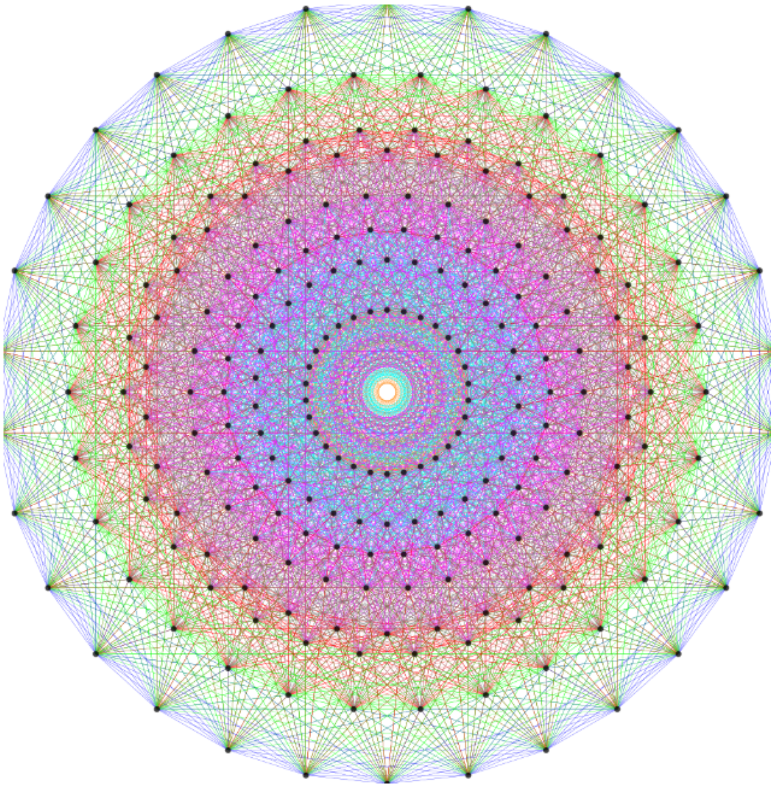

Notes: These are notes based on Kevin's video lectures, which are not quite a transcription or an attempt to capture all of the information, but rather to jot down some reference material for myself to refer back to later!

MSRI Summer School: Automorphic Forms and the Langlands Program

MSRI Summer School 2017, Lectures by Kevin Buzzard

D. Zack Garza
University of Georgia

dzackgarza@gmail.com

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1 Lecture 1: Overview and what questions we're interested in

Remark 1.0.1: References:

- Kevin's website: https://www.ma.imperial.ac.uk/~buzzard/MSRI/
- Youtube Playlist: https://www.youtube.com/watch?v=Rv59aRUMfio&list=PLhsb6tmzSpiysoRR0bZozu

Remark 1.0.2: This course: adapted from handwritten notes from a course by Richard Taylor in 1992 at Caltech. The original course focused on GL_2, we'll discuss GL_n. About a year before Wiles-Taylor!

Ways to learn:

- Engage with the material
- Type it up.[1] Better than hours of videos!
- Complete the exercises and fill in the gaps. See also the problem set on the course website.
- Talk to the experts at the talks
- Complete the project on the website on the abelian p-adic Langlands correspondence. This mimics the classical correspondence, and is very much in its infancy at the moment.

In all cases, it's useful to work with other people, communicate, interact, etc.

Remark 1.0.3: The story begins with Dirichlet characters – let $N \in \mathbb{Z}_{\geq 1}$ and consider a character

$$\chi : C_n^\times \to \mathbb{C} \qquad \in \mathsf{Grp}.$$

Attached to χ is a Galois representation

$$\rho_\chi : G := \mathsf{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_1(\mathbb{C}).$$

Note that G is an infinite group. This arises as the following composition:

$$\begin{array}{ccc} G & \xrightarrow{\rho_\chi} & \mathrm{GL}_1(\mathbb{C}) \\ \downarrow & & \uparrow \chi \\ \mathsf{Gal}(\mathbb{Q}(\zeta_N)/\mathbb{Q}) & \xrightarrow{=} & C_N^\times \end{array}$$

Link to Diagram

Here the equality denotes a canonical isomorphism, and is given by the map $n \mapsto (\zeta_N \mapsto \zeta_N^n)$.

[1] DZG: Like me!

Remark 1.0.4: There is a version of this for GL_2. Let f be a cuspidal modular form which is an eigenform for the Hecke operators T_p, which are endomorphisms of the space of modular forms. So $T_p f = \lambda f$ for some $\lambda \in \mathbb{C}$, and it turns out that the subfield $\langle \lambda_p \rangle \mathbb{C}$ (which could be infinitely generated) is a number field – it is equipped with an embedding into $\mathbb{C}$ and has finite degree over $\mathbb{Q}$.

Theorem 1.0.5 ***(Deligne, 60s/70s).***
Let $\ell \in \mathbb{Z}$ is prime and $\lambda \mid \ell$ is a prime of E_f, i.e. a nonzero prime ideal of $\mathcal{O}_{E_f}$. Following an observation due to Serre, Deligne constructs a map

$$\rho_f : G \to \mathrm{GL}_2(\mathrm{cl}_{\mathsf{Alg}}(E_{f\widehat{\lambda}})),$$

where the RHS is completing E_f at λ and taking an algebraic closure. This is an ℓ-adic representation, and formally resembles the $\mathbb{C}$-representation above in the sense that ρ_f is "attached" to f. Deligne's construction uses étale cohomology.

Remark 1.0.6: For f a modular form, there is a level $N \geq 1$, a weight $k \geq 1$, and a Dirichlet character χ. It turns out that ρ_f is unramified away from $N\ell$ – note that there is no analog of ℓ for the GL_1 case. If p is a prime not dividing $N\ell$, then $\rho_f(\mathrm{Frob}_p)$ has characteristic polynomial

$$x^2 + \lambda_p x + p^{k-1}\chi(p).$$

Since $p \nmid N\ell$, $p \nmid N$ and thus $\chi(p) \neq 0$. Note that this is the trace of the representation ρ, and it turns out that the conjugacy classes Frob_p are dense in G. By **Chebotarev density**, there is at most one semisimple ρ_f with this property, and Deligne's theorem is that there is *at least* one. These days, N-dimensional ℓ-adic Galois representations are common precisely because we now know they are the right things to look at. Historically, number theorists may not have considered modular forms number-theoretic objects – instead, they were considered objects of harmonic analysis, and number theory likely focused on things like class numbers and Iwasawa's main conjecture.

Remark 1.0.7: A word on Deligne's construction – how does he find a 2-dimensional ℓ-adic representation of G? He constructs ρ_f using étale cohomology with nontrivial coefficients. A modern take is that it would come from a motive, and étale cohomology produces motives, although one would normally take trivial coefficients. There is a slight issue in constructing ℓ-adic sheaves on the modular curve. Deligne later uses trivial coefficients and takes cohomology on some power of a universal elliptic curve. Thus Deligne's proof is a partial proof that the ρ_f are motivic. This all assumes $k \geq 2$, and the $k = 1$ case was handled by Deligne-Serre in the 70s.

#todo What is k

Question 1.0.8
Some questions arising from Deligne's construction:

- What do the representations locally look like at ramified primes p, i.e. $p \mid N\ell$? There is a formal meaning here: the global Galois group contains the local one, and G is embedded canonically and only ambiguous up to conjugacy. One can restrict the global representation to this local representation In the unramified case, we know the characteristic polynomial, and this essentially determines the local behavior, although there is a semisimplicity issue involving the Tate conjecture – specifying the characteristic polynomial of a matrix doesn't uniquely determine it, due to possible multiplicity in eigenvalues.

- Case 1: if $p \mid N$ and $p \neq \ell$, the answer is given by the conjectured **local Langlands correspondence**. These are theorems for $\mathrm{GL}_2(\mathbb{Q}_p)$ and $\mathrm{GL}_n(K)$ for any $K \in \mathsf{LocalField}$. This correspondence aims to relate (possibly infinite dimensional) representations of groups like $\mathrm{GL}_n(\mathbb{Q}_p)$ to Weil-Deligne representations, which are similar to Galois representations. We'll soon explain what this has to do with modular forms.

- Case 2: if $p = \ell$, so we have a p-adic representation of G, there should be a **p-adic local Langlands correspondence**. This is essentially the boundary of what we currently know, and in a sense we don't even know what the right question should be. This is a theorem for GL_2, but unknown for GL_3 and above. Ask Rebecca about this!

Remark 1.0.9: An easier variant: instead of asking for ρ_f, ask for $\bar{\rho}_f : \mathsf{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\overline{\mathbb{F}}_\ell)$, i.e. reduce $\bmod \ell$. Identify $\overline{\mathbb{F}}_\ell$ with the residue field of $\overline{E}_\lambda$ at λ. So reduce Deligne's ℓ-adic representation to get a $\bmod \ell$ representation. These reps are easier to understand since there are tricks, e.g. looking for $\bar{\rho}_f$ in $A[\ell]$ for some $A \in \mathsf{AbVar}$. Note that ρ_f involves étale cohomology.

Question 1.0.10
Questions Taylor asked at the time:

Are ρ_χ, ρ_f special cases of a general story? Note that this relates Dirichlet characters to modular forms.

Answer: yes, kind of. There is the following theorem from 2013, which was reproved by Scholze using perfectoid magic.

Theorem 1.0.11 ***(?)*.**
Let E be a totally real (so all embeddings $E \overset{\sigma}{\hookrightarrow} \mathbb{C}$ have $\sigma(E) \subseteq \mathbb{R}$) or CM number field (totally imaginary extension of a totally real number field) and let π be a *cuspidal automorphic representation* of $\mathrm{GL}_n(\mathbb{A}_E)$. Suppose that π is "cohomological", which is a condition on the weights and the PDEs that the automorphic forms come from and is a strong algebraicity condition. Then there is a representation $\rho_\pi : \mathsf{Gal}(\overline{E}/E) \to \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$ attached to π in a canonical way, which is the analog of $\operatorname{ch poly} \rho_f(\mathrm{Frob}_p)$

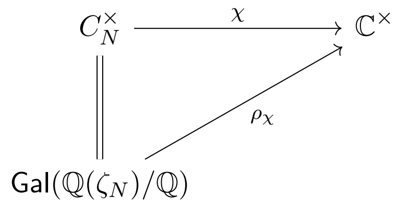
Remark 1.0.12: Automorphic forms will be solutions to elliptic PDEs Unrelated, but see Frank Calegari's blog.

Remark 1.0.13: Meta-theorem: Galois representations come from étale cohomology groups and their p-adic deformations, say of an algebraic variety defined over a number field e.g. a Shimura variety. General idea: given an algebraic or analytic object like χ, f, π (or more generally *motivic* objects), some technical machinery produces representations of Galois groups. Can we classify the image of this correspondence? Which Galois representations come from such things? I.e. given $\rho : G \to \mathrm{GL}_n(k)$ for *some* $k \in \mathsf{Field}$, is ρ isomorphic to some ρ' coming from an algebraic variety?

Dimension 1: let $K/\mathbb{Q}$ be finite Galois and $\rho : \mathsf{Gal}(K/\mathbb{Q}) \to \mathrm{GL}_1(\mathbb{C})$. Is $\rho \cong \rho_\chi$ for χ a Dirichlet character? We can take an epi-mono factor any group morphism and Galois theory works better with epis, so replace K with a subfield $L \leq K$ to make ρ *injective*. This is because $\rho : \mathsf{Gal}(K/\mathbb{Q}) \twoheadrightarrow \mathsf{Gal}(L/\mathbb{Q}) \hookrightarrow \mathrm{GL}_1(\mathbb{C})$. So we assume

$$\rho : \mathsf{Gal}(K/\mathbb{Q}) \hookrightarrow \mathbb{C}^\times,$$

hence $\mathsf{Gal}(K/\mathbb{Q}) \in \mathsf{AbGrp}$. A reminder of what a ρ_χ will look like:



Link to Diagram

By the same trick, we can factor ρ_χ to assume it is injective from some $L \leq \mathbb{Q}(\zeta_N)$ a subfield of a cyclotomic field. So the question reduces to the following:

Question 1.0.14
If $K \in \mathsf{NumberField}$ is Galois over $\mathbb{Q}$ with $\mathsf{Gal}(K/\mathbb{Q}) \in \mathsf{AbGrp}$, does there exist an $N \geq 1$ with $K \hookrightarrow \mathbb{Q}(\zeta_N)$.

Remark 1.0.15: Answer: yes, but this takes some work and has a name, the Kronecker-Weber theorem (an explicit special case of global CFT). Note that the converse is clear: subfields of cyclotomic fields will yield Galois groups which are subgroups of a cyclic group and hence abelian.

So for all $\rho : \mathsf{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_1(\mathbb{C})$ (so the image is finite order), there is a $\chi : C_N^\times \to \mathbb{C}^\times$ with $\rho \cong \rho_\chi$. Thus is the proposed correspondence, the image is everything, and the proof is class field theory.

Remark 1.0.16: What about GL_2? Let f be a cuspidal modular eigenform as before, there is a Galois representation $\rho_f : \mathsf{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\overline{\mathbb{Q}}_\ell)$ such that

- ρ_f is absolutely irreducible
- ρ_f is "odd", i.e. writing $\in \mathsf{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ for complex conjugation, $\det \rho_f(\bar{}) = -1$.
- $\rho_f$is unramified away from a finite set of primes and carries some p-adic Hodge theory and is *potentially semistable*.

Étale cohomology always has the third property, which is sometimes called "being geometric", and the conjecture is that geometric representations come from geometry.

Conjecture 1.0.17.
Fontaine-Mazur asked in the 1990s if all ρ satisfying these properties are of the form ρ_f for some f, which became the Fontaine-Mazur conjecture. This is basically known now, see Emerton and Kisin.

Remark 1.0.18: Similarly, we can ask that if $\rho : \mathsf{Gal}(\overline{E}/\mathbf{E}) \to \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$ satisfies some assumptions, is it the case that $\rho \cong \rho_\pi$ for some π. State of the art: BLGGT, proves this in many cases using very technical p-adic Hodge theory. See also the 10-author paper.

2 Lecture 2: Part 1, Classical Local Langlands

References: Serre's *Local Fields*

https://www.youtube.com/watch?v=1mhSFioQInU&list=PLhsb6tmzSpiysoRR0bZozub-MM0k3mdFR&index=2

Remark 2.0.1: Vaguely stated local Langlands for GL_n over $K/\mathbb{Q}_p$ a finite extension: there is a bijective correspondence

$$\left\{ \substack{\text{Certain } \infty\text{-dimensional irreducible} \\ \text{reps of } \mathrm{GL}_n \text{ over } \mathbb{C}} \right\} \rightleftharpoons \left\{ \substack{\text{Certain } n\text{-dimensional reps of a group} \\ A \text{ related to } \mathsf{Gal}(\overline{K}/K)} \right\}.$$

LHS: complicated reps of a simple group, vs RHS: relatively simple reps of a complicated group. For $n = 1$, this recovers local CFT. This is an entirely local statement, and is in fact a theorem now but the proof is global. For $n \geq 2$, local Langlands for GL_n over K is a theorem due to Harris-Taylor, and the proofs are again global (i.e. working with number fields). Oddly there isn't quite a "local" statement of the above bijection. This statement really should be categorified, which has happened on the function field side via the geometric Langlands correspondence.

Remark 2.0.2: Working toward infinite Galois groups, recall the finite case: let L/K be a finite extension, then L is Galois iff normal and separable. Recall that separable only needs to be checked in characteristic p, and normal means being a splitting field. Then $\mathsf{Gal}(L/K) \in \mathsf{FinGrp}$ (which is a theorem) are the auts of L fixing K pointwise, and there is a correspondence

$$\begin{aligned} \mathsf{Gal}(L/-) : \left\{ \text{Subfields } K \leq M \leq L \right\} &\rightleftharpoons \left\{ \text{Subgroups } 0 \leq \mathsf{Gal}(L/M) \leq \mathsf{Gal}(L/K) \right\} \\ M &\mapsto \left\{ \sigma \in \mathsf{Gal}(L/M) \;\middle|\; \sigma|_M = \mathrm{id}_M \right\}. \end{aligned}$$

Definition 2.0.3 (Galois Extensions)
For the infinite case, let $K \in \mathsf{Field}$ and let L/K be algebraic, possibly with $[L:K] = \infty$. We

say L is again Galois iff normal and separable.

Warning 2.0.4
Note that the cardinalities need not match up here: L can be a countably infinite extension with an uncountable Galois group.

Remark 2.0.5: For a fixed element $\lambda \in L$ and $\sigma \in \mathsf{Gal}(L/K)$, one doesn't need to actually work in the infinite group: there exists a finite M with $K \leq M \leq L$ and M finite Galois over L with $\lambda \in M$ – e.g. one can take M to be the splitting field of $\min_{\lambda, L}(x) \in K[x]$. Moreover there is a canonical map $\operatorname{Res}_{L/M} : \mathsf{Gal}(L/K) \to \mathsf{Gal}(M/K)$ and $\sigma(\lambda)$ is determined by $\tilde{\sigma}(\lambda)$ for $\tilde{\sigma} = \operatorname{Res}_{L/M}(\sigma)$. Note that if L/K is separable then M/K is separable – normality may not descend this way, but will if M is a splitting field. This argument shows that there is an injection

$$\mathsf{Gal}(L/K) \hookrightarrow \prod_{\substack{K \subseteq M \subseteq L \\ M/K \text{ finite Galois}}} \mathsf{Gal}(M/K) \implies \mathsf{Gal}(L/K) \cong \varprojlim_{\substack{K \subseteq M \subseteq L \\ M/K \text{ finite Galois}}} M \in \mathsf{Grp},$$

and it turns out that $\mathsf{Gal}(L/K)$ is a closed subspace of $\prod \mathsf{Gal}(M/K)$ in the subspace topology.

Warning 2.0.6
Each $\mathsf{Gal}(M/K)$ has the discrete topology, but $\mathsf{Gal}(L/K)$ need not be discrete in the product topology! The basic opens for the product topology on $\prod X_i$ are of the form $\prod U_i$ with only finitely many $U_i \neq X_i$.

Proposition 2.0.7 ***(FTGT).***
If L/K is Galois, i.e. algebraic normal and separable, with $\mathsf{Gal}(L/K)$ given the projective limit topology, there is a bijection

$$\mathsf{Gal}(L/-) : \left\{\text{Subfields } K \leq M \leq L\right\} \rightleftharpoons \left\{\text{Closed subgroups of } \mathsf{Gal}(L/K)\right\}$$

One might convince themselves that "restricting to the identity on M" is a closed condition. Note that everything is discrete in the finite case, so all subgroups were closed!

Example 2.0.8 ***(?):*** Take $K = \mathbb{Q}$ and $L = \bigcup_{n \geq 1} L_n$ where $L_n := \mathbb{Q}(\zeta_{p^n}) \subseteq \mathbb{C}$. There is a canonical isomorphism $\mathsf{Gal}(L_n/\mathbb{Q}) = C_{p^n}^\times$, so $G_L \hookrightarrow \prod_{n \geq 1} C_{p^n}^\times$. Note that this is not a surjection in general: note that the L_i are filtered, and we can restrict $\mathsf{Gal}(L/K) \to \mathsf{Gal}(L_n/K)$ by just restricting automorphisms. So writing $\varphi_n := \varphi|_{L_n}$ for any $\varphi \in \mathsf{Gal}(L/K)$, knowing φ_n implies knowing $\varphi_{\leq n}$ (by restriction). Under the canonical identification, $\varphi_n \in C_{p^n}^\times$, and for $m \leq n$ we have $\varphi_m = \varphi_n \bmod p^m$. In this case, there is a homeomorphism

$$\mathsf{Gal}(L/\mathbb{Q}) = \varprojlim C_{p^n}^\times := \mathbb{Z}_p{}^\times \hookrightarrow \mathbb{Z}_p,$$

noting that $\mathbb{Z}_p{}^\times \hookrightarrow \mathbb{Z}_p$ is a topological group in a topological ring, and the subspace topology works here since inversion is continuous.

Example 2.0.9*(?)*: Let K be finite, say $\sharp K = q$, and $L = \overline{K}$. Noting that $L_n := \mathbb{F}_{q^n} \subseteq L_m := \mathbb{F}_{q^m} \iff n \mid m$, we can form the filtered colimit

$$L = \underset{\underset{n}{\longrightarrow}}{\mathrm{colim}}\, L_n = \bigcup_{n \geq 1} L_n.$$

Recall that $\mathsf{Gal}(L_n/\mathbb{F}_q) = C_n = \langle \mathrm{Frob}_q \rangle$ where $\mathrm{Frob}_q(x) := x^q$ has order n. So $\mathsf{Gal}(L/K) \hookrightarrow \prod C_n$; which subset is it? If g_n is in the image, then $g_n \bmod m = g_m$ for all $m \mid n$, so

$$\mathsf{Gal}(L/K) = \varprojlim_n C_n = \widehat{\mathbb{Z}},$$

the profinite integers.

⚠ Warning 2.0.10

$$\mathbb{F}_q \subseteq \mathbb{F}_{q^2} \not\subseteq \mathbb{F}_{q^3}$$

for dimension reasons, so one can't form the usual directed system here.

2.1 Local Fields

Remark 2.1.1: For us, the examples of local fields will be $K/\mathbb{Q}_p$ finite extensions. To be happy with $\mathbb{Q}_p$, see exercises in Cassels' book on elliptic curves. Choose an algebraic closure $\overline{K}/K$, which is unique but not up to unique isomorphism. Slight issue with isomorphisms here: picking two algebraic closures $\overline{K}, \overline{K}'$ and two random isomorphisms $i : \overline{K} \to \overline{K}'$ and $j : \overline{K}' \to \bar{k}$, it may not be that $ij = \mathrm{id}$. On $\mathsf{Gal}(\overline{K}/K)$, this induces an inner automorphism, so we should avoid trying to work with explicit elements of this group (which in some sense are not well-defined).

Remark 2.1.2: Recall that there is a (normalized) valuation $v : \mathbb{Q}_p \to \mathbb{Z} \cup \{\infty\}$ where $v(p^n u) = n$ for $u \in {\mathbb{Z}_p}^\times$. Similarly $K \supseteq \mathcal{O}_K \supseteq \mathfrak{p}_K$ its unique maximal/prime ideal, which is principal and generated by a uniformizer $\mathfrak{p}_K = \langle \pi_K \rangle$. Thus for every $k \in K$ we can write $k = \pi_K^n u$, and we have a valuation

$$\begin{aligned} v : K^\times &\to \mathbb{Z} \\ \pi_K &\mapsto 1 \\ \pi_K^n u &\mapsto n. \end{aligned}$$

Definition 2.1.3 (Unramified extensions)
Let L/K be algebraic, possibly infinite. The valuation v_K extends to $L^\times \to \mathbb{Q}$, and $L \supseteq \mathcal{O}_L = \{0\} \cup \left\{ \lambda \in L \;\middle|\; v(\lambda) \geq 0 \right\}$. Note $\mathcal{O}_L \supseteq \mathfrak{p}_L$ its unique maximal ideal, which need not be principal here. Write $\kappa_L = \mathcal{O}_L/\mathfrak{p}_L$, which is an algebraic extension of $\kappa_K := \mathcal{O}_K/\mathfrak{p}_K$, which here is a

finite field. If L/K is Galois, there is a surjective map

$$\mathsf{Gal}(L/K) \twoheadrightarrow \mathsf{Gal}(\kappa_L/\kappa_K).$$

This need not be injective in general, so we say L/K is **unramified** iff this is a bijection.

Proposition 2.1.4 ***(?).***
Let $K/\mathbb{Q}_p$ be a finite extension and $\mathfrak{p}_K \in \operatorname{Spec} \mathcal{O}_K$ with $\mathfrak{p}_K = \langle \pi_K \rangle$ generated by a uniformizer. TFAE:

- $\mathfrak{p}_L = \pi_K \mathcal{O}_L$, so the same element generates the maximal ideal of L,
- $v_K : L^\times \to \mathbb{Q}$ has image $\mathbb{Z}$.
- L/K is unramified.

Remark 2.1.5: The compositum of two unramified extensions of K is again unramified, and if L/K there is a unique unramified subextension M with $L \supseteq M \supseteq K$. Moreover $\mathsf{Gal}(M/K) = \mathsf{Gal}(\kappa_M/\kappa_L)$, since all unramified extensions are Galois, and this will be a procyclic group.

3 | Lecture 3

<!–> Video–>

Remark 3.0.1: Plans: group theoretic properties of $G(\overline{K}/K)$ for K a p-adic field, tamely ramified extensions, and the theorems of local CFT.

Remark 3.0.2: Setup: $K/\mathbb{Q}_p$ finite with L/K algebraic, normal, and separable (automatic if we assume $\operatorname{ch} k = 0$), so Galois. The goal is to understand $G_K := \mathsf{Gal}(\overline{K}/K)$, we'll first show it surjects onto an easier group and investigate the kernel:

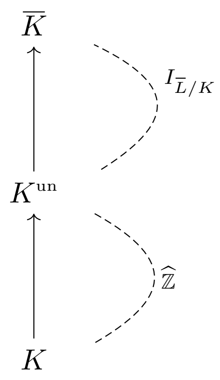
Definition 3.0.3 (Inertia subgroup)
The **inertia subgroup** is defined as the kernel of the reduction map to residue fields:

$$\begin{array}{ccc}
0 & \longrightarrow & I_{L/K} := \ker \kappa \\
 & & \downarrow \\
 & & G(L/K) \\
 & & \downarrow \kappa \\
0 & \longleftarrow & G(\kappa_L, \kappa_K) \cong \mathsf{Gal}(L/K)/I_{L/K}
\end{array}$$

Link to Diagram

Remark 3.0.4: Note that κ_L is finite, so $G(\kappa_L/\kappa_L)$ is an easier group to understand, and in fact is procyclic (topologically monogenic). It's clear that $I_{L/K} \leq G_{L/K}$ is a closed subgroup, and thus corresponds to some M with $K \subseteq M \subseteq L$ where $G(L/M) = I_{L/K}$ and $\mathsf{Gal}(M/K) \cong \mathsf{Gal}(\kappa_L/\kappa_K)$. In fact, M will be the union of all subfields of L containing K which are unramified over K.

Remark 3.0.5: A special interesting case is when $L = \overline{K}$, then $M = K^{\text{un}}$ will be the maximal unramified extension:



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Note that

$$G(K^{\text{un}}/K) \xrightarrow[\text{can}]{\sim} \mathsf{Gal}(\overline{\kappa}_K/\kappa_K) \xrightarrow[\text{can}]{\sim} \widehat{\mathbb{Z}}.$$

Question 3.0.6

Two stories in CFT: what are the groups, and what are the corresponding fields?

Remark 3.0.7: These are two genuinely different stories, e.g. for $K \in \mathsf{Field}_{/\mathbb{Q}}$, for K^{ab} the maximal abelian extension it's precisely known what the Galois group is, but what is the actual extension (the Hilbert class field)? If $K = \mathbb{Q}_p$, then

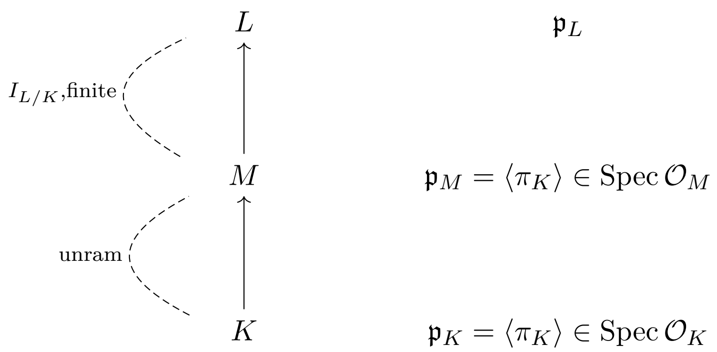
$$K^{\text{un}} = \bigcup_{m \geq 1, p \nmid m} \mathbb{Q}_p(\zeta_m),$$

being careful because $\mathbb{Q}_p(\zeta_p)$ is ramified. This might also be true for other fields K.

Remark 3.0.8: Setup: let L/K be Galois, but now assume $I_{L/K} \in \mathsf{FinGrp}$, e.g. when L/K is a finite extension. We know $I_{L/K} \trianglelefteq G(L/K)$ is normal since it is a kernel, and if L/K is finite then the quotient is finite cyclic. Put a filtration on the inertia group and we'll look at the filtered pieces: note that if $\sigma \in I_{L/K}$, then $\sigma : L \to L$ descends to a local morphism $\sigma : \mathcal{O}_L \to \mathcal{O}_L$ preserving $\mathfrak{p}_L$.

Question 3.0.9
How much does σ disrupt these local pieces?

The claim is that since $I_{L/K}$ finite implies $\mathfrak{p}_L = \langle \pi_L \rangle$ is principal, where the idea is that the uniformizer in the base extends all the way to L:



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Moreover the discrete valuation $v_L : L \twoheadrightarrow \mathbb{Z}$ satisfies $v_L = \sharp I_{L/K} v_K$ on $K^\times$. So write $\mathfrak{p}_L = \langle \pi_L \rangle$ for some uniformizer. Define a filtration by

$$I_{L/K,i} = \left\{ \sigma \in I_{L/K} \;\middle|\; \frac{\sigma(\pi_L)}{\pi_L} \in 1 + \mathfrak{p}_L^i \right\}, \qquad I_{L/K,0} \coloneqq I_{L/K},$$

where we note that $\sigma(\pi_L)$ will still be a uniformizer, so the quotient is a unit, and we are measuring how far it is from 1. All of the subquotients turn out to be abelian.

Setting $i = 1$ should recover the Sylow subgroup.

One can check that this defines a series of normal subgroups of $G(L/K)$:

$$I_{L/K} = I_{L/K,0} \supseteq I_{L/K,1} \supseteq I_{L/K,2} \supseteq \cdots .$$

Note that since $L = M\langle \pi_L \rangle$, so if σ fixes π_L it fixes all of L and must be the identity. Thus if $\sigma \neq \mathrm{id}$, then σ moves π_L and $\sigma(\pi_L)/\pi_L \neq 1$, so for $i \gg 1$ one has $\sigma(\pi_L)/\pi_L \neq 1 \bmod \mathfrak{p}_L^i$. The conclusion is that $I_{L/K,i} = 1$ for $i \gg 1$, making this a finite filtration.

We can write down an embedding

$$\begin{aligned} I_{L/K}/I_{L/K,1} &\hookrightarrow \kappa_L^\times \\ \sigma &\mapsto \sigma(\pi_L)/\pi_L. \end{aligned}$$

Jackie is now happy about this.

In particular, the domain is cyclic of order prime to p. So given the full Galois group, we factored out the cyclic unramified part and were left with inertia, and now we've factored something simple out of the remaining inertia part. Bad news: $I_{L/K,1}$ is complicated!

Definition 3.0.10 (Lower Numbering)
For $i \geq 1$, define an embedding

$$\begin{aligned} I_{L/K,i}/I_{L/K,i+1} &\hookrightarrow \mathfrak{p}_L^i/\mathfrak{p}_L^{i+1} \xrightarrow[\sim]{\mathsf{AbGrp}} \mathbb{G}_a(\kappa_L) \\ \sigma &\mapsto \frac{\sigma(\pi_L)}{\pi_L} - 1. \end{aligned}$$

Note that the RHS could be an infinite field, but turns out to be a finite-dimension $\mathbb{F}_p$-vector space and thus the domain is isomorphic as a group to C_p^n for some n. We call this the filtration the **lower numbering**. In particular, its order is a power of p.

Remark 3.0.11: Upshot: $I_{L/K,1}$ has p-power order, is a normal subgroup, and the quotient is cyclic of order prime to p, making it the unique Sylow p-subgroup of $I_{L/K}$. In particular, this is a p-group, hence solvable.

Definition 3.0.12 (Tame and wild ramification)
We say L/K is **tamely ramified** iff $I_{L/K,1} = 1$. Otherwise, we say L/K is **wildly ramified**.

⚠ **Warning 3.0.13**
This seems to mean "ramified but not too badly", but unramified extensions are also tamely ramified! However, wildly ramified does imply ramified.

Remark 3.0.14: We're interested in $L = \overline{K}$ where $I_{L/K}$ is not finite, so we'll need to glue using a limit. Note that the lower numbering doesn't behave well with respect to extensions of L since extensions L'/L change the uniformizer. More precisely, if $L'/L/K$ with L/K and L'/K Galois with $I_{L'/K}$ finite, there is a map $I_{L'/K} \twoheadrightarrow I_{L/K}$ but $I_{L'/K,i}$ does not get identified with $I_{L/K,i}$ – the issue is that π_L may have nothing to do with $\pi_{L'}$, and one is trying to control things $\operatorname{mod} \pi_L^i$ and $\operatorname{mod} \pi_{L'}^i$.

The fix: introduce a relabeling.

Definition 3.0.15 (Upper numbering)
Set $g_i = \sharp I_{L/K,i}$ so $g_0 \geq g_1 \geq \cdots \geq g_M = 1$ for some $M \gg 0$. Define a PL continuous function φ, a scaling factor. Some facts about it:

- φ will be linear on $(i, i+1)$ for all i
- $\varphi(0) = 0$
- The slope of φ on $(i, i+1)$ will be $\dfrac{g_{i+1}}{g_0}$,
- $\varphi : [0, \infty) \to [0, \infty)$ is a strictly increasing bijection.

Note that the slopes are decreasing, and limit to a constant slope g_M/g_0. For $v \in \mathbb{R}_{\geq 0}$, define

$$I_{L/K,v} \coloneqq I_{L/K,\lceil v \rceil}.$$

Then define the renormalized **upper numbering**

$$I_{L/K}^u \coloneqq I_{L/K,\varphi^{-1}(u)}.$$

Remark 3.0.16: The point: the upper numbering will now lift to infinite extensions.

Proposition 3.0.17 ***(?).***
If $L'/L/K$ are all Galois with $I_{L'/K}$ finite, then

$$I_{L/K}^u = \operatorname{im} I_{L'/K}^u.$$

Proof (?).
Omitted, see Serre's *Local Fields*, or Cassels-Froehlich. ∎

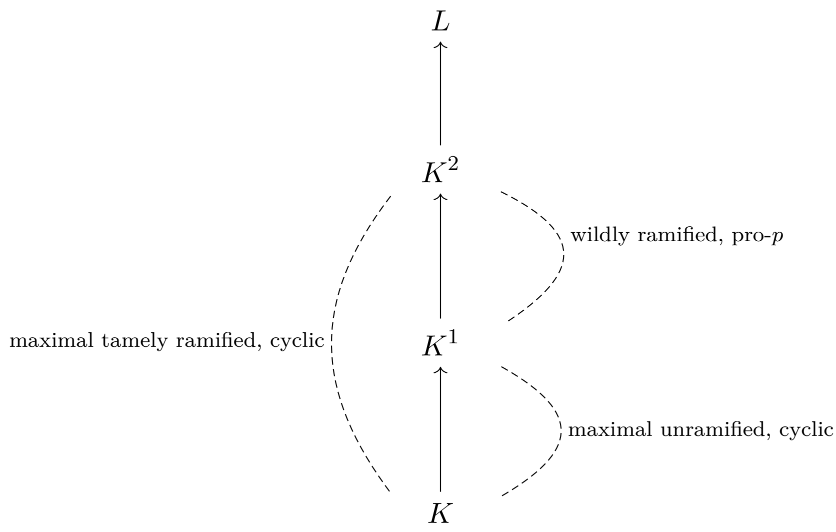
Remark 3.0.18: If one graphs $v \mapsto \sharp I_{L/K,v}$, there are jump discontinuities at random $\mathbb{Z}$-points. After the rescaling, the graph of $u \mapsto \sharp I_{L/K,i}^u$ still jumps, but now at $\mathbb{Q}$-points instead of on $\mathbb{Z}$. The denominators of all of the jumps will all divide g_0.

Theorem 3.0.19 ***(Hasse-Arf).***
If L/K is abelian, then the jumps in $I_{L/K}^u$ are in fact in $\mathbb{Z}$.

Remark 3.0.20: If L/K is any Galois extension, define $I_{L/K}^u$ by gluing $I_{M/K}^u$ for all M/K algebraic and $I_{M/K}$ finite.

Remark 3.0.21: It turns out that L/K is tamely ramified $\iff$ $I_{L/K,\varepsilon} = 1$ for all $\varepsilon > 0$ $\iff$ $I_{L/K}^\delta = 1$ for all $\delta > 0$, and this last condition makes sense for any (possibly infinite) Galois extensions.

Note that compositing tame extensions is still tame, so L/K contains a maximal tamely ramified extensions. So we've split off some easier parts of the full extension:



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Recall that $G(K_2/K_1) \cong I_{L/K}/I_{L/K,1}$ was cyclic of order m with $p \nmid m$. By Kummer theory, one can generally write down cyclic extensions of order m over some field as long as the field contains mth roots of unity.

See Birch's article in Cassels-Froehlich.

Since $p \mid m$, $K^{\text{un}} \supseteq \mu_m$ the mth roots of unity in $\overline{K}$. So if K^2/K^{un} is Galois with $G(K^2/K^{\text{un}}) \cong C_m$, then by Kummer theory, this extension arises by taking an mth root, so $K^2 \cong \mathsf{K}^{\text{un}}(a^{\frac{1}{m}})$ for some $a \in K^{\text{un}}$. E.g. since $\mathbb{Q}$ contains square roots of unity (± 1), every quadratic extension $K/\mathbb{Q}$ is obtained as $K = \mathbb{Q}(\sqrt{a})$. Here K^{un} only has one prime, so $K^2 = K^{\text{un}}(\pi_K^{\frac{1}{m}})$ since mth roots of units give unramified extensions and $a = \pi_K^{\ell} u$ for some unit.

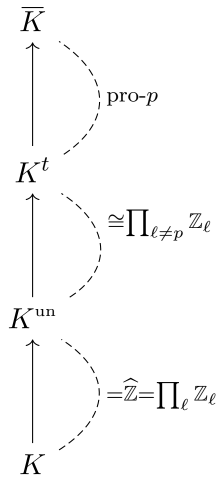
Thus the tamely ramified extension $K^2 = K^t := \bigcup_{m \geq 1, p \nmid m} K^{\text{un}}(\pi_K^{\frac{1}{m}})$. Moreover there is a canonical isomorphism

$$G(K^{\text{un}}(\sqrt[m]{\pi_K})/K^{\text{un}}) \xrightarrow[\text{can}]{\sim} \mu_m$$
$$\sigma \mapsto \sigma(\sqrt[m]{\pi_K})/\sqrt[m]{\pi_K}.$$

Thus

$$G(K^t/K^{\text{un}}) = \varprojlim \mu_m \cong \varprojlim \left\{ C_m \;\middle|\; p \nmid m \right\} = \prod_{\ell \neq p} \mathbb{Z}_\ell.$$

Remark 3.0.22: Our decomposition of the absolute Galois group is now:



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Note that if the top field is a finite extension, this is just factoring the full Galois group as

- a cyclic group with a canonical generator,
- a cyclic group with with a non-canonical generator, and
- a finite p-group.

The story thus far packages all of this together for all finite extensions at once by taking direct limits.

Remark 3.0.23: Note that $G(K^t/K)$ is unknown at this point, and will in general be a semidirect product of the two (pro)cyclic group and its (pro)cyclic quotient. Take a canonical generator to write

$$\langle \mathrm{Frob} \rangle = G(K^{\mathrm{un}}/K) = \mathsf{Gal}(\kappa(K^{\mathrm{un}})/\kappa(K))$$

which contains the map $x \mapsto x^q$ where $q := \sharp\kappa(K)$, and Frob is defined by pulling this back to $\mathsf{Gal}(K^{\mathrm{un}}/K)$ along the canonical isomorphisms. We can lift Frob to $G(K^t/K)$ (which we want to understand), then it acts on the normal subgroup $\mathsf{Gal}(K^t/K^{\mathrm{un}})$ by conjugation, i.e. $\sigma \mapsto \mathrm{Frob} \circ \sigma \circ \mathrm{Frob}^{-1}$. This is independent of choice of lift, since another lift is of the form $\psi\,\mathrm{Frob}$ where ψ is in the abelian part and thus commutes with σ. So we need to specify this action to say what $G(K^t/K)$ is. We have a canonical isomorphism

$$G(K^t/K^{\mathrm{un}}) \xrightarrow[\text{can}]{\sim} \varprojlim_m \mu_M(\overline{K}),$$

so it suffices to give an automorphism of this group. The following exercise yields the glue:

Exercise 3.0.24 *(?)*
Note that Frobenius acts on the RHS, and $x \mapsto x^q$ makes sense here since q is a power of p and m is prime to p. Check that the map induced by Frob is $\zeta \mapsto \zeta^q$.

Remark 3.0.25: Next: statements of local CFT.

4 Lecture 4: Statements of Local Class Field Theory

Remark 4.0.1: Idea: LCFT offers an interpretation of $G(\overline{K}/K)^{\mathrm{ab}}$, and a vast number of group cohomology computations.

Remark 4.0.2: Just the statements today, the proofs require clever tricks like Lubin-Tate groups.

We've reduced studying $G(\overline{K}/K)$ to looking at the inertia group I. The obstacle: the Sylow p-subgroup for I is complicated. We'll try to understand instead $\mathsf{Ab}(G(\overline{K}, K))$, its abelianization.

Definition 4.0.3 (Weil Group)
Let $K/\mathbb{Q}_p$ be finite, so we have a SES

$$1 \to I_{\overline{K}/K} \to G(\overline{K}/K) \to \widehat{\mathbb{Z}} = \langle \mathrm{Frob} \rangle \to 1.$$

Note that Frob isn't quite a well-defined element in $G(\overline{K}/K)$ – any two lifts differ by inertia. Consider

$$\mathrm{Frob}^{\mathbb{Z}} := \left\{ \cdots, \mathrm{Frob}^{-2}, \mathrm{Frob}^{-1}, \mathrm{id}, \mathrm{Frob}, \mathrm{Frob}^{2}, \cdots \right\} \cong \mathbb{Z} \subseteq \widehat{\mathbb{Z}}.$$

Define the **Weil group** W_K by the following diagram:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & I_{\overline{K}/K} & \longrightarrow & W_K & \longrightarrow & \mathrm{Frob}^{\mathbb{Z}} & \longrightarrow & 1 \\
 & & \Vert & & \big\downarrow & & \big\downarrow & & \\
1 & \longrightarrow & I_{\overline{K}/K} & \longrightarrow & G(\overline{K}/K) & \longrightarrow & \widehat{\mathbb{Z}} & \longrightarrow & 1
\end{array}$$

Link to Diagram

More precisely,

$$W_K := \left\{ g \in G(\overline{K}/K) \;\middle|\; \mathrm{im}(g) \in \widehat{\mathbb{Z}} = \mathsf{Gal}(K^{\mathrm{un}}/K) \text{ is in } \mathrm{Frob}^{\mathbb{Z}} \right\}.$$

Topologize W_K in the following way: $I_{\overline{K}/K} \subseteq W_K$ is open, but not an open subgroup of $G(\overline{K}/K)$ since 0 is closed in $\widehat{\mathbb{Z}}$ and preimages send closed sets to closed sets, making $I_{\overline{K}/K}$ closed in $\mathsf{Gal}(\overline{K}/K)$. Thus $W_K/I_{\overline{K}/K} = \mathbb{Z}^{\mathrm{disc}}$.
Using that the embedding $\mathbb{Z}^{\mathrm{disc}} \to \widehat{\mathbb{Z}}$ is continuous, we can equivalently define W_K by a pullback in TopGrp:

$$\begin{array}{ccc}
W_K & \longrightarrow & \mathbb{Z}^{\mathrm{disc}} \\
\big\downarrow & \lrcorner & \big\downarrow \\
G(\overline{K}/K) & \longrightarrow & \widehat{\mathbb{Z}}
\end{array}$$

Link to Diagram

⚠ Warning 4.0.4
W_K does *not* use the subspace topology.

Remark 4.0.5: We haven't lost much by passing to W_K, since its image is dense in $G(\overline{K}/K)$. Note that at every finite stage L/K, $W_K \twoheadrightarrow G(L/K)$.

Remark 4.0.6: For $G \in \mathsf{TopGrp}$, define $G^c \in \mathsf{TopAbGrp}$ to be the (topological) closure $\mathrm{cl}_G \left\langle \left\{ ghg^{-1}h^{-1} \;\middle|\; g, h \in G \right.\right.$
Note that $G^{\mathrm{ab}} := G/G^c$ is the maximal abelian Hausdorff quotient of G. Why take the topological closure: if id is not closed, then $\mathrm{cl}_G(\mathrm{id}) \trianglelefteq G$ is closed, as are all of its cosets, so quotienting by this makes points closed and yields a Hausdorff space.

Theorem 4.0.7 ***(Main Theorem of LCFT).***
If $K/\mathbb{Q}_p$ is finite, then there is a canonical isomorphism, **the Artin map**:

$$r_K : K^\times \xrightarrow[\mathrm{can}]{\sim} W_K^{\mathrm{ab}}.$$

Remark 4.0.8: Note that W_K is not a profinite group, and is mapping to a discrete group $\mathbb{Z}$. This is what $K^\times$ looks like: it contains $\mathcal{O}_K^\times$, which is profinite, and the quotient is $\mathbb{Z}^{\mathrm{disc}}$. So $K^\times$ also has a discrete and profinite part.

4.1 Properties of the Artin Map

Remark 4.1.1: See Serre's article in Cassels-Froehlich for proofs!

4.1.1 The Canonical Isomorphism

Proposition 4.1.2 ***(Properties of the Artin map).*** • r_K restricts to maps:

$$\begin{array}{ccc}
K^\times & \xrightarrow{\sim} & W_K^{\mathrm{ab}} \\
\big\uparrow & & \big\uparrow \\
\mathcal{O}_K^\times & \xrightarrow{\sim} & \mathrm{im}(I_{\overline{K}/K}) \\
\big\uparrow & & \big\uparrow \\
1 + \mathfrak{p}_K^i,\ i \geq 1 & \xrightarrow{\sim} & \mathrm{im}(I_{\overline{K}/K}^i)
\end{array}$$

Link to Diagram

- Note that only the upper numbering makes sense here, since these are infinite extensions. This is weird because of the difference in jumps for the filtrations, but is explained by the **Hasse-Arf theorem**.
- $r_k(\pi_K) \in \mathrm{Frob}^{-1} \cdot \mathrm{im}(I_{\overline{K}/K})$, which is a coset in a quotient.
  - Note that π_K is a *choice* of uniformizer, which e.g. for $\mathbb{Q}_p$ is p, but a random extension one can take any root of an Eisenstein polynomial. So there is some ambiguity on both sides of this map, but we use the one that sends π_K to a Frob^{-1}.

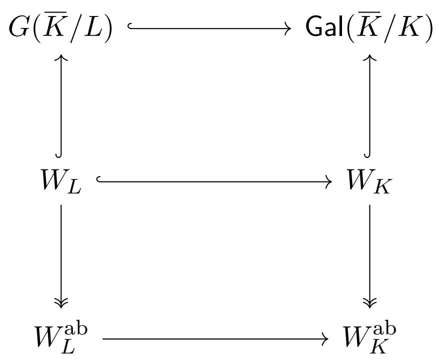
Remark 4.1.3: Confusing comment: if $\varphi : A \xrightarrow[\text{can}]{\sim} B$ is a canonical isomorphism in AbGrp, then $\varphi \circ \iota$ where $\iota(x) = x^{-1}$ is just as canonical! This occurs e.g. for the Weil pairing on an elliptic curve $E[n]^{\times^2} \to \mu_n$, which are maps between abelian groups. This is an issue here since there are really two canonical isomorphisms we could call r_K, by composing with inversion on $K^\times$, but the last isomorphism $r_k(\pi_K) \in \mathrm{Frob}^{-1}\, \Im(I_{\overline{K}/K})$ singles out *which* isomorphism it is since if π_K is a uniformizer, π_K^{-1} is not. So we can tell the difference: one isomorphism sends Frobenius to a uniformizer, the other to the *inverse* of a uniformizer.

Definition 4.1.4 (Geometric and Arithmetic Frobenius)
The **arithmetic Frobenius** is Frob, and **geometric Frobenius** is Frob^{-1}.

Remark 4.1.5: Deligne's convention: associate the uniformizer with *geometric* Frobenius.

4.1.2 Abelian Extensions

Remark 4.1.6: If L/K is finite, then $G(\overline{K}/L) \hookrightarrow G(\overline{K}/K)$ is a subgroup, and there is a map $W_L \hookrightarrow W_K$. These are injections by TFTGT, and there is a diagram:



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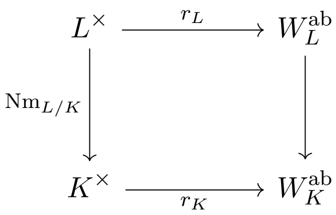
Warning 4.1.7
Note that (topological) abelianization may not preserve monomorphism! E.g. consider G with $G^{\mathrm{ab}} = 1$ and a map $C_n \hookrightarrow G$ from a cyclic group, then $C_n^{\mathrm{ab}} = C_n \twoheadrightarrow 1$.

Remark 4.1.8: One can fill in the following diagram with the norm:



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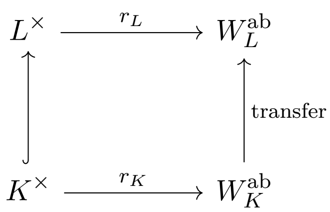
Remark 4.1.9: For $H \leq G$ a finite index subgroup, there is a **transfer** map on the Weil groups, the *Verlagerung transfer*

$$V : G^{\mathrm{ab}} \to^{\mathrm{ab}} g \mapsto \prod_{1 \leq i \leq n} \gamma_i g \gamma_i^{-1}, \qquad \gamma_i \in G/H \text{ coset representatives.}$$

which behaves like a norm. It fits into a diagram:



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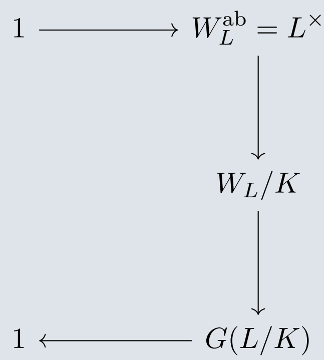
Remark 4.1.10: If L/K is finite Galois, noting that $W_L \leq W_K$ is a finite index normal subgroup with $W_K/W_L = G(L/K)$. Recall that W_L^c is the closure of $[W_L, W_L]$ and is a characteristic subgroup, so $W_L^c \trianglelefteq W_K$. Define the quotient

$$W_{L/K} \coloneqq W_K/W_L^c.$$

Definition 4.1.11 (Fundamental Class in Galois Cohomology)
There is a canonical SES



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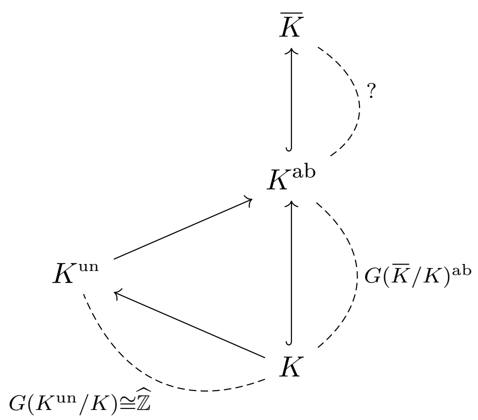
This extension gives a **fundamental class**

$$\alpha_{L/K} \in H^2(G(L/K); L^\times),$$

and is in fact a nontrivial element (so not a semidirect product). It turns out that this H^2 is cyclic of order $n = [L : K]$ and generated by α, and $(-) \smile \alpha_{L/K}$ is a map between cohomology groups and is often an isomorphism.

Remark 4.1.12: Say L/K is an abelian extension iff it is Galois with abelian Galois group. There is a maximal abelian extension K^{ab}, which exists since compositing preserves abelian extensions. We now essentially understand $G(K^{ab}/K)$:



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We now have a diagram:

$$\begin{array}{ccccc} I_{K^{\mathrm{ab}}/K} & \longrightarrow & G(K^{\mathrm{ab}}/K) & \longrightarrow & \mathsf{Gal}(K^{\mathrm{un}}/K) = \widehat{\mathbb{Z}} \\ \| & & \uparrow & & \uparrow \\ I_{K^{\mathrm{ab}}/K} & \longrightarrow & W_K^{\mathrm{ab}} \xrightarrow[CFT]{\sim} K^\times & \longrightarrow & \mathbb{Z} \end{array}$$

Link to Diagram

Note that $G(K^{\mathrm{ab}}/K)$ is a profinite topological group, so can't quite be $K^\times$, but is some kind of completion.

Remark 4.1.13: Conclusion: after choosing a uniformizer/Frobenius there is a non-canonical isomorphism

$$G(\overline{K}/K)^{\mathrm{ab}} = \mathsf{Gal}(K^{\mathrm{ab}}/K) \xrightarrow{\sim} \mathcal{O}_K^\times \times \widehat{\mathbb{Z}} \cong I_{K^{\mathrm{ab}}/K} \times \mathsf{Gal}(K^{\mathrm{un}}/K).$$

We know $G(\overline{K}/K)$ is solvable, since it factors as cyclic and pro-p parts, but the difficulty lies in how they extend. CFT is enough to understand the abelianization, but then we were stuck for 50 years! Next: Langlands ideas generalizing this story.

5 | Lecture 5

Remark 5.0.1 ***(A neat trick):*** A neat trick for pre/post-multiplying by diagonal matrices:

$$\begin{bmatrix} | & | \\ \mathbf{v}_1 & \mathbf{v}_2 \\ | & | \end{bmatrix} \cdot \begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix} = \begin{bmatrix} | & | \\ d_1\mathbf{v}_1 & d_2\mathbf{v}_2 \\ | & | \end{bmatrix}.$$

Similarly

$$\begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix} \cdot \begin{bmatrix} - & \mathbf{v}_1 & - \\ - & \mathbf{v}_2 & - \end{bmatrix} = \begin{bmatrix} - & d_1\mathbf{v}_1 & - \\ - & d_2\mathbf{v}_2 & - \end{bmatrix}.$$

I.e. post-multiplication by a diagonal matrix acts by scaling the columns, and pre-multiplying scales the rows. A useful consequence:

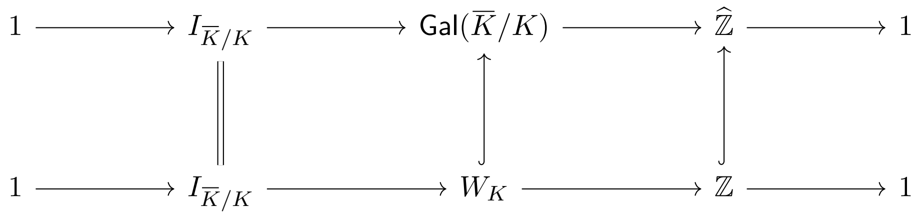
$$\begin{bmatrix} l_1 & 0 \\ 0 & l_2 \end{bmatrix} \cdot \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} r_1 & 0 \\ 0 & r_2 \end{bmatrix} = \begin{bmatrix} l_1 r_1 a & l_1 r_2 b \\ l_2 r_1 c & l_2 r_2 d \end{bmatrix},$$

i.e. entry a_{ij} gets hit by the nonzero entries in the ith row on the left and jth column on the right.

5.1 Conductor

Remark 5.1.1: Next goal: statements of local Langlands conjectures (LLC). For GL_n/K with $K/\mathbb{Q}_p$ finite, there will be some correspondence between representations of some "Galois group" and representations of $\mathrm{GL}_n(K)$. Here the "Galois group" is the Weil-Deligne group, which is related to the Weil group, so we'll discuss n-dimensional representations of W_K.

Remark 5.1.2: Recall we have a diagram:



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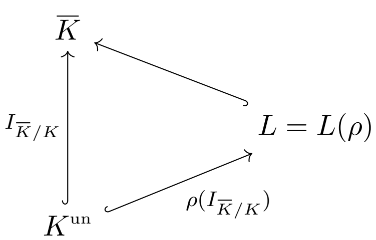
We're really interested in reps of G_K here, where e.g. étale cohomology or the Tate module of an elliptic curve induce (ℓ-adic) representations. Note that if one *starts* with a G_K representation on $\widehat{\mathbb{Z}}$, one gets a W_K representation on $\mathbb{Z}$, but the converse is not true since reps may not lift, despite the fact that $\mathbb{Z} \hookrightarrow \widehat{\mathbb{Z}}$ is dense.

Remark 5.1.3: Let E be a field with the discrete topology and put the discrete topology on $\mathrm{GL}_n(E)$. Consider continuous representations $\rho : W_K \to \mathrm{GL}_n(E)$. Note that

$$\rho \text{ is continuous} \iff \ker \rho \leq W_K \text{ is open.}$$

Note that $\rho(I_{\overline{K}/K}) \subseteq \mathrm{GL}_n(E)$ is the continuous image of a compact space, thus a compact subset of a discrete group, and thus finite since $\mathrm{GL}_n(E)$ is covered by its open points and this must admit a finite subcover. So we've reduced the hard part, inertia, to something more finite. We'll now try to attach numerical invariants by pushing the upper/lower filtrations into the image of ρ. Since $\rho(I_{\overline{K}/K})$ is a finite quotient, we have a factoring of Galois groups:



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So write $\rho(I_{\overline{K}/K}) = I_{L/K} \supseteq I_{L/K,1} \supseteq \cdots$.

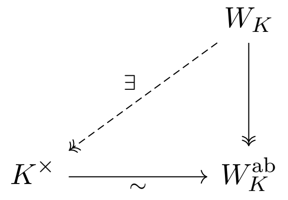
Definition 5.1.4 (Conductor)
Let $\rho : W_K \to \mathrm{GL}_N(E) = \underset{E}{\mathrm{Aut}}(V)$ be a representation as above where $V = E^n$ as a vector space, and define the **conductor of** ρ as

$$f(\rho) := \sum_{i \geq 0} \frac{1}{[I_{L/K} : I_{L/K,i}]} \dim(V/V^{I_{L/K,i}}) \in \mathbb{Q}_{\geq 0},$$

where for $H \to \mathrm{Aut}(V)$ we define the fixed points as $V^H := \left\{ v \in V \;\middle|\; hv = v \;\middle|\; \forall h \in H \right\}$. Note that this sum is finite, since $i \gg 0 \implies I_{L/K,i} = \{1\}$ and the dimension appearing is zero.

Remark 5.1.5: It should be easy to show that $f(\rho) = 0 \iff \rho$ is unramified $\iff \rho(I_{\overline{K}/K}) = \{1\}$. One can also get a trivial lower bound by examining the first term, which is $\dim(V/V^{I_{L/K}})$.

Example 5.1.6 ***(?):*** Representations of $K^\times$ can be used to get reps of W_K:



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We have a valuation $v_K : K^\times \to \mathbb{Z}$ where $v(\pi_K) = 1$, so define a norm $\|\lambda\| := \varepsilon^{-v(\lambda)}$ for some $\varepsilon \in (0,1)$, e.g. taking $\varepsilon := 1/p$ for $K = \mathbb{Q}_p$. Note that changing ε just yields an equivalent norm, and e.g. in non-Archimedean geometry we only use equivalence classes of norms anyway!

Remark 5.1.7: If $K \in \mathsf{Field}$ is complete with respect to a nontrivial non-Archimedean norm, there is a good theory of *rigid geometry*. E.g. for $K = \mathbb{C}[[t]]$, one can take $v(t^n + \cdots) := n$. Note that $\mathbb{C}[[t]]$ is not locally compact, so doesn't admit a Haar measure.

If $K/\mathbb{Q}_p$ is finite, there is a canonical choice for this norm. Since $\kappa(K)$ is a finite field, K is locally compact since $\mathcal{O}_K \cong {\mathbb{Z}_p}^d$ which is a compact open neighborhood of the identity. So K is a locally compact Hausdorff abelian group, and thus admits an additive (translation-invariant) Haar measure $\mu : K \to \mathbb{R}_{\geq 0} \cup \{\infty\}$. We'll have $\mu(\mathcal{O}_K) < \infty$, so we can normalize to $\mu(\mathcal{O}_K) = 1$. Note that $\mu(X) = \mu(X+a)$ for any $a \in K$ and any subset $X \subseteq K$, and $X \cap Y = \emptyset \implies \mu(X \cup Y) = \mu(X) + \mu(Y)$.

We can then write

$$\mathcal{O}_K = \coprod_{\lambda \in \kappa(K)} \tilde{\lambda} + \mathfrak{p}_K$$

where $\tilde{\lambda}$ is a lift of $\lambda \in \kappa(K) = \mathcal{O}_K/\mathfrak{p}_K$ to $\mathcal{O}_K$. Thus

$$\mu(\mathcal{O}_K) = q\mu(\mathfrak{p}_K) \implies \mu(\mathfrak{p}_K) = 1/q, \qquad q := \sharp\kappa(K).$$

5.2 Norms

Remark 5.2.1: If $a \in K^\times$, one can define $\|a\|$ as the factor by which multiplication scales the Haar measure. E.g. for $X = [0,1]$ and the Lebesgue measure, for $\lambda \in \mathbb{R}^\times$ this yields $\|\lambda\| = \mu(\lambda X)$. Noting that $\lambda[0,1] = [0,\lambda]$ for $\lambda > 0$, this recovers $\|\lambda\| = |\lambda|$, the usual absolute value. In general, one can define

$$\|\lambda\| := \frac{\mu(\lambda X)}{\mu(X)}.$$

Remark 5.2.2: If we want to set $v(\pi_K) = 1$ and $\|\pi_K\| = \varepsilon^1$, what should one choose for ε? Using the Haar measure trick above, if $\mu(\mathcal{O}_K) = 1$,

$$\|\pi_K\| = \mu(\pi_K \mathcal{O}_K) = \mu(\mathfrak{p}_K) = 1/q,$$

so take $\varepsilon := 1/q$. Note that this needs to be chosen carefully to get a global product formula on adeles later.

Definition 5.2.3 (Norm)
So for $v_K : K^\times \to \mathbb{Z}$ and $\lambda \neq 0$, we define

$$\begin{aligned} \|-\| : K &\to \mathbb{Q}_{\geq 0} \\ \|\lambda\| &:= \left(\frac{1}{q}\right)^{v_K(\lambda)} = q^{-v_K(\lambda)}, \end{aligned}$$

which restricts to

$$\|-\| : K^\times \to \mathbb{Q}_{>0}.$$

We also set $\|0\| = 0$.

Remark 5.2.4: Under the identification, for $E = \mathbb{Q}$ or any field of characteristic zero, we now have an induced representation

$$\|-\| : W_K \to \mathrm{GL}_1(E).$$

Moreover, $\|-\|^m$ for $m \in \mathbb{Z}_{\geq 1}$ all produce representations of W_K.

Exercise 5.2.5 (?)
Show that

$$f(\|-\|^m) = 0.$$

Remark 5.2.6: What is $\left\|\widetilde{\mathrm{Frob}}\right\|$, where $\widetilde{\mathrm{Frob}} \in W_K$ is a lift of Frob $\in \mathbb{Z}$ along $W_K \twoheadrightarrow \mathbb{Z}$? Use that $r_K : K^\times \to W_K^{\mathrm{ab}}$ sends $\mathcal{O}_K^\times \pi_K$ to $\mathrm{Frob}^{-1} I_{K^{\mathrm{ab}}/K}$, and $\|\pi_K\| = 1/q$, so $\left\|\mathrm{Frob}^{-1}\right\| = 1/q$ and thus $\|\mathrm{Frob}\| = q$.

See Tate's article, "Number Theoretic Background"

5.3 Weil-Deligne Representations

Remark 5.3.1: We'll assume $\operatorname{ch} E = 0$ since we'll want to take norms of elements in W_K and regard them as elements in E, or really elements of $\mathbb{Q}_{>0}$.

Definition 5.3.2 (Weil-Deligne representations)
A **Weil-Deligne** representation is a pair (ρ_0, N) where

$$\rho_0 : W_K \to \operatorname*{Aut}_E(V) = \mathrm{GL}_n(E)$$

is a continuous representation where $V \cong E^n$, and $N \in \operatorname*{End}_E(V)$ is nilpotent. We also require a compatibility condition:

$$\rho_0(\sigma) N \rho_0(\sigma)^{-1} = \|\sigma\| N \qquad \forall \sigma \in W_K.$$

Remark 5.3.3: Note that this recovers the previous notion of representations by setting $N = 0$. For $N \neq 0$, taking $\sigma = \widetilde{\mathrm{Frob}}$ this equation reduces to saying that N is conjugate to qN for q a scalar. Since $\operatorname{Spec} qN = \left\{ q\lambda \;\middle|\; \lambda \in \operatorname{Spec} N \right\}$, so if there is any nonzero eigenvalue there are infinitely many, so the only eigenvalue is zero and this forces N to be nilpotent.

Example 5.3.4 *(?)*: Are there examples where $N \neq 0$ is nilpotent? We need dimension at least 2, so let $E = \mathbb{Q}$ and $V = \operatorname*{span}_{\mathbb{Q}} \{e_1, e_0\}$. Then

$$\rho_0(\sigma) e_1 = \|\sigma\|^1 e_1$$
$$\rho_0(\sigma) e_0 = \|\sigma\|^0 e_0$$

$$\rho_0(\sigma) = \begin{bmatrix} \|\sigma\| & 0 \\ 0 & 1 \end{bmatrix},$$

which is a direct sum of two 1-dimensional representation. Here $Ne_0 = e_1$ and $Ne_1 = 0$, so

$N = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Checking the equation:

$$\begin{bmatrix} \|\sigma\| & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} \|\sigma\|^{-1} & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & \|\sigma\| \\ 0 & 0 \end{bmatrix} = \|\sigma\| N.$$

Example 5.3.5 ***(The Steinberg representation):*** Let $V = \mathbb{Q}^n = \operatorname{span}_{\mathbb{Q}} \{e_0, \cdots, e_{n-1}\}$ and define $\rho_0(\sigma)e_i = \|\sigma\|^i e_i$ and $Ne_i = e_{i+1}$ with $Ne_{n-1} = 0$ yields a Weil-Deligne representation. This is the most basic indecomposable example, and the rest arise from tensoring with reps of W_K and taking direct sums.

Definition 5.3.6 (Semisimple Weil-Deligne Representations)
A Weil-Deligne representation (ρ_0, N) is F**-semisimple** if $\rho_0(\widetilde{\mathrm{Frob}})$ is a semisimple matrix (i.e. diagonalizable over $\overline{E}$), independent of the choice of lift of Frob.

6 Lecture 6: Representations of $\mathrm{GL}_n(k)$ and the Local Langlands Conjecture

Remark 6.0.1: The LLC will take the following form:

$$\left\{ \begin{matrix} n\text{-dimensional } F\text{-semisimple} \\ \text{Weil-Deligne representations of } W_k \end{matrix} \right\} / \sim \rightleftharpoons \left\{ \begin{matrix} \text{Irreducible admissible reps} \\ \text{of } \mathrm{GL}_n(K) \end{matrix} \right\},$$

where we're working toward describing the RHS.

⚠ **Warning 6.0.2**
Most representations here will be infinite dimensional.

Remark 6.0.3: Setup:

- $E \in \mathsf{TopField}$ with the discrete topology and any characteristic,
- $V \in \mathsf{E\text{-}Mod}$ an infinite-dimensional E-vector space,
- $K/\mathbb{Q}_p$ a finite extension,
- $\mathrm{GL}_n(K)$ has the p-adic topology, described by a bases of open neighborhoods of the identity:

$$\mathcal{B} := \left\{ M := \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{GL}_n(\mathcal{O}_K) \;\middle|\; M \equiv I_n \bmod \mathfrak{p}_K^m \,\forall m \in \mathbb{Z}_{\geq 1} \right\}.$$

- $\pi \in \mathsf{Grp}(\mathrm{GL}_n(K), \operatorname{Aut}_E(V))$, where we're not using the topology on E yet.

We want a sensible notion of continuity for π.

Definition 6.0.4 (Smooth representations)
A representations π is **smooth** iff stabilizers are open, i.e.

$$\forall v \in V,\ \mathrm{Stab}_\pi(v) := \left\{ g \in \mathrm{GL}_n(K) \;\middle|\; g.v := \pi(g)(v) = v \right\} \text{ is open in } \mathrm{GL}_n(K).$$

Definition 6.0.5 (Admissible representations)
A smooth representation π is **admissible** iff for all $U \leq \mathrm{GL}_n(K)$ open subgroups, the fixed set V^U is finite dimensional.

Remark 6.0.6: On terminology: admissible will always imply smooth.

Example 6.0.7 ***(?)***: If $\dim V = 1$ and $\pi(g) = 1$, then π is admissible. If $\dim V = \infty$ instead, this is smooth but not admissible, and in some sense is an infinite direct sum of trivial 1-dimensional reps.

Fact 6.0.8
Irreducible and smooth implies admissible.

Definition 6.0.9 (Irreducible representations)
A representation $\pi : G \to \mathrm{GL}(V)$ for $G = \mathrm{GL}_n(K)$ is **irreducible** iff there are only 2 G-invariant subspaces: 0 and V itself. Note that this still holds in infinite-dimensions in this case, i.e. we don't have to require the invariant subspaces to be closed.

Remark 6.0.10: By LCFT, we know $K^\times \xrightarrow{\sim} W_k^{\mathrm{ab}}$, but how do we understand W_K? Langlands' insight: reinterpreting this isomorphism in terms of representations. If the groups are isomorphic, their reps are isomorphic, so

$$\left\{ \substack{\text{Irreducible 1-dimension reps} \\ \text{of } K^\times = \mathrm{GL}_1(K)} \right\} / \sim \ \rightleftharpoons \left\{ \substack{\text{Irreducible 1-dimensional reps} \\ \text{of } W_K} \right\} / \sim$$

So the conjecture is that there should be a canonical bijection, say of sets, of the following form:

$$\mathsf{IrrRep}^{\dim = n}\, \mathrm{GL}_1(K) \rightleftharpoons \mathsf{IrrRep}^{\dim = n} W_k.$$

Next time we'll see how to generalize this to $n = 2$, where the $n = 1$ case is class field theory, and we'll see how to match things up.

7 | Lecture 7: Local Langlands

7.1 Statement of LLC

Conjecture 7.1.1 ***(The Local Langlands Correspondence for*** GL_n***).***
There is a canonical bijective correspondence of sets

$$\coprod_{n\geq 1} F\text{-}\mathsf{ssWDRep}_{/\sim} \rightleftharpoons \coprod_{n\geq 1} \mathsf{smIrrAdmRep}(\mathrm{GL}_n(K))_{/\sim}$$

$$\rho \rightleftharpoons \pi$$

The objects on the left are F-semisimple Weil-Deligne reps, on the right are *admissible* reps. Note that we've not yet seen interesting objects showing up naturally on either side! Both sides are somewhat pathological at this point.
Compare to the Taniyama-Shimura conjecture, where elliptic curves come from modular forms, and both show up naturally. See also the BSD conjecture – if you don't know an elliptic curve is modular, you can't meromorphically extend the L function.

Remark 7.1.2: Here "canonical" means that the bijection satisfies a long list of properties. Some examples of what should match up:

- Duality. A representation V will have a left G-action, so $V^\vee$ will have a right G-action, and composing with inversion makes it a left action again – typically this yields a different representation.
- Conductors. The conductor on the RHS is harder to define.
- L-functions.
- ε factors.
- ε factors of pairs – these turn out to be crucial!
- $\cdots$

This list of properties became so long that there is now a theorem showing that there is at most one bijection satisfying all of them. Another theorem shows that there is *at least* one bijection.

- In the function field case, this was established in the 80s (Laumm, Rapoport, Stuhler).
- In the p-adic field case, this was proved in 2000 (Harris-Taylor)

All of the proofs are global, i.e. use number fields, despite the fact that Galois groups of number fields are more complicated than Galois groups of local fields.

Remark 7.1.3: Philosophy: this gives a direction to go after class field theory, and highlights that G_K isn't the fundamental object, but rather the category $\mathsf{Rep}(G_K)$. In analogy, note that $\pi_1 X \in \mathsf{Grp}$ is not canonically defined, but $\Pi_1 X \in \mathsf{Grpd}$ is. Defining the group depends on choice of a base point, and a path $x_0 \to x_1$ produces an isomorphism $\pi_1(X, x_0) \to \pi_1(X, x_1)$, but e.g. on a torus the paths can wind around genus so that two choices of path need not be homotopic. However, there is a theorem that says $\mathsf{Rep}(\pi_1 X) \xrightarrow{\sim} \mathsf{LocSys}(X)$. In this analogy, choosing a point for π_1 is like choosing an algebraic closure $\overline{K}/K$.

Remark 7.1.4: Next goal: finding natural examples of the LHS of the correspondence, and at least some global occurrences on the RHS, and show that the LLC is useful. We'll consider

- The $n = 1$ case, recovering CFT,
- Weil-Deligne representations showing up naturally,
- Examples of smooth irreducible admissible reps π, e.g. in the cohomology of Shimura varieties.

7.2 The $n = 1$ case

Conjecture 7.2.1.
If $G \in \mathsf{AlgGrp}_{/K}$ is connected and reductive, i.e. over $\overline{K}$, G is isomorphic to one of

$$\mathrm{GL}_n, \mathrm{SL}_n, \mathrm{PGL}_n, \mathrm{O}_n, \mathsf{Sp}_{2g}, E_6, E_7, E_8, F_4, G_2,$$

then there is a LLC for G: there is a canonical surjection

$$F \text{ finite fibers} \hookrightarrow \left\{ \text{Certain Weil-Deligne reps}(\rho, N) : W_K \to {}^L G(\mathbb{C}) \right\} \twoheadrightarrow \mathsf{smIrrRep}(G(K))$$

where ${}^L G$ is an L-group for G and the fibers are referred to as **L-packets**. This surjection is similarly supposed to satisfy a big list of properties, but it is not known if these uniquely characterize the surjection.

Remark 7.2.2: Consider $n = 1$ dimensional reps, so on the LHS we have pairs $(\rho, N) : W_K \to \mathrm{GL}_1(\mathbb{C})$ where N being 1×1 nilpotent forces $N = 0$. Note that $\ker \rho_0$ is closed and the quotient is Hausdorff and abelian, so ρ_0 factors through W_K^{ab}. So the LHS reduces to 1-dimensional continuous reps of W_K^{ab} over $\mathbb{C}$.

A coincidence in dimension 1: the RHS reads smooth admissible irreducible reps of $K^\times$, and one can show that admissible and irreducible implies $\dim \pi < \infty$ and further $\dim \pi = 1$. This needs that we have compact open normal subgroups, and this fails quite seriously for GL_2. In the $n = 1$

case, continuity is equivalent to smooth and admissible, so we're considering

$$\pi \in \mathsf{TopGrp}(K^\times, \mathbb{C}^\times), \qquad \rho \in \mathsf{TopGrp}(W_L^{\mathrm{ab}}, \mathbb{C}^\times).$$

and these are equal by local class field theory.

Remark 7.2.3: A source of Weil-Deligne reps: **ℓ-adic representations**. Let $K/\mathbb{Q}_p$ be finite, and suppose $\rho \in \mathsf{TopGrp}(G_K, \mathrm{GL}_n(\mathbb{Q}_\ell))$ where G_K is a profinite topological group with the profinite topology, and $\mathbb{Q}_\ell$ has ℓ-adic topology. Assume also $\ell \neq p$.

Note that the target is now *not* discrete, so there won't be finite inertia. The wild part of inertia will be pro-p and end up being finite, while the tame part will have a large ℓ-adic component. In the discrete case, we knew the image of inertia was finite by a compactness argument, but that may not hold here.

Example 7.2.4*(of where Weil-Deligne reps show up):* These show up in nature, e.g.

- In the ℓ-adic Tate module $T_\ell(E/K)$ of an elliptic curve E/K.
- In ℓ-adic etale cohomology $H^i_{\text{ét}}(X_{\overline{K}}; \mathbb{Q}_\ell)$. The vast majority of Weil-Deligne representations come from here.
- In ℓ-adic deformations of the above, i.e. taking an ℓ-adic representation, reducing mod ℓ, and try to deform back into characteristic zero. Some families come reps of $\mathrm{GL}_n(R)$ for R an affinoid object. See also eigencurves.

These are slightly more difficult because we can no longer control inertia.

Remark 7.2.5: Given a ρ, one can construct a WD rep. One example of where $T_\ell(E/K)$ is easy to compute: if E is an elliptic curve with split (bad) multiplicative reduction, then E can be uniformized to obtain

$$E(\overline{K}) \cong \overline{K}^\times / q^{\mathbb{Z}} \qquad q \in K, |q| < 1.$$

This

$$T_\ell E|_{E_{\overline{K}/K}} = \begin{bmatrix} \chi & * \\ 0 & \mathrm{id} \end{bmatrix},$$

where χ can be a nontrivial infinite cyclotomic character. Note that this yields infinite inertia.

Remark 7.2.6: Recall that if ρ is an ℓ-adic representation as above, $\rho(I_{\overline{K}/K})$ can be infinite but can't be too bad, e.g. $\rho(I^{\varepsilon}_{\overline{K}/K})$ must be finite for $\varepsilon > 0$ since it is pro-p. The tame inertia isn't so bad: we have

$$\mathsf{Gal}(K^t/K^{\mathrm{un}}) \cong \prod_{r \neq p \text{ prime}} \mathbb{Z}_r,$$

and we should worry about the $\mathbb{Z}_\ell$ component. To isolate this part, fix $t \in \mathsf{Gal}(K^t/K^{\mathrm{un}}) \twoheadrightarrow \mathbb{Z}_\ell$ and $\varphi \in G_K$ lifting $\mathrm{Frob} \in G_{K^{\mathrm{un}}}$. The full G_K now breaks into three stages:

- Unramified: controlled by what happens to φ,
- Tamely ramified: controlled by t,
- Wildly ramified: pro-p and hence finite.

Proposition 7.2.7 *(Grothendieck)*.
If $\rho \in \mathsf{TopGrp}(G_K, \mathrm{GL}_n(E))$ with $E/\mathbb{Q}_\ell$ is finite, then there exists a unique Weil-Deligne representation $(\rho, N) \in \mathsf{TopGrp}(W_K, \mathrm{GL}_n(E)^{\mathrm{disc}})$. This satisfies the following: if $\sigma \in I_{\overline{K}/K}$, for every $m \in \mathbb{Z}_{\geq 0}$,

$$\rho(\varphi^m \sigma) = \rho_0(\varphi^m \sigma) \exp\left(N \cdot t(\sigma)\right),$$

where nilpotence of N guarantees that the series expansion for $\exp(-)$ is finite.

Remark 7.2.8: Note that we can always restrict ρ using that $W_K \leq G_K$, but this may not be continuous. Since we dropped the topology on $\mathrm{GL}_n(E)$, it makes it harder for ρ_0 to be continuous.

Remark 7.2.9: Not all WD reps show up this way, since $\rho(\varphi)$ has restrictions on its eigenvalues – if (ρ_0, N) arises this way, eigenvalues of $\rho(\varphi)$ will be in $\mathbb{Q}_\ell{}^\times$. However, this is essentially the only obstruction: given (ρ_0, N) with $\lambda_i \in \mathbb{Q}_\ell{}^\times$, then it comes from a ρ.

7.3 Smooth admissible reps π of $\mathrm{GL}_n(K)$

Remark 7.3.1 *(on conductors)*: Given $\pi : K^\times \to \mathbb{C}^\times$ smooth admissible irreducible, define

$$f(\pi) = \begin{cases} 0 & \pi|_{\mathcal{O}_K^\times} = 1 \\ r & \text{if } r \in \mathbb{Z}_{>0} \text{ is minimal such that } \pi|_{1+\mathfrak{p}_K^r \mathcal{O}_K} = 1. \end{cases}.$$

As r increases, this gives a basis of neighborhoods of the identity. In the LLC for $n = 1$, if $\rho_0 = (\rho_0, N = 0) \rightleftharpoons \pi$, it turns out that $f(\rho_0) = f(\pi)$.

7.4 The $n = 2$ case

Remark 7.4.1: Constructing a π: given a pair of characters, one can construct a representation of $\mathrm{GL}_2(K)$. Let $\chi_1, \chi_2 \in \mathsf{TopGrp}(K^\times \to \mathbb{C}^\times)$ and define

$$I(\chi_1, \chi_2) := \left\{ \varphi : \mathrm{GL}_2(K) \to \mathbb{C} \;\middle|\; \varphi \text{ is locally constant}, \varphi\left(\begin{bmatrix} a & b \\ 0 & d \end{bmatrix} g \right) = \chi_1(a)\chi_2(d) \left\| \frac{a}{d} \right\|^{\frac{1}{2}} \varphi(g) \right\}.$$

Remark 7.4.2: The norm term is a fudge factor!

Note that since the source is totally disconnected, there can be many locally constant but non-constant functions, e.g. $\chi_{\mathbb{Z}_p}(x)$ on $\mathbb{Q}_p$. This is a vector space under pointwise operations in the target, and is supposed to look like $\mathrm{Ind}_B^G V$ for V a 1-dimensional representation of $B = \begin{bmatrix} * & * \\ 0 & * \end{bmatrix}$. Now define

$$\pi : \mathrm{GL}_2(K) \to \operatorname*{Aut}_{\mathbb{C}\text{-Mod}}(I(\chi_1, \chi_2))$$
$$g \mapsto \varphi \mapsto (h \mapsto \varphi(hg)),$$

so that $(\pi(g)\varphi)(h) = \varphi(hg)$ for all $g, h \in \mathrm{GL}_2(K)$. One can check that this defines an action – write this as $g.\varphi(h) = \varphi(hg)$, then one needs to check that $(g_1 g_2)\varphi = g_1(g_2\varphi)$. Consider instead writing $(x)f := f(x)$, then the equation reads $(h)(g\varphi) = (hg)\varphi$ and the check is $(h)(g_1 g_2(\varphi)) = (hg_1)(g_2\varphi)$, and one can rearrange the brackets by definition. Thus this yields a representation of $\mathrm{GL}_2(K)$.

8 | Lecture 8

Question 8.0.1
Is this representation π smooth, admissible, and irreducible?

Lemma 8.0.2 *(?).*
Let $B(K) := \left\{ \begin{bmatrix} * & * \\ 0 & * \end{bmatrix} \in \mathrm{GL}_2(K) \right\} \leq \mathrm{GL}_2(K)$, then there is a decomposition into upper triangular and integer matrices:

$$\mathrm{GL}_2(K) = B(K) \cdot \mathrm{GL}_2(\mathcal{O}_K) := \left\{ bg \;\middle|\; b \in B(K), g \in \mathrm{GL}_2(\mathcal{O}_K) \right\}.$$

Remark 8.0.3: Why this is useful: $\mathrm{GL}_2(\mathcal{O}_K)$ is compact, so locally constant functions $\varphi : \mathrm{GL}_2(\mathcal{O}_K) \to \mathbb{C}$ will only take finitely many values (using that continuous images of compact sets are compact and compact subsets of discrete spaces are finite). So $\varphi(\mathrm{GL}_2(\mathcal{O}_K))$ is a finite set, and $\varphi(B(K))$ is controlled by $I(\chi_1, \chi_2)$.

Proof (Specializing a proof for GL_n*).*
Let $\gamma := \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{GL}_2(K)$, then we want to produce β, κ with $\gamma = \beta\kappa$.

- Without loss of generality $\gamma \in \mathrm{SL}_2(K)$ by left-multiplying by $\begin{bmatrix} (\det\gamma)^{-1} & 0 \\ 0 & 1 \end{bmatrix} \in B(K)$.
- Without loss of generality, $c, d \in \mathcal{O}_K$ and at least one is a unit: To scale c, d, choose $\alpha \in K^\times$ so that $ac, ad \in \mathcal{O}_K$ and at least one is in $\mathcal{O}_K^\times$ since not both of c, d are zero. Left-multiply by $\begin{bmatrix} \alpha^{-1} & 0 \\ 0 & \alpha \end{bmatrix} \in B(K)$ to send $c \to \alpha c, d \to \alpha d$.
- Without loss of generality, $c \in \mathcal{O}_K^\times$. If $d \in \mathcal{O}_K^\times$ and $c \notin \mathcal{O}_K^\times$, right-multiply by $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \in$

$\mathrm{SL}_2(\mathcal{O}_K)$ to swap c, d

- Check $\begin{bmatrix} 1 & -a/c \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & -1/c \\ c & d \end{bmatrix} \in \mathrm{GL}_2(\mathcal{O}_K)$, noting that the first matrix is in $B(K)$.

■

Remark 8.0.4: More generally, for GL_n the Weyl group S_n is involved.

Exercise 8.0.5 (A really good one.)
Show that $I(\chi_1, \chi_2)$ is smooth and admissible.

Observation 8.0.6
The norm term in $I(\chi_1, \chi_2)$ is a fudge factor, and

$$\left\| \frac{a}{d} \right\|^{\frac{1}{2}} = \frac{\|a\|^{\frac{1}{2}}}{\|d\|^{\frac{1}{2}}}.$$

One could redefine

$$\tilde{\chi}_1(x) \coloneqq \chi_1(x)\|x\|^{-\frac{1}{2}}$$
$$\tilde{\chi}_2(x) \coloneqq \chi_2(x)\|x\|^{\frac{1}{2}}$$

to make the formula read

$$\varphi \begin{bmatrix} a & b \\ c & d \end{bmatrix} g = \tilde{\chi}_1(a)\tilde{\chi}_2(d)\varphi(g).$$

⚠ **Warning 8.0.7**
The fudge factor is needed here: taking the trivial characters will yield a subspace of constant functions in $I(\chi_1, \chi_2)$ when the χ_i are trivial reps. When $\chi_1 = \chi_2$, there is an invariant 1-dimensional subrepresentation, and is thus not irreducible.

Remark 8.0.8 ***(on group representations):*** Recall that for $H \leq G \in \mathsf{Grp}$ with $[G : H] < \infty$, one can induce H-representations to G-representations. If χ is a character of H, then $\mathrm{Ind}_H^G \chi$ is a character of G. If χ is trivial, its induction will be unlikely to be irreducible since it contains the 1-dimensional trivial representation.

Exercise 8.0.9 (?)
Show that if $\chi_1/\chi_2 = \|-\|_K^{\pm 2}$, then the naive definition of $I(\chi_1, \chi_2)$ admits a 1-dimensional quotient.

Remark 8.0.10: What's really going on with $I(\chi_1, \chi_2)$: the pair (χ_1, χ_2) is a 1-dim representation ψ of B, and we're writing down an explicit model for $\mathrm{Ind}_B^{\mathrm{GL}_2} \psi$. Moreover there is a pairing involving an integral on G and B with respect to a Haar measure:

$$I(\chi_1, \chi_2) \times I(\chi_1^{-1}, \chi_2^{-1}) \to \mathbb{C}.$$

Where the fudge factor comes from: at some point one changes a left Haar measure to a right one, and there is a fudge factor of $\|a/d\|$, so one splits it between the two terms to yield a duality $I(\chi_1, \chi_2)^\vee \cong I(\chi_1^{-1}, \chi_2^{-1})$.

Fact 8.0.11
$I(\chi_1, \chi_2)$ is irreducible if $\chi_1/\chi_2 \neq \|-\|^{\pm 1}$.

Exercise 8.0.12 (?)
If $\chi_1/\chi_2 = \|-\|^{-1}$, show that there is a 1-dimensional representation Ψ of $\mathrm{GL}_2(K)$ given by $(\chi_1 \cdot \|-\|^{\frac{1}{2}}) \circ \det$ fitting into a SES

$$0 \to \psi \to I(\chi_1, \chi_2) \to S(\chi_1, \chi_2) \to 0,$$

where S is irreducible. If instead $\chi_1/\chi_2 = \|-\|^1$, then there is a SES

$$0 \to S(\chi_1, \chi_2) \to I(\chi_1, \chi_2) \to (\chi_2 \cdot \|-\|^{\frac{1}{2}}) \circ \det \to 0.$$

Remark 8.0.13: We've induced characters to get reps of $\mathrm{GL}_2(K)$, how do we get more? An observation due to Weil: if K is any field,

$$\left\langle T := \begin{bmatrix} t & 0 \\ 0 & t^{-1} \end{bmatrix}, U := \begin{bmatrix} 1 & u \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \;\middle|\; \cdots \right\rangle \cong \mathrm{SL}_2(K),$$

where there are some explicit relations. So we can cook up $\mathrm{SL}_2(K)$ reps by specifying these generators and checking that the relations hold. Note that $\mathrm{GL}_2(K)$ is an extension of $\mathrm{SL}_2(K)$ by an abelian group, so this can be used to construct GL_2 reps.

Remark 8.0.14: A large source of $\mathrm{SL}_2(K)$ and hence $\mathrm{GL}_2(K)$ reps: let $\mathrm{SL}_2(K) \curvearrowright L^2(K; \mathbb{C})$ by

$$\begin{aligned}(Tf)(x) &= f(tx) \\ (Uf)(x) &= f(u + x) \\ (Wf)(x) &= \widehat{f}(x),\end{aligned}$$

i.e. W acts by the Fourier transform and is given by an explicit integral.

Fact 8.0.15
From Jacquet-Langlands: if

- L/K is a quadratic extension,
- $\chi : L^\times \to \mathbb{C}^\times$ is admissible,
- $\chi \neq \chi \circ \sigma$ for any nontrivial $\sigma \in \mathsf{Gal}(L/K)$ (i.e. χ is not its own Galois conjugate),

Jacquet-Langlands construct an irreducible infinite-dimensional representation of $\mathrm{GL}_2(K)$ using $L^2(L)$ and Fourier transforms. We'll call this representation $\mathrm{BC}_L^K(\psi)$, for *base change*.

Fact 8.0.16
We have a collection of infinite-dimensional smooth admissible irreducible representations of $\mathrm{GL}_2(K)$.

- $I(\chi_1, \chi_2) \cong I(\chi_2, \chi_1)$ which is irreducible when $\chi_1/\chi_2 \neq \|-\|^{\pm 1}$
- $S(\chi, \chi \cdot \|-\|)$
- $\mathrm{BC}_L^K(\psi)$.

Moreover, for $\operatorname{ch} \kappa_K = p > 2$, these exhaust all such representations. For $p = 2$, there are exceptional representations. So we know all possibilities for π.

⚠ **Warning 8.0.17**
The obvious representations of $\mathrm{GL}_2(K)$ like the 2-dimensional representation ${K_{\mathrm{disc}}^{\times}}^2$ work, due to the p-adic topology – the usual action has stabilizers which are closed but not open. The only finite-dimensional representations are of the form $\chi \circ \det$ for $\chi : K^\times \to \mathbb{C}^\times$ a character.

9 | Lecture 9

Remark 9.0.1: Recall that we've seen the following reps of $\mathrm{GL}_2(K)$ for $K \in \mathsf{Field}_{p\text{-adic}}$:

- $I(\chi_1, \chi_2) \subseteq \mathsf{Grp}(\mathrm{GL}_2(K), \mathbb{C}_{\mathrm{disc}})$ with $\chi_1/\chi_2 \neq \|-\|^{\pm 1}$,
- $S(\chi, \chi \times \|-\|)$, a subset/quotient of I,
- $\mathrm{BC}_L^K(\psi) \in \mathsf{Grp}(L^\times, \mathbb{C}^\times)$ admissible with $\psi \neq \psi \circ \sigma$,
- $\chi \circ \det$, a subset/quotient of I which is 1-dimensional.

For $p > 2$, these are all of the smooth admissible irreducible reps.

Exercise 9.0.2 (?)
Prove that I is smooth and admissible.

9.1 Conductors

Remark 9.1.1: Let π be a smooth admissible irreducible representation of $\mathrm{GL}_2(K)$ of infinite dimension. Note that these definitions generalize to G any connected reductive algebraic group, and there's a notion of genericity for π – it turns out that for GL_2, π is generic iff $\dim \pi = \infty$.

Theorem 9.1.2 ***(Casselman).***
For $n \geq 0$, define

$$G := U_1(\mathfrak{p}_K^n) = \left\{ M := \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{GL}_2(\mathcal{O}_K) \;\middle|\; M \equiv \begin{bmatrix} * & * \\ 0 & 1 \end{bmatrix} \bmod \mathfrak{p}^n \right\} \leq \mathrm{GL}_2(K),$$

which is a local analog of $\Gamma_1(p^n)$ for modular forms. These are all compact and open, and therefore by admissibility

$$d(\pi, n) := \dim \pi^G < \infty,$$

where the RHS denotes the G-invariants. There is a *conductor* $\mathfrak{f}(\pi) \in \mathbb{Z}_{\geq 0}$ such that

$$d(\pi, n) = \max(0, 1 + n - \mathfrak{f}(\pi)) \qquad n \geq 0,$$

i.e. the sequence is zeros for a fixed number of n depending on π and increases linearly beyond that.

⚠ **Warning 9.1.3**
Conductors are delicate!

Remark 9.1.4: For a complicated representation, one would expect the first few of these invariants to be zero, since this is asking for invariants under the action of a maximal compact subgroup in the case $n = 0$ (which recover $\mathrm{GL}_2(\mathcal{O}_K)$), but the groups get smaller as n increases so $d(\pi, n)$ generally increases.

Remark 9.1.5: A glimpse at the proof: for interesting reps of $\mathrm{GL}_2(K)$, one can use *Whittaker models* that realize π as a set of functions on K.

Exercise 9.1.6 (?)
Compute the conductors for I and S:

- $\mathfrak{f}(I(\chi_1, \chi_2)) = \mathfrak{f}(\chi_1) + \mathfrak{f}(\chi_2)$
- $\mathfrak{f}(S(\chi, \chi\|\|)) = 1$ if $\mathfrak{f}(\chi) = 0$ (the unramified case) and is $2\mathfrak{f}(\chi)$ if $\mathfrak{f}(\chi) > 0$ (the ramified case).

9.2 Central Characters

Exercise 9.2.1 (?)
Show that Schur's lemma holds: if π is an irreducible admissible representation of $\mathrm{GL}_2(K)$ then there is an admissible character $\chi_\pi : K^\times \to \mathbb{C}^\times$ such that for all $\lambda \in K^\times = Z(\mathrm{GL}_n(K))$,

using the diagonal embedding

$$\begin{aligned} K^\times &\to \mathrm{GL}_n(K) \\ \lambda &\mapsto \mathrm{diag}(\lambda, \lambda, \cdots, \lambda), \end{aligned}$$

the action is given by

$$\lambda.\pi = \chi(\pi)\lambda$$

where $\chi(\pi)$ is a scalar called the **central character**.

Exercise 9.2.2 (?)
Show that

- $\chi_{I(\chi_1,\chi_2)} = \chi_1\chi_2$.
- $\chi_{S(\chi,\chi\|\|)} = \chi_1\chi_2$.
- $\chi_{\varphi\circ\det} = \varphi^2$.

9.3 Local Langlands for GL_2

Remark 9.3.1: We'll assume LLC for GL_1. Some notation: for $\chi_i \in \mathsf{Grp}(K^\times, \mathbb{C}^\times)$, attach $\rho_i \in \mathsf{Grp}(W_K, \mathbb{C}^\times)$. Note that $I(\chi_1, \cdots, \chi_n)$ makes sense for $\mathrm{GL}_n(K)$, and the fudge factor here is ρ, the half-sum of positive weights. There would be a corresponding semisimple Galois representation given by $\rho_1 \oplus \cdots \rho_n$.

Remark 9.3.2: We'll match up infinite dimensional reps of $\mathrm{GL}_2(K)$ with 2-dimensional reps of G_K:

Weil-Deligne Representations	Galois Representations
π	ρ
$I(\chi_1, \chi_2)$	$(\rho_0 = \rho_1 \oplus \rho_2, N = 0)$
$S(\chi_1, \chi_1 \cdot \|-\|)$	$\left(\rho_0 = \begin{bmatrix} \|-\|\rho_2 & 0 \\ 0 & \rho_1 \end{bmatrix}, N = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right)$
$\chi_1 \circ \det$	$\left(\rho_0 = \begin{bmatrix} \rho_1 \cdot \|-\|^{\frac{1}{2}} & 0 \\ 0 & \rho_1 \cdot \|-\|^{-\frac{1}{2}} \end{bmatrix}, N = 0\right)$
$\mathrm{BC}_L^K(\psi), \psi \in \mathsf{Grp}(L^\times, \mathbb{C}^\times) \rightsquigarrow_{\mathrm{CFT}} \sigma \in \mathsf{Grp}(W_L, \mathbb{C}^\times)$	$\left(\rho_0 = \mathrm{Ind}_{W_L}^{W_K} \sigma, N = 0\right)$
Conductors $\mathfrak{f}(\pi)$	$\mathfrak{f}(\rho_0, N)$
Central characters χ_π	$\det \rho_0$

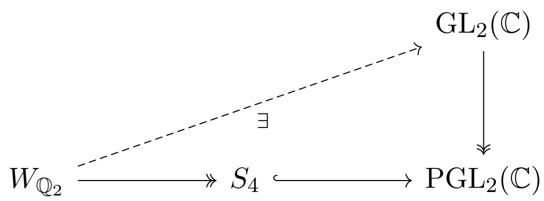
Note that many of these are reducible, and an irreducible π can be matched with a reducible ρ. In general, irreducible WD reps get matched with the most complicated π reps, the **supercuspidal** ρ (here BC).

For $p = 2$, there are more things on both sides – on the right, it turns out there is an S_4 extension of $\mathbb{Q}_2$. This can't happen for $\mathbb{Q}_p$ for $p > 2$:

- Wild inertia must be trivial, since it is pro-p,
- Tame inertia must be cyclic
- The unramified extension must also be cyclic

So this would force S_4 to have a cyclic subgroup with a cyclic quotient, which doesn't exist. Thus no extension of a p-adic field for $p \geq 3$ can be an S_4 extension.

Using such an S_4, one can find a lift of the following form:



Link to Diagram

The indicated lift exists by a theorem of Tate, and is measured by an element of $H^2(W_{\mathbb{Q}_2}; \mathbb{C}^\times) = 1$. Using this, one gets a 2-dimensional representation $W_{\mathbb{Q}_2} \to \mathrm{GL}_2(\mathbb{C})$ which will be irreducible due to the S_4, and won't be on the list above since the induced representation will be of generalized dihedral type, i.e. the image will have an index 2 normal (and thus abelian) subgroup, while S_4 has no such subgroup.

Remark 9.3.3: A good way of getting irreducible reps is inducing up.

Remark 9.3.4: The ρ side is definitely the less complicated story! So far, we haven't seen any natural π reps – the ones we'll build will come from modular forms.

Definition 9.3.5 (Conductors and determinants of a Weil-Deligne representation)
The **conductor** of a Weil-Deligne representation (ρ_0, N) is defined as

$$\mathfrak{f}(\rho_0, N) = \mathfrak{f}(\rho_0) + \dim\left(\frac{V^S}{(\ker N)^S}\right), \ S := I_{\overline{K}/K},$$

where the second term is 0 when $N = 0$. The **determinant** is the 1-dimensional representation

$$\det(\rho_0) \in \mathsf{Grp}(W_K, \mathbb{C}^\times).$$

Exercise 9.3.6 (A computation)
Check that if the LLC for GL_2 hold, there are matchings

- $\pi \rightleftharpoons (\rho_0, N)$
- $\mathfrak{f}(\pi) \rightleftharpoons f(\rho_0, N)$
- $\chi_\pi \rightleftharpoons \det \rho$ by LCFT

Exercise 9.3.7 (?)
Check that if $p > 2$, all of the F-semisimple 2-dimensional WD reps of W_K are listed above. Use Tate's "Number Theoretic Background" 2.2.5.2, which says that all irreducible representations of W_K that aren't inductions must have dimension p^k for some k.

Remark 9.3.8: Conductors measure to what extent something is ramified. Here $\mathfrak{f}(\pi)$ controls a dimension.

:::{.remark "Title on the unramified case"} Consider the $\mathfrak{f}(\pi) = 0$ unramified case. This forces

- $\pi = I(\chi_1, \chi_2)$ (these are called **principal series representations**), and the characters factor: $\chi_i : K^\times \to K^\times / \mathcal{O}_K^\times \to \mathbb{C}^\times$. The χ_i are 1-dimensional and unramified.

- $\pi = \det \circ \chi$ where χ factors as above. On the W_K side, this forces $\rho = \rho_1 \oplus \rho_2$ and $\rho_i : W_K \to W_K / I_{\overline{K}/K}$ and $N = 0$.

:::

Remark 9.3.9: Supposing $\dim \pi = \infty$ and $\mathfrak{f}(\pi) = 0$, the invariants π^G for $G = \mathrm{GL}_2(\mathcal{O}_K)$ satisfy $\dim \pi^G = 1$. So from an infinite-dimensional rep, we isolate a nontrivial 1-dimensional subspace which carries information about π.

This construction works extremely generally: let $G \in \mathsf{AlgGrp}_{/K}$ be connected reductive and unramified, e.g. $G = \mathrm{GL}_n$, and let π be a smooth admissible representation of $G(K)$. We say π **is unramified** if there exists a hyperspecial maximal compact subgroup $H \leq G(K)$ (e.g. $H = \mathrm{GL}_n(\mathcal{O}_K)$) with $\pi^H \neq 0$.

Note that G being unramified means there is a model over $\mathcal{O}_K$, so a group scheme over $\operatorname{Spec} \mathcal{O}_K$, and $G(\mathcal{O}_K)$ is a hyperspecial maximal compact subgroup.

Remark 9.3.10: The invariants π^H here will be the finite piece we pick out of an infinite dimensional representation – unfortunately it is not $\mathrm{GL}_2(K)$ invariant The trick: use Hecke operators. Let $G = \mathrm{GL}_2(K)$, or any locally compact totally disconnected topological group and π be an admissible representation of G. If $U, V \leq G$ are compact open subgroups, e.g. $U = U_1(\mathfrak{p}_K^n)$ or $\mathrm{GL}_2(\mathcal{O}_K)$, and if $g \in G$ then there exists a Hecke operator

$$\begin{aligned} [UgV] \in \mathbb{C}\text{-}\mathsf{Mod}(\pi^V \to \pi^U) \\ x \mapsto \sum_{1 \leq i \leq r} g_i x, \end{aligned}$$

where we take a coset decomposition $UgV = \coprod_{1 \leq i \leq r} g_i V$, which must be finite using compactness and that V is open. This is a type of averaging process or a trace.

Exercise 9.3.11 (?)
Show $UgV \in \pi^U$ and is independent of the choices of g_i.

Remark 9.3.12: Now just consider $\mathrm{GL}_2(K)$ with $\mathfrak{f}(\pi) = 0$ and $U = V = \mathrm{GL}_2(\mathcal{O}_K)$. Define a map

$$T := [UgV] \in \mathbb{C}\text{-Mod}(\pi^U \to \pi^U), \qquad g := \begin{bmatrix} \pi_K & 0 \\ 0 & 1 \end{bmatrix} S := [UhV] \in \mathbb{C}\text{-Mod}(\pi^U \to \pi^U), \qquad h := \begin{bmatrix} \pi_K & 0 \\ 0 & \pi_K \end{bmatrix}.$$

Since $\dim \pi^U = 1$, T and S act by scalars $t, s \in \mathbb{C}$.

Exercise 9.3.13 (Forces unraveling definitions)
If $\pi = I(\lambda_1, \lambda_2)$, $\chi_1/\chi_2 = \|{-}\|^{\pm 1}$, and $\mathfrak{f}(\pi) = 0$, find the values of t and s. Possible solutions:

- $t = \sqrt{q_K} \cdot (\alpha + \beta)$ where q_K is the size of the residue field
- $s = \chi_\pi(\pi_K) = \alpha\beta$ where $\alpha = \chi_1(\pi_K), \beta = \chi_2(\pi_K)$.

Also do the 1-dimensional unramified case. As a consequence, show that if π is an admissible irrep of $\mathrm{GL}_2(K)$ and $\mathfrak{f}(\pi) = 0$, then π is 1-dimensional or $\pi = I(\chi_1, \chi_2)$. You can use that BC don't have conductor zero. By LLC, π should match up with $N = 0$, $\rho_0 : W_K \to W_K / I_{\overline{K}/K} \cong \mathbb{Z}$, which contains Frob, and we can further embed $\mathbb{Z} \hookrightarrow \mathrm{GL}_2(\mathbb{C})$. Then $\rho_0(\mathrm{Frob})$ has characteristic polynomial

$$x^2 - -\frac{t}{\sqrt{q_K}} x + s.$$

Remark 9.3.14: What's the point? In the unramified case, we can construct the LLC in a more conceptual way: isolated a 1-dim space on which Hecke operators act, choose them carefully and look at their eigenvalues t, s, and some magic machine produces the punchline above.

Exercise 9.3.15 (?)
More generally for $G = \mathrm{GL}_n(K)$, one can take $T_i = U \operatorname{diag}(\pi_K, \cdots, \pi_K, 1, \cdots, 1) U$ with i copies of π_K where $U := \mathrm{GL}_n(\mathcal{O}_K)$. Then the T_i eigenvalue is t_i, what is $\mathrm{charpoly}_{\rho_0(\mathrm{Frob})}(x)$?

Remark 9.3.16: If $G = G(K)$ with K unramified with π an unramified representation of G, the Langlands reinterpretation of the **Satake correspondence** associates to π a semisimple conjugacy class C in ${}^L G(\mathbb{C})$. In this case, we'll have $\pi \to \rho_0 \cdot \rho_0(\mathrm{Frob}) = C$.

So even though the LLC for a general G is not well-defined, there is a conjectural map from π to ρ, and in the simplest case where everything is unramified, LLC sends an unramified π to an unramified ρ, which is the same data as specifying the image of Frobenius, i.e. a choice of conjugacy class.

10 | Lecture 10: Part 2, Global Langlands

Remark 10.0.1: Global means number fields, which are harder than local fields but perhaps more familiar. From now on, $K \in \mathsf{NumberField}$, so $K/\mathbb{Q}$ is finite. We'll study the global groups $\mathsf{Gal}(L/K)$ and its relation to local Galois groups – taking limits will produce structure on G_K. There is an analog of the Weil group in the global case, but it's much more complicated – similar to how we needed to replace local Weil groups with WD reps, we'll need to replace global Weil groups with "global Langlands groups", which won't quite be defined.

The same machines for producing ℓ-adic representations of G_K in the local setting will work here:

- $T_\ell X$ the ℓ-adic Tate module of X an elliptic curve or abelian variety,
- $H_*^{\text{ét}}(X; \mathbb{Q}_\ell)$ for $X \in \mathsf{sm\,Proj\,Var}_{/K}$,

yielding a ρ side. The π side will be **automorphic representations**, which we'll define.

Conjecture 10.0.2 ***(Global Langlands).***
Every automorphic representation of $\mathrm{GL}_2(K)$ corresponds to a 2-dimensional representation of the global Langlands group.

Remark 10.0.3: Locally, given an ℓ-adic representation of G_K, a proposition of Grothendieck let us construct a WD rep, but we don't get all such WD reps this way since there was an ℓ-adic unit issue. Here, they'll give us a source of automorphic reps, but perhaps not all of them. However, reps that are "algebraic" (so those with a good p-adic Hodge theory) should match on both sides.

Conjecture 10.0.4.
There is a correspondence between automorphic reps π of $\mathrm{GL}_n(K)$ and motivic n-dimensional representations $\rho : G_K \to \mathrm{GL}_n(\overline{\mathbb{Q}_\ell})$ which are unramified away from a finite set of places which support a good de Rham and p-adic Hodge theory. Both should come from motives, and this should be compatible with the LLC in the following sense: global ℓ-adic reps ρ of G_K should restrict to local ℓ-adic representations, and global algebraic automorphic representations π should be related to local πs.

Remark 10.0.5: Note that the global Langlands conjecture above is essentially uncheckable, except perhaps for GL_1 where it suffices to understand the abelianization of the global Langlands group $(?)^{\mathrm{ab}} \cong K^\times \backslash \mathbb{A}_K^\times$. This can be handled with global class field theory.

10.1 Galois Groups

Remark 10.1.1: Let $K/\mathbb{Q}$ be a finite Galois extension, then $K \supseteq \mathcal{O}_K$ – note that when K is local, $\mathcal{O}_K$ is a DVR and has a unique prime ideal, but e.g. if $K = \mathbb{Q}$ then $\mathcal{O}_K = \mathbb{Z}$ has infinitely many primes. Pick $\mathfrak{p} \in \operatorname{Spec} K \subseteq \operatorname{mSpec} K$ nonzero, so $\kappa(\mathfrak{p}) := \mathcal{O}_K/\mathfrak{p}$ is a finite field. Take the $\mathfrak{p}$-adic completion of K:

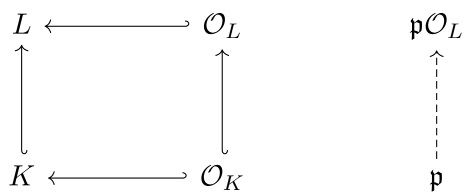
$$\mathcal{O}_{\widehat{K,\mathfrak{p}}} := \varprojlim_n \mathcal{O}_K/\mathfrak{p}^n, \qquad K_{\widehat{\mathfrak{p}}} := \operatorname{ff}(\mathcal{O}_{K,\mathfrak{p}}).$$

Equivalently, pick $\lambda \in K^\times$ and consider the fractional ideal $\lambda\mathcal{O}_K$ it generates, which is a finitely generated $\mathcal{O}_K$-submodule of $\mathcal{O}_K$. It factors into principal fractional ideals:

$$\lambda\mathcal{O}_K = \mathfrak{p}^{v_\mathfrak{p}(\lambda)} \times \prod_{\mathfrak{q}_i \neq \mathfrak{p}} \mathfrak{q}_i^{e_i}.$$

where $v_\mathfrak{p} : K^\times \twoheadrightarrow \mathbb{Z}$ and $\|\lambda\|_\mathfrak{p} := \varepsilon^{-v_\mathfrak{p}(\lambda)}$ where $\varepsilon := q_\mathfrak{p} := \sharp\kappa(\mathfrak{p})$. This norm induces a metric $d(x,y) := \|x-y\|_\mathfrak{p}$, and one can take the Cauchy completion to define the local field $K_{\widehat{\mathfrak{p}}}$. Note that $K_{\widehat{\mathfrak{p}}}/\mathbb{Q}_p$ is a finite extension where $\mathfrak{p} \cap \mathbb{Z} = \langle p \rangle$.

Remark 10.1.2: Let L/K be a finite Galois extension of number fields and $\mathfrak{p} \in \operatorname{Spec} \mathcal{O}_K$ nonzero as above. Then $\mathsf{Gal}(L/K) \in \mathsf{FinGrp}$ is the global Galois group we'll study. We can then extend $\mathfrak{p}$:



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⚠ Warning 10.1.3
The ideal $\mathfrak{p}\mathcal{O}_L \in \operatorname{Id}(L)$ need not be prime! E.g. take $L/K = \mathbb{Q}(\sqrt{p})/\mathbb{Q}$, then p becomes a square in L.

We can factor

$$\mathfrak{p}\mathcal{O}_L = \prod_{1 \leq i \leq g} P_i^{e_i}, \qquad P_i \in \operatorname{Spec} \mathcal{O}_L.$$

Since $\mathsf{Gal}(L/K) \curvearrowright L$, it acts on $\mathcal{O}_L$ and fixes $\mathfrak{p} \subseteq \mathcal{O}_K \subseteq K$. This action fixes $\mathfrak{p}\mathcal{O}_L$ as a set and permutes its elements, and permutes the $P_i \in \operatorname{Spec} \mathcal{O}_L$ appearing in the factorization above since $\sigma(P_i)$ is again prime and divides $\mathfrak{p}$. This is clearly true by transport of structure, since $\sigma : L \to L$ is an isomorphism of fields.

Fact 10.1.4
Galois acts transitively, so there is only one orbit and thus all of the e_i are preserved by σ and all of the P_i are isomorphic to $\sigma(P_i)$. As a corollary, all of the completions are isomorphic:

$$L_{\widehat{P_1}} \cong L_{\widehat{P_2}} \cong \cdots L_{\widehat{P_g}}.$$

Remark 10.1.5: Let $P := P_1 \in \operatorname{Spec} L$ be a fixed choice of a prime above $\mathfrak{p}$. Note that if $\mathcal{P} := \{P_1, P_2, \cdots, P_g\}$ then by Orbit-Stabilizer,

$$\mathsf{Gal}(L/K)/\mathrm{Stab}(P_1) \xrightarrow{\sim} \mathrm{Orb}(\mathsf{Gal}(L/K) \curvearrowright \mathcal{P}) = \mathcal{P}.$$

Define the **decomposition group**

$$D_P := \left\{ \sigma \in \mathsf{Gal}(L/K) \;\middle|\; \sigma(P) = P \right\} \implies \mathsf{Gal}(L/K)/D_P \cong \mathcal{P}.$$

Note that this need not be a normal subgroup. Now $\sigma : L \to L$ descends to a map $\sigma : (\mathcal{O}_L, P) \to (\mathcal{O}_L, P)$ and thus to the completions $\sigma : L_{\widehat{P}} \to L_{\widehat{P}}$ by transport of structure.

Fact 10.1.6
$L_{\widehat{P}}/\kappa(\mathfrak{p})$ is Galois and $D_P \cong \mathsf{Gal}(L_{\widehat{P}}/\kappa(\mathfrak{p}))$, which is the local Galois group, and $D_P \hookrightarrow \mathsf{Gal}(L/K)$.

Remark 10.1.7: So given $\mathsf{Gal}(L/K)$, choose $\mathfrak{p} \in \operatorname{Spec} \mathcal{O}_K$ and $P \mid \mathfrak{p}\mathcal{O}_L$ with $P \in \operatorname{Spec} \mathcal{O}_L$ to produce $D_P \subseteq \mathsf{Gal}(L/K)$ with $D_P \cong \mathsf{Gal}(L_{\widehat{P}}/\kappa(\mathfrak{p}))$. Note that there is an inertia subgroup $I \subsetneq \mathsf{Gal}(L_{\widehat{P}}/\kappa(\mathfrak{p}))$, and the quotient by I is generated by Frob. It turns out that I is almost always trivial:

$$\mathfrak{p} \nmid \operatorname{disc}(L/K) \implies I = 1,$$

so $L_{\widehat{P}}$ is an unramified extension of $\kappa(\mathfrak{p})$ and there is a distinguished element $\mathrm{Frob}_P \in D_P \subseteq \mathsf{Gal}(L/K)$. Note that changing L might change Frob_P, so this depends on $\mathfrak{p}$, the extension L, and $P \mid \mathfrak{p}\mathcal{O}_L$. Consider choosing a different P': since Galois acts transitively, there is some $\sigma \in \mathsf{Gal}(L/K)$ with $\sigma(P) = P'$ and thus

$$D_{P'} = \sigma D_P \sigma^{-1}, \qquad \mathrm{Frob}_{P'} = \sigma\, \mathrm{Frob}_P\, \sigma^{-1}$$

by transport of structure. So this yields a well-defined *conjugacy class*

$$\mathrm{Frob}_{\mathfrak{p}} := \left\{ \mathrm{Frob}_{P'} \;\middle|\; P' \mid \mathfrak{p}\mathcal{O}_L \right\},$$

which works for all $\mathfrak{p} \nmid \operatorname{disc}(L/K)$.

11 | Lecture 11

Look up Fontaine's B_{dR}!

11.1 The Frobenius Machine

Remark 11.1.1: Last time: global Galois groups, i.e. Galois groups of number fields. The setup:

- L/K finite Galois global,
- $\mathfrak{p} \in \operatorname{Spec} \mathcal{O}_K$ a nonzero prime,
- $\mathfrak{p}\mathcal{O}_L \in \operatorname{Id}(\mathcal{O}_L)$, probably not in $\operatorname{Spec} \mathcal{O}_L$,
- A fixed prime $P \mid \mathfrak{p}$ and a factorization

$$\mathfrak{p}\mathcal{O}_L = P^e \times \prod_{\mathfrak{q} \mid \mathfrak{p}} \mathfrak{q}_i^{e_i}.$$

- The decomposition group.

$$G(L/K) \supseteq D_{P/\mathfrak{p}} := \left\{ \sigma \in G(L/K) \;\middle|\; \sigma P = P \right\}.$$

Fact 11.1.2
If $\sigma \in D_{P/\mathfrak{p}}$, then $\sigma : L \to L$ descends to $\sigma : \mathcal{O}_L \to \mathcal{O}_L$ restricts to the identity on K, $\mathcal{O}_L$, and $\mathfrak{p}$. Moreover $\sigma(P) = P$, so there is an induced map $\mathcal{O}_L/\mathfrak{p}^n \circlearrowleft$ and thus a map on the completion $\sigma : L_{\widehat{P}} \circlearrowleft$ fixing $K_{\widehat{\mathfrak{p}}}$, yielding an element

$$\sigma \in G(L_{\widehat{P}}, K_{\widehat{\mathfrak{p}}}) = D_{P/\mathfrak{p}} \supseteq I_{P/\mathfrak{p}} = I_{L_{\widehat{P}}/K_{\widehat{\mathfrak{p}}}}.$$

So the global group contains the local group.

For global fields, we have a discriminant $\Delta = \operatorname{disc}(L/K) \in \operatorname{Id}(\mathcal{O}_K)$, and $p \nmid \Delta$ then $I_{P/\mathfrak{p}} = 1$ for all $P \mid \mathfrak{p}$, making $\mathfrak{p}$ unramified. Note that changing P yields isomorphic inertia groups. Common situation: things in NT tend to be unramified outside a finite set of primes S; here $S = \left\{ p \in \operatorname{Spec} \mathcal{O}_K \;\middle|\; p \nmid \Delta \right\}$. For all $P \mid \mathfrak{p}$, we get a cyclic group

$$D_{P/\mathfrak{p}}/I_{P/\mathfrak{p}} = D_{P/\mathfrak{p}} = \langle \operatorname{Frob}_P \rangle = G(\kappa(P)/\kappa(\mathfrak{p}))$$

since this is a local Galois group where the inertia is trivial. So each P yields an element $\operatorname{Frob}_P \in G(L/K)$, and varying P yields a conjugacy class $\operatorname{Frob}_{\mathfrak{p}} := \left\{ \operatorname{Frob}_P \;\middle|\; P \mid \mathfrak{p} \right\}$.

Theorem 11.1.3 ***(Chebotarev).***
Every conjugacy class $C \subseteq G(L/K)$ is of the form $C = \operatorname{Frob}_{\mathfrak{p}}$ for infinitely many $\mathfrak{p} \in \operatorname{Spec} \mathcal{O}_K$, which is a weak form of **Chebotarev density**. Moreover the density of such $\mathfrak{p}$ is exactly $\sharp C / \sharp G(L/K)$.

Remark 11.1.4: A useful consequence: it suffices to specify representations on $\operatorname{Frob}_{\mathfrak{p}}$ for all $\mathfrak{p}$. There is a variant for infinite extensions, which we'll always assume to be cofinitely unramified. Fix an algebraic closure $\overline{K} := \operatorname{cl}_{\mathsf{Alg}}(K)/K$; then G_K is ramified at every prime $\mathfrak{p} \in \operatorname{Spec} K$, so the reduction to Frob above no longer works.

Let $S \subseteq \operatorname{mSpec} \mathcal{O}_K$ be a finite set of primes, we'll show that $\operatorname{Frob}_{\mathfrak{p}}$ makes sense for $\mathfrak{p} \notin S$. Suppose $K \subseteq L_1, L_2 \subseteq K$ with L_i/K finite and unramified outside of S, then the compositum $L_1 L_2$ is again unramified outside of S. So define

$$K_S := \bigcup_{L \in \tilde{S}} L, \qquad \tilde{S} := \left\{ L/K \;\middle|\; L \text{ is finite Galois unramified outside of } S \right\}.$$

Note that every number field other than $\mathbb{Q}$ is ramified at some prime.

Question 11.1.5
Consider $K = \mathbb{Q}$ and let $S = \{p\}$ be a fixed prime. What is K^S? Note that $\mathbb{Q}(\zeta_{p^n}) \subseteq K^S$ for all $n \geq 1$.

Example 11.1.6 *(?)*: Like $K = \mathbb{Q}$ and fix $N \in \mathbb{Z}_{\geq 1}$, and take $S = \left\{ p \in \operatorname{Spec} \mathbb{Z} \;\middle|\; p \mid N \right\}$. Then $\mathbb{Q}(\zeta_N)/\mathbb{Q}$ is unramified outside of S and has Galois group $C_N^\times$. Thus if $p \in \mathbb{Z}$ is prime and $p \notin S$, so $p \nmid N$, there is a canonical conjugacy class $\operatorname{Frob}_p \in C_N^\times$ – since this group is abelian, this is in fact an element. It could be p or p^{-1}, and one can work out that it must be p.

> **Corollary 11.1.7** *(?)*.
> There are infinitely many primes in any arithmetic progression.

Remark 11.1.8: Fixing $K = \mathbb{Q}$ and a model of $\overline{\mathbb{Q}}$, for $S = \{p\}$ one has $K^S \supseteq \mathbb{Q}(\zeta_{p^n})$ for all n and thus their union

$$\mathbb{Q}(\zeta_{p^\infty}) := \bigcup_{n \geq 1} \mathbb{Q}(\zeta_{p^n}) \subseteq \overline{\mathbb{Q}}.$$

. Therefore there is a surjection

$$\begin{aligned} G(K^S/K) &\twoheadrightarrow G(\mathbb{Q}(\zeta_{p^\infty})/\mathbb{Q}) \\ &\xrightarrow{\sim} \varprojlim_n G(\mathbb{Q}(\zeta_{p^n})/\mathbb{Q}) \\ &\xrightarrow[\text{can}]{\sim} \varprojlim_n C_n^\times \\ &\xrightarrow{\sim} {\mathbb{Z}_p}^\times. \end{aligned}$$

We can get Frobenius elements in such infinite extensions: if $r \neq p$ is prime, since the groups are abelian we have elements $\operatorname{Frob}_r \in C_{p^n}^\times = G(\mathbb{Q}(\zeta_{p^n})/\mathbb{Q})$ which gets identified with r as a residue class. Taking the inverse limit yields a Frobenius element $\operatorname{Frob}_r \in {\mathbb{Z}_p}^\times$.

Remark 11.1.9: Upshot: for $K \in \mathsf{NumberField}, S \subseteq \operatorname{Places}(K)$ finite,

$$G(K^S/K) = \varprojlim \left\{ G(L/K) \;\middle|\; L/K \text{ finite, Galois, unramified outside of } S \right\}$$

and for all $\mathfrak{p} \notin S$, we get conjugacy classes $\operatorname{Frob}_{\mathfrak{p},L/K} \subseteq G(L/K)$ which glue in the limit to a conjugacy class

$$\operatorname{Frob}_{\mathfrak{p}} = \operatorname{Frob}_{\mathfrak{p},K^S/K} \subseteq G(K^S/K).$$

So we have names for conjugacy classes in this group, although not for elements. By Chebotarev density, the following map is surjective

$$\begin{aligned} \left\{ \mathfrak{p} \in \operatorname{mSpec} K \;\middle|\; \mathfrak{p} \notin S \right\} &\twoheadrightarrow G(L/K)/\operatorname{Inn} G(L/K) \\ \mathfrak{p} &\mapsto \operatorname{Frob}_{\mathfrak{p}}, \end{aligned}$$

where $G \curvearrowright G$ by inner automorphisms, and the RHS denotes conjugacy classes.

Corollary 11.1.10 ***(Density of Frobenii).***
If L/K is infinite Galois and unramified outside of $S \subseteq \text{Places}(K)$, then

$$\left\{ \text{Frob}_{\mathfrak{p}} \;\middle|\; \mathfrak{p} \notin S \right\} \underset{\text{dense}}{\hookrightarrow} G(L/K).$$

Corollary 11.1.11 ***(?).***
If $F : G(L/K) \to X$ is continuous and factors through $G(L/K)/\operatorname{Inn} G(L/K)$, so is constant on conjugacy classes, it may be possible to recover F from $\left\{ F(\text{Frob}\,\mathfrak{p}) \;\middle|\; \mathfrak{p} \in \text{mSpec}\, K \setminus S \right\}$.
Upshot: if F is the characteristic polynomial of a representation, then it may suffice to know the characteristic polynomials of Frobenius.

11.2 Brauer-Nesbitt

Theorem 11.2.1 ***(Brauer-Nesbitt).***
Let $G \in \mathsf{Grp}$ and $E \in \mathsf{Field}$ be arbitrary. Recall that $\rho : G \to \text{GL}_n(E)$ is semisimple iff ρ is a direct sum of irreducible representations. Note that irreducibles and direct sums of irreducible are semisimple. If $\rho_1, \rho_2 : G \to \text{GL}_n(E)$ are two semisimple representations, then

$$\text{charpoly}\rho_1(g) = \text{charpoly}\rho_2(g)\, \forall g \in G \implies \rho_1 \cong \rho_2.$$

⚠ **Warning 11.2.2**
If one replaces $GL_n(E)$ with an arbitrary group and asks for the ρ_i just to be conjugate, the analogous theorem will no longer be true. So automorphic representations may not be determined by their local components.

Remark 11.2.3: If $\operatorname{ch} K = 0$, then $\text{tr}\rho_1 = \text{tr}\rho_2 \implies \rho_1 \cong \rho_2$ for semisimple ρ, so one doesn't need the entire characteristic polynomial – one can compute the entire polynomial from $\rho(g^i)$ for $1 \leq i \leq n$, but this uses division by $n!$.

Exercise 11.2.4 (Non-example)
Let $G = C_3$, $E = \bar{\mathbb{F}}_2$, and $n = 2$. Find non-isomorphic ρ_i semisimple and reducible with identical traces. The issue will be the 2! is not invertible in E.

Remark 11.2.5: Upshot: if $\rho \in \mathsf{TopGrp}(G(K^S/K) \to \text{GL}_n(E))$ is a continuous semisimple irrep, say where $E/\mathbb{Q}_p$, and one knows $F_{\mathfrak{p}} \coloneqq \text{charpoly}\rho(\text{Frob}_{\mathfrak{p}}) \in E[x]$ for all $\mathfrak{p} \notin S$, then this determines ρ uniquely. This is because the function $g \mapsto \text{charpolycharpoly}\rho(g)$ will be continuous, then Brauer-Nesbitt says this determines ρ. If we know $\text{charpoly Frob}_{\mathfrak{p}}$ (as is often the case in NT), by continuity there will be at most one way to extend this to a map, so at most one ρ.

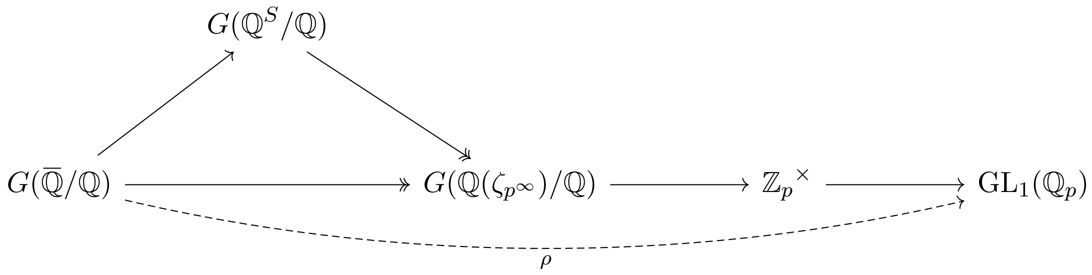
Example 11.2.6 ***(?):*** Let $K = \mathbb{Q}$ and $S = \{p\}$ for a fixed prime. Let $L = \mathbb{Q}(\zeta_{p^\infty})$, then

$$G(L/K) = \varprojlim_n C_{p^n}^\times = {\mathbb{Z}_p}^\times = \mathrm{GL}_1(\mathbb{Z}_p) \subseteq \mathrm{GL}_1(\mathbb{Q}_p).$$

We can then define a representation

$$\rho : G_\mathbb{Q} \twoheadrightarrow G(\mathbb{Q}(\zeta_{p^\infty})/\mathbb{Q}) = {\mathbb{Z}_p}^\times \to \mathrm{GL}_1(\mathbb{Q}_p).$$

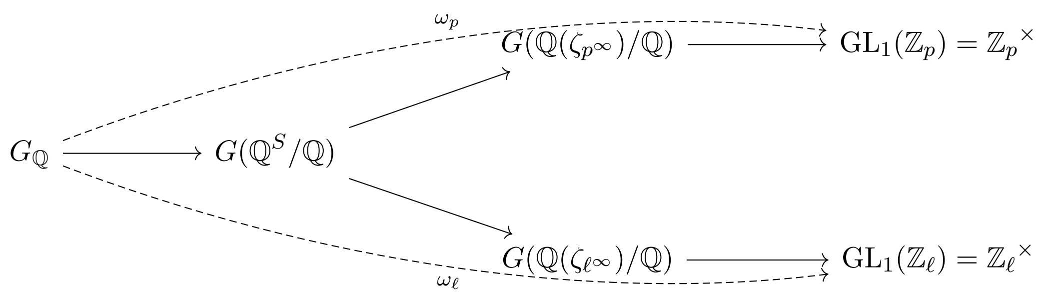
Note that this factors:



Link to diagram

So there is a map $G_\mathbb{Q} \to G(\mathbb{Q}^S/\mathbb{Q})$, which is still complicated but contains conjugacy classes Frob_r for $r \notin S$. This composite is the p-adic **cyclotomic character**, which we'll notate ω_p (noting that it's often notated χ_p). By Brauer-Nesbitt and Chebotarev, ρ is determined by the fact that $\rho(\mathrm{Frob}_r) = r$ for all $r \neq p$.

Warning 11.2.7
A confusing fact: let p, ℓ be two different primes and set $S = \{p, \ell\}$. We can then produce two Galois representations that factor through $G(\mathbb{Q}^S/\mathbb{Q})$:



Link to Diagram

Note that $\omega_p(\mathrm{Frob}_r) = \omega_\ell(\mathrm{Frob}_r) = r$ for all $r \notin S$ and

$$\left\{ \mathrm{Frob}_r \;\middle|\; r \notin S \right\} \underset{\text{dense}}{\hookrightarrow} G(\mathbb{Q}^S/\mathbb{Q}),$$

so why aren't these representations equal since they have the same traces? The catch is that Brauer-Nesbitt doesn't apply, since it requires to representations to $\mathrm{GL}_n(E)$ where E is a single, fixed field. In fact, these two reps are extremely different: $\ker \omega_p = \mathbb{Q}(\zeta_{p^\infty})$ which is unramified away from p and totally ramified at p, while $\ker \omega_\ell = \mathbb{Q}(\zeta_{\ell^\infty})$. Moreover these are two completely disjoint extensions of $\mathbb{Q}$, since anything in both fields generates an extension that's unramified away from p and away from ℓ, and thus unramified everywhere and in $\mathbb{Q}$.

12 | Lecture 12, ℓ-adic Representations

12.1 Families of ℓ-adic reps

Remark 12.1.1: See Néron-Shafarevich criterion, Tate modules of AVs.

Example 12.1.2 ***(?):*** Let $K \in \mathsf{Field}_{/\mathbb{Q}}$, say $K = \mathbb{Q}$, and let E/K be an elliptic curve. Let S_0 be a finite set of places of K, so points in $\operatorname{mSpec} \mathcal{O}_K$, where E has bad reduction. For ℓ a prime, consider the ℓ-adic Tate module: since $E[\ell^n](\overline{K})$ receives an action of G_K, setting $T_\ell E := \varprojlim_n E[\ell^n]$ produces a representation

$$\rho_{E,\ell} : G_K \to \mathrm{GL}_2(\mathbb{Z}_\ell),$$

which is well-defined up to conjugation that factors through $G(K^{\tilde{S}_0}/K)$ where $\tilde{S}_0 := S_0 \cup \left\{ p \;\middle|\; p \mid \ell \right\}$.

If $p \notin S_0$ and $p \nmid \ell$,

$$\operatorname{charpoly} \rho_{E,\ell} \operatorname{Frob}_p = x^2 - a_p x - \operatorname{Nm}(p) \in \mathbb{Q}[x] \hookrightarrow \mathbb{Q}_\ell[x], \qquad a_p := 1 + \operatorname{Nm}(p) - \sharp E(\kappa(p)).$$

Note that this no longer depends on ℓ, so

$$\operatorname{Tr} \rho_{E,\ell} \operatorname{Frob}_p = a_p,$$

independent of ℓ. However, $\rho_{E,\ell} \not\cong \rho_{E,p}$ for $\ell \neq p$, since $\rho_{E,\ell}$ is infinitely ramified at ℓ and has wild inertia, while $\rho_{E,p}$ will have finite wild inertia at ℓ.

Definition 12.1.3 (ℓ-adic representations of absolute Galois groups)
Setup:

- $K \in \mathsf{Field}_{/\mathbb{Q}}$,
- $E/\mathbb{Q}_\ell$ a finite extension,
- S a finite set of places in $\operatorname{mSpec} \mathcal{O}_K$

Then a morphism of topological groups

$$\rho \in \mathsf{TopGrp}(G(K^S/K) \to \mathrm{GL}_n(E))$$

where the LHS has the profinite topology and the RHS has the ℓ-adic topology is an **ℓ-adic representation** of G_K. In this situation we say ρ is *unramified outside of S*.

Remark 12.1.4: Note that we regard this as G_K representation despite having $G(K^S/K)$ as the source, since we implicitly require reps to be unramified away from S since $G_K \twoheadrightarrow G(K^S/K)$.

Definition 12.1.5 (Rational representations)
We say ρ is **rational over** E_0 for $E_0 \subseteq E$ if $\operatorname{charpoly} \rho\, \operatorname{Frob}_p \in E[x]$ in fact lies in $E_0[x]$.

Example 12.1.6 *(?):*

- For ρ the cyclotomic character $\rho : G_{\mathbb{Q}} \to \operatorname{GL}_1(\mathbb{Q}_\ell)$ satisfies $\rho\, \operatorname{Frob}_r = r$ for all $r \neq \ell$ and is thus rational over $\mathbb{Q}$.
- $T_\ell E$ for E an elliptic curve is rational over $\mathbb{Q}$.
- $\rho = H^1_{\text{ét}}(X_{\overline{K}}; \mathbb{Q}_\ell)$ the ℓ-adic etale cohomology of $X \in \mathsf{smAlgVar}_{/K}$ with X proper is rational over $\mathbb{Q}$.

Definition 12.1.7 (Pure reps and their weights)
The representation ρ is **pure of weight** w iff ρ is rational over some $E_0 \in \mathsf{Field}_{/\mathbb{Q}}$ and for all embeddings $i \in \mathsf{Field}(E_0, \mathbb{C})$ and for all eigenvalues α of $\rho\, \operatorname{Frob}_p$, the magnitude satisfies

$$|i(\alpha)| = q_p^{-\frac{w}{2}}, \qquad q_p := \sharp \mathcal{O}_K/p.$$

Theorem 12.1.8 *(Deligne).*
For $X \in \mathsf{sm\,proj\,Var}_{/K}$ proper, $H^i(X_{\overline{K}}; \mathbb{Q}_\ell)$ is pure of weight i.

Example 12.1.9 *(?):*

- $H^2(\mathbb{P}^1_{/K}; \mathbb{Q}_\ell) = \omega_\ell^{-1}$ is the inverse of the cyclotomic character, making it pure of weight -2.
- $T_\ell E$ is pure of weight -1, so the roots of $x^2 - a_p x + \operatorname{Nm}(p)$ are complex conjugates, which yields the Hasse bound

$$|a_p| \leq 2\sqrt{\operatorname{Nm}(p)}.$$

Remark 12.1.10: We'll now let ℓ vary, and formalize the notion that the cyclotomic characters for $\ell \neq p$ are distinguished.

Definition 12.1.11 (Compatible Systems)
Setup:

- $K \in \mathsf{Field}_{/\mathbb{Q}}$,
- $E_0 \in \mathsf{Field}_{/\mathbb{Q}}$ (e.g. $\mathbb{Q}$),
- $S_0 \subseteq \operatorname{Places}(K)$ a finite set of places,
- For all $p \notin S_0$, a polynomial $F_p(x) \in E_0[x]$ (e.g. primes of good reduction and $F_p(x) = x - \operatorname{Nm}(p)$),

- For all finite places $\lambda \in \mathcal{O}_{E_0}$, an ℓ-adic representation

$$\rho_\lambda : G(K^{\tilde{S}_0}/K) \to \mathrm{GL}_n\left(\mathrm{cl}_{\mathsf{Alg}}\left((E_0)_{\widehat{\lambda}}\right)\right), \qquad \tilde{S}_0 := S_0 \cup \left\{p \in \mathrm{Places}(K) \;\middle|\; p \mid \ell\right\}.$$

We'll say ρ_λ is a **compatible system** of λ-adic representations iff for all $\lambda \mid \ell$ and for all $p \notin \tilde{S}_0$, so $p \notin S_0$ and $p \nmid \ell$,

$$\mathrm{charpoly}\rho_\lambda \,\mathrm{Frob}_p$$

is independent of λ.

Remark 12.1.12: Note that $(E_0)_{\widehat{\lambda}}/\mathbb{Q}_\ell$ is a finite extension for any $\lambda \mid \ell$. If one assumes $E_0 = \mathbb{Q}$, the above is the data of an ℓ-adic representation for every rational prime ℓ whose traces all agree.

Example 12.1.13*(?)*: Examples that are known to be compatible systems:

- The cyclotomic characters $F_p(x) = x - p$,
- $T_\ell E$ for all ℓ, where $E_0 = \mathbb{Q}$ and $F_p(x) = x^2 - a_p x - \mathrm{Nm}(p)$, where S_0 are the primes of bad reduction,
- $H^i_{\text{ét}}(X; \mathbb{Q}_\ell)$.

Remark 12.1.14: Instead of asking for traces to agree, one can apply LLC: the ρ_λ are **strongly compatible** iff for all $p \in S_0$ with $p \mid P$, for all $\lambda \in E_0$ with $\lambda \not| P$, consider the restricted representation $\rho_\lambda|_{G(\overline{K}_p/K_P)}$ yields a WD representation by Grothendieck, and by LLC a representation π of $\mathrm{GL}_n(K_p)$ on some infinite dimensional vector space. One could then define them to be *strongly compatible* iff π only depends on λ. This is unknown for $H^i_{\text{ét}}$.

*See the **Weight-Monodromy conjecture**.*

12.2 Adeles and Global CFT

Question 12.2.1
Motivating question: what is G_K^{ab}?

Definition 12.2.2 (Infinite Places)
Let $K \in \mathsf{Field}_{/\mathbb{Q}}$ with $[K : \mathbb{Q}] = d$, so there are d field embeddings $K \xrightarrow{\sigma} \mathbb{C}$. These split into two types:

- Let r_1 be the number of σ such that $\sigma(K) \subseteq \mathbb{R}$, i.e. the number of totally real embeddings.
- If $\sigma(K) \not\subseteq \mathbb{R}$, writing $\cong (z) := \bar{z}$ for conjugation in $\mathbb{C}$, σ and $\bar{\circ}\sigma$ are distinct embeddings that induce the same norm since $|z| = |\bar{z}|$. These non-real embeddings come in pairs, so let $2r_2$ be the number of such embeddings.

Then

$$r_1 + 2r_2 = d.$$

An **infinite place** v of K is either

- A real place $v = \sigma \in \mathsf{Field}(K \to \mathbb{R})$, or
- A complex place, which is the pair of maps $\{\sigma, (z \mapsto \bar{z}) \circ \sigma\} \in \mathsf{Field}(K \to \mathbb{C})$ whose image isn't totally real.

Further define

$$K_\infty := \prod_{v \mid \infty} K_{\widehat{v}},$$

the product of all infinite places, noting that $K_{\widehat{v}} \cong \mathbb{R}$ or $\mathbb{C}$ so that $K \hookrightarrow K_{\widehat{v}}$.

Example 12.2.3*(?)*: For $K = \mathbb{Q}(2^{\frac{1}{3}})$, note that

- $r_1 = 1$
- $r_2 = 1$

Remark 12.2.4: It turns out that

$$K_\infty \cong \mathbb{R}^{r_1} \times \mathbb{C}^{r_2} = K \otimes_{\mathbb{Q}} \mathbb{R}, \quad K_\infty^\times \cong (\mathbb{R}^\times)^{r_1} \times (\mathbb{C}^\times)^{r_2},$$

and $K_\infty^\times$ is generally disconnected since $\mathbb{R}^\times$ is disconnected. The connected component of the identity satisfies

$$(K_\infty^\times)^0 \cong (\mathbb{R}_{\geq 0})^{r_1} \times (\mathbb{C}^\times)^{r_2}.$$

We'll define the **adeles** as a restricted product in the category of topological rings:

$$\mathbb{A}_L = \prod_{p < \infty} K_{\widehat{p}} \times K_\infty,$$

and the **ideles** as $\mathbb{A}_L^\times$. It will be locally compact, has an explicit compact subspace, and is related to class groups and unit groups.

13 | Lecture 13

Remark 13.0.1: For $\sigma \in \mathsf{Field}(K, \mathbb{C})$, we define an equivalence relation by $\sigma \sim (z \mapsto \bar{z}) \circ \sigma$, and get equivalence classes of size exactly 1 or 2. The point of this: such embeddings induce the same norm. The equivalence classes are the **infinite places** and we sometimes write $v \mid \infty$. We saw that

$$K_\infty := \bigoplus_{v \mid \infty} K_{\widehat{v}} \cong K \otimes_{\mathbb{Q}} \mathbb{R}.$$

Example 13.0.2 *(?):* An example of the above isomorphism, in the simplest example for which we know nearly nothing about the GLC for $\mathrm{GL}_2(K)$, namely $K = \mathbb{Q}(2^{\frac{1}{3}})$. For elliptic curves over CM or totally real fields, one can hope to prove potential modularity and there are statement for GL_n over such fields, but here K is neither CM nor totally real. Given a random elliptic curve E/K here, the chances of proving it is modular are virtually zero!

In this case, note that $x^3 - 2$ is irreducible over $\mathbb{Q}$ by Eisenstein, or by passing to the local extensions $\mathbb{Q}_2(2^{\frac{1}{3}})$ has degree at least three since the valuation increases by a factor of 3. Write $x^3 - 2 = (x-\alpha)(x-w)(x-\overline{w})$, we can then compute

$$\begin{aligned} K_\infty &= \frac{\mathbb{Q}[x]}{\langle x^3-2\rangle} \otimes_{\mathbb{Q}} \mathbb{R} \\ &\cong \frac{\mathbb{R}[x]}{\langle x^3-2\rangle} \\ &\cong \frac{\mathbb{R}[x]}{\langle x-\alpha\rangle} \oplus \frac{\mathbb{R}[x]}{x^2+\alpha x+\alpha^2} \\ &\cong \mathbb{R} \oplus \mathbb{C} \\ &= K_v \oplus K_{[v']}, \end{aligned}$$

where we've used that the factors are irreducible and coprime over $\mathbb{R}$. Note that there are two isomorphisms to $\mathbb{C}$ here, and

$$\begin{aligned} v : K &\to \mathbb{R} \\ \sqrt{2} &\mapsto \alpha \end{aligned}$$

$$\begin{aligned} v' : K &\to \mathbb{R} \\ \sqrt{2} &\mapsto w \text{ or } \overline{w}. \end{aligned}$$

Definition 13.0.3 (Finite Adeles)
Let $K \in \mathsf{Field}_{/\mathbb{Q}}$, so that $\mathrm{Places}(K) \cong \mathrm{mSpec}\, \mathcal{O}_K$ consists of finitely many places. For p a finite place, we can complete to obtain $K_{\widehat{p}}, \mathcal{O}_{k,\widehat{p}}$, generalizing $\mathbb{Q}_p$ and $\mathbb{Z}_p$. Define the **finite adeles of** K as

$$\begin{aligned} \mathbb{A}_{K,\mathsf{Fin}} &:= \prod_{p \in \mathrm{Places}(K)}^{\mathrm{res}} K_{\widehat{\mathfrak{p}}} \\ &:= \left\{ (x_p) \in \prod_{p\in\mathrm{Places}(K)} K_{\widehat{p}} \;\middle|\; x_p \in \mathcal{O}_{K,\widehat{p}} \text{ for all but finitely many } p \right\} \\ &\subseteq \prod_{p\in\mathrm{Places}(K)} K_{\widehat{\mathfrak{p}}}. \end{aligned}$$

Equivalently,

$$\mathbb{A}_{K,\mathsf{Fin}} := \left\{ (x_p) \in \prod_p K_{\widehat{p}} \;\middle|\; \exists S \text{ finite where } x_p \in \mathcal{O}_{K,\widehat{p}} \forall p \notin S \right\},$$

where one thinks of S as a set of bad primes depending on (x_p). This forms a ring under pointwise operations, but is topologized in the following way: use that $\mathcal{O}_{K,\widehat{p}}$ is compact and thus the subring $R := \prod_p \mathcal{O}_{K,\widehat{p}}$ is compact by Tychonoff, and declare R to be an open neighborhood of zero. This yields a basis of opens about zero, which can now be translated.

Remark 13.0.4: Note that the naive product is not locally compact, so does not admit a good theory of Haar measures The motivation for the restricted product: an element of K has only finitely many primes involved in the denominator, so only maps into finitely many factors. There is also a diagonal embedding

$$\begin{aligned}\Delta : K &\to \mathbb{A}_{K,f} \\ \lambda = \frac{a}{b} &\mapsto \cdots, S := \left\{ p \in \mathrm{Places}(K) \;\middle|\; p \mid b \right\} = \left\{ p \in \mathrm{Places}(K) \;\middle|\; v_p(\lambda) < 0 \right\},\end{aligned}$$

which is always a finite set.

Definition 13.0.5 (Adeles)
The full ring of **adeles of** K is defined as

$$\mathbb{A}_K := \mathbb{A}_{K,\mathsf{Fin}} \times K_\infty = \prod_{v \in \mathrm{Places}(K)}^{\mathrm{res}} K_{\widehat{v}}.$$

Remark 13.0.6: Note that K is a 1-dimensional ring and contains $\mathcal{O}_K$, and in analogy there is a ring of meromorphic functions $\mathbb{C}(t) \supseteq \mathbb{C}[t]$ and one could take a Laurent expansion about any point v to get an element of $\prod_{v \in \mathbb{C}} \mathbb{C}((t-v))$. This contains a subring $\prod_{v \in \mathbb{C}} \mathbb{C}[[t-v]]$ of holomorphic functions, so

$$\mathbb{C}(t) \hookrightarrow \prod_{v \in \mathbb{C}} \mathbb{C}((t-v)) \supseteq \prod_{v \in \mathbb{C}} \mathbb{C}[[t-v]].$$

Idea: $\mathbb{C}(t)$ embeds into the restricted product, since a meromorphic function has only finitely many poles, so this product encodes informations about meromorphic functions in a way such that modifying the function at one place won't change its behavior at another.

Remark 13.0.7: More generally, the GLC would apply to global fields such as $K = \mathbb{F}_q(t)$, and has essentially been proved by the Lafforgues using moduli spaces of shtukas. This are sort of like elliptic curves… but also not.

Exercise 13.0.8 (?)
Note that $K_\infty \cong \otimes_{\mathbb{Q}} \mathbb{R} \cong K \otimes_{\mathbb{Q}} \mathbb{Q}_\infty$, and show

$$\begin{aligned}\mathbb{A}_{K,\mathsf{Fin}} &\cong K \otimes_{\mathbb{Q}} \mathbb{A}_{\mathbb{Q},\mathsf{Fin}} \\ \mathbb{A}_K &\cong K \otimes_{\mathbb{Q}} \mathbb{A}_{\mathbb{Q}}.\end{aligned}$$

Lemma 13.0.9 *(?)*.

$$\mathbb{A}_{\mathbb{Q},\mathsf{Fin}} \cong \mathbb{Q} \oplus \prod_p \mathbb{Z}_p,$$

i.e. for all $x := (x_p) \in \mathbb{A}_{\mathbb{Q},\mathsf{Fin}}$ (so $x_p \in \mathbb{Q}_p$ for all primes and $x_p \in \mathbb{Z}_p$ for all but finitely many p) there exists a $\lambda \in \mathbb{Q}$ such that $x = \lambda + \mu$ for some $\mu \in \prod_p \mathbb{Z}_p$.

Remark 13.0.10: The analogy: for a fixed meromorphic function, one can find a rational function to subtract off to remove all of the poles. So an adele can be modified by a globally meromorphic function to produce an entire function, which is a Riemann-Roch type of statement (associated to local functions to some global function with the same poles).

Proof (of lemma).
By induction on $\sharp S$, where for the base case when $S = \emptyset$ one can take $\mu = x$. Let S be finite such that $x_p \in \mathbb{Z}_p$ for all $p \notin S$. For a general x, choose $p \in S$ with $x_p \in \mathbb{Q}_p$ and take a p-adic expansion as a Laurent series with a finite tail:

$$x_p = \lambda + f := \left(a_{-n}p^{-n} + \cdots + a_{-1}p^{-1}\right) + (a_0 + \cdots), \qquad a_i \in \{0, \cdots, p-1\},$$

so that λ is the finite tail and satisfies $\lambda = N/p^{-n}$ for some N. Thus has a single pole at p, or more rigorously $v_r(\lambda) \geq 0$ for all $r \neq p$. Now write $x - \lambda = (y_p)$, where the bad set for (y_p) is $S \setminus \{p\}$, so we're done by induction. ■

Exercise 13.0.11 (Challenging)
Show that

$$\mathbb{A}_K \cong K + \prod_p \mathcal{O}_{K,\widehat{\mathfrak{p}}}.$$

The above proof doesn't quite work if p is a prime that is not principal, so one needs input from class groups.

13.1 Ideles

Remark 13.1.1: We're interested in $\mathbb{A}_K^\times$, and more generally $\mathrm{GL}_n(\mathbb{A}_K)$.

Definition 13.1.2 (Ideles)

One can check that

$$\mathbb{A}_{K,\mathsf{Fin}}^{\times} = \prod_{p}^{\mathrm{res}} K_{\widehat{p}}^{\times} := \left\{ (x_p) \in \prod_{p} K_{\widehat{p}}^{\times} \;\middle|\; x_p \in \mathcal{O}_{K,\widehat{p}}^{\times} \text{ for almost all } p \right\},$$

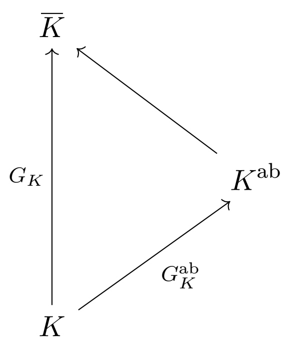
and $\mathbb{A}_K^{\times} = \prod_{v}^{\mathrm{res}} K_{\widehat{v}}^{\times}$. The topology on $\mathbb{A}_{K,\mathsf{Fin}}^{\times}$ is such that $\prod_{p} \mathcal{O}_{K,\widehat{p}}^{\times}$ is open; this is not the subspace topology from $\mathbb{A}_{K,\mathsf{Fin}}$.

Warning 13.1.3
$\mathbb{A}_K^{\times} \hookrightarrow \mathbb{A}_K$ but the topology is not the subspace topology.

Example 13.1.4*(?)*: An element in $\mathbb{A}_K \setminus \mathbb{A}_K^{\times}$: take $x_p = p$ and set $x = (x_p) \in \mathbb{A}_{\mathbb{Q},\mathsf{Fin}}$ and in fact $x \in \prod_p \mathbb{Z}_p$, however $1/x \in \prod_p \mathbb{Q}_p$ but $1/x \notin \mathbb{A}_{\mathbb{Q},\mathsf{Fin}}$ since there are problems at infinitely many primes.

13.2 Global Class Field Theory

Remark 13.2.1: Recall that $G_K/G_K^c = G_K^{\mathrm{ab}}$ is the maximal Hausdorff abelian quotient of G_K, where G_K^c is the topological closure of the commutator subgroup. This yields a factorization:



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After choosing $\overline{K} \supseteq K$, one can form the infinite extension

$$K^{\mathrm{ab}} = \bigcup_S L, \qquad S := \left\{ L \;\middle|\; K \le L \le \overline{K}, L/K \text{ finite }, G(L/K) \in \mathsf{AbGrp} \right\}.$$

Remark 13.2.2: Note that for $K = \mathbb{Q}$, Kronecker-Weber yields

$$\mathbb{Q}^{\mathrm{ab}} = \bigcup_{N \ge 1} \mathbb{Q}(\zeta_N).$$

Class field theory tells you what Galois group for K^{ab}/K is, but not necessarily what K^{ab} is itself. It exists for $\mathbb{Q}$ by Kronecker-Weber, and for imaginary quadratic fields for reasons involving j-invariants of elliptic curves, but for real quadratic fields this is completely open.

Question 13.2.3 (An open problem)
What is K^{ab} for $K = \mathbb{Q}(2^{\frac{1}{3}})$?

Theorem 13.2.4 ***(GCFT).***
There is a surjection r_K, the **global Artin map**, which is a continuous group morphism:

$$K^\times \backslash \mathbb{A}_K^\times \xrightarrow{r_K} G(K^{\text{ab}}/K) \longrightarrow 0$$

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Remark 13.2.5: The Artin map can't be an isomorphism since the RHS is profinite, but the LHS contains $K_\infty^\times \cong (\mathbb{R}^\times)^{r_1} \times (\mathbb{C}^\times)^{r_2}$, so any connected component must be in the kernel of r_K since the RHS is totally disconnected and continuous images of connected sets are connected. This happens to be the entire kernel when K has no units, $K = \mathbb{Q}$, and imaginary quadratic number fields. Since the identity is closed in the image, the kernel must be closed in the source by continuity.

Theorem 13.2.6 ***(from CFT).***
If C_K is defined as the image of $(K_\infty^\times)^0)$ in $K^\times \backslash \mathbb{A}_K^\times$, then

$$\ker r_K = \text{cl}_{\mathrm{T}}(C_K),$$

the topological closure.

Remark 13.2.7: It turns out that for $K = \mathbb{Q}$ or an imaginary quadratic field, C_K is already closed, but there are examples where it is not. Thus $G(K^{\text{ab}}/K) \cong K^\times \backslash \mathbb{A}_K^\times / \text{cl}_{\mathrm{T}}(C_K)$.

Remark 13.2.8: Some properties of r_K:

$$\begin{array}{ccccc}
[1, 1, \cdots, 1, x_p, 1, \cdots] & \quad & K^\times \backslash \mathbb{A}_K^\times & \xrightarrow{r_K \text{ global Artin}} & G(K^{\text{ab}}/K) \\
\uparrow & & \uparrow & & \uparrow \\
x_p & & K_{\widehat{p}}^\times & \xrightarrow{r_K \text{ local Artin}} & G_{K_{\widehat{p}}}^{\text{ab}}
\end{array}$$

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So the global Artin maps glues all of the local Artin maps.

If L/K is finite, $G_L \hookrightarrow G_K$ and there is an induced map $G(L^{\mathrm{ab}}/L) \to G(K^{\mathrm{ab}}/K)$ which need not be injective. There is a commuting square involving norms and transfers:

$$\begin{array}{ccc} L^\times \backslash \mathbb{A}_K^\times & \xrightarrow{\ r_L\ } & G(L^{\mathrm{ab}}/L) \\ \big\downarrow \mathrm{Nm}_{L/K} & & \big\downarrow \ \big\uparrow \text{Transfer} \\ K^\times \backslash \mathbb{A}_K^\times & \xrightarrow{\ r_K\ } & G(K^{\mathrm{ab}}/K) \end{array}$$

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Remark 21.0.1: Some additional resources:

- https://math.stackexchange.com/questions/48981/the-langlands-program-for-beginners

Remark 21.0.2: Some vague definitions:

- The congruence subgroups $\Gamma(N), \Gamma_0(N), \Gamma_1(N)$.

$$\Gamma(N) := \left\{ M \in \mathrm{SL}_2(\mathbb{Z}) \;\middle|\; M \equiv I \bmod N \right\}$$
$$\Gamma_0(N) := \left\{ M \in \mathrm{SL}_2(\mathbb{Z}) \;\middle|\; M \equiv \begin{bmatrix} * & * \\ 0 & * \end{bmatrix} \bmod N \right\}.$$

  - A **congruence subgroup of level** N is any $H \supseteq \Gamma(N)$
  - The **level** is the smallest N such that $H \supseteq \Gamma(N)$.

  - Recovers $\mathrm{SL}_2(\mathbb{Z}) = \Gamma_0(1)$.
  - Principal congruence subgroups of **level** N:

$$1 \to \Gamma(N) \to \mathrm{SL}_2(\mathbb{Z}) \xrightarrow{\bmod N} \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z}) \to 1.$$

- Fuchsian groups: discrete subgroups of $\mathrm{SL}_2(\mathbb{R})$.

- $Y(\Gamma)$: modular curves of the form $\Gamma \diagdown \mathbb{H}$ for Γ a Fuchsian group of the first kind. For congruence subgroups, abbreviated $Y(N)$.

- $X(\Gamma)$: the compactification of $Y(\Gamma)$ obtained by adding cusps. For congruence subgroups, abbreviated $X(N)$.

- Shimura-Taniyama-Weil theorem: for E an elliptic curve, there is a cover $X_0(N) \to E$ where N is the conductor of E.

- Bad reduction for an elliptic curve: primes p for which the equation reduces $\bmod p$ to a singular curve.

- Factors of automorphy: for $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma$, $j(\gamma, \tau) \coloneqq (c\tau + d)$.

- Slash operators: $f \,\Big|\, [\gamma]_k \coloneqq f(\gamma(-)) \cdot j(\gamma, -)^{-k}$

- Classical automorphic forms of weight k and level N:

  - $f \in \mathrm{Mero}(\mathbb{H}, \mathbb{C})$,
  - $f \,\Big|\, [\gamma]_k = f$ for all $\gamma \in \Gamma$ a congruence subgroup of level N,
  - Meromorphic at cusps, so the corresponding Fourier expansion at the cusps has a finite tail.
  - Note that these conditions guarantee the corresponding L function will be meromorphic with known poles, or holomorphic for cusp forms.

- Classical modular forms as automorphic forms:

- $f \in \mathrm{Hol}(\mathbb{H}, \mathbb{C})$

- $f \,\Big|\, [\gamma]_k = f$ for all γ,

- Holomorphic at cusps, so Fourier expansion as no negative terms.

- **Cusp form**: modular forms with vanishing constant Fourier coefficient at every cusp.

- Automorphic forms f of weight $2k$ correspond to meromorphic differential forms $\omega = f(z)(dz)^{\otimes_{\mathbb{C}}^k}$.

- Weakly modular forms of **weight** k for $\mathrm{SL}_2(\mathbb{Z})$: $f(\gamma(\tau)) = (c\tau+d)^k f(\tau)$ for all $\gamma \in \Gamma$ and $\tau \in \mathbb{H}$.
  - Automorphic forms: meromorphic at ∞, form spaces $\mathcal{A}_k(\Gamma)$
  - Modular forms: holomorphic on $\mathbb{H} \cup \{\infty\}$, form spaces $\mathcal{M}_k(\Gamma)$
  - Cusp forms: vanishing at cusp points, i.e. $f(\infty) = 0$, form spaces $\mathcal{S}_k(\Gamma)$.
  - The containment:
$$\mathcal{A}_k(\Gamma) \supseteq \mathcal{M}_k(\Gamma) \supseteq \mathcal{S}_k(\Gamma).$$
  - The Eisenstein space:
$$0 \to \mathcal{S}_K(\Gamma) \to \mathfrak{m}_k(\Gamma) \to \mathcal{E}_k(\Gamma) \to 0.$$
- Eisenstein series:
$$G_k(\tau) := \sum_{(c,d)\in\mathbb{Z}^2}' \frac{1}{(c\tau+d)^k},$$
where $G_k(\infty) = 2\zeta(k)$.
  - Normalization:
$$E_k := \frac{G_k}{2\zeta(k)} \in \mathcal{M}_k(\Gamma).$$
- The discriminant form:
$$\Delta : \mathbb{H} \to \mathbb{C}$$
$$\Delta = g_2^3 - 27g_3^2, \quad g_2 := 60G_4, \quad g_3 := 14 - G_6.$$
  - Facts: $\Delta/(2\pi)^{12} \in \mathcal{M}_{12}(\mathrm{SL}_2(\mathbb{Z}))$ and has a Fourier expansion $\sum_k \tau(k)q^k$ for $q = \exp(2\pi i\tau)$ where τ is the Ramanujan τ function. Moreover $\Delta \in \mathcal{S}_{12}(\mathrm{SL}_2(\mathbb{Z}))$
- Modular curves $Y(N)$: ?
  - $X(N)$: ?
- Geometric interpretations:
  - Modular forms of weight k and level N: meromorphic differential forms on $X_0(N)$ which are multiples of a certain divisor, i.e. holomorphic sections of lines bundles on modular curves, so $\mathcal{M}_k(\Gamma) \cong H^0(X; \mathcal{L}_{\Gamma,k})$ for some line bundle.

  - Automorphic form: for G an algebraic group, a function $f : G \to \mathbb{C}$ which is invariant with respect to some subgroup $\Gamma \leq G$

- Petersson inner product: for f, g modular forms of weight k for Γ, at least one cuspidal, and F a fundamental domain for Γ,

$$\langle f,\ g\rangle := \iint_F f\bar{g}y^k \frac{dx\,dy}{y^2}.$$

- Mellin transform:

$$M(f)(s) := \int_{\mathbb{R}_{\geq 0}} f(y)y^s \frac{dy}{y}.$$

  - Note that $M(e^{-x})(s) = \Gamma(s)$

- Riemann zeta is the L function associated to the Jacobi theta function, which is modular of weight 1/2 with respect to $\left\langle z \mapsto z+2, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \right\rangle$.

- First instance of Langlands: Taniyama-Shimura-Weil theorem. For $E_{/\mathbb{Q}}$ an elliptic curve of conductor N, there is a weight 2 cusp form which is a Hecke eigenform on $\Gamma_0(N)$ with $L(E, s) = L(f, s)$. More generally, all L functions attached to algebraic varieties should arise as L functions coming from automorphic forms.

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