AGGITATE

D. ZACK GARZA

ABSTRACT. Some rough notes from the AGGITATE 2024 summer school on moduli theory.

Contents

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1. 2024-07-22

1A. Valuations. Let $k = \mathbf{C}$ and X be a proper variety over k. Write $k(X)$ for its function field. Recall that a valuation is a group morphism $v : k(X)^\times \to (\mathbf{R}, +)$ where

- $v(fg) = v(f) + v(g)$
- $v(f + g) \geq \max\{v(f), v(g)\}$
- $v(k) = 0$

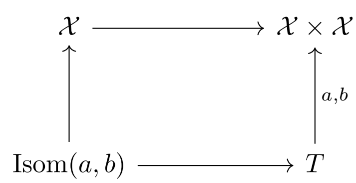
We generalize this slightly to *quasi-monomial valuations.* For X of dimension n, let (Y, E) be a (log smooth) minimal resolution. Write $E = \sum E_i$, and e.g. $p = \bigcap_i E_i$, and write $\widehat{\mathcal{O}_{Y,p}} = k[\![y_1, \cdots y_r]\!]$. Then any $f \in \mathcal{O}_{Y,p}$ can be written as $f = \sum_\alpha f_\alpha y^\alpha$. A quasi-monomial valuation v_m is given by $v_m(f) = \min\{m \cdot \alpha\}$ ranging over α. Write $QM(Y, E)$ for the set of quasi-monomials depending on the choice of (Y, E).

For $Y \xrightarrow{\varphi} X$ a resolution with exceptional divisor E, write $A_X(E) = \mathrm{Ord}_E(K_{Y/X}) + 1$ for the log discrepancy. For $v_\beta \in QM_p(Y, E)$ with $p = \bigcap E_i$, define $A_X(v_\beta) \coloneqq \sum \beta_i A_X(E_i)$.

2. 2024-07-23-14-30-14

2A. Review.

Remark 2.1. Recall from yesterday that an algebraic stack is a stack $\mathcal{X}$ over the big etale site $\mathsf{Sch}_{\text{Ét}}$ such that there exists a scheme U and a representable, smooth, surjective map $U \to \mathcal{X}$, i.e. a smooth presentation. A DM stack replaces smooth with étale, and an algebraic space is an algebraic stack where all stabilizers are trivial. Fact: the diagonal is representable.



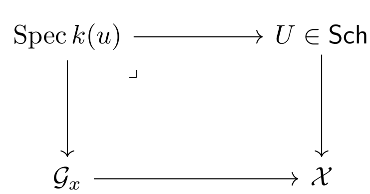
> Link to Diagram

The stabilizer of $x : \operatorname{Spec} k \to \mathcal{X}$ is $G_x = \mathrm{Isom}_k(x, x) \coloneqq \mathrm{Aut}_k(x)$.

Proposition 2.2. *If $\mathcal{X}$ is a Noetherian algebraic stack and $x \in |\mathcal{X}|$ is finite type, then $\exists \mathcal{G}_x \hookrightarrow \mathcal{X}$ a locally closed substack with $|\mathcal{G}_x| = \{x\}$ and $\mathcal{G}_x$ representable.*

Remark 2.3. See *residual gerbes $\mathcal{G}_x$.* Letting $[C] \in \mathcal{M}_g$ be a curve class, the residual gerbe is the classifying stack $\mathcal{G}_{[C]} = \mathbf{B}\,\mathrm{Aut}(C) \coloneqq [\operatorname{Spec} k / \mathrm{Aut}(C)]$.

Proposition 2.4 (Minimal presentations)**.** *Let $\mathcal{X}$ be a Noetherian algebraic stack and $x \in |\mathcal{X}|$ a finite type point with smooth stabilizer. Then there exists a diagram*



> *Link to Diagram*
Here $U \to \mathcal{X}$ is smooth of relative dimension $\dim G_x$.

Corollary 2.5. *$\mathcal{X}$ is DM iff stabilizers are finite and reduced.*

Proof. Set the top-left corner to an orbit $\mathcal{O}(u)$, show $\mathcal{O}(u) \to \mathcal{G}_x$ is smooth, and use the flat slicing criterion. □

Theorem 2.6 (Local structure)**.** *Let $\mathcal{X}$ be a separated Noetherian DM stack and $x \in |\mathcal{X}|$ a finite type point with geometric stabilizer G_x. Then there exists a nice etale presentation: $([\operatorname{Spec} A/G_x], w) \to (\mathcal{X}, x)$ which is etale and affine, a quotient of an affine by a finite group, inducing an isomorphism of stabilizer groups at w.*

Definition 2.7. A morphism $\pi : \mathcal{X} \to X$ from an algebraic stack to an algebraic space is a coarse moduli space if

1. π is universal for maps to algebraic spaces, so $\mathcal{X} \to Y \implies \exists! X \to Y$, and
2. $\forall k = \overline{k}$, $\mathcal{X}(k)_{/\sim} \to X(k)$ is bijective.

Remark 2.8. Idea: remove stabilizers from stack in exchange for giving up a universal family. Under mild conditions, (2) implies a bijection on topological spaces. Condition (1) is analogous to being a categorical quotient, while (2) is analogous to X being an orbit space.

Theorem 2.9. *If $G \curvearrowright \operatorname{Spec} A$ is a finite group action on an affine scheme, then $[\operatorname{Spec} A/G] \to \operatorname{Spec} A^G$ is a coarse moduli space.*

Theorem 2.10 (Keel-Mori)**.** *Let $\mathcal{X}$ be a separated and finite type DM stack over a Noetherian ring. Then $\exists \pi : \mathcal{X} \to X$ a coarse moduli space such that*

1. *π is a proper universal homeomorphism,*
2. *$\mathcal{O}_X \cong \pi_* \mathcal{O}_{\mathcal{X}}$,*
3. *stable under flat base change.*

Remark 2.11. An application: consider $\overline{\mathcal{M}_g}$. Because $\operatorname{Aut}(C)$ is finite and reduced, this is a DM stack. Semistable reduction implies it is proper. The Keel-Mori theorem implies existence of a coarse moduli space which is a proper algebraic space. Proving projectivity is substantially more difficult.

2B. **Quasicoherent sheaves.**

Remark 2.12. In particular for algebraic stacks, one can form $\mathsf{QCoh}(\mathcal{X})$ and the standard adjunctions f^* and f_* where the latter is proper pushforward. If $G \curvearrowright \operatorname{Spec} A$ is an algebraic group action, then there is a diagram

$$\begin{array}{ccccc} \operatorname{Spec} A & \xrightarrow{p} & [\operatorname{Spec} A/G] & \xrightarrow{q} & \mathbf{B}G \\ & & \big\downarrow{\scriptstyle \pi} & & \\ & & \operatorname{Spec} A^G & & \end{array}$$

> Link to Diagram
Then

- $q_* M = M$ forgets the A-module structure,

- $p^* M = M$ forgets the G-action,
- $\pi_* M = M^G$ recovers the invariants.

Definition 2.13. A good moduli space is a map $\pi : \mathcal{X} \to X$ from an algebraic stack to an algebraic space such that

1. π_* is exact on quasicoherent sheaves, and
2. $\pi_* \mathcal{O}_{\mathcal{X}} \cong \mathcal{O}_X$.

Remark 2.14. If $\mathcal{X} = [U/G]$ with G finite reductive, this is equivalent to $U \to X$ being a good quotient.

Example 2.15. If $G \curvearrowright \operatorname{Spec} A$ is linearly reductive, then $[\operatorname{Spec} A/G] \to \operatorname{Spec} A^G$ is a good moduli space.

Theorem 2.16. *If $\mathcal{X}$ is an algebraic space with mild hypotheses, then $\pi : \mathcal{X} \to X$ is a good moduli space iff etale locally it looks like taking invariants, i.e. there are diagrams*

$$\begin{array}{ccc} [\operatorname{Spec} A/G] & \longrightarrow & \mathcal{X} \\ \big\downarrow & \lrcorner & \big\downarrow \\ \operatorname{Spec} A^G & \longrightarrow & X \end{array}$$

> *Link to Diagram*

Theorem 2.17. *Let $\pi : \mathcal{X} \to X$ be a good moduli space. Then*

- *π is surjective.*
- *If Z_i are closed and disjoint substacks of $\mathcal{X}$, their images under π are closed and disjoint.*
- *Closed points of X correspond to closed points of $\mathcal{X}$. Two orbits are identified iff their closures intersect.*
- *If $\mathcal{X}$ is finite type over a Noetherian scheme then X is finite type.*
- *π is universal for maps to algebraic spaces.*

Department of Mathematics, University of Georgia, Athens GA 30602, USA
Email address: zack@uga.edu