $(A \leftarrow_{p_1} A \times A^\vee \to_{p_2} A^\vee)$, so

$$\mathcal{F} \mapsto (p_2)_* \left((p_1)^* \mathcal{F} \otimes P_A\right).$$

Remark 43.2.5: The next in line: K3 surfaces. An easy example: take Kummer surfaces, so $A \to A/\pm 1$ and then blow up the 16 nodes.

44 | Monday, May 02

44.1 Calabi-Yau Categories

Remark 44.1.1: Recall that a collection $\mathcal{E}_i$ is *exceptional* iff $[\mathcal{E}_j, \mathcal{E}_i[n]] = 0$ if $n \geq 0$ and $j > i$. If there exists a full exceptional collection, $\mathbf{D}X \cong \mathbf{D}(\mathsf{R\text{-}Mod})$ for some R. Recall that a variety is *Fano* if $-K_X$ is ample.

Question 44.1.2
Do full exceptional collections exist for Fano n-folds for $n = 3$ or 4?

Answer 44.1.3
Typically no.

Remark 44.1.4: Let

$$X_3 := V(f_3(x_0, \cdots, x_5)) \subseteq \mathbb{P}^5,$$

then which X_3 are rational? Note that $K_{X_3} = \mathcal{O}(-6+3) = \mathcal{O}(-3)$. Kuznatsov shows that $H^i(\mathcal{O}_X) = \mathbb{C}[0]$ and $\mathrm{Ext}^i(\mathcal{L}, \mathcal{L}) = H^i(\mathcal{O}_X)$. One could look for exceptional collections of line bundles, so $\mathrm{Ext}^i(\mathcal{L}_j, \mathcal{L}_i) = H^m(\mathcal{L}_i \otimes \mathcal{L}_j^{-1}) = 0$ for all m. On $\mathbb{P}^n$, take $\mathcal{O}(-k)$ for $1 \leq k \leq n$ since $K_{\mathbb{P}^n} = \mathcal{O}(-n-1)$. For X_3, there is enough vanishing that $\mathcal{O}, \mathcal{O}(1), \mathcal{O}(2)$ are exceptional (everything below the index 3 from above). Kuznatsov shows that the "Kuznatsov component" $K = \langle \mathcal{O}, \mathcal{O}(1), \mathcal{O}(2) \rangle^\perp$ is a **Calabi-Yau category** of dimension 2.

Remark 44.1.5: If Y is a Calabi-Yau variety of dimension n, so $K_Y = 0$, there is a Serre functor

$$S : \mathbf{D}^b(Y) \to \mathbf{D}^b(Y)$$
$$F \mapsto F \otimes \omega_Y[n].$$

Then (probably) $S = T^n$, a shift by n. A category is a Calabi-Yau category of dimension n iff

- It has a Serre functor
- $S = T^n$

One can also define fractional dimension using S^q.

Conjecture 44.1.6.
X is rational iff $K = \mathbf{D}^b(Y)$ for $Y \in \mathsf{K3}$.

Remark 44.1.7: A technique due to Clemens-Griffith for cubic threefolds. Let $X \subseteq \mathbb{P}^4$ be a smooth non-nodal curve. Consider the intermediate Jacobian $J_3(X)$, which is a PPAV for any smooth 3-fold. Basic operations: blowing up a point p or a curve C, since blowing up a surface is the identity. Blowing up a point: $J_3(\mathrm{Bl}_p X) = J_3(X)$, so it doesn't change. For a curve, $J_3(\mathrm{Bl}_C X) = J(C) \oplus J_3(X)$. As a corollary, if X is rational then $J_3(X) = \bigoplus J(C_i)$ for some curves C_i. For non-rationality, show it's not the Jacobian of a curve by considering the theta divisor.

Remark 44.1.8: For 4-folds X, one can now also blow up surfaces. The intermediate cohomology carries a Hodge structure. Conjecture: X is rational iff its Hodge structure looks like a K3.

Remark 44.1.9: Older techniques for checking rationality: see log thresholds, generally birational geometry e.g. due to Manin. E.g. groups of birational automorphism for quartic 4-folds are small. See another approach due to Mumford using torsion in cohomology.

44.2 T-Structures and Hearts

Remark 44.2.1: Note that it's possible for $\mathsf{A}, \mathsf{B} \in \mathsf{AbCat}$ to satisfy $\mathbf{D}\mathsf{A} \xrightarrow{\sim} \mathbf{D}\mathsf{B}$.

Example 44.2.2 ***(?):*** Some examples:

- In the presence of a full exceptional collection $\mathbf{D}^b(\mathsf{Coh}X) \xrightarrow{\sim} \mathbf{D}^b(\mathsf{R\text{-}Mod})$.
- Fourier-Mukai: $\mathbf{D}^b(\mathsf{Coh}A) \xrightarrow{\sim} \mathbf{D}^b(\mathsf{Coh}A^\vee)$ for dual AVs.

Example 44.2.3 ***(Perverse sheaves (BBD)):*** Start with $X \in \mathsf{Var}_{/\mathbb{C}}$ Hausdorff paracompact and constructible sheaves which come with stratifications into closed subsets on which they restrict to locally constant sheaves. Note that one can realize these sheaves as pullbacks from a poset associated to the stratification. There are categories Const and Perv with $\mathbf{D}^b(\mathsf{Const}) \xrightarrow{\sim} \mathbf{D}^b(\mathsf{Perv})$ – here perverse sheaves are complexes of constructible sheaves with support conditions $h^j(\mathcal{F}^\bullet) \leq -j$ and $h^j(D\mathcal{F}^\bullet) \leq -j$ for D the Verdier dual; this is a category closed under duality.

Remark 44.2.4: On T-structures: write $D = \mathbf{D}\mathsf{A}$, then there are subcategories

- $D^{\leq n} = D^{\leq 0}[n]$, complexes such that $H^{>n} = 0$.
- $D^{\geq n} = D^{\geq 0}[n]$, complexes such that $H^{<n} = 0$.

Then $D^{\geq 0} \cap D^{\leq 0} = A$ is a category equivalent to complexes supported in degree zero, since any such bounded complex is quasi-isomorphic to such a complex. Some properties:

- $A_0 \in D^{<0}$ and $A_1 \in D^{>1}$ satisfy $[A_0, A_1] = 0$.
- For all $C \in \mathbf{D}^b(\mathsf{A})$, there exists $A_0 \to C \to A_1 \to A_0[1]$.

Note that there is a canonical truncation

$$\tau_{\leq 0}(\cdots \to C^{-1} \xrightarrow{d^0} C^0 \to C^1 \to \cdots) = (\cdots \to C^{-1} \to \ker d^0 \to 0).$$

44.3 Bridgeland stability

Remark 44.3.1: Take X a smooth projective curve, let $D = \mathbf{D}^b(\mathsf{Coh}X) = \mathbf{D}^b\mathsf{Bun}\,(\mathrm{GL}_r)$. There is a notion of a semistable sheaf (all subsheaves have smaller slopes $\mu(\mathcal{F}) := \deg \mathcal{F}/\operatorname{rank}\mathcal{F}$) and an HN filtration where the quotients are semistable and the slopes decrease. Bridgeland observed there is a central charge

$$Z : \mathsf{Coh}X \to \mathbb{C}$$
$$\mathcal{F} \mapsto -\deg \mathcal{F} + i \operatorname{rank}\mathcal{F},$$

which can be used to recovered the heart $\mathsf{A} = \mathsf{Coh}X$. Idea: vary Z to get different hearts, and $\{Z_i\}$ form a complex analytic variety, and one can form a new category of tilted complexes (complexes sitting in two degrees).

45 | Useful Facts

45.1 Category Theory

Remark 45.1.1:

- Products: a collection of maps into factors $Y \to X_i$ is the same as a map $Y \to \prod X_i$. Products are easy to map *into*. Products have projections $\prod X_i \to X_i$.
  - Products are limits.
- Coproducts: a collection of out of factors $X_i \to Y$ is the same as a map $\coprod X_i \to Y$. Coproducts are easy to map *out* of. Coproducts have injections $X_i \to \coprod X_i$.
  - Coproducts are colimits.
- If C has a zero object, there is a canonical map $\coprod_{i \in I} X_i \to \prod_{j \in I} X_j$ given by assembling maps δ_{ij}.
- $\varinjlim_{i \in I}(-)$ is generally not exact, but is exact if the colimit is filtered.
  - In any case, the functor of taking stalks $(-)_x : \mathsf{Sh}(X; \mathsf{AbGrp}) \to \mathsf{AbGrp}$ is always exact.
- Left adjoints/colimits are characterized by morphisms on $F(x)$, and right adjoints/limits by morphisms *into* it.
- Why RAPL and LAPC:

$$[\varinjlim_i L(x_i), -] \cong \varprojlim_i [L(x_i), -] \cong \varprojlim_i [x_i, R(-)] \cong [\varinjlim_i x_i, R(-)] = [L(\varinjlim_i x_i), -].$$

- Right-derived functors are left Kan extensions.
- Colimits are quotients of coproducts and **receive** maps from objects (i.e. they are cocones). Taking colims is right exact. Limits send maps.

45.2 Tor and Ext

Consider a commutative ring R and some R-modules M and N. One can compute the R-modules $\mathrm{Tor}_i^R(M,N)$ in essentially two ways.

1.) Begin with a projective resolution of N, e.g., $P_\bullet : \cdots \to P_2 \to P_1 \to P_0 \to N \to 0$; then, apply the functor $M \otimes_R -$ to this to obtain a chain complex

$$T_\bullet : \cdots \to M \otimes_R P_2 \to M \otimes_R P_1 \to M \otimes_R P_0 \to 0.$$

We define $\mathrm{Tor}_i^R(M,N)$ as the homology of this chain complex, i.e., $\mathrm{Tor}_i^R(M,N) = H_i(T_\bullet)$.

2.) Begin with a short exact sequence of R-modules $0 \to K \to N \to I \to 0$. By applying the right-exact functor $M \otimes_R -$ to this exact sequence, we obtain a long exact sequence of Tor, i

$$\cdots \to \mathrm{Tor}_1^R(M,K) \to \mathrm{Tor}_1^R(M,N) \to \mathrm{Tor}_1^R(M,I)$$

$$\to \mathrm{Tor}_0^R(M,K) \to \mathrm{Tor}_0^R(M,N) \to \mathrm{Tor}_0^R(M,I) \to$$

Remark 45.2.1:

Tor:

- Tor commutes with arbitrary direct sums, colimits (direct limits), localization.
- If M is flat over R, $\mathrm{Tor}_i^R(M \otimes A, B) \cong M \otimes_R \mathrm{Tor}_i^R(A,B)$.
- If S is a flat R-algebra, $S \otimes_R \mathrm{Tor}_i^R(A,B) \cong \mathrm{Tor}_i^S(S \otimes_R A, S \otimes_R B)$.
- $\mathrm{Tor}_1^R(R/I, R/J) \cong \dfrac{I \cap J}{IJ}$
- If I is an R-regular sequence $I = \langle x_1, \cdots, x_n \rangle$, then $\mathrm{Tor}_n^R(R/I, M) = (0 :_M I)$ is a colon ideal.
- If $A \in \mathsf{R\text{-}Mod}^\flat$ then $\mathrm{Tor}_{\geq 1}^R(A,B) = 0$.
- $\mathrm{Tor}(A,B) \cong \mathrm{Tor}(B,A)$.

Ext:

- $\tau_{\geq 1}\mathrm{Ext}^\bullet{}_R(A,B) = 0$ if either A is injective or B is projective.
- The Koszul complex for $k[x,y]$: $K_x \otimes K_y = 0 \to k[x,y] \to k[x,y]^{\times 2} \to k[x,y] \to k \to 0$ where $K_x = 0 \to k[x,y] \xrightarrow{\cdot x} k[x,y] \to 0$.
- $\mathrm{Ext}^\bullet{}_{k[x,y]}(k) = k \oplus \Sigma k^{\times 2} \oplus \Sigma^2 k$.

46 Problem Set 1

46.1 Problem 1

Problem 46.1.1 (1.1)
Recall that:

- A topology on a set X is T_0 if any two points $x, y \in X$ can be topologically distinguished (by open sets).
- A topology is an Alexandrov if an intersection of any, possibly infinite, collection of open sets is open.
- The **order topology** on a poset $(X, \leq)$ is defined in the following way: the open sets are the *upper sets*, which satisfy the property

$$x \in U, x \leq y \Longrightarrow y \in U$$

  The closed sets are **lower sets**, which satisfy

$$x \in Z, x \geq y \Longrightarrow y \in U$$

Prove that a topology on X is an order topology $\iff$ it is T_0 and Alexandrov. As a corollary conclude that any T_0 topology on a finite set is an order topology.

Proposition 46.1.1.
A topology τ on X is an order topology $\iff$ τ is T_0 and Alexandrov.

Proof .
$\Longleftarrow$: Suppose X is a topological space and τ is a T_0 Alexandrov topology on X. For $U \subseteq X$, write $\mathrm{cl}_X(U)$ for the closure in X of U with respect to τ, define a poset $(P, \leq)$ where $P := X$ with an ordering defined by

$$x \leq y \iff x \in \mathrm{cl}_X(y).$$

Regarding τ now as a topology on $(P, \leq)$, the claim is that this is an order topology on a poset. That this ordering defines a poset is clear, since the ordering is:

- **Reflexive**: since x is contained in its closure, $x \leq x$.
- **Antisymmetric**: if $x \leq y$ and $y \leq x$, then x is a limit point of $\{y\}$ and vice-versa. So every neighborhood of y contains x and similarly every neighborhood of x contains y. Since X is T_0 and topologically distinguishes points, this can only occur if $x = y$.
- **Transitive**: if $x \leq y$ and $y \leq z$, then $x \in \mathrm{cl}_X(y)$ and $y \in \mathrm{cl}_X(z)$. Since $\mathrm{cl}_X(z)$ is a closed set containing y and $\mathrm{cl}_X(y)$ is the *smallest* closed set in X containing y, we have $x \in \mathrm{cl}_X(y) \subseteq \mathrm{cl}_X(z)$, so $x \leq z$.

It thus suffices to show that if $U \ni x$ is a neighborhood of x and $x \leq y$, then $y \in U$ so that U is an upper set. By definition of the closure of a set,

$$x \leq y \iff x \in \mathrm{cl}_X(y) \iff \text{every neighborhood of } x \text{ intersects } \{y\},$$

so if $U_\alpha \ni x$ is any neighborhood of x, then $y \in U_\alpha$. Write $\tilde{U} := \cap_\alpha U_\alpha$ for the neighborhood basis at x, the intersection of all neighborhoods of x. Note that by construction, since $y \in U_\alpha$ for all α, $y \in \tilde{U}$. Since τ is T_0, $\tilde{U}$ is an open set. Moreover, since U is a neighborhood of x, $\tilde{U} \subseteq U$, so $y \in U$.
$\implies$: Suppose $(X, \leq)$ is a poset with an order topology τ, so U is open iff whenever $x \in U$ and $x \leq y$ then $y \in U$. To see that τ defines a T_0 topology, let $x \neq y$ in X. If x and y are not comparable, there is nothing to show, so suppose either $x < y$ or $y < x$ – without loss of generality, relabeling if necessary, we can assume $x < y$. Now every neighborhood of x contains y by definition, but for example

$$U_{\geq y} := \left\{ z \in X \;\middle|\; z \geq y \right\}$$

is neighborhood of y not containing x, topologically distinguishing x and y.
To see that τ is Alexandrov, it suffices to show that arbitrary intersections of open sets are open. This follows from the fact that any intersection of upper sets is again an upper set – if $\{U_i\}_{i\in I}$ is an arbitrary family of upper sets, set $U := \cap_{i\in I} U_i$. Then if $x \in U$ with $x \leq y$, $x \in U_i$ for every i and so $y \in U_i$ for every i, and thus $y \in U$.

■

Corollary 46.1.2 ***(?).***
If X is a finite set and τ is a T_0 topology on X, then τ is an order topology.

Proof (of cor).
By the exercise, it suffices to show that any finite space is Alexandrov. Let (X, τ) be a T_0 space and let $\{U_i\}_{i\in I} \subseteq \tau$ be an arbitrary collection of open sets – we'll show $U := \cap_{i\in I} U_i \in \tau$ is again open. This follows immediately, since finite intersections of open sets are open in any topology, and since X is finite and $\tau \subseteq 2^X$ is finite, I can only be a finite indexing set.

■

46.2 Problem 2

Problem 46.2.1 (1.2)
Recall that:

- A paracompact space is a topological space in which every open cover has an open refinement that is locally finite.
- A partition of unity of a topological space X is a set f_α of continous functions $f_\alpha : X \to [0, 1]$ such that for every point $x \in X$ there exists an open neighborhood of x where all but finitely many $f_\alpha \equiv 0$, and such that $\sum f_\alpha = 1$.

Prove that

- Any Hausdorff space is paracompact iff it admits a partition of unity subordinate to any open cover.
- Any metric space is paracompact.

A sketch would suffice.

Proposition 46.2.1 *(?).*
X is Hausdorff $\iff$ X admits a partition of unity subordinate to any open cover $\mathcal{U} \rightrightarrows X$.

Proof (?).
?

■

Proposition 46.2.2 *(?).*
Metric spaces are paracompact.

Proof (?).
Let $\mathcal{U} \rightrightarrows X$ be an open cover of a metric space X; we'll show $\mathcal{U}$ admits a locally finite refinement. Without loss of generality, writing $\mathcal{U} = \{U_j\}_{j \in J}$ for some index set J, we can assume the U_i are disjoint – this follows by invoking the axiom of choice to well-order the index set J and setting

$$\tilde{U}_j := U_j \setminus \bigcup_{k<j} U_k.$$

Then $\tilde{\mathcal{U}} := \left\{\tilde{U}_j\right\}_{j \in J}$ refines $\mathcal{U}$ since $\tilde{U}_j \subseteq U_j$, and still covers X. Moreover, we note that for every $x \in X$, we can now produce a *minimal* index $j(x)$ such that $x \in U_{j(x)}$.
The idea is to now refine $\mathcal{U}$ to a cover $\mathcal{V}$ by filling each disjoint annulus U_j with balls of small enough radius. For ease of notation and to more clearly demonstrate the following construction, suppose $J \cong \{0, 1, \cdots\}$ is countable. For each $n \in \mathbb{Z}_{\geq 0}$, let $\delta_n < \varepsilon_n$ be small to-be-determined real numbers depending on n, and define the following subsets of X:

$$\begin{aligned}
X_{0,n} &:= \left\{x \in U_0 \;\middle|\; B_{\varepsilon(n)}(x) \subseteq U_0\right\} & V_{0,n} &:= \bigcup_{x \in X_{0,n}} B_{\delta_n}(x) \subseteq X_{0,n} \subseteq U_0 \\
X_{1,n} &:= \left\{x \in U_1 \;\middle|\; B_{\varepsilon(n)}(x) \subseteq U_1\right\} \setminus \bigcup_{\ell<n} V_{0,\ell} & V_{1,n} &:= \bigcup_{x \in X_{1,n}} B_{\delta_n}(x) \subseteq X_{1,n} \subseteq U_1 \\
X_{2,n} &:= \left\{x \in U_2 \;\middle|\; B_{\varepsilon(n)}(x) \subseteq U_2\right\} \setminus \bigcup_{\ell<n} V_{0,\ell} \setminus \bigcup_{\ell<n} V_{1,n} & V_{2,n} &:= \bigcup_{x \in X_{2,n}} B_{\delta_n}(x) \subseteq X_{2,n} \subseteq U_2 \\
&\vdots & &\vdots \\
X_{j,n} &:= \left\{x \in U_j \;\middle|\; B_{\varepsilon(n)}(x) \subseteq U_j\right\} \setminus \bigcup_{k<j} \bigcup_{\ell<n} V_{k,\ell} & V_{j,n} &:= \bigcup_{x \in X_{j,n}} B_{\delta_n}(x) \subseteq X_{j,n} \subseteq U_j.
\end{aligned}$$

Note that the last line prescribes a general formula which depends only on the ordering and not on the countability of J.
In other words, for each fixed $j_0 \in J$, we consider all of those $x \in X$ such that $j(x) = j_0$, so that for each such x we have $x \in U_{j_0}$ but $x \notin U_k$ for any $k < j_0$. For a fixed n, we then consider those $x \in U_{j_0}$ that are not too close to the boundary, so that a ball of radius ε_n fits entirely in U_{j_0}. We then shrink these balls to a smaller radius δ_n and take their union to form an open set $V_{j_0,n}$ in the new cover, and as $n \to \infty$ these balls get smaller and fill out all of U_{j_0}. However, at each stage $V_{j,n}$ we remove redundancies by discarding sets $V_{k,\ell}$ for $k < j$ and $\ell < n$.

Claim: $\mathcal{V} := \{V_{j,n}\}_{j \in J, n \in \mathbb{Z}_{\geq 0}}$ is a locally finite refinement of $\mathcal{U}$.

Proof (of claim).
There are several things that are clear from the construction:

- Each $V_{j,n}$ is open, as they are arbitrary unions of open balls in a metric space.
- $\mathcal{V}$ refines $\mathcal{U}$, since in fact $V_{j,n} \subseteq U_j$ for every n and every j.
- $\mathcal{V}$ is a cover, since for any x one can pick $j(x)$ minimally so that $x \in U_{j(x)}$, and since x is an interior point and open balls form a basis for a metric space, for some n large enough we have $x \in B_{\delta_n}(x) \subseteq V_{j(x),n}$.

So the content of this statement is that each $x \in X$ is contained in only finitely many opens from $\mathcal{V}$. Fix $x \in X$ and pick $j(x)$ minimally as above, so $x \in V_{j(x),n}$ for every n and $j(x)$ is the first such j where x is added. Then $x \in B_{\delta_n}(x')$ for some x' near x, so choose n and k so that $V_x := B_{\frac{1}{2^k}}(x) \subset B_{\delta_n}(x) \subseteq V_{j,n}$. The claim now is that V_x intersects only finitely many elements of $\mathcal{V}$. A proof of this follows from using the triangle inequality to show that $B_{1/2^{n+k}}(x)$ does not intersect any $V_{\beta,\ell}$ for $\ell \geq n+k$, and for $\ell < n+k$ it intersects these for at most one β, leaving only finitely many such ℓ.

■

■

46.3 Problem 3

Problem 46.3.1 (1.3)
Let $A = \mathbb{Z}$ be an abelian group. Let $\mathcal{F}$ be a sheaf on X such that every stalk $\mathcal{F}_x = A$. Does it follow that $\mathcal{F}$ is a constant sheaf?

- Show that the answer is no in general.
- What if X is an irreducible algebraic variety with Zariski topology?
- What if $X = [0,1]$ with classical topology?

Proposition 46.3.1 ***(Part 1).***
There is a sheaf $\mathcal{F}$ on a space X with stalks satisfying $\mathcal{F}_x = \mathbb{C}$ for every $x \in X$, but $\mathcal{F}$ is not isomorphic to the constant sheaf $\underline{\mathbb{C}_X}$.

Proof (Part 1).
Let $X = S^1$ in the Euclidean topology, $U = X \setminus \{1\}$, $Z = \{1\}$ and let the two inclusions be

$$\begin{aligned} j &: U \to X, \\ i &: Z \to X. \end{aligned}$$

Now set

$$\mathcal{F} := j_! \underline{\mathbb{C}_U} \oplus i_* \underline{\mathbb{C}_Z}$$

We can then compute the stalks:

$$\begin{aligned} (j_! \underline{\mathbb{C}_U})_x &= \varinjlim_{W \ni x} (j_! \underline{\mathbb{C}_U})(W) \\ &= \varinjlim_{W \ni x} \begin{cases} \mathbb{C} & W \subseteq U \\ 0 & \text{else.} \end{cases} \\ &= \varinjlim \begin{cases} \cdots \to \mathbb{C} \to \mathbb{C} \to \mathbb{C} \to \cdots & x \in U \\ 0 \to 0 \to 0 \to 0 \to \cdots & x \notin U \end{cases} \\ &= \begin{cases} \mathbb{C} & x \in U \\ 0 & x \notin U, \end{cases} \\ &= \begin{cases} \mathbb{C} & x \neq \{1\} \\ 0 & x = \{1\}. \end{cases} \end{aligned}$$

where we've used that since U is open, if $x \in U$ then there is an open neighborhood $W \ni x$ completely contained in U, making the directed system eventually constant. Otherwise, if $x \notin U$, then no neighborhood of x is completely contained in U, and the sections here are zero for every $W \ni X$.

Note: this uses that the colimit of an eventually constant diagram is isomorphic to whatever that constant object is, i.e. it satisfies the correct universal property.

Similarly, noting that for $W \subseteq X$,

$$i^{-1}(W) = \begin{cases} \{1\} & \{1\} \in W \\ \emptyset & \{1\} \notin W, \end{cases}$$

we have

$$
\begin{aligned}
i_*\underline{\mathbb{C}_Z} &= \varinjlim_{W \ni x} \underline{\mathbb{C}_Z}(i^{-1}(W)) \\
&= \varinjlim_{W \ni x} \underline{\mathbb{C}_Z}\left(\begin{cases} \{1\} & \{1\} \in W \\ \emptyset & \{1\} \notin W \end{cases}\right) \\
&= \varinjlim \begin{cases} \mathbb{C} \to \mathbb{C} \to \cdots & x = \{1\} \\ \cdots \to 0 \to 0 \to \cdots & x \neq \{1\} \end{cases} \\
&= \begin{cases} \mathbb{C} & x = \{1\} \\ 0 & x \neq \{1\}, \end{cases}
\end{aligned}
$$

where we've used that if U is open and $x \in U$ with $x \neq \{1\}$, there eventually all small enough neighborhoods $W \ni x$ will not intersect $X \setminus U = \{1\}$. Thus

$$
\mathcal{F}_x = \begin{cases} 0 \oplus \mathbb{C} & x = \{1\} \\ \mathbb{C} \oplus 0 & x \neq \{1\}, \end{cases}
$$

and all of the stalks are one copy of $\mathbb{C}$, as in the constant sheaf $\underline{\mathbb{C}_X}$ on X. However, $\mathcal{F}$ and $\underline{\mathbb{C}_X}$ do not have the same sections: take $W \subseteq S^1$ to be a connected open neighborhood of $\{1\}$, then

$$
\{1\} \in W \implies i_*\underline{\mathbb{C}_Z}(W) = \underline{\mathbb{C}_Z}(i^{-1}(W)) = \underline{\mathbb{C}_Z}(\{1\}) = \mathbb{C}.
$$

Note that $j^{-1}(W) = W \setminus \{1\} = W_1 \coprod W_2$ breaks into two connected components, so

$$
j_!\underline{\mathbb{C}_U}(W) = \underline{\mathbb{C}_U}(W_1 \coprod W_2) = \mathbb{C}^{\oplus^2},
$$

so

$$
\mathcal{F}(W) = \mathbb{C}^{\oplus^2} \oplus \mathbb{C} \neq \mathbb{C} = \underline{\mathbb{C}_X}(W) \implies \mathcal{F} \not\cong \underline{\mathbb{C}_X}.
$$

■

Proposition 46.3.2 ***(Parts 2 and 3).***
If X is an irreducible algebraic variety or $X = [0, 1]$ in the Euclidean topology, the answer is still generally no.

Proof (Parts 2 and 3).
The previous example shows this, noting that $S^1 \cong \operatorname{Spec} \dfrac{\mathbb{R}[x, y]}{\langle x^2 + y^2 - 1 \rangle}$ and $f(x, y) = x^2 + y^2 - 1$ does not factor in $\mathbb{R}[x, y]$, making S^1 irreducible in the Zariski topology.
For $X = [0, 1]$, a modification of the previous example yields the same conclusion: set $Z = \left\{\frac{1}{2}\right\}$ and $U = X \setminus Z$; the same argument with the same sheaf goes through.

■

46.4 Problem 4

Problem 46.4.1 (1.4)
Let X be a space with a poset topology (with increasing open sets).

- Prove that a sheaf F on X is the same as a collection F_x for $x \in X$ and maps $r_{x,y} : F_x \to F_y$ for all $x \leq y$ satisfying $r_{y,z} \circ r_{x,y} = r_{x,z}$.
- Prove that for an open $U \subset X$, the set of sections $F(U)$ is $\varprojlim_{U \ni x} F_x$.

Proposition 46.4.1 *(?).*
Let $(X, \leq)$ be a poset with the order topology, then a sheaf $\mathcal{F} \in \mathsf{Sh}(X)$ on X is is equivalently a functorial assignment on the corresponding poset category

$$\begin{aligned} \mathcal{F} : \mathsf{Poset}(X) &\to \mathsf{C} \\ x &\mapsto \mathcal{F}_x, \end{aligned}$$

where the objects of $\mathsf{Poset}(X)$ are elements $x \in X$ where the hom spaces are defined as

$$\underset{\mathsf{Poset}(X)}{\mathrm{Hom}}(x, y) := \begin{cases} \{\mathrm{pt}\} & x \leq y \\ \emptyset & \text{else.} \end{cases}$$

Here $\mathsf{C} = \mathsf{AbGrp}, \mathsf{Ring}, \mathsf{R\text{-}Mod}, \mathsf{Alg}_{/k}$, etc.

Proof (?).
$\implies$: Suppose that $\mathcal{F}$ is a sheaf on $(X, \leq)$, and define

$$\begin{aligned} \mathcal{G} : \mathsf{Poset}(X) &\to \mathsf{C} \\ x &\mapsto \mathcal{G}_x := \mathcal{F}(U_{\geq x}) \\ (\iota_{xy} : x \to y) &\mapsto (f_{xy} : \mathcal{G}_x \to \mathcal{G}_y), \end{aligned}$$

where we define f_{xy} using that

$$x \to y \in \mathsf{Poset}(X) \iff x \leq y \in X \implies U_{\geq y} \hookrightarrow U_{\geq x} \in \mathsf{Open}(X),$$

and since $\mathcal{F}$ is a contravariant functor, the latter inclusion induces an morphism

$$f_{xy} : \mathcal{F}(U_{\geq x}) \to \mathcal{F}(U_{\geq y}) \qquad \in \mathsf{C}.$$

Compatibility of the f_{xy} for $\mathcal{G}$ follow immediately from the fact that $\mathcal{F}$ is a functor.
$\impliedby$: Given a functorial assignment

$$\begin{aligned} \mathcal{G} : \mathsf{Poset}(X) &\to \mathsf{C} \\ x &\mapsto \mathcal{G}_x, \end{aligned}$$

we want to construct an associated sheaf

$$\mathcal{F} : \mathsf{Open}(X)^{\mathrm{op}} \to \mathsf{C}.$$

By a result from class, it suffices to specify the sheaf on a basis $\mathcal{B}$ for the order topology on X, so let $\mathcal{B} := \{U_{\geq x}\}_{x \in X}$ be the basis of up-sets. Define a presheaf by

$$\mathcal{F}^{-}(U_{\geq x}) := \mathcal{G}_x,$$

and take $\mathcal{F} := (\mathcal{F}^{-})^{+}$.

■

Proposition 46.4.2 ***(?).***
For $(X, \leq)$ a poset in the order topology, $U \subseteq X$ open, and $\mathcal{F}$ a sheaf on X,

$$F(U) \cong \varprojlim_{x \in U} \mathcal{F}_x.$$

Proof (?).
It suffices to show that if $\mathcal{B}$ is a basis for a topology,

$$\mathcal{F}(U) = \varprojlim_{V \in \mathcal{B}, V \subseteq U} \mathcal{F}(V),$$

which follows because this precisely describes a continuous section of the *espace étalé* over $U \subseteq X$ as a compatible collection of sections on U decomposed in a basis as $U = \cup B_i$. With this, we can then directly compute

$$\begin{aligned}
\mathcal{F}(U) &= \varprojlim_{V \in \mathcal{B}, V \subseteq U} \mathcal{F}(V) \\
&= \varprojlim_{V_{\geq x} \subseteq U} \mathcal{F}(V_{\geq x}) && \text{by the definition of } \mathcal{B} \\
&= \varprojlim_{V_{\geq x} \subseteq U} \mathcal{F}_x && \text{since } V_{\geq x} \text{ is the smallest open containing } x \\
&= \varprojlim_{x \in U} \mathcal{F}_x && \text{since } V_{\geq x} \subseteq U \implies x \in U.
\end{aligned}$$

■

47 | Problem Set 2

47.1 Problem 1

Proposition 47.1.1 *(1.1).*
The global sections functor is left-exact.

Proof .
We'll use the fact that a sequence of sheaves is exact if and only if the induced sequence on stalks is exact. Given this, let $\mathcal{F}, \mathcal{G}, \mathcal{H}$ be sheaves of abelian groups on X, and consider the diagram induced by restriction morphisms:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{F} & \xrightarrow{f} & \mathcal{G} & \xrightarrow{g} & \mathcal{H} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \mathcal{F}(U) & \xrightarrow{f_U} & \mathcal{G}(U) & \xrightarrow{g_U} & \mathcal{H}(U) & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{F}_p & \xrightarrow{f_p} & \mathcal{G}_p & \xrightarrow{g_p} & \mathcal{H}_p & \longrightarrow & 0
\end{array}$$

Link to Diagram

Note that we can take $U = X$ in this diagram. If the top sequence of sheaves is exact, there are isomorphisms of sheaves:

- $\ker f = 0$ and
- $\operatorname{im} f = \ker g$.

Claim:

$$\ker f_X = 0,$$

making f_X injective and yielding exactness at the first position.

Proof (?).
Since the presheaf $\ker f$ is in fact a sheaf, writing $\mathbf{0}$ for the sheaf $U \mapsto 0$, we have

$$\ker \left(\mathcal{F}(X) \xrightarrow{f_X} \mathcal{G}(X) \right) = (\ker f)(X) = \mathbf{0}(X) = 0$$

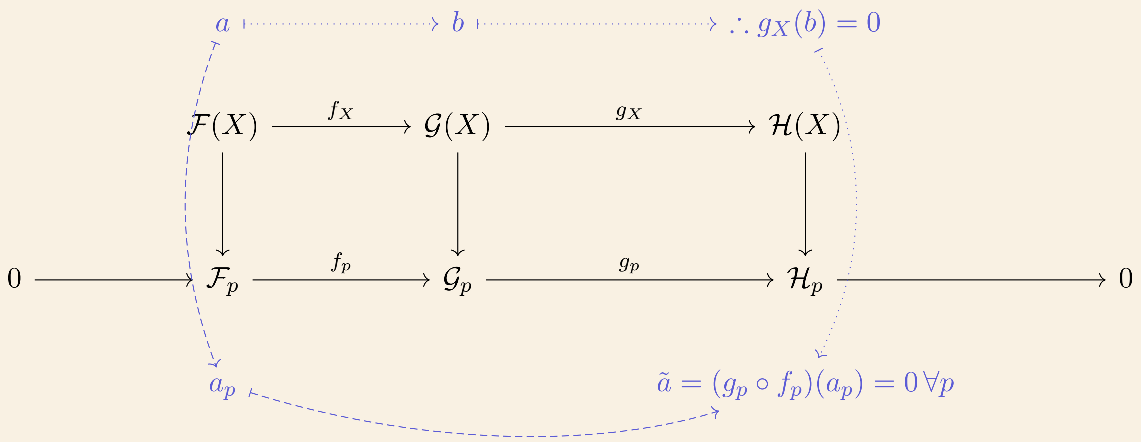
■

Claim:

$$\operatorname{im} f_X = \ker g_X,$$

yielding exactness at the middle position.

Proof (?).
im $f_X \subseteq \ker g_X$ follows from a diagram chase:



Link to Diagram

- Fix $b \in \operatorname{im} f_X \subseteq \mathcal{G}(X)$, then by surjectivity choose a lift $a \in \mathcal{F}(X)$.
- Map a along $\mathcal{F}(X) \to \mathcal{F}_p \to \mathcal{G}_p \to \mathcal{H}_p$; by exactness the result is zero in C_p.
- By commutativity of the diagram, mapping a along $\mathcal{F}(X) \to \mathcal{G}(X) \to \mathcal{H}(X) \to \mathcal{H}_p$ also yields zero in C_p.
- Write $\tilde{b} := g_X(b)$; since the above argument holds for all $p \in X$, $g_X(b)$ is a section of $\mathcal{H}$ that is zero in every stalk. Thus by the sheaf property for $\mathcal{H}$, the section $g_X(b)$ must be zero, and $b \in \ker g_X$.

Similarly, $\ker g_X \subseteq \operatorname{im} f_X$ follows from a diagram chase:

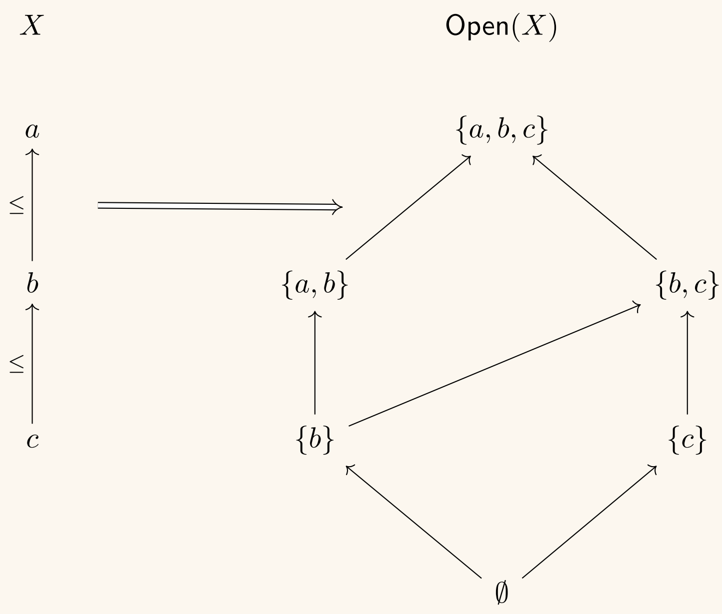
- Fix $b \in \ker g_X$, so the image of g_X in $\mathcal{H}(X)$ is zero. Then its image c_p along $\mathcal{G}(X) \to \mathcal{H}(X) \to \mathcal{H}_p$ is also zero.
- By commutativity of the right square, $g_p(b_p) = 0$ and so $b_p \in \ker g_p = \operatorname{im} f_p$ by exactness of the bottom row.
- Choose a lift $a_p \in \mathcal{F}_p$ along f_p, so $f_p(a_p) = b_p$ Since a_p is a germ of $\mathcal{F}$, pick any global section $a \in \mathcal{F}(X)$ restricting to a_p and making the square commute.
- Since $f_X(a)_p = f_p(a_p) = b_p$ for all p, by uniqueness of gluing for $\mathcal{G}$ we have $f_X(a) = b$ and $b \in \operatorname{im} f_X$.

■

■

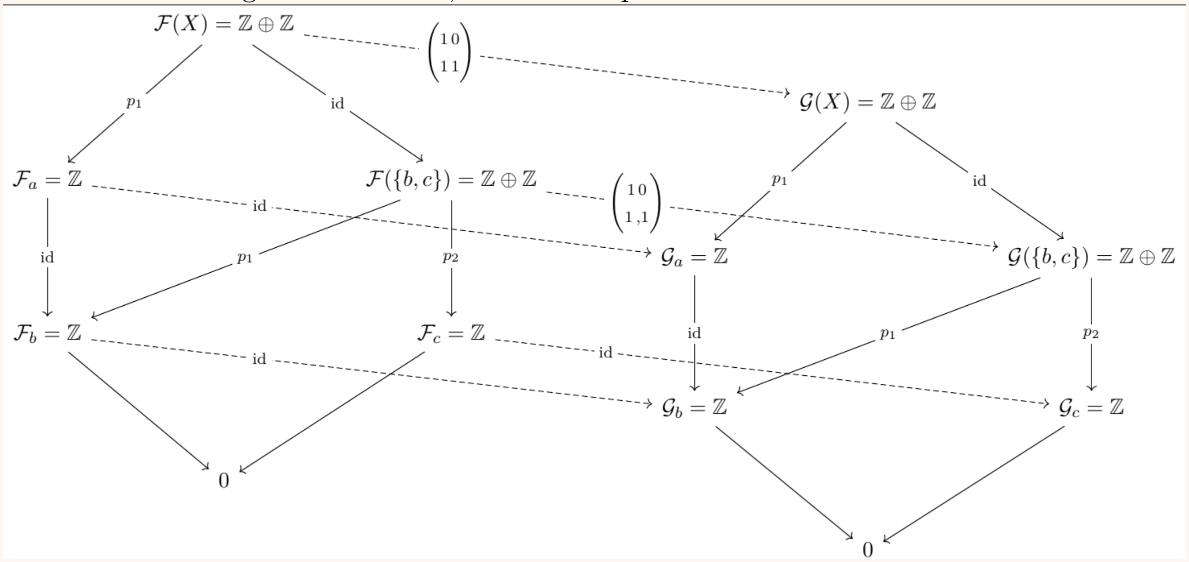
Proposition 47.1.2 ***(1.2).***
Taking global sections may fail to be right-exact.

Proof (?).
Consider the following poset and its corresponding category of open sets:



Link to Diagram

Define the following two sheaves $\mathcal{F}, \mathcal{G}$ and a morphism between them:



Link to diagram

Note that there are only three stalks to consider, none of which coincide with global sections, so we can take the sheaf morphism to be the identity on these to get a surjection on stalks. We then choose a non-surjective map $\mathcal{F}(X) \to \mathcal{G}(X)$ given by $(a, b) \mapsto (a, a + b)$, where e.g. the image does not contain the element $(1, 1)$.
One can check that the individual diagrams for $\mathcal{F}$ and $\mathcal{G}$ commute, yielding a presheaf, and that existence and uniqueness of gluing hold for both. Moreover, all of the squares formed by the map $\mathcal{F} \to \mathcal{G}$ commute, so this does in fact yield a morphism of sheaves.

■

47.2 Problem 2

Proposition 47.2.1 *(?).*
If a map $f : X \to Y$ between posets is continuous, it is order-preserving, i.e. if $x_1 \leq x_2$ then $f(x_1) \leq f(x_2)$.

Proof (?).
Continuity can be checked on a basis, so let $U_b = \left\{ y \in Y \;\middle|\; y \geq b \right\}$ be a basic open upper set. Then f is continuous iff $f^{-1}(U_a)$ is an open set in X. Being open means that for every $x_0 \in f^{-1}(U_a)$, $x_1 \geq x_0 \implies x_1 \in f^{-1}(U_a)$.

$$\begin{aligned}
f \text{ is continuous } &\iff \forall U \text{ open in } Y,\ f^{-1}(U) \text{ is open in } X \\
&\iff \forall U_a \text{ a basic open in } Y,\ f^{-1}(U_a) \text{ is open in } X \\
&\iff \forall a \in Y,\ \forall x_0 \in f^{-1}(U_a),\ x_1 \geq x_0 \implies x_1 \in f^{-1}(U_a) \\
&\iff \forall a \in Y,\ \forall x_0 \in f^{-1}(U_a),\ x_1 \geq x_0 \implies f(x_1) \in U_a \\
&\iff \forall a \in Y,\ \forall x_0 \in f^{-1}(U_a),\ x_1 \geq x_0 \implies f(x_1) \geq a \\
&\iff \forall a \in Y,\ \forall x_0 \in X \text{ s.t. } f(x_0) \geq a,\ x_1 \geq x_0 \implies f(x_1) \geq a.
\end{aligned}$$

Now taking $x_0 = f^{-1}(a)$ for $a \in \operatorname{im} f$ yields

$$\implies \forall a \in \operatorname{im} f, \quad x_1 \geq f^{-1}(a) \implies f(x_1) \geq a.$$

Relabeling $x_1 = f^{-1}(b)$,

$$\begin{aligned}
&\implies \forall a \in \operatorname{im} f, \quad f^{-1}(b) \geq f^{-1}(a) \implies b \geq a \\
&\implies \forall \tilde{a} \in f^{-1}(Y), \quad \tilde{b} \geq \tilde{a} \implies f(\tilde{b}) \geq f(\tilde{a}).
\end{aligned}$$

■

Proposition 47.2.2 *(?).*
For $\mathcal{F} \in \mathsf{Sh}_X, \mathcal{G} \in \mathsf{Sh}_Y, \mathcal{H} \in \mathsf{Sh}_U$ with $U \subseteq X$, $X \xrightarrow{f} Y$, and $U \xhookrightarrow{j} X$,

- $f_* \mathcal{F}$ is no additional data
- $f^{-1}\mathcal{G} = \left\{ \mathcal{G}_{f(x_0)}, \varphi_{f(x_0), f(x_1)} \;\middle|\; x_0, x_1 \in X, x_0 \leq x_1 \right\}$.
- $j_! \mathcal{H} = ?$

Proof .
We'll use that $\mathcal{F} \in \mathsf{Sh}(X, \mathsf{AbGrp})$ is the same as the data of $\{\mathcal{F}_x, \varphi_{xy}\}$ where $\mathcal{F}_x$ is a collection of groups and $\varphi_{xy} : \mathcal{F}_x \to \mathcal{F}_y$ are group morphisms for every $x \leq y$. Thus the values of a sheaf on posets are entirely determined by a functorial assignment of groups to the stalk at each point, i.e. an assignment of a group to each point. So it suffices to determine what the stalks

of these three sheaves are.

- For $f_*\mathcal{F}$, noting that

$$(f_*\mathcal{F})(U_{\geq a}) = \mathcal{F}(f^{-1}(U_{\geq a})) = \varprojlim_{x \in f^{-1}(U_{\geq a})} \mathcal{F}_x,$$

  we see that this sheaf is completely determined by the data for $\mathcal{F}$.

- For $f^{-1}\mathcal{G}$, we can use the fact that for any sheaf, there is a formula on stalks:

$$(f^{-1}\mathcal{G})_p \cong \mathcal{G}_{f(p)},$$

  and so $f^{-1}\mathcal{G}$ is the data $\{\mathcal{G}_x, \psi_{xy}\}$ for every $x \leq y$ with $x, y \in \operatorname{im} f$.

- For $j_!\mathcal{H}, \cdots$?

Attempts to approach this: the general definition involves sheafification, which seems hard to describe in general. On the other hand, I haven't been able to work out what the sheaf space for a poset should look like.

■

47.3 Problem 3

Proposition 47.3.1 *(?).*
Let $\mathcal{F} \in \mathsf{Sh}(X)$ and let $\text{Ét}(\mathcal{F}) \xrightarrow{\pi} X$ be its corresponding sheaf space, so $\mathcal{F} = \underset{\text{cts}}{\operatorname{Sec}}(\pi)$, and let $\mathcal{G} = \operatorname{Sec}(\pi)$. Then

$$\mathcal{G} = \prod_{x \in X} x_* \mathcal{F}_x$$

where $x : \{x\} \hookrightarrow X$ is the inclusion of a point and $\mathcal{F} \in \mathsf{Sh}(\{x\})$ is regarded as a sheaf on a one-point space.

Proof .
We'll use the fact that as a set, $\text{Ét}(\mathcal{F}) = \coprod_{x \in X} \mathcal{F}_x$ is the coproduct of all of the stalks of $\mathcal{F}$. We

can compute the sections of this sheaf as follows:

$$\begin{aligned}
\mathcal{G}(U) &= \left(\prod_{x\in X} x_*\mathcal{F}_x\right)(U) \\
&= \prod_{x\in X} (x_*\mathcal{F}_x)(U) \\
&= \prod_{x\in X} \mathcal{F}_x(x^{-1}(U)) \\
&= \prod_{x\in X} \mathcal{F}_x\left(\begin{cases}\{x\} & x\in U \\ \emptyset & x\notin U.\end{cases}\right) \\
&= \prod_{x\in X} \begin{cases}\mathcal{F}_x & x\in U \\ 0 & x\notin U.\end{cases} \\
&= \prod_{x\in U} \mathcal{F}_x.
\end{aligned}$$

We can now simply regard $\mathcal{G}(U)$ as the set of set-valued functions $s : U \to \prod_{x\in U} \mathcal{F}_x \subseteq \text{Ét}(\mathcal{F})$ by setting $s(x) = \pi_x\left(\prod_{x\in U} \mathcal{F}_x\right)$ to be the x-coordinate in the direct product, where $\pi_x : \prod_{x\in U} \mathcal{F}_x \to \mathcal{F}_x$ is projection onto the x-coordinate.
On the other hand, the data of a set-valued section $s \in \text{Sec}(\text{Ét}(\mathcal{F}) \xrightarrow{\pi} U)$ is the following: for every $x \in X$, a choice of an element

$$s(x) \in \pi^{-1}(x) = \mathcal{F}_x \subseteq \text{Ét}(\mathcal{F}),$$

with no other compatibility conditions, which is precisely the same as the set-valued functions specified by $\mathcal{G}(U)$ above.

■

Proposition 47.3.2*(?)*.
The stalks $\mathcal{G}_p$ are given by

$$\mathcal{G}_p = \underset{U\ni p}{\underrightarrow{\text{colim}}} \prod_{x\in U} \mathcal{F}_x,$$

the direct limit of the product of stalks of $\mathcal{F}$ along neighborhoods of p.

Proof .

$$\begin{aligned}
\mathcal{G}_p &:= \varinjlim_{U \ni p} \mathcal{G}(U) \\
&:= \varinjlim_{U \ni p} \left(\prod_{x \in X} (\iota_x)_* \mathcal{F}_x \right)(U) \\
&= \varinjlim_{U \ni p} \prod_{x \in X} \left((\iota_x)_* \mathcal{F}_x \right)(U) \\
&:= \varinjlim_{U \ni p} \prod_{x \in X} \mathcal{F}_x(\iota_x^{-1}(U)) \\
&= \varinjlim_{U \ni p} \prod_{x \in X} \mathcal{F}_x \left(\begin{cases} \{x\} & x \in U \\ \emptyset & \text{else.} \end{cases} \right) \\
&= \varinjlim_{U \ni p} \prod_{x \in X} \left(\begin{cases} \mathcal{F}_x & x \in U \\ 0 & \text{else.} \end{cases} \right) \\
&= \varinjlim_{U \ni p} \prod_{x \in U} \mathcal{F}_x .
\end{aligned}$$

■

Proposition 47.3.3 *(?).*
There is an injective morphism of sheaves $\mathcal{F} \hookrightarrow \mathcal{G}$.

Proof .
For every open $U \subseteq X$, define a map of sets on the function spaces:

$$\Psi_U : \underset{\text{cts}}{\operatorname{Sec}}(\text{Ét}(\mathcal{F}) \xrightarrow{\pi} U) \to \operatorname{Sec}(\text{Ét}(\mathcal{F}) \xrightarrow{\pi} U)$$
$$f \mapsto f,$$

which does nothing more than a forgetful map that regards a continuous section as a set-valued section. This is evidently an injective map of sets, since if f_1, f_2 are continuous sections and $f_1 = f_2$ as set-valued functions, they continue to be equal when regarded as continuous sections, so $\Psi_U(f_1) = \Psi_U(f_2) \implies f_1 = f_2$.
These Ψ_U assemble to a morphism of sheaves $\Psi : \mathcal{F} \to \mathcal{G}$, and since $(\ker \Psi)^- = \mathbf{0}$ vanishes as a presheaf and the kernel presheaf is a sheaf, we have $\ker \Psi = \mathbf{0}$.

■

48 | Problem Set 3

48.1 Problem 1

Problem 48.1.1 (Problem 1)
Let I be an index category, $\mathcal{A}$ an abelian category, and $\mathcal{A}^I$ be the category of functors $F : I \to \mathcal{A}$. Prove that the functor

$$\varprojlim_{i\in I} : \mathcal{A}^I \to \mathcal{A}, \quad F \mapsto \varprojlim_{i\in I} F_i$$

is left exact. (By duality, the functor $\varinjlim_{i\in I}$ is right exact.)
What is this functor in the case when I is a poset and F_i is a collection of stalks on the space $X = I$ with poset topology?

Solution (Part 1):
It suffices to show that $\varprojlim_{i\in I}$ is a right adjoint functor, and right adjoints are left exact by general homological algebra.

Claim: There is an adjunction

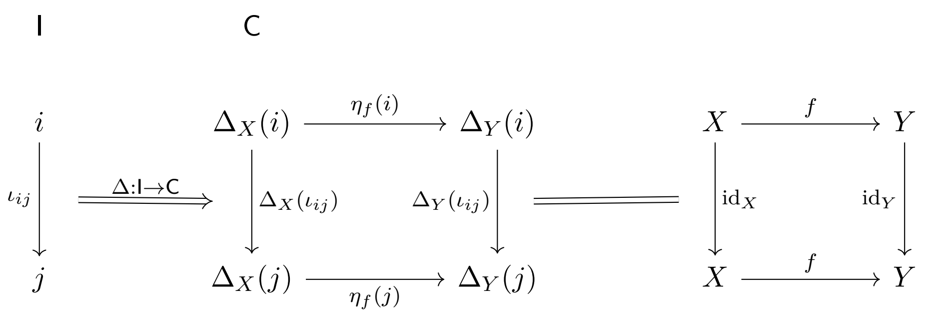
$$\mathsf{A} \underset{\varprojlim_{i\in I}}{\overset{\Delta}{\rightleftarrows}} \mathsf{A}^{\mathsf{I}},$$

where Δ is the diagonal functor:

$$\begin{aligned} \Delta : \mathsf{A} &\to \mathsf{A}^{\mathsf{I}} \\ X &\mapsto \Delta_X \\ (X \xrightarrow{f} Y) &\mapsto (\Delta_X \xrightarrow{\eta_f} \Delta_Y) \end{aligned}$$

where

- The constant functor $\Delta_X : \mathsf{I} \to \mathsf{C}$ is defined on objects $i \in \mathsf{I}$ as $\Delta_X(i) \coloneqq X$ and on morphisms $i \xrightarrow{\iota_{ij}} j$ as $\Delta_f(\iota_{ij}) = X \xrightarrow{\mathrm{id}_X} X$.
- η_f is a natural transformation of functors with components given by f:

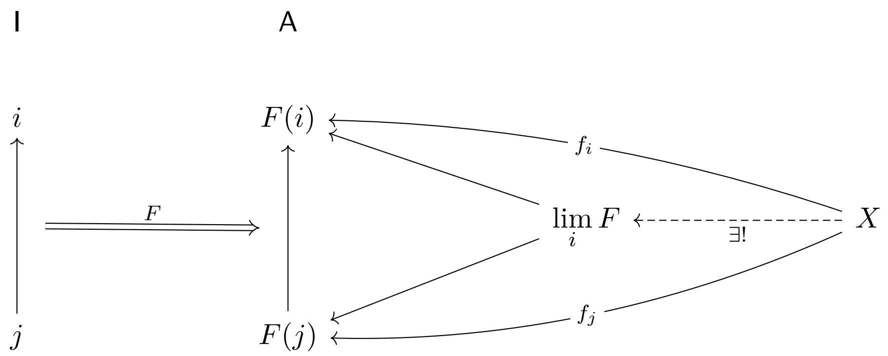


Link to Diagram

Why this claim is true: this follows immediately from the fact that there is a natural isomorphism

$$\operatorname*{Hom}_{\mathsf{A}}(X, \lim F) \xrightarrow{\sim} \operatorname*{Hom}_{\mathsf{A}^{\mathsf{I}}}(\Delta_X, F),$$

i.e. maps from an object X into the limit of F are equivalent to natural transformations between the constant functor Δ_X and F. This follows from the fact that a morphism $X \to \lim F$ in A is the data of a family of compatible maps $\{f_i\}_{i \in \mathsf{I}}$ over the essential image of F:



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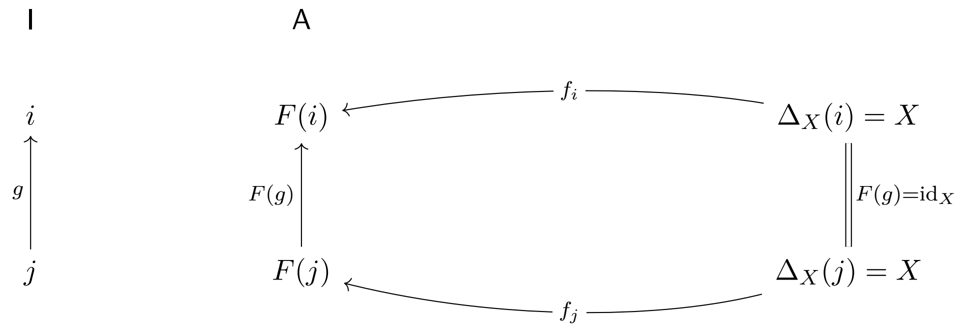
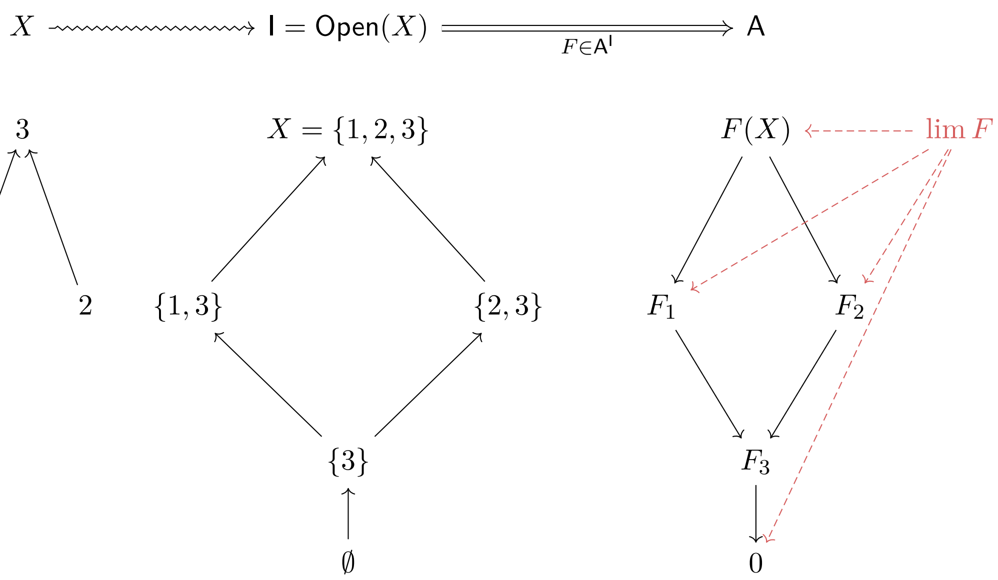
On the other hand, a natural transformation $\Delta_X \to F$ is precisely the same data:



Link to Diagram

Solution (Part 2):
If $\mathsf{I} = \mathsf{Open}(X)$ where X is given the order topology and $F : \mathsf{Open}X \to \mathsf{A}$ is a functor specified by stalks, lim sends F to the universal object $\lim F$ living over the essential image of F in A:



Link to Diagram

The object corresponding to global sections $F(X) \in \mathsf{A}$ seems to also satisfies this universal property, so a conjecture would be that this construction recovers $\lim F \cong F(X) := \Gamma(X; F)$.

48.2 Problem 2

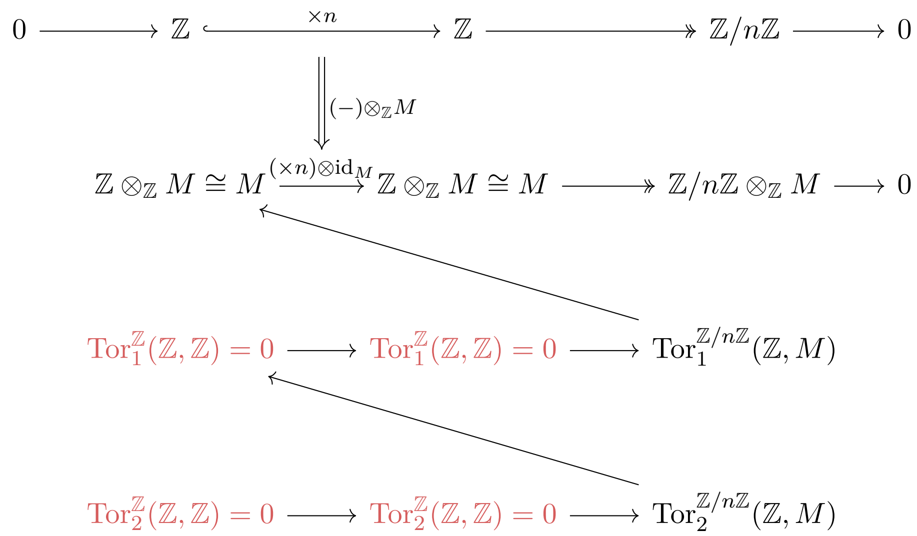
Problem 48.2.1 (Problem 2)
In the category of abelian groups compute $\mathrm{Tor}_i^{\mathbb{Z}}(\mathbb{Z}_n, M)$, the left derived functors of $N \mapsto N \otimes_{\mathbb{Z}} M$.

Solution:

Claim:

$$\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, M) \cong \ker(M \xrightarrow{\times n} M) \cong \left\{ m \in M \;\middle|\; nm = 0_M \right\},$$

which is the kernel of multiplication by n, and $\mathrm{Tor}_{\mathbb{Z}}^{i>1}(\mathbb{Z}/n\mathbb{Z}, M) = 0$.
Why this is true: in $\mathsf{R}\text{-}\mathsf{Mod}$, free implies flat, and Tor is balanced and can thus be resolved in either variable, so this can be computed by tensoring a free resolution of $\mathbb{Z}/n\mathbb{Z}$ and using the long exact sequence in Tor:



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In the resulting long exact sequence, since $\mathbb{Z}$ is free, thus flat, thus tor-acyclic, the first two columns vanish in degrees $d \geq 1$. As a result, in degrees $d \geq 2$, the terms $\operatorname{Tor}_d^{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, M)$ are surrounded by zeros and thus zero, meaning that only Tor_1 survives. By exactness, $\operatorname{Tor}_1(\mathbb{Z}/n\mathbb{Z}, M)$ is isomorphic to the kernel of the next map in the sequence, which is precisely $\ker(M \xrightarrow{\times n} M)$ after applying the canonical isomorphism

$$\begin{aligned} \mathbb{Z} \otimes_{\mathbb{Z}} M &\to M \\ n \otimes m &\mapsto nm. \end{aligned}$$

48.3 Problem 3

Problem 48.3.1 (Problem 3)
Let k be a field and $R = k[x, y]$. In the category of R-modules compute

- $\operatorname{Ext}_R^n(R, m)$
- $\operatorname{Ext}_R^n(m, R)$, and
- $\operatorname{Tor}_n^R(m, m)$,

where $m = (x, y)$ is the maximal ideal at the origin.

Solution (Problem 3):
Note that R is a free R-module, and so $\operatorname{Ext}_R^n(R, M) = 0$ for any R-module M. This is because Ext can be computed using a free resolution of either variable. For $\operatorname{Ext}_R^n(R, m)$, compute this

as $\mathbb{R}\underset{R}{\operatorname{Hom}}(-,m)$ evaluated at R. Take the free resolution

$$\cdots \to 0 \to R \xrightarrow{\mathrm{id}_R} R \to 0,$$

delete the augmentation and apply the contravariant $\underset{R}{\operatorname{Hom}}(-,m)$ to obtain

$$0 \to \underset{R}{\operatorname{Hom}}(R,m) \cong m \to 0 \to \cdots,$$

and take homology to obtain

$$\operatorname{Ext}^0_R(R,m) \cong m, \qquad \operatorname{Ext}^{>0}_R(R,m) = 0.$$

Compute $\operatorname{Ext}_R(m,R)$ as $\mathbb{R}\operatorname{Hom}(m,-)$ applied to R proceeds similarly: using the same resolution, applying covariant $\underset{R}{\operatorname{Hom}}(m,-)$ yields

$$0 \to \underset{R}{\operatorname{Hom}}(m,R) \to 0 \to \cdots,$$

and taking homology yields

$$\operatorname{Ext}^0_R(m,R) \cong \underset{R}{\operatorname{Hom}}(m,R) \qquad \operatorname{Ext}^{>0}_R(m,R) = 0.$$

For the Tor calculation, we can use the Koszul resolution of m:

$$0 \to k[x,y] \xrightarrow{\cdot[x,y]} k[x,y] \oplus k[x,y] \xrightarrow{\cdot\, {}^t([-y,x])} \langle x,y \rangle \to 0,$$

so the differentials are $t \mapsto [tx,ty]$ and $[u,v] \mapsto -uy+vx$ respectively. More succinctly, this resolution is

$$0 \to R \xrightarrow{d_1} R^{\oplus^2} \xrightarrow{d_2} m \to 0,$$

so we can delete m and apply $(-) \otimes_R m$ to obtain

$$0 \to R \otimes_R m \xrightarrow{d_1 \otimes \mathrm{id}_m} R^{\oplus^2} \otimes_R m \to 0$$

which simplifies to

$$C_\bullet \coloneqq 0 \to m \xrightarrow{\tilde{d}_1 \coloneqq [x,y]} m \oplus m \to 0$$

and thus we can compute Tor as the homology of this complex. We have

$$\begin{aligned}
\operatorname{Tor}_0^R(m,m) &= H^0(C_\bullet) \\
&= \operatorname{coker} \tilde{d}_1 \\
&= \frac{m \oplus m}{xm \oplus ym} \\
&\cong \frac{m}{xm} \oplus \frac{m}{ym} \\
&= \frac{\langle x,y\rangle}{\langle x^2,y\rangle} \oplus \frac{\langle x,y\rangle}{\langle x,y^2\rangle} \\
&= \left\{ f(x,y) := c_1 x \in k[x,y] \;\middle|\; c_1 \in k \right\} \oplus \left\{ g(x,y) := c_1 y \in k[x,y] \;\middle|\; c_1 \in k \right\} \\
&\cong k \oplus k
\end{aligned}$$

$$\begin{aligned}
\operatorname{Tor}_1^R(m,m) &= H^1(C_\bullet) \\
&= \ker \tilde{d}_1 \\
&= \left\{ t \in \langle x,y\rangle \;\middle|\; [tx,ty] = [0,0] \right\} \\
&= 0
\end{aligned}$$

$$\begin{aligned}
\operatorname{Tor} \geq 2^R(m,m) &= H^{\geq 2}(C_\bullet) \\
&= 0.
\end{aligned}$$

48.4 Problem 4

Problem 48.4.1 (Problem 4)
Let $0 \to F' \to F \to F'' \to 0$ be a short exact triple of sheaves and assume that F' is flasque. Prove that the sequence

$$0 \to \Gamma\left(F'\right) \to \Gamma(F) \to \Gamma\left(F''\right) \to 0$$

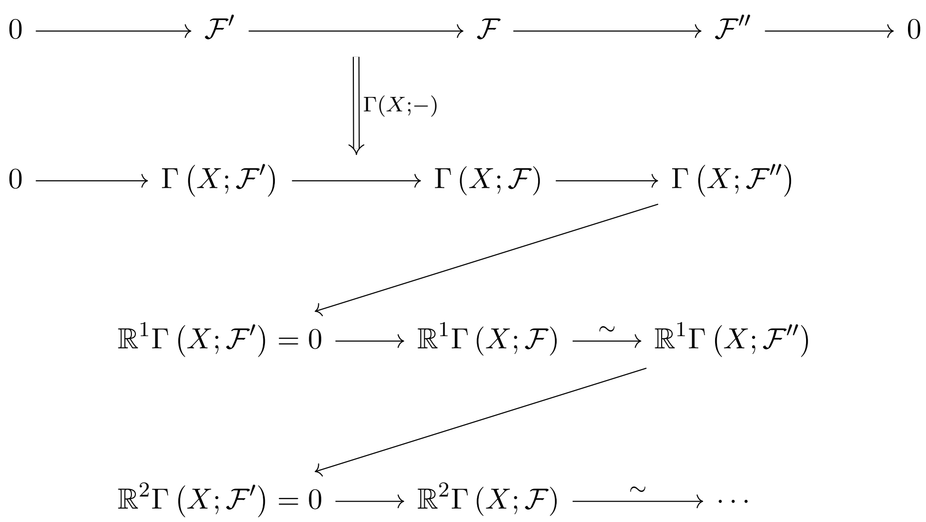
of the spaces of global sections is exact.

Solution (Using cohomology):

Claim: Flasque sheaves are F-acyclic for the functor global sections functor $F(-) := \Gamma(X; -)$.

Proof (of claim).
Proved in class. ∎

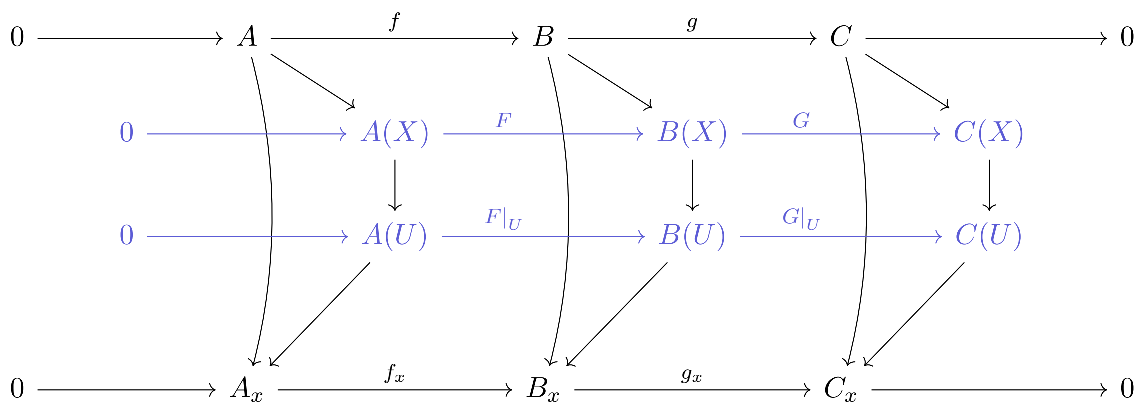
Applying the functor $\Gamma(X; -)$ to the given short exact sequence of sheaves produces a long exact sequence of abelian groups in its right-derived functors. Using the claim above, we have $\mathbb{R}^i\Gamma(X; \mathcal{F}') = 0$ for $i \geq 1$, and thus we have the following:



Link to Diagram

In particular, since $\mathbb{R}^1\Gamma(X; \mathcal{F}') = 0$, the first row forms the desired short exact sequence. As a corollary, we also obtain $\mathbb{R}^i\Gamma(X; \mathcal{F}) \cong \mathbb{R}^i\Gamma(X; \mathcal{F}'')$ for all $i \geq 1$.

Solution (Direct):
First, we'll modify the notation slightly and give names to the maps involved. We'll use the following convention for restrictions of sheaf morphisms to opens and stalks:



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Given $c \in C(X)$, our goal is to produce a $b \in B(X)$ such that $g(b) = c$, and the strategy will be to use surjectivity at stalks to produce a maximal section of B mapping to c, and argue that it must be a section over all of X. This will proceed by showing that if a lift is not maximal, sections over open sets that are missed can be extended using that A is flasque, contradicting maximality.

Write $c|_x$ for the image of c in the stalk C_x; by surjectivity of $g_x : B_x \twoheadrightarrow C_x$ we can find a germ b_x with $g_x(b_x) = c_x$. The germ lifts to some set $U \ni x$ and some $b \in B(U)$ with $b \mapsto c|_U$ under $F|_U : B(U) \to C(U)$. So define a poset of all such lifts:

$$P := \left\{ (U, b \in B(U)) \;\middle|\; F|_U(b) = c|_U \right\}$$
$$\text{where } (U_1, b_1) \le (U_2, b_2) \iff U_1 \subseteq U_2 \text{ and} b_2|_{U_1} = b_1.$$

As noted above, P is nonempty, and every chain $\{(U_i, b_i)\}_{i \in I}$ has an upper bound given by $(\tilde{U}, \tilde{b})$ where $\tilde{U} := \cup_{i \in I} U_i$ and $\tilde{b}$ is the unique glued section of B restricting to all of the b_i, which exists by the sheaf property for B. Thus Zorn's lemma applies, and (reusing notation) we can assume (U, b) is maximal with respect to this property.
The claim is that U must be all of X. Toward a contradiction, suppose not – then pick any $x \in X \setminus U$, and again using surjectivity on stalks at x, produce an open set $V \ni x$ and a section $b' \in B(V)$ with $G|_V(b') = c|_V$. Now on the overlap $W := U \cap V$, both b and b' map to $c|_W$, and so

$$G|_W(b|_W - b'|_W) = c|_W c|_W = 0 \implies b - b' \in \ker G|_W = \operatorname{im} F|_W,$$

where we've used exactness in the middle spot in the exact sequence $A(W) \to B(W) \to C(W)$. So there is some $\alpha \in A(W)$ with $F|_W(\alpha) = b|_W - b'|_W$, and since A is flasque this can be extended to a global section $\tilde{\alpha} \in A(X)$. Write $\tilde{\beta} := F(\tilde{\alpha}) \in B(X)$ with $\tilde{\beta}\Big|_W = b|_W - b'|_W$ in $B(W)$. We can now glue $\tilde{\beta}$ to a section over $U \cup V$ which extends the original section b: setting $\widehat{b} := \tilde{\beta} + b'$ yields

$$\widehat{b}\Big|_W = (b|_W - b'|_W) + b' = b|_W,$$

so this section over $U \cup V$ agrees with b on the overlap $W = U \cap V$, and thus by existence and uniqueness of gluing (using the sheaf property of B) $\widehat{b} \in B(U \cup V)$ is a section extending b over a set that strictly contains U. This contradicts the maximality of the pair (U, b).

48.5 Problem 5

Problem 48.5.1 (Problem 5)
For a sheaf F on X, let

$$S(F) = \prod_{x \in X} (i_x)_* F_x, \quad i_x : x \to X$$

be the sheaf of all, possibly discontinuous section of the étale space of F. The canonical flasque resolution of F is

$$\underline{S}(F) := 0 \to F \to S(F_0) \to S(F_1) \to S(F_2) \to \dots$$

where $F_0 = F$ and F_i are defined inductively as $F_{i+1} = S(F_i)/F_i$. Some books define cohomology groups $\mathbf{H}^n(X, F)$ as the cohomology groups of the complex

$$0 \to \Gamma(S(F_0)) \to \Gamma(S(F_1)) \to \Gamma(S(F_2)) \to \dots$$

Prove that they coincide with the cohomology defined by other means by showing that this gives an exact δ-functor and that $\mathbf{H}^n$ are effaceable for $n > 0$ through the following steps:

(1) A homomorphism $F \to G$ induces a canonical homomorphism of resolutions $\underline{S}(F) \to \underline{S}(G)$.

(2) A short exact triple $0 \to F' \to F \to F'' \to 0$ induces a short exact triple of complexes $0 \to \underline{S}(F') \to \underline{S}(F) \to \underline{S}(F'') \to 0$.

(3) Applying Γ to it gives a short exact triple of complexes, i.e. $0 \to S(F_n') \to S(F_n) \to S(F_n'') \to 0$ is exact. (You can assume the previous problem.)

(4) $(\mathbf{H}^n)$ is an exact δ-functor.

(5) For $n > 0, \mathbf{H}^n(F) \to \mathbf{H}^n(S(F))$ is the zero map.

Conclude by Grothendieck's universality theorem.

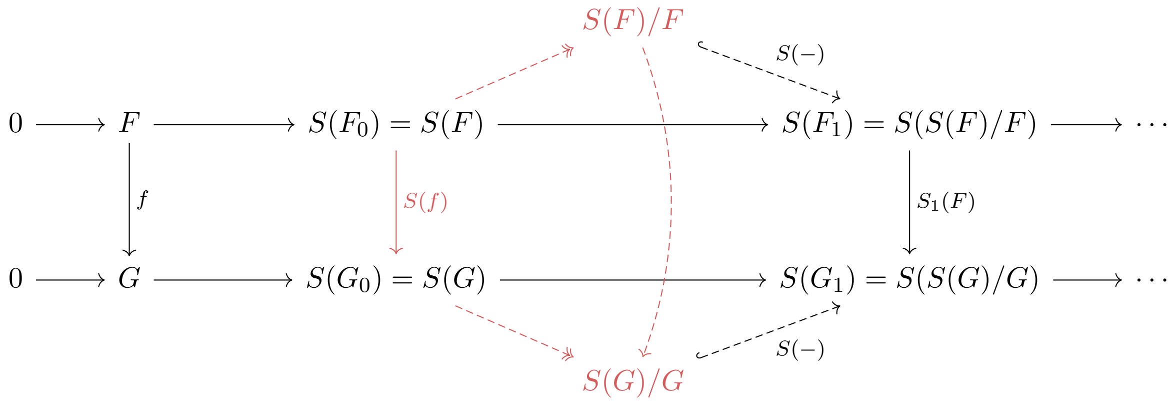
Solution (Part 1):
This follows readily from the fact that a morphism $f : F \to G$ of sheaves on X induces group morphisms $f_x : F_x \to G_x$ on stalks for every $x \in X$. Letting $y \in X$ be arbitrary, there is a morphism

$$\varphi_y : \prod_{x \in X} F_x \xrightarrow{\pi_y} F_y \xrightarrow{f_y} G_y$$

where π_y is the canonical projection out of the product. By the universal property of the product, the φ_y assemble to a morphism

$$S(f) : \prod_{x \in X} F_x \to \prod_{y \in X} G_y.$$

So there is a morphism $S(F_0) \to S(G_0)$ at the first stage of the complex. This induces a morphism on the quotient sheaves $S(F_0)/F_0 \to S(G_0)/G_0$, and thus by the same argument as above, a morphism on the second stage $S(S(F_0)/F_0) \to S(S(G_0)/G_0)$, i.e. a morphism $S(F_1) \to S(G_1)$. Continuing inductively yields levelwise morphisms $S(F_i) \to S(G_i)$. The claim is that these assemble to a chain map



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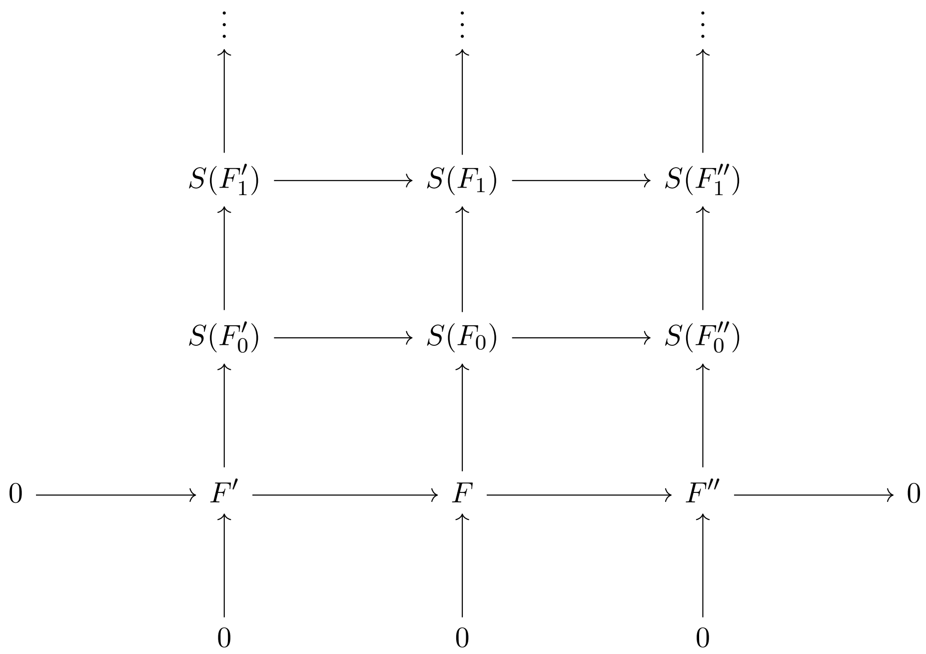
To see this is true, it is enough to show that the first square commutes, i.e. that applying $S(-)$ to a morphism of sheaves produces a commuting square. This is because every other square has a factorization as indicated, where the square in red naturally commutes since it involves canonically induced maps on quotients/cokernels, and the other half of the square arises by applying the S construction to some morphism of sheaves.

However, this square can be readily seen to commute using the following: first regard the sections of $\mathcal{F}$ as continuous sections of its espace étale $\text{Ét}_F \xrightarrow{\pi} X$ and regarding sections of $S(F)$ as arbitrary (potentially discontinuous) sections of π. Then $\mathcal{F} \leq S(F)$ is clearly a subsheaf and $F \to S(F)$ is an inclusion of spaces of sections.

Solution (Part 2):
By part 1, it is clear there are morphisms $\underline{S}(F') \to \underline{S}(F) \to \underline{S}(F'')$ of complexes of sheaves, yielding a double complex:



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It suffices to show injectivity, exactness, and surjectivity respectively along each horizontal row. Exactness is a local condition, so it suffices to show exactness on stalks.

Claim: For any open U, the following sequence at the first stage of the complex is exact:

$$0 \to S(F')(U) \to S(F)(U) \to S(F'')(U) \to 0.$$

Proof (of claim).
This follows because $S(F')(U) = \prod_{x \in U} F'_x$ and similarly for F, F'', and so if $f : F' \to F$ is injective on sheaves, then $f_x : F'_x \to F_x$ is injective on stalks. ∎

Now apply the functor $\underset{\overrightarrow{U \ni p}}{\text{colim}}(-)$ to this exact sequence and use that taking stalks is exact (despite not generally being a *filtered* colimit) to conclude

$$0 \to S(F')_x \to S(F)_x \to S(F'')_x \to 0.$$

is exact for all $x \in X$, thus making the following sequence exact:

$$0 \to S(F'_0) \to S(F_0) \to S(F''_0) \to 0$$

Our double complex is now the following:

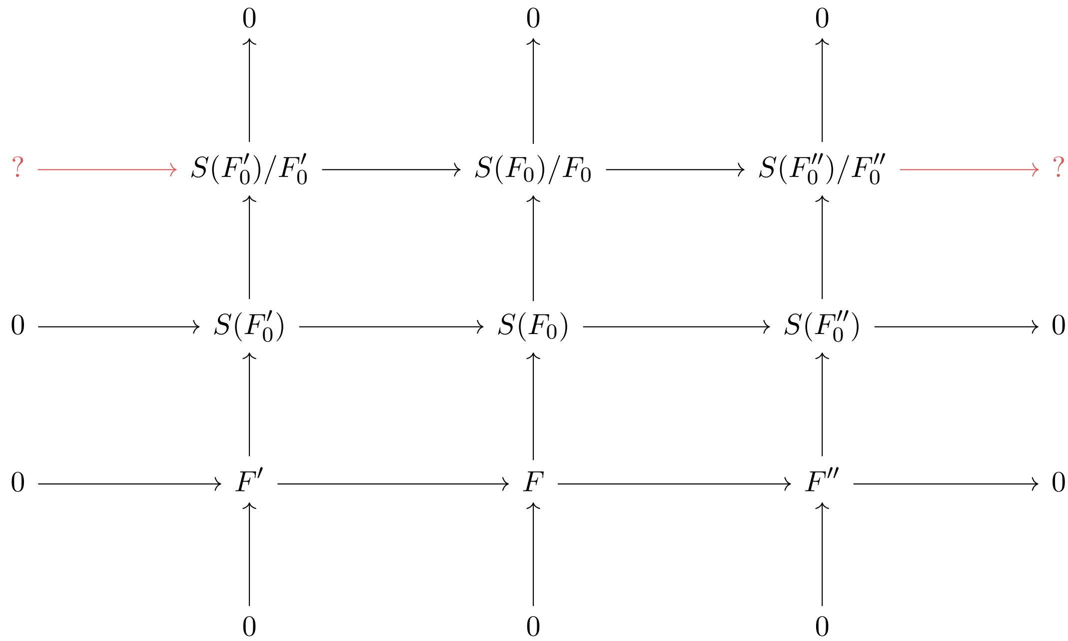
$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
? & \longrightarrow & S(F'_1) & \longrightarrow & S(F_1) & \longrightarrow & S(F''_1) & \longrightarrow & ? \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & S(F'_0) & \longrightarrow & S(F_0) & \longrightarrow & S(F''_0) & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & F' & \longrightarrow & F & \longrightarrow & F'' & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Link to Diagram

To see that

$$0 \to S(F'_k) \to S(F_k) \to S(F''_k) \to 0$$

is exact for all k, we can truncate this complex:



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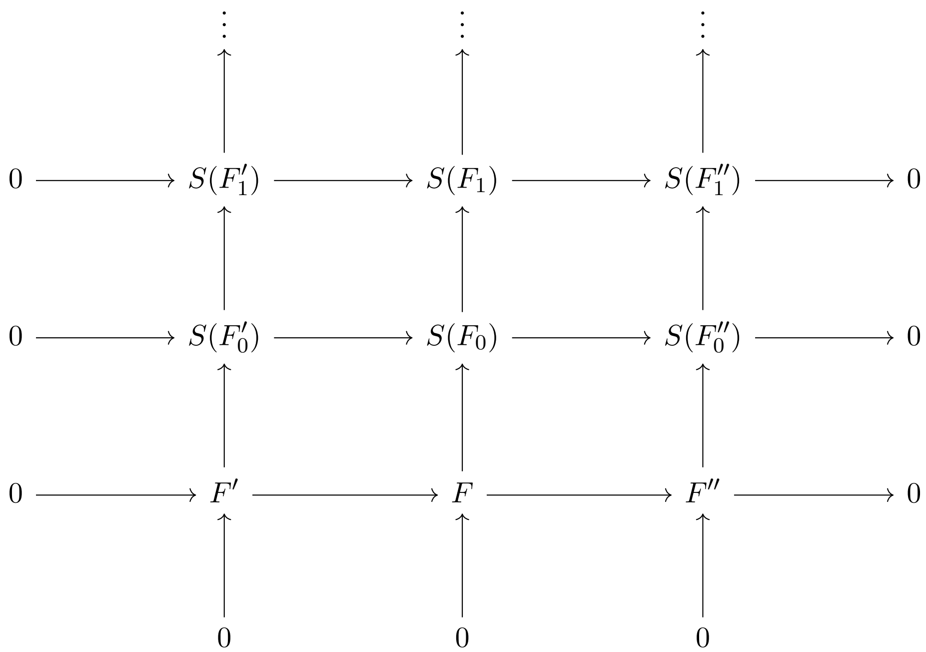
The row highlighted in red is exact by the Nine Lemma, regarding each row as a chain complex, and since applying $S(-)$ is exact, by applying this to the top row we obtain



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The remaining rows are exact by repeating this argument inductively, and regarding the columns as complexes, we obtain the desired exact sequences of complexes by deleting the first row.

Solution (Part 3):
Note: there may be a typo in the statement of this problem, so what I will show is that the following sequence of complexes is exact:

$$0 \to \Gamma\left(X; \underline{S}(F')\right) \to \Gamma\left(X; \underline{S}(F)\right) \to \Gamma\left(X; \underline{S}(F'')\right) \to 0.$$

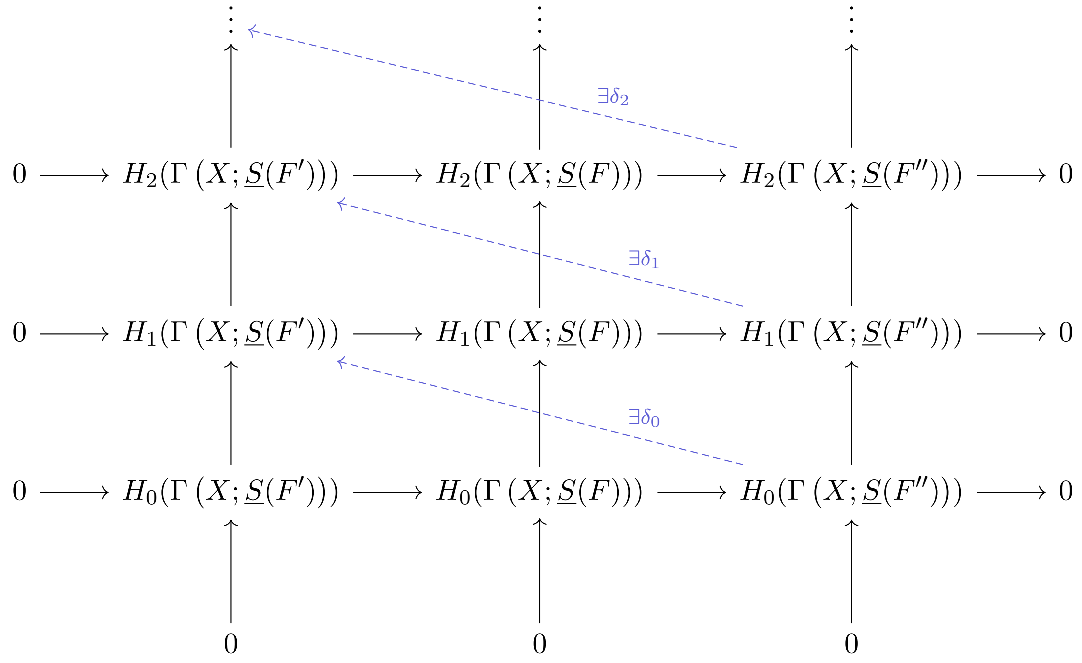
Take the double complex from part (2) and apply the functor $\Gamma(X; -)$ to obtain the following double complex:

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \Gamma\left(X; S(F_1')\right) & \longrightarrow & \Gamma\left(X; S(F_0)\right) & \longrightarrow & \Gamma\left(X; S(F_1'')\right) & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \Gamma\left(X; S(F_0')\right) & \longrightarrow & \Gamma\left(X; S(F_0)\right) & \longrightarrow & \Gamma\left(X; S(F_0'')\right) & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \Gamma\left(X; F'\right) & \longrightarrow & \Gamma\left(X; F\right) & \longrightarrow & \Gamma\left(X; F''\right) & \longrightarrow & \mathbb{R}^1\Gamma\left(X; F'\right) \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Link to Diagram

Here the bottom row continues in the long exact sequence for the right-derived functors of $\Gamma(X; -)$, i.e. sheaf cohomology. Since the desired sequence of complexes involved truncating this double complex by deleting the first row, consider everything from row two upward. That these levelwise maps assemble to a map of complexes is just a consequence of functoriality of $\Gamma(X; -)$, and left exactness preserves the zeros in the left-most column, so it suffices to show that the right-most column (highlighted in red) is zero as claimed.
However, this follows from the previous problem if the sheaves $S(F_n')$ are all flasque. This is immediate since they are sheaves of discontinuous sections, and such a section on U can always be extended to a global section by simply assigning any other values on $X \setminus U$ – any choice works, since no compatibility (e.g. continuity) is required.

Solution (Part 4):
It is a general theorem in homological algebra that a short exact sequence of chain complexes induces a long exact sequence in cohomology. In this case, if we take the vertical homology of the above double complex, by the snake lemma there are connecting morphisms:



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Solution (Part 5):
This holds because flasque sheaves are F-acyclic for $F(-) = \Gamma(X; -)$, so we can conclude that $\mathbf{H}^n(S(F)) = 0$ for $n > 0$ since the sheaves $S(F)$ are always flasque for any sheaf F.

Note: I realized at the last minute that this argument may not actually work, since this $\mathbf{H}^n$ a priori has nothing to do with $\mathbb{R}\Gamma(X; -)$ computed via injective resolutions.

ToDos

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Theorems

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