Notes: These are notes live-tex'd from a graduate course in Hochschild Homology taught by Tekkin at the University of Georgia in Spring 2023. As such, any errors or inaccuracies are almost certainly my own.

Hochschild Homology

Lectures by Tekkin. University of Georgia, Spring 2023

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Remark 1.0.1: Some review:

- $M \in {}_k\mathsf{Alg} \iff M \in {}_k\mathsf{Mod} \cap \mathsf{Ring}$ and $\exists m : M \otimes_k M \to M$ a multiplication map.
- $M \in \mathsf{Lie\text{-}Alg} \iff \exists [-,-] : M \otimes_k M \to M$ satisfying the usual identities.
  - E.g. $\operatorname{End}_k(V) \in \mathsf{Lie\text{-}Alg}$ when $V \in {}_k\mathsf{Mod}$.
- $\operatorname{Der}_k(M)$ is *not* closed under composition, but is a Lie algebra under $[\delta_1 \delta_2] \coloneqq \delta_1 \circ \delta_2 - \delta_2 \circ \delta_1$.
  - Counterexample: $\operatorname{Der}_k(k[x]) = k[x]\frac{\partial}{\partial x} \cong k[D]$ but $\frac{\partial}{\partial x} \circ \frac{\partial}{\partial x}$ s not a derivation.
- $\operatorname{HH}(M)$ will make meaningful higher analogs of derivations, $\delta^n : A^{\otimes_k^n} \to A$.
  - 1-cocycles are derivations
  - 2-cocyles are $\delta : M^{\otimes_k^2} \to M$ such that $\delta(ab, c) - \delta(a, bc) = a\delta(b, c) - \delta(a, b)c$.
  - n-cocycles will be $\delta : M^{\otimes_k^n} \to M$ satisfying

$$\sum_{i=1}^{n} (-1)^i \delta\left(a_1, a_2, \cdots, a_i a_{i+1}, \cdots, a_{n+1}\right) = -a_1 \delta\left(a_2, \cdots, a_{n+1}\right) + (-1)^n \delta\left(a_1, \cdots, a_n\right) a_{n+1}..$$

- Define $Z^n(M)$ to be n-cycles – this is not a Lie algebra for $n \geq 2$ unless the bracket is trivial.
- Gerstenhaber's idea: define a new bracket $[-,-] : Z^m(M) \otimes_k Z^n(M) \to Z^{n+m-1}(M)$ with for $m = n = 1$ is the commutator; this makes $Z^*(M)$ into a graded Lie algebra.
- Define boundaries $B_n(M)$ and $\operatorname{HH}^n(M) \coloneqq Z^n(M)/B^n(M)$.
  - $\operatorname{HH}^1(M) = Z(M)$ is the center.
  - $\operatorname{HH}^2(M) = \operatorname{Der}(M)$ when M is commutative.
- Recall the definitions of chain complexes and their morphisms.
- Recall the different formulations of projectives P in ${}_R\mathsf{Mod}$:
  - $\exists F \in {}_R\mathsf{Mod}^{\text{free}}$ with $F \cong P \oplus T$ for some $T \in {}_R\mathsf{Mod}$ (not necessarily free).
  - Every $B \twoheadrightarrow P$ and $B' \to P$ lifts to $B' \to B$.
  - Every SES $A \hookrightarrow B \twoheadrightarrow P$ splits.
- Some useful resolutions:
  - $\mathbf{Z} \xrightarrow{\cdot n} \mathbf{Z} \xrightarrow{\varepsilon} \mathbf{Z}/n\mathbf{Z}$ for $R = \mathbf{Z}$ where ε is the quotient and $\ker \varepsilon = n\mathbf{Z}$.

- $k[x] \overset{\cdot x}{\hookrightarrow} k[x] \xrightarrow{\varepsilon(x)=0} k \in {}_R\mathsf{Mod}$ for $R = k[x]$, where the kernels are all $\langle x \rangle$.
- $\cdots \to k[x] \xrightarrow{\cdot x} k[x] \xrightarrow{\cdot x} k[x] \xrightarrow{\varepsilon(x)=0} k$ for $R = k[x]/\langle x^2 \rangle$ where the kernels are all $\langle x \rangle$. Note that this is an infinite periodic resolution.

2 | Monday, January 23

Remark 2.0.1: Recall

- $(-) \otimes_R B$ is right-exact for any $B \in {}_R\mathsf{Mod}$ and $\mathrm{Hom}_R(-, B)$ is left-exact.
- For $A \in \mathsf{Mod}_R$ and $B \in {}_R\mathsf{Mod}$, define $\mathrm{Tor}_*^R(A, B)$ as $H_*(P_A \otimes_R B)$ where $P_A \rightrightarrows A$ is a projective resolution.
- $\mathrm{Tor}_0^R(A, B) = A \otimes_R B$. Here $B[n] := \left\{ b \in B \;\middle|\; nb = 0 \right\}$.
- $\mathrm{Ext}_R^*(A, B) = H_*(\mathrm{Hom}_R(P_A, B))$.

Example 2.0.2*(?)*:

$$\mathrm{Tor}_*^R(C_n, B) \cong B/nB \cdot t^0 + B[n] \cdot t^1$$

for any $B \in {}_{\mathbf{Z}}\mathsf{Mod}$ using $\mathbf{Z} \overset{\cdot n}{\hookrightarrow} \mathbf{Z} \twoheadrightarrow C_n$ to get $P_B = (0 \to B \to B \to 0)$. Similarly,

$$\mathrm{Ext}_{\mathbf{Z}}^*(C_m, B) = B[m] \cdot t^0 + B/mB \cdot t^1.$$

3 | Wednesday, February 01

> **Exercise 3.0.1** (?)
> Show $\mathrm{HH}^* k[x] = k[x]^{\oplus^2}$ and find $\mathrm{HH}_* k[x]$. Use the complex
>
> $$k[x]^{\oplus^2} \hookrightarrow k[x]^{\oplus^2} \twoheadrightarrow k[x].$$

Example 3.0.2*(?)*: Let $A = k[x]/\langle x^n \rangle$ and consider

$$\cdots \xrightarrow{v} A^e \xrightarrow{u} A^e \xrightarrow{v} A^e \xrightarrow{u} A^e \xrightarrow{\pi} A \to 0$$

where $u = (x \otimes 1 - 1 \otimes x)\cdot$ and $v = ((x^{n-1} \otimes x^0) + (x^{n-2} \otimes x^1) + (x^{n-3} \otimes x^2) + \cdots + (x^0 \otimes x^{n-1}))\,\cdot$. Compute $uv(x^i \otimes x^j) = 0$ and $vu = 0, \pi u = 0$ to verify that this is a complex. Show it is exact using the contracting homotopy $s_{-1}(1) = 1 \otimes 1$ and

$$s_{2m}(1 \otimes x^j) = -\sum_{\ell=1}^{j} x^{j-\ell} \otimes x^{\ell-1}, \qquad s_{2m-1}(1 \otimes x^j) = \delta_{j,n-1} \otimes 1.$$

Apply $\mathrm{Hom}_{A^e}(-, A)$ to get

$$0 \to A \xrightarrow{u^*} A \xrightarrow{v^*} A \xrightarrow{u^*} \cdots,$$

using $\mathrm{Hom}_{A^e}(A^e, A) \cong A$ via $f \mapsto f(1 \otimes 1)$. Show that $u^*(a) = 0$ for $a \in A$ corresponding to f_a where $f_a(1 \otimes 1) = a$:

$$u^*(a) = u^*(f_a(1 \otimes 1)) = (u^* f_a)(1 \otimes 1) = f_a(u(1 \otimes 1)) = f_a(x \otimes 1 - 1 \otimes x) = x f_a(1 \otimes 1) - f(1 \otimes 1)x = xa - ax$$

and similarly

$$v^*(a) = v^*(f_a(1 \otimes 1)) = (v^* f_a)(1 \otimes 1) = f_a v(1 \otimes 1) = f_a(x^{n-1} \otimes 1 + \cdots + 1 \otimes x^{n-1}) = x^{n-1} f_a(1 \otimes 1) + \cdots + f_a$$

This yields

$$0 \to A \xrightarrow{0} A \xrightarrow{nx^{n-1}\cdot} A \xrightarrow{0} A \to \cdots.$$

So the homology depends on if $\mathrm{ch}\, k \mid n$:

- If so, $HH^*A = A + \sum_{n \geq 0} \langle x \rangle t^{2n+1} + \sum_{n \geq 0} A / \langle x^{n-1} \rangle$.
- If not, check!

Exercise 3.0.3 (?)
How can you interpret $\mathrm{HH}(A; M)$ in low degrees?

4 | Wednesday, February 15

Definition 4.0.1 (Gerstenhaber bracket)
For $f \in \mathrm{Hom}_k(A^{\otimes_k^m}, A)$ and $g \in \mathrm{Hom}_k(A^{\otimes_k^n}, A)$, set

$$[f, g] := f \circ g - (-1)^{m-1} g \circ f$$

where

$$(f \circ g)(a_1 \otimes \cdots \otimes a_{m+n-1}) := \sum_{i=1}^{m} (-1)^{(n-1)(m-1)} f(a_1 \otimes \cdots \otimes a_{i-1} \otimes g(a_i \otimes \cdots \otimes a_{n+i-1}) \otimes a_{n+i} \otimes \cdots \otimes a_{m+n-1}).$$

Lemma 4.0.2 *(?)*.
Let f, g as above and $h \in \mathrm{Hom}_k(A^{\otimes_k^p}, A)$. Then

1. Graded anticommutativity: $[f, g] = (-1)^{(m-1)(n-1)}[g, f]$

2. Graded Jacobi identity:

$$(-1)^{(m-1)(p-1)}[f,[g,h]]+(-1)^{(n-1)(m-1)}[g,[h,f]]+(-1)^{(p-1)(n-1)}[h,[f,g]].$$

3. Graded derivation: $d^*([f,g])=(-1)^{n-1}[d^*(f),g]+[f,d^*(g)]$.

Proof (?).
Define $|f|=m-1, |g|=n-1, |h|=p-1$ and $fg:=f\circ g$.
Part 1:

$$\begin{aligned}[f,g]=fg-(-1)^{|f||g|}gf &= -(-1)^{|f||g|}(gf-(-1)^{|f||g|}fg)\\ &= -(-1)^{|f||g|}[gf].\end{aligned}$$

Part 2:

$$\begin{aligned}&(-1)^{|f||h|}[f,gh-(-1)^{|g||h|}hg]\\ &+(-1)^{|g||f|}[g,hf-(-1)^{|h||f|}fh]\\ &+(-1)^{|h||g|}[h,fg-(-1)^{|f||g|}gf]\end{aligned}$$

$$\begin{aligned}=&(-1)^{|f||h|}[fgh-(-1)^{|g||h|}fhg-(-1)^{|f|\cdot(|g|+|h|)}\left(ghf-(-1)^{|g||h|}hgf\right)\\ &(-1)^{|g||f|}[ghf-(-1)^{|h||f|}gfh-(-1)^{|g|\cdot(|f|+|h|)}\left(hfg-(-1)^{|h||f|}fhg\right)\\ &(-1)^{|h||g|}[hfg-(-1)^{|f||g|}hgf-(-1)^{|h|\cdot(|f|+|g|)}\left(fgh-(-1)^{|f||g|}gfh\right)\end{aligned}$$

$$\begin{aligned}=&(-1)^{(m-1)(p-1)}fgh-(-1)^{(p-1)(m+n-2)}fhg-(-1)^{(m-1)(n+2p-3)}ghf+(-1)^{mn+np-m-p}hgf\\ &(-1)^{(n-1)(m-1)}ghf-(-1)^{(m-1)(n+p-2)}gfh-(-1)^{(n-1)(p+2m-3)}hfg+(-1)^{mp+np-m-n}hfg\\ &(-1)^{(p-1)(n-1)}hfg-(-1)^{(n-1)(m+p-2)}hfg-(-1)^{(p-1)(m+2n-3)}fgh+(-1)^{mp+mn-p-n}fgh,\end{aligned}$$

and everything cancels.

■

Exercise 4.0.3 (?)
Check part 3, this shows why the bracket is generally difficult to compute.

Remark 4.0.4: Properties 1 and 2 make $\bigoplus_{i\geq 0}\mathrm{Hom}_k(A^{\otimes_k^i},A)$ into a graded Lie algebra, and property 3 makes it into a DGLA with graded derivation δ: for f as above,

$$\delta(f):=(-1)^{|f|}d^*(f),\quad \delta([f,g])=[\delta(f),g]+(-1)^{|f|}[f,\delta(g)].$$

Thus $\mathrm{HH}^*(A)$ is a graded Lie algebra.

Lemma 4.0.5 *(?)*.
Let f,g as above, then

1.
$$(-1)^{(|f|+1)(|g|+1)} f \smile g - g \cup f = d^*(g) \circ f + (-1)^{|f|+1} d^*(g \circ f) + (-1)^{|f|} g \circ d^*(f).$$

2. $[f, \pi] = -d^*(f)$ where π is multiplication.

Proof (?).
Follows from a direct calculation. ∎

Theorem 4.0.6 ***(?).***
Let $A \in \mathsf{Assoc}_k\mathsf{Alg}$ for $k \in \mathsf{CRing}$. Then the cup product on $\mathrm{HH}^*(A)$ is graded commutative, so $a \smile b = (-1)^{|a||b|} b \cup a$ for $a \in \mathrm{HH}^m(A), b \in \mathrm{HH}^n(A)$ and $|a| := m, |b| := n$.

Proof (?).
Let a, b be images of cocycles f, g in $\mathrm{Hom}_k(A^{\otimes_k^m}, A)$ and $\mathrm{Hom}_k(A^{\otimes_k^n}, A)$ respectively. By part 1 of the lemma,
$$(-1)^{|a||b|} f \circ g - g \circ f = d^*(g) \circ f - (-1)^{|a|} d^*(g \circ f) + (-1)^{|a|-1} g \circ d^*(f).$$
Since f, g are cocycles, $d^*(f) = d^*(g) = 0$, so
$$(-1)^{|a||b|} f \smile g = g \smile f + (-1)^{|a|} d^*(g \circ f).$$
The error term vanishes in homology yielding
$$(-1)^{|a||b|} a \smile b = b \smile a \quad \in \mathrm{HH}^*(A).$$
∎

Lemma 4.0.7 ***(?).***
Let $a \in \mathrm{HH}^m(A)$ and $b \in \mathrm{HH}^n(A)$ and $g \in \mathrm{HH}^p(A)$. Then
$$[g, a \smile b] = [g, a] \smile b + (-1)^{|a| \cdot (|g|-1)} a \smile [g, b].$$

Proof (?).
See *The Cohomology Structure of an Associative Algebra*, Gerstenhaber 1963. ∎

Definition 4.0.8 (Gerstenhaber algebras)
A **Gerstenhaber algebra** or G**-algebra** $(H, \smile, [])$ is a free $\mathbf{Z}$-graded k-module H where $(H, \smile)$ is a commutative associative algebra and $(H, [])$ is a graded Lie algebra, where the two operations are compatible as in the lemma above.

Theorem 4.0.9 ***(?).***
$\mathrm{HH}^*(A)$ is a Gerstenhaber algebra.

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