ALGEBRAIC SURFACES

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ABSTRACT. Notes from a course on algebraic surfaces taught by Valery Alexeev in Fall 2024 at the University of Georgia.

CONTENTS

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1. 2024-08-15-12-47-25 Introduction

Main topic: algebraic surfaces, always assumed projective. Can we classify all such surfaces? Either

- Up to isomorphism (biregular classification)
- Up to birational isomorphism (birational classification, weaker)

Recall that $X_1 \overset{\sim}{\dashrightarrow} X_2$ iff they share a Zariski open subset, noting that open sets are dense. Equivalently, there is an isomorphism of function fields $k(X_1) \cong k(X_2)$, so varieties up to birational isomorphism are equivalent to finitely generated field extensions. We will take $k = \overline{k}$ characteristic zero, and assume $k = \mathbf{C}$ and apply the Lefschetz principle – any such field admits an embedding $k \hookrightarrow \mathbf{C}$. Fields of other characteristics will appear. Note that $\dim X = d$ corresponds to fields of transcendence degree d over k.

For $d = 1$, we are considering smooth projective curves. However, $C_1 \cong C_2 \iff C_1 \overset{\sim}{\dashrightarrow} C_2$, so there is no birational geometry to speak of in this dimension. The basic invariant is the genus $g(C) \geq 0$.

- If $g = 0$ then $C \cong \mathbf{P}^1$,
- if $g = 1$ then C is an elliptic curve.
- If $g \geq 2$ (which includes most curves), there exists a moduli space $\mathcal{M}_g$ of such curves.

Points of $\mathcal{M}_g$ correspond to isomorphism classes of such curve – there are infinitely many such classes, but they are organized into a variety. This space is almost smooth but is not complete. There exists a Deligne-Mumford compactification $\overline{\mathcal{M}_g}$, the boundary curves are so-called *stable curves.*

1A. Basic tools.

We will introduce tools which will be black-boxed in order to apply them to the classification problem. For S a surface, divisors D are formal linear combinations of curves. For any divisor D, there is an invertible (rank 1) locally free sheaf $\mathcal{O}_S(D)$. The sections are locally regular functions. Divisors modulo linear equivalence correspond to invertible sheaves up to isomorphism, we call this group $\mathrm{Pic}(S)$.

We introduce numerical invariants to generalize the genus. The first is the irregularity $q = h^0(\Omega^1_S)$. For curves, one recovers $q = g$. Another is the geometric genus $p_g = h^0(\det \Omega_S) = h^0(\omega_S)$, the dimension of the space of sections of the top-degree differentials. Note that $\omega_S \cong \mathcal{O}_S(K_S)$ for a canonical divisor K_S on S. For curves, $\deg K_S = 2g-2$. This splits the classification into three cases:

- $g = 0 \iff K_S < 0 \iff \kappa = -\infty$,
- $g = 1 \iff K_S = 0 \iff \kappa = 0$,
- $g \geq 2 \iff K_S > 0 \iff \kappa = 1$. This is the *general type* case.

The plurigenus is defined as $p_m = h^0(mK_S)$. As $m \to \infty$, it is a fact that $p_m \sim m^k$ for some k, we define $\kappa := k$. One generally has $\kappa \in \{-\infty, 1, 2\}$ for surfaces, where $\kappa = -\infty$ iff $p_m = 0$ for all $m \gg 0$. The classification of surfaces similarly is by Kodaira dimension.

1B. **Birational geometry of surfaces.** Let $p \in S$ be a point of a smooth surface. Consider $\mathrm{Bl}_p S$, which replaces p with $E \cong \mathbf{P}^1$ and $E^2 = -1$. To define the intersection pairing – look at the normal bundle $N_{E/S} = \mathcal{O}_S(-1)$. There is a map $\mathrm{Bl}_p(S) \to S$ which is an isomorphism away from p and E. Castelnuovo's criterion gives a converse: any such curve can be blown down. Note that E is referred to as an exceptional curve of the first kind, i.e. a (-1)-curve. Exceptional curves of higher kinds are rarely used. Fact: any birational isomorphism of surfaces factors as a composition of blowups and blowdowns.

We say S is *minimal* if there are no (-1)-curves. Otherwise just blow them down, i.e. contract those curves. We will introduce a numerical invariant that decreases under blowdowns, so this process terminates.

Recall that there is an intersection product C_1C_2 which, if they intersect transversally, counts intersection points. In more singular situations, it takes into account intersection multiplicity. Note that everything is oriented when over $\mathbf{C}$, so $C_1C_2 \geq 0$ for any curves.

How does one compute E^2? Replace one copy of E by $D_1 - D_2$ and compute $E(D_1 - D_2)$ instead.

New goal: classify minimal surfaces.

- $\kappa = -\infty$: $S = \mathbf{P}^2$ or geometrically ruled surfaces (fibrations over a curve C with $\mathbf{P}^1$ fibres). If $C = \mathbf{P}^1$ in this fibration, one obtains a Hirzebruch surface F_n for $n \geq 0$ and $n \neq 1$ (which is not minimal). These are rational surfaces. Note that "minimal" is classical terminology and is no longer used for $\kappa = -\infty$: these are instead called Mori Fano fibrations.
- $\kappa = 0$: abelian surfaces and K3 surfaces ($K_S = 0$), Enriques surfaces ($K_S \neq 0, 2K_S = 0$) and $p_g = q = 0$, or bielliptic surfaces (products $E_1 \times E_2/\mathbf{Z}_n$ where $n = 2, 3, 4, 6$ and E_i are elliptic curves).
- $\kappa = 1$: elliptic surfaces (admit fibrations over curves whose general fiber is an elliptic curve).
- $\kappa = 2$: general type (most surfaces).

Note that if $\kappa \geq 0$ and S is minimal then K_S is nef, i.e. $K_S \cdot C \geq 0$ for any curve C. One can take this as the definition of minimal.

Note that one has $K_S = mF$. If $m < 0$ then $C \cong \mathbf{P}^1$, if $m = 0$ then S is K3 or Enriques, and if $m > 0$ then $\kappa = 1$.

The general type surfaces are hard to classify, except for special cases. E.g. $p_g = q = 0$ which give counterexamples to these invariants determining rationality. Examples include Godeaux, Campadelli, Balrov, Inuoe, Burniat surfaces. These are sometimes referred to as fake projective planes, famously there are 100 of these.

Note Castelnuovo's criterion: S is rational iff $p_2 = q = 0$.

One can consider the moduli of surfaces of general type $M_{c_1^2, c_2}$ where $c_1^2 = K_S^2$ and c_2 is another numerical invariant. These are sometimes called Gieseker moduli space. There are KSBA compactifications, since GIT does not work well in the setting of surfaces.

1C. **Other subjects.** We can discuss:

- K3s
- Characteristic p, mainly $p = 2, 3$
- Complex-analytic surfaces (non-algebraic), algebraic dimension $a \in \{0, 1, 2\}$. Here $a = 2$ is the algebraic case, and the other cases are non-algebraic.
- Non-closed fields $k \neq \overline{k}$
- Singular surfaces

Why studying smooth surfaces suffices: any variety admits a normalization, and any normal surface has a unique minimal resolution in any characteristic. In higher dimensions, existence of resolutions is generally an open problem, and are almost never unique. Thus birational geometry for threefolds is significantly harder.

2. 2024-08-20-12-47-17: Algebraic Tools

2A. **Divisors and** Pic**.**

Remark 2.1. The tools we'll use: divisors, line bundles, sheaves, cohomology, intersection theory, and Riemann-Roch. Almost all surfaces we consider in this class will be smooth. Let X be a smooth algebraic variety. A divisor $D = \sum n_i Z_i$ with $n_i \in \mathbf{Z}$ and Z_i codimension 1 subvarieties. On a curve, divisors are weighted sums of points, and on a surface they are weighted sums of curves. We say $D_1 \sim D_2$ if $D_1 - D_2 = (\phi)$ with $\phi \in \mathbf{C}(X)$ a rational function. These are principal divisors. We define $\operatorname{Pic}(X) \coloneqq \operatorname{Div}(X)/\sim$.

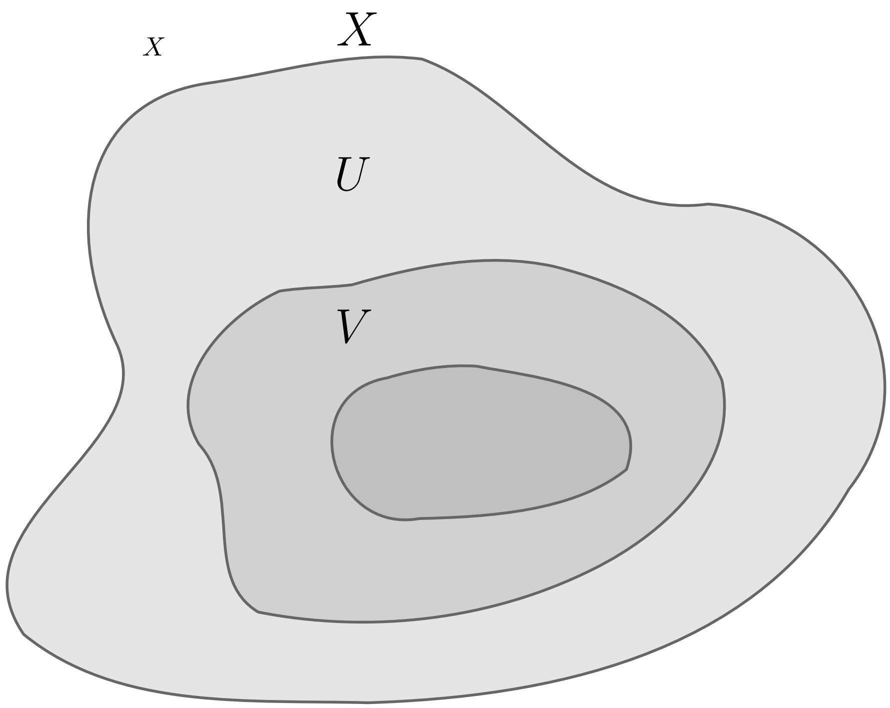
Example 2.2. Fact: $\operatorname{Pic}(\mathbf{P}^n) = \mathbf{Z}$. Write $\mathbf{P}^n = \{[x_0 : \cdots : x_n]\}/x \sim \lambda x$ and $H_i \coloneqq \{x_i \neq 0\}$. We have $\mathbf{P}^n \setminus H_0 = \{[1 : x_1 : \cdots : x_n]\}$. The claim is that for any D, $D \sim dH_0$ for some d. For $Z \subseteq \mathbf{P}^n$ a codimension 1 subvariety, we have $Z|_{H_0} = V(f)$ since $\mathbf{C}[x_1, \cdots, x_n]$ is a PID. Write $\phi = f_d\left(\frac{x_1}{x_0}, \cdots, \frac{x_n}{x_0}\right) = \frac{g(x)}{x_0^d}$, which has a pole of order d at x_0. Thus $(\phi) = Z - dH_0$, so $Z \sim dH_0$. Moreover, $dH_0 \sim 0 \iff d = 0$.

Remark 2.3. Over $\mathbf{C}$, one has the GAGA principle: if $X \subseteq \mathbf{CP}^n$ is a closed analytic subset, by the Chow lemma it is algebraic. One can then compute cohomology using analytic tools or algebraic tools.

Remark 2.4. For example, one can identify $\operatorname{Pic}(X)$ as line bundles modulo isomorphism, i.e. $H^1(\mathcal{O}_X^\times)$. Note that $\mathcal{O}_X(U)$ are regular functions, and $\mathcal{O}_X^\times(U)$ are nowhere vanishing regular functions – these are the first examples of sheaves.

2B. **Sheaves.**

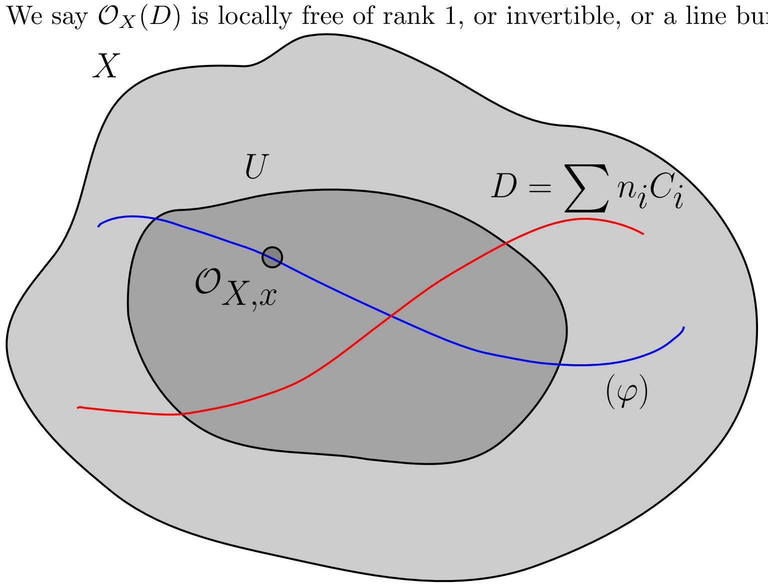
Remark 2.5. Recall that a sheaf F of abelian groups is an assignment of open sets U to sections $F(U)$ with appropriate restriction maps.



Recall that $\mathcal{O}_X(D)$ is locally isomorphic to $\mathcal{O}_X$, and sections of $\mathcal{O}_X(D)$ are rational functions ψ such that $(\psi) + D \geq 0$. Recall that $D_1 \sim D_2 \implies \mathcal{O}_X(D_1) \cong \mathcal{O}_X(D_2)$ by writing $D_1 - D_2 = (\zeta)$ and mapping $s \mapsto s\zeta$. Checking $(\zeta) + D_1 \geq 0$, one has

$$(\zeta\psi)+D_2 \geq 0 \iff (\zeta)+(\psi)+D_2 \geq 0 \iff D_1-D_2+(\psi)+D_2 \geq 0 \iff D_1+(\psi)+D_1 \geq 0.$$

We say $\mathcal{O}_X(D)$ is locally free of rank 1, or invertible, or a line bundle.



Note that $\mathrm{Aut}(\mathcal{O}_X) = \mathcal{O}_X^\times$.

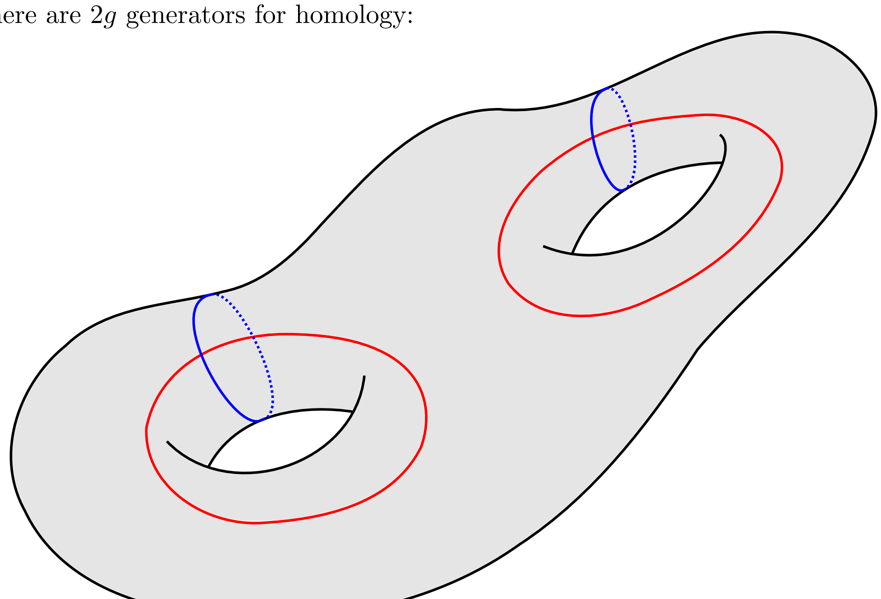
Remark 2.6. There is a SES

$$0 \to I \to \mathcal{O}_X \to \mathcal{O}_C \to 0$$

where $I \cong \mathcal{O}_X(-C)$. These are examples of coherent sheaves, those which can locally be written $\mathcal{O}^{\oplus J} \to \mathcal{O}^{\oplus I} \to F \to 0$. An example of a non-coherent sheaf is $\mathcal{O}_X^\times$, since one can not multiply a section by a function with a zero. Other examples include locally constant sheaves $\underline{\mathbf{Z}}, \underline{\mathbf{C}}$.

Remark 2.7. Given $A \hookrightarrow B \twoheadrightarrow C$ there is a LES $H^i(A) \to H^i(B) \to H^i(C) \to H^{i+1}(A) \to \cdots$. By Grothendieck, $H^i(F) = 0$ for $i > d = \dim X$ if F is coherent and X is a Noetherian topological space. If X is projective and F is coherent, $H^i(F)$ all form finite-dimensional vector spaces. We write these dimensions as $h^i(F)$, which are numerical invariants. Define $\chi(F) = \sum_i (-1)^i h^i(F)$. One has $\chi(B) = \chi(A) + \chi(C)$, since χ of a LES is zero.

Remark 2.8. Recall that the Betti numbers are defined as $\beta_i \coloneqq \dim_{\mathbf{Q}} H^i(X; \mathbf{Q})$ and $\chi(X) \coloneqq \sum (-1)^i \beta_i$. Note that $\chi(X) \neq \chi(\mathcal{O}_X)$ in general: let C be a smooth projective curve of genus g over $\mathbf{C}$. Check that $\chi(X) = 1 - 2g + 1 = 2 - 2g$, since there are $2g$ generators for homology:



However, recall $h^1 = h^{1,0} + h^{0,1} = g + g$, and $\chi(\mathcal{O}_X) = h^0(\mathcal{O}_X) - h^1(\mathcal{O}_X) = 1 - g = \frac{1}{2}(2 - 2g)$ instead.

2C. **Cohomology.**

Remark 2.9. Recall the exponential exact sequence $0 \to \mathbf{Z} \xrightarrow{\cdot 2\pi i} \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^\times \to 1$. Taking the LES yields:

$$\begin{array}{ccccc} H^0(\mathbf{Z}) = \mathbf{Z} & \longrightarrow & H^0(\mathcal{O}_X) = \mathbf{C} & \longrightarrow & H^0(\mathcal{O}_X^\times) = \mathbf{C}^\times \\ \\ H^1(\mathbf{Z}) = \mathbf{Z}^{2g} \oplus \text{tors} & \longrightarrow & H^1(\mathcal{O}_X) = \mathbf{C}^g & \longrightarrow & H^1(\mathcal{O}_X^\times) = \text{Pic}(X) \\ \\ H^2(\mathbf{Z}) = \mathbf{Z}^n \oplus \text{tors} & \longrightarrow & H^2(\mathcal{O}_X) & \longrightarrow & \cdots \end{array}$$

> Link to Diagram

We thus get a SES $0 \to \mathbf{C}^g/\mathbf{Z}^{2g} \to \text{Pic}(X) \to F \to 0$ where F is some finitely generated abelian group. We define $\text{Pic}^0(X) = \text{Jac}(X) = \mathbf{C}^g/\mathbf{Z}^{2g}$, which is an algebraic torus over $\mathbf{C}$ of dimension g, and is the continuous part. We identify $F \coloneqq \text{NS}(X)$ as the Néron-Severi group. Note that divisors in $\text{Pic}^0(X)$ are numerically zero, since intersection numbers are integers that vary continuously.

Example 2.10. $\text{NS}(\mathbf{P}^n) = \mathbf{Z}$, and $\text{NS}(C) = \mathbf{Z}$ for a genus g curve since the only numerical invariant of continuously varying divisors is $\sum n_i$.

Remark 2.11. Note that $D_1, D_2 \in \text{Pic}^0(X)$ means D_1 is algebraically equivalent to D_2, i.e. can be varied continuously. For linear equivalence, one uses $\mathbf{P}^1$ as the base, since rational functions on X are equivalent to maps to $\mathbf{P}^1$. Algebraic equivalence allows for the base to be an arbitrary curve.

3. 2024-08-22-12-46-47: Divisors and Intersection Theory

3A. Divisors.

Remark 3.1. Let X be a smooth projective variety, and recall $\text{Pic}(X) = \text{Div}(X)/\sim$ or the group of rank 1 locally free sheaves F where $F|_U = \mathcal{O}_X|_U$. Both are isomorphic to $H^1(\mathcal{O}_X^\times) \cong \check{H}^1(\mathcal{O}_X^\times)$, the Čech cohomology. Let $\mathcal{U}$ be an open cover of X, so $X = \cup_i U_i$. Then a divisor D on U_i locally has an equation f_i for some rational function, i.e. $(f_i) = D$. Consider $\{f_i \in K^\times(U_i)\}$. Then on U_{ij}, write $g_{ij} = \frac{f_i}{f_j}$. This has no zeros and no poles, so $(g_{ij}) = 0$. Thus g_{ij} and $1/g_{ij}$ are both regular and give sections $g_{ij} \in \mathcal{O}^\times(U_{ij})$. We get a collection $\left\{ g_{ij} \in \mathcal{O}^\times(U_{ij}) \;\middle|\; g_{ij} \cdot g_{jk} \cdot g_{ki} = 1 \text{ on } U_{ijk} \right\}$ modulo $f_i' = f_i g_i$ for $g_i \in \mathcal{O}^\times(U_i)$. This is precisely $\check{C}^1/\check{B}^1 = \check{H}^1(\mathcal{U}, \mathcal{O}_X^\times)$. One defines $\check{H}^1(X, F) = \text{colim}_{\mathcal{U}} \check{H}^1(\mathcal{U}, F)$ for a sheaf F.

Example 3.2. Let $F = \mathcal{O}_{\mathbf{P}^1}(H)$ where $H \coloneqq p_\infty \coloneqq \{x_0 = 0\}$ in coordinates $[x_0 : x_1]$. Write $\mathbf{P}^1 = \mathbf{A}^1_{\frac{x_1}{x_0}} \cup \mathbf{A}^1_{\frac{x_0}{x_1}}$. The equations of D are 1 and $\frac{x_0}{x_1}$ respectively, and the transition function is $f_{12} = \frac{x_0}{x_1}$. Thus $F = \mathcal{O}(1)$.

Letting $D = dH$ yields $\mathcal{O}(D) = \mathcal{O}(d)$. Note that $\mathbf{P}^n \setminus H \cong \mathbf{A}^n$ where $H = \{x_0 = 0\}$.

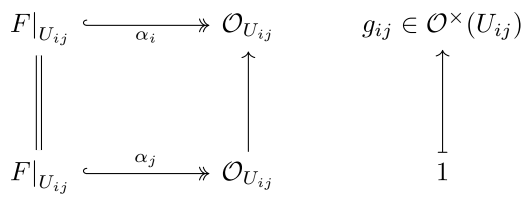
Remark 3.3. For any projective variety, there are three important sheaves:

- $\mathcal{O}_X$,

- $\mathcal{O}_{\mathbf{P}^n}(1)\big|_X = \mathcal{O}_X(1)$ for $X \hookrightarrow \mathbf{P}^N$ a projective embedding,
- $\omega_X = \mathcal{O}_X(K_X)$.

On a normal variety, $\mathrm{CDiv}(X) \hookrightarrow \mathrm{Div}(X)$ is a subgroup. There is always such a map, even for non-normal varieties, but generally $\mathrm{CDiv}(X)$ is bigger.

Remark 3.4. Letting $\mathcal{U}$ be an open cover, consider U_{ij}. We have $F\big|_{U_i} \cong \mathcal{O}_{U_i}$, and similarly we get a diagram



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Remark 3.5. If $\dim X = n$, then $\omega_X := \Omega_X^n$ which has sections $\Omega_X(U) = \{f dx_1 \wedge \cdots \wedge dx_n\}$ where f is regular. This is an invertible rank 1 locally sheaf. We have $\Omega_X^1 = \{f_1 dx_1 + \cdots + f_n dx_n\}$ which is rank n, locally isomorphic to $\mathcal{O}_X^{\oplus n}$ since the dx_i form a basis. Similarly we have $\operatorname{rank} \Omega_X^k = \binom{n}{k}$.

Example 3.6. Consider again $\mathbf{P}^1 = \mathbf{A}_u^1 \cup \mathbf{A}_v^1$, where $u = \frac{x_0}{x_1}$ and $v = \frac{x_1}{x_0}$, so $v = 1/u$. On the intersection, $dv = d(1/u) = -\frac{du}{u^2}$ and thus $\omega_{\mathbf{P}^1} = \mathcal{O}_{\mathbf{P}^1}(-2)$ and $K_{\mathbf{P}^1} = -2H$ where H is a point. More generally, $K_{\mathbf{P}^n} = -(n+1)H$ and $\omega_{\mathbf{P}^n} = \mathcal{O}(-n-1)$.

Similarly, $K_{\mathbf{P}^1 \times \mathbf{P}^1} = p_1^*(-2H) + p_2^*(-2H)$ where p_i are the two projections. More generally, $K_{X \times Y} = p_1^* K_X + p_2^* K_Y$. Note that $\mathrm{Pic}(\mathbf{P}^1 \times \mathbf{P}^1) = \mathbf{Z} \oplus \mathbf{Z}$, but $\mathrm{Pic}(X \times Y) \neq \mathrm{Pic}(X) \oplus \mathrm{Pic}(Y)$ in general. For a counterexample, take E an elliptic curve, then $\mathrm{Pic}(E \times E) \supseteq \mathbf{Z}^3$. We write $\mathcal{O}(a,b) \in \mathrm{Pic}(\mathbf{P}^1, \mathbf{P}^1)$ to denote bidegree a, b curves.

Remark 3.7. To compute for other varieties, we do this indirectly using the adjunction and Hurwitz formula. For $Y \subseteq X$, we have $K_Y = (K_X + Y)\,\Big|_Y$, or $\omega_Y = (\omega_X \otimes \mathcal{O}_X(Y))\,\Big|_Y$. For $C \hookrightarrow \mathbf{P}^2$ a curve of degree d, this yields $K_C = (d-3)H\,\Big|_C$. One checks $2g - 2 = \deg K_C = d(d-3)$, which implies $g = \frac{1}{2}(d-1)(d-2)$.

Exercise 3.8. For $C \times C$, show that $\Delta^2 = 2 - 2g$. On the other hand, show $(f_1 + f_2)^2 = 2$, so $\Delta \not\sim a f_1 + b f_2$ for any fibers f_i.

Remark 3.9. Consider $S_d \hookrightarrow \mathbf{P}^3$ a hypersurface of degree d cut out by some f_d. Then $K_S = (d-4)H\,\Big|_S$, and $d = 4 \implies K_S = 0$ and this gives a K3 surface. Similarly $d < 4 \implies K_S > 0$ and $d < 4 \implies K_S < 0$.

Remark 3.10. The Hurwitz formula: let $\pi : X \to Y$ be a map of curves, then $K_X = \pi^* K_Y + R$ where $R = \sum (n_i - 1) p_i$ where n_i is the ramification number – given in local equations as $y = x^{n_i}$. Note that if $y = x^n$ then $dy = n x^{n-1} dx$. This equation generalizes to ramified maps of surfaces with the same formula.

Generally for a 2-to-1 cover $\pi : S \to \mathbf{P}^2$, one has $R = \frac{1}{2}\pi^* B$ and

$$K_S = \pi^* K_{\mathbf{P}^2} + R = \pi^*(K_{\mathbf{P}^2} + \frac{1}{2}B) = \pi^*(-3 + d/2)H.$$

Remark 3.11. Recall that $h^i(D) = h^{n-i}(K - D)$ by Serre duality. Define

- $q \coloneqq h^1(\mathcal{O})$ the irregularity,
- $p_g \coloneqq h^2(\mathcal{O})$ the geometric genus,
- $h^1(\omega) = h^1(\mathcal{O}) = q$,
- $h^2(\omega) = h^0(\mathcal{O}) = 1$,
- $h^0(\omega) = h^2(\mathcal{O}) = p_g$,
- $h^0(mK) = p_m$ the plurigenera,
- Consider $\lim_{m\to\infty} p_m \sim m^\kappa$, then κ is the Kodaira dimension.

3B. **Intersection Theory.**

Remark 3.12. There is a symmetric pairing $\mathrm{NS}(X) \times \mathrm{NS}(X) \to \mathbf{Z}$ given by intersecting curves and counting points. Recall there was a SES

$$0 \to \frac{H^1(\mathcal{O}_X)}{H^1(X;\mathbf{Z})} \to \mathrm{Pic}(X) \to \mathrm{NS}(X) \coloneqq \ker\left(H^2(X;\mathbf{Z}) \to H^2(\mathcal{O}_X)\right) \to 0.$$

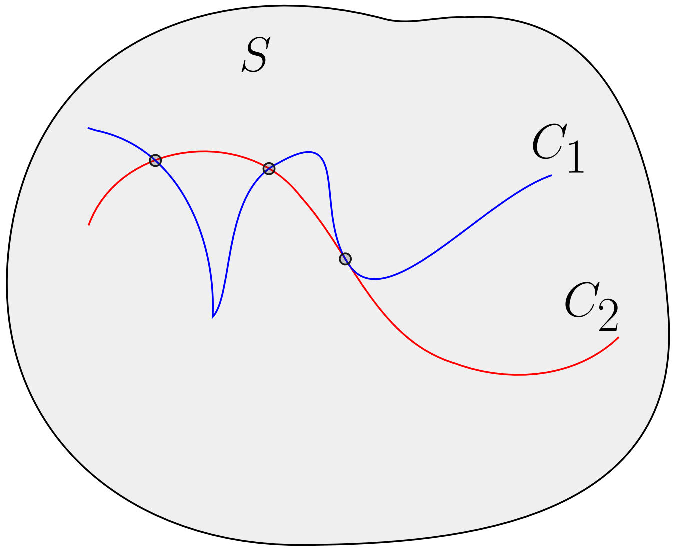
We have $\mathrm{NS}(X)/\mathrm{tors} \cong \mathbf{Z}^\rho$ where ρ is the Picard rank. Note that this is compatible with the cup product pairing when working with varieties over $\mathbf{C}$. A definition that works over any field: $C_1 \cdot C_2 = \sum_{p\in C_1\cap C_2} \mu_p(C_1, C_2)$ where $\mu_p(C_1, C_2) = \dim_{\mathbf{C}} \mathcal{O}_{S,p}/\langle f, g\rangle$ where f, g are local equations for the C_i near p. This works when $\sharp(C_1 \cap C_2) < \infty$.

Remark 3.13. The moving lemma: $\forall C_2 = A - B$ where A, B are general hyperplanes for some $S \hookrightarrow \mathbf{P}^{n_i}$. Note that if $C \hookrightarrow S$ is smooth, then $C^2 = \deg N_{C/S}$, the degree of the normal bundle.

Upcoming: Hirzebruch-Riemann-Roch.

4. 2024-08-27-12-46-20: Intersection theory on surfaces

Remark 4.1. Recall that $\mathrm{NS}(X)$ carries an integral intersection form. For curves without common components, it counts the intersection points with multiplicities.



For all $f : X \to Y$ maps of varieties, define $f^* : \mathrm{Pic}(Y) \to \mathrm{Pic}(X)$ and $\mathrm{NS}(Y) \to \mathrm{NS}(X)$, defined by pulling back equations. Define $f_* : \mathrm{Pic}(X) \to \mathrm{Pic}(Y)$ by

$$f_* C = \begin{cases} 0 & f(C) = \mathrm{pt} \\ dC' & f(C) = C', d = \deg(C \to C') \end{cases}.$$

Some properties:

1. If $f : S' \to S$ is generically degree d then
$$(f^* D_1) \cdot (f^* D_2) = d D_1 \cdot D_2.$$
2. Projection formula:
$$f^* D_1 \cdot D_2 = D_1 \cdot f_* D_2.$$
3. Hirzebruch-Riemann-Roch: let X be a smooth projective of dimension n and $L = \mathcal{O}_X(D)$ a line bundle on X. Then
$$\chi(L) = \int_X e^D \mathrm{Td}_X.$$
Here $e^D = \sum_k D^k / k!$ and $\mathrm{Td}_X = 1 + \frac{1}{2} c_1 + \frac{1}{12}(c_1^2 + c_2) + \cdots$ and c_i are the Chern numbers of T_X.

Recall that $c_1 : \mathrm{Pic}(X) \to H^2(X; \mathbf{Z})$. Note that $c_1 = -K_X$ and $c_n = \chi_{\mathsf{Top}}(X)$. Here $\int_X (-)$ is the degree of a cycle of codimension n, hence dimension 0, and counts its number of points.

Example 4.2. Let $n = 1$, so C is a curve of genus g. Then $c_1 = -K_C$ has degree $2 - 2g$, and $c_n = c_1 = \chi_{\mathsf{Top}}(C) 1 - 2g + 1 = 2 - 2g$. Thus HRR yields

$$\chi(D) = \int_X (1 + D)(1 + \frac{1}{2} c_1) = \deg\left(D + \frac{c_1}{2}\right) = \deg D + 1 - g.$$

Setting $D = 0$ yields $\chi(\mathcal{O}_C) = \deg\left(\frac{c_1}{2}\right) = 1 - g$.

Example 4.3.

$$\chi(\mathcal{O}_S(D)) = \int_S \left(1 + D + \frac{1}{2} D^2\right)\left(1 + \frac{1}{2} c_1 + \frac{1}{12}(c_1^2 + c_2)\right) = \frac{1}{2} D^2 + \frac{1}{2} D c_1 + \frac{1}{12}(c_1^2 + c_2),$$

thus $\chi(\mathcal{O}_S) = \frac{1}{12}(c_1^2 + c_2)$ by setting $D = 0$, which recovers Noether's formula. Note that $c_1 = -K_S$, so $\chi(\mathcal{O}_S) = \frac{K^2+c^2}{12}$. We can then rewrite this as

$$\chi(D) = \frac{D(D-K)}{2} + \chi(\mathcal{O}_S)$$

where $\chi(\mathcal{O}_S) = 1 - q + p_g$.

Remark 4.4. Recall that for a curve D on a surface S, one has $\deg K_D = 2p_a(D) - 2$ and $K_D = (K_S + D)\Big|_D$. From the genus formula, $p_a(D) = \frac{D(D+K)}{2} + 1$.

Example 4.5. Consider $S = \mathbf{P}^2$, so $K_S = -3H$ and $K_S^2 = 9$. Check that $\chi_{\mathsf{Top}}(S) = 3$ by counting torus fixed points in the standard polytope, and thus $\frac{c_1^2+c_2}{12} = \frac{9+3}{12} = 1$. Check that

$$h^0(\mathcal{O}_{\mathbf{P}^2}(d)) = \begin{cases} \binom{d+2}{2} & i = 0 \\ h^0(-3-d) & i = 2 \\ 0 & i \neq 0, 2 \end{cases}.$$

Thus $\chi(\mathcal{O}_S) = 1$.

Example 4.6. Let $S = \mathbf{P}^1 \times \mathbf{P}^1$, so $K = \mathcal{O}(-2,-2) = 2e - 2f$ and $K^2 = 8$ since $ef = 1$. Moreover $\deg c_2 = 4$ by considering the toric polytope and counting points. Noether's formula yields $\frac{8+4}{12} = 1$.

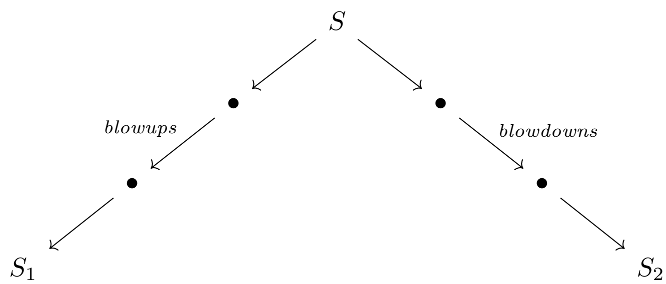
Remark 4.7. Note that $\chi_{\mathsf{Top}}(\mathcal{O}_S(D))$ can be computed easily for toric varieties. The Euler characteristic is additive, and every such variety decomposes as $(\mathbf{C}^\times)^2 \coprod \mathbf{C}^\times \coprod \{p_i\}$, and $\chi(\mathbf{C}^\times) = 0$, so only points p_i contribute.

4A. **Birational geometry of surfaces.**

Remark 4.8. Recall that rational maps on irreducible varieties are regular functions on open subsets, and correspond to maps $X \to \mathbf{A}^1$. Note that any two open subsets of an irreducible variety intersect. One can add, multiply, and invert nonzero rational functions by throwing out closed sets of zeros, so these form a field. Say $X \overset{\sim}{\dashrightarrow} Y$ are birationally isomorphic if they share a common open set $X \supseteq U \subseteq V$. Say X is rational iff $X \overset{\sim}{\dashrightarrow} \mathbf{P}^n$ or $\mathbf{A}^n$, and X is unirational if there exists a dominant rational map $\mathbf{P}^n \dashrightarrow X$ (covered by a rational variety). Dominant morphisms: closure of the image is the entire space. Note that dominant rational maps $X \to Y$ yields an embedding of fields $\mathbf{C}(Y) \hookrightarrow \mathbf{C}(X)$. Thus rational varieties of dimension n satisfy $\mathbf{C}(X) = \mathbf{C}(x_1, \cdots, x_n)$, while unirational means $\mathbf{C}(X)$ is a subfield.

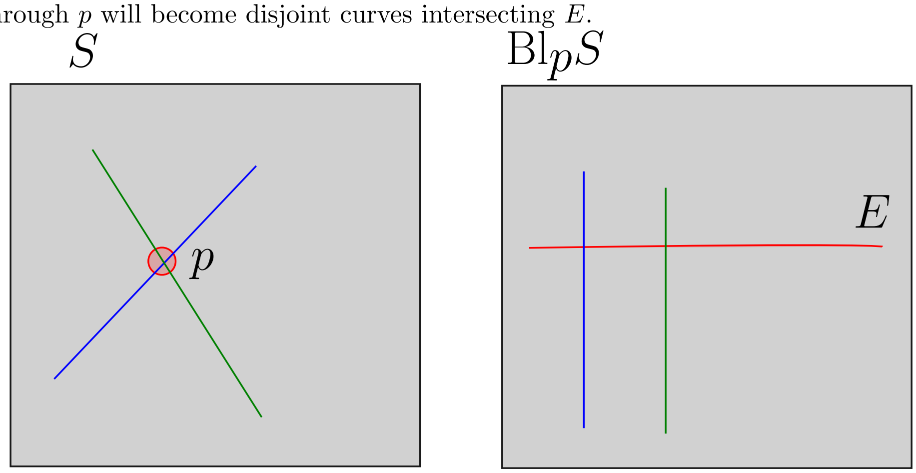
Remark 4.9. Luroth problem: does unirational imply rational? True for $n = 1, 2$, we will prove that $n = 2$ case in this course. Generally false for $n \geq 3$.

Theorem 4.10. *Suppose that $S_1 \overset{\sim}{\dashrightarrow} S_2$ are birationally isomorphic smooth projective surfaces. Then there exists a diagram of blowups and blowdowns forming a correspondence:*



> *Link to Diagram*

Remark 4.11. We will define a blowup of a smooth surface at a smooth point $\pi : \mathrm{Bl}_p S \to S$. We start with the example of $S = \mathbf{A}^2$ and p an arbitrary point. This will be birational because $\mathbf{A}^2 \setminus \{p\} \cong \mathrm{Bl}_p \mathbf{A}^2 \setminus E$. Different lines passing through p will become disjoint curves intersecting E.



In equations, write $\mathrm{Bl}_p \mathbf{A}^2 = \{xY = yX\} \subseteq \mathbf{A}^2 \times \mathbf{P}^1$ with coordinates x, y and X, Y respectively. Note that $x/y = X/Y$ if $y, Y \neq 0$.

Remark 4.12. In general, for $\widetilde{S} \to S$, let x, y be local parameters for $p \in S$. One takes $\mathcal{O}_{S,p} \supseteq \mathfrak{m}_p = \langle x, y \rangle$ with $T_p{}^\vee X = \langle dx, dy \rangle$ and $\widehat{\mathcal{O}_{S,p}} = \mathbf{C}[\![x, y]\!]$. So any local regular function s can be approximated by a power series.

Remark 4.13. Claim: $E^2 = -1$. Write $\mathbf{P}^1 = \mathbf{A}^1 \cup \mathbf{A}^1$ with $u = X/Y$ and $v = Y/X$. Write $\mathrm{Bl}_p \mathbf{A}^2 = \mathbf{A}^2_{u,y} \cup \mathbf{A}^2_{v,x}$. This is a change of coordinates

- $u, y = \frac{x}{y}, y$ where $x = uy$,
- $v, x = \frac{y}{x}, x$ where $y = vx$.

If C is a line through $p \in \mathbf{A}^2$, which can be replaced with a curve C that is smooth at p, then $f^*C = C' + E$ where $C' = \overline{f^{-1}(C \setminus p)}$ is the strict transform. Write $C = \{g = 0\} = \{y + \alpha x = 0\}$. Then $f^*g = vx + ax = x(v + a)$ where $x = 0$ is the equation of E and $v + a$ is the equation of C'. Conclude using the projection formula: $f^*C \cdot E = C \cdot f_*E = C \cdot 0$ on one hand, and is equal to $(C' + E)E = C'E + E^2 = 1 + E^2$ on the other hand, so $E^2 = -1$.

5. 2024-08-29-12-46-42

Remark 5.1. Recall the construction of the blowup at a point $\pi : \widetilde{S} \to S$. We proved $E^2 = -1$.

Lemma 5.2. *Some facts:*

- $\operatorname{Pic} \widetilde{S} = \operatorname{Pic} S \oplus \mathbf{Z}E$
- $\operatorname{NS}(\widetilde{S}) = \operatorname{NS}(S) \oplus \mathbf{Z}E$

Thus every line bundle on $\widetilde{S}$ is of the form $\pi^* L + mE$ for some m and some $L \in \operatorname{Pic}(S)$.

Proof. Write $\operatorname{Pic}(S)$ as divisors modulo principal divisors, where the latter are $(\varphi) = (\varphi)_0 - (\varphi)_\infty$ for some $\varphi \in \mathbf{C}(S)$. The numerator includes an additional divisor E, while the denominator is the same since they are birational. □

Lemma 5.3. *If $p \in C$, then $\pi^* C = C' + mE$ where C' is the strict transform of C, $C' = \overline{\pi^{-1}(C \setminus p)}$.*

Proof. Cover $\widetilde{S}$ by $S \times \mathbf{A}^1_u$ and $S \times \mathbf{A}^1 \times v$. On the first we have coordinates x, u where $y = xu$, on the second we have y, v where $x = yv$. Then the map is $(x, u) \mapsto (x, xu)$, the affine blowup. Note that $x = 0 \mapsto (0, 0)$ is the exceptional curve. If $x \neq 0$ this map is invertible, since $u = y/x$. Write $\pi^* f_m(x, y) = f_m(x, xu) = x^m g(x, u)$, and this is $mE + C'$. □

Lemma 5.4.

$$\widetilde{K}_S = \pi^* K_S + E.$$

Proof. Write $\mathcal{O}(K_S) = \Omega^2_S = \langle dx \wedge dy \rangle_{\mathcal{O}_S}$. Check dx, dy in the coordinates $y = xu$ to get

$$dx \wedge dy = dx \wedge d(xu) = dx \wedge (udx + xdu) = xdx \wedge du.$$

Conclusion: $\pi^* \Omega^2_S \subseteq \Omega^2_{\widetilde{S}}$, and $\pi^* \Omega^2_S = \Omega^2_{\widetilde{S}}(E)$ since these forms vanish along E. This says $\pi^* K_S = K_{\widetilde{S}} - E$. □

Lemma 5.5. *Claim:*

- $(C')^2 = C^2 - m^2$.
- $C'_1 C'_2 = C_1 C_2 - m_1 m_2$.
- $p_a(C') = p_a(C) - \frac{m(m-1)}{2}$.

Proof. We have $\pi^* C_1 \pi^* C_2 = C_1 C_2$ since π is generically 1-to-1. By the projection formula, $\pi^* C E = C \pi_* E = 0$.

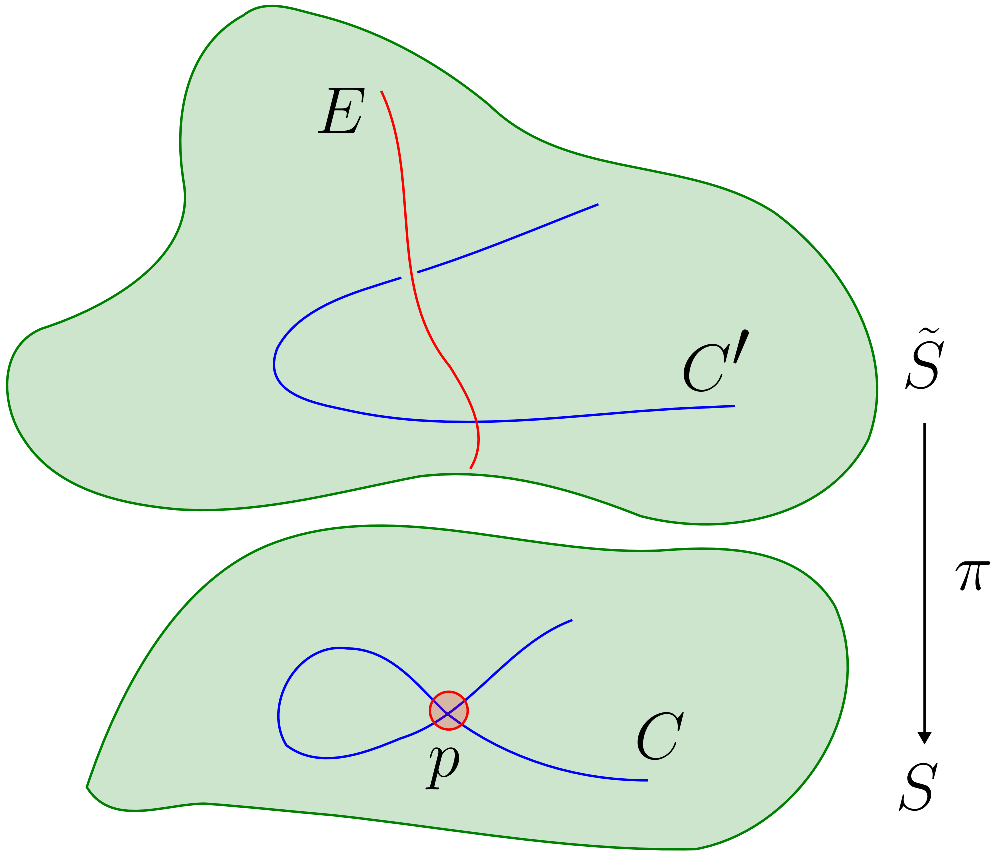
Recall

$$p_a(C) = \frac{1}{2}(K_S C + C^2) + 1,$$

so write

$$p_a(C') = \frac{(\pi^* K_S + E)(\pi^* C - mE) + (\pi^* C - mE)^2}{2} + 1.$$

□



Remark 5.6.

Remark 5.7. Blow up the nodal curve to get $p_a' = p_a - 1$. Blow up the cuspidal curve to get. . . .something.

Note that all cubics have the same p_a, since it only depends on the linear equivalence class of C.

Consider resolving C to C' by ν and use the fundamental SES

$$0 \to \mathcal{O}_C \to \nu^* \mathcal{O}_{\widetilde{C}} \to F \to 0.$$

Then $\chi(\mathcal{O}_{\widetilde{C}}) = \chi(\mathcal{O}_C) + \chi(F)$ where $\chi(F) = h^0(F) = d > 0$. Recall $p_a = 1 - \chi(\mathcal{O}_C)$. If C is smooth then $p_a(C) = g$.

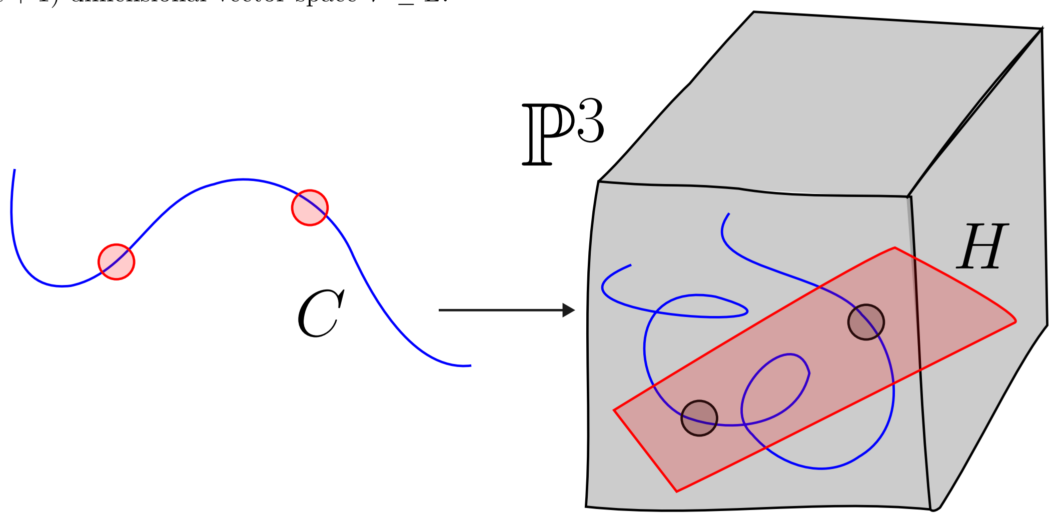
5A. **Castelnuovo.**

Theorem 5.8. *Suppose $\widetilde{S}$ is a smooth surface containing $E \cong \mathbf{P}^1$ where $E^2 = -1$. Then $\exists \pi : \widetilde{S} \to S$ where $E \mapsto \mathrm{pt}$ such that $\widetilde{S} \setminus E \cong S \setminus \mathrm{pt}$ and $\widetilde{S} \cong \mathrm{Bl}_p S$.*

Remark 5.9. On linear systems: let $V \subseteq H^0(L)$ where $L = \mathcal{O}(C)$. Let $s \in V$, then $(s) = D$ an effective divisor. Write $\mathbf{P}V = \left\{ (s) \;\middle|\; s \neq 0 \right\}$, this is equivalent to the set of effective divisors $D \sim C$. Recall $|C| = H^0(\mathcal{O}(C)) = \left\{ \phi \text{ rational} \;\middle|\; (\varphi) + C = D \geq 0 \right\}$, i.e. those (ϕ) such that $(\phi) + C$ has no poles. This is a complete linear system. Note that $(\varphi_1) = (\varphi_2) \iff \left(\frac{\varphi_1}{\varphi_2} \right) = 0$. Here s and φ are in bijection.

Remark 5.10. Let $f : S \to \mathbf{P}^n$ such that $f(S)$ is not contained in a hyperplane. This yields a linear system. In projective coordinates x_i of $\mathbf{P}^n$, write a hyperplane

H as $\sum c_i x_i \in \mathbf{P}V$ where $\mathbf{P}^n = \mathbf{P}V^\vee$. Then f^*H is a divisor on S. Define a line bundle $L = f^*\mathcal{O}_{\mathbf{P}^n}(1)$ on S. Each H defines a section s. This yields an $(n+1)$-dimensional vector space $V \subseteq L$.



Remark 5.11. Write the base locus of $\mathbf{P}V$ as $\bigcap_{D\in\mathbf{P}V} D$. A base component is a divisor in the base locus. Say $\mathbf{P}V$ is basepoint free if $\mathrm{Base}(\mathbf{P}V) = \emptyset$. We then get a bijection between linear systems without base components for smooth varieties S and rational maps $S \dashrightarrow \mathbf{P}^n$.

Lemma 5.12. *If C is a smooth curve and $C \dashrightarrow \mathbf{P}^n$, then this map can be extended to a regular map.*

Remark 5.13. Write the uniformizer of $\mathcal{O}_{C,p}$ as t, let $\phi_0, \cdots, \phi_n$ be rational functions. Write $\phi_i = ut^{m_i}$, take $d = \min m_i$, and divide by t^d.

Example 5.14. Consider the linear system L of lines through a point in $\mathbf{P}^2$. Then $L \cong \mathbf{P}^1$, since each such line is specified by a point in $\mathbf{P}^1$. Write $H^0(\mathcal{O}_{\mathbf{P}^2}(1)) = \langle x_0, x_1, x_2\rangle$, so $L = \langle x_1, x_2\rangle$. Then take the rational map

$$\mathbf{P}^2 \to \mathbf{P}^1$$
$$[x_0 : x_1 : x_2] \mapsto [x_1 : x_2]$$

which is undefined at $[1 : 0 : 0]$. This single point is the base locus of L.

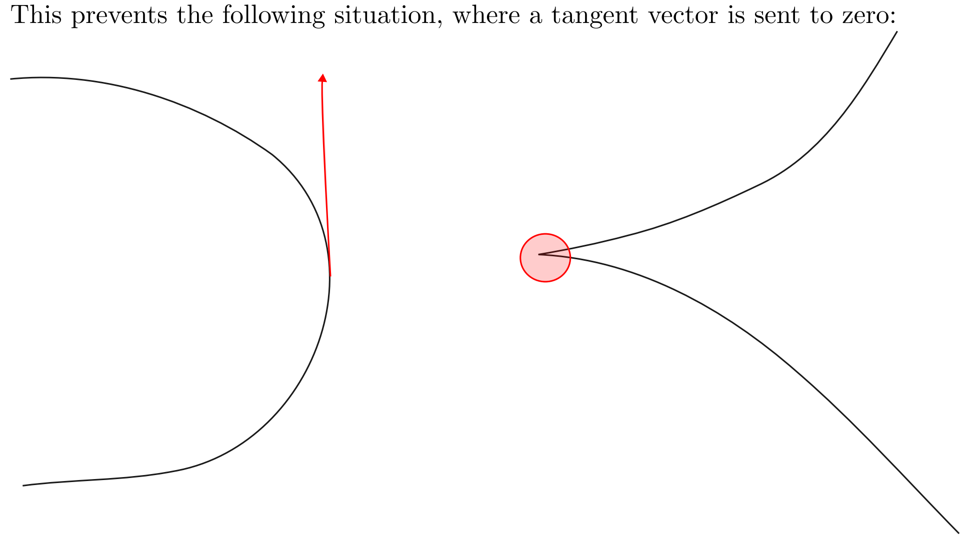
Example 5.15. Take $H^0(\mathcal{O}_{\mathbf{P}^2}(2)) \supseteq L = \langle x_1^2, x_1x_2\rangle$. Write $L' = L + E$ where $E = \{x_1 = 0\}$.

Example 5.16. Let $S = \mathbf{P}^2$ and $L = |2H - p|$, quadrics passing through the origin. A basis for all quadrics is given by $\langle x_0^2, x_1^2, x_2^2, x_0x_1, x_0x_2, x_1x_2\rangle$. Imposing $x_0^2 = 0$ yields $L \cong \mathbf{P}^4$. What is the image? It is the blowup of $\mathbf{P}^2$ at the origin. Note that there are no basepoints on the blowup. The pullbacks of all such quadrics contain E, generically with multiplicity one. The pulled back linear system on $\widetilde{S}$ is $|2H - E|$; this gives an embedding $\widetilde{S} \to \mathbf{P}^4$.

Remark 5.17. Next time:

- f is injective if L separates points: $\forall p, q, \exists D$ such that $p \in D$ but $q \notin D$.
- f is a closed embedding if L separates tangent vectors.

This prevents the following situation, where a tangent vector is sent to zero:



6. 2024-09-03-12-46-25: Linear Systems

Remark 6.1. Recall that a linear system is defined by $L = \mathbf{P}(V)$ where $V \subseteq H^0(F)$ is a finitely-generated vector space. We define $\mathrm{Base}(L) = \cap(s)$ for $s \in V$. Recall that if F is a line bundle then $F = \mathcal{O}_X((s))$ for any effective (s).

Theorem 6.2. *There is a bijection between maps $X \xrightarrow{f} \mathbf{P}^n$ with $f(X)$ not contained in a hyperplane and linear systems L with $\mathrm{Base}(L) = \emptyset$. One sets $F := f^*\mathcal{O}_{\mathbf{P}^n}(1)$ in one direction. In the other direction, writing $s = \sum c_i x_i$ with $x_0, \cdots, x_n$ local coordinates on $\mathbf{P}^n$ and sets $D = (s)$.*

Remark 6.3. Recall that

- f is injective $\iff$ L separates points.
- f is an isomorphism $\iff$ additionally L separates tangent directions.

Writing $\mathbf{T}_{X,p} = (\mathfrak{m}_p/\mathfrak{m}_p^2)^\vee$, the latter condition means $T_p \to T_{f(p)}$ is injective, or equivalently $T_{f(p)}^\vee \to T_p^\vee$ surjective. This is satisfied if the pullbacks $f^*y_1, \cdots, f^*y_n$ generate $\mathfrak{m}_p/\mathfrak{m}_p^2$, where $y_1, \cdots, y_n$ are local coordinates in $\mathbf{P}^n$. Note that a surjection on maximal ideals corresponds to a surjection on the completions of local rings $\widehat{\mathcal{O}_{f(p)}} \twoheadrightarrow \widehat{\mathcal{O}_p}$.

Remark 6.4. For X a smooth curve, these conditions simplify:

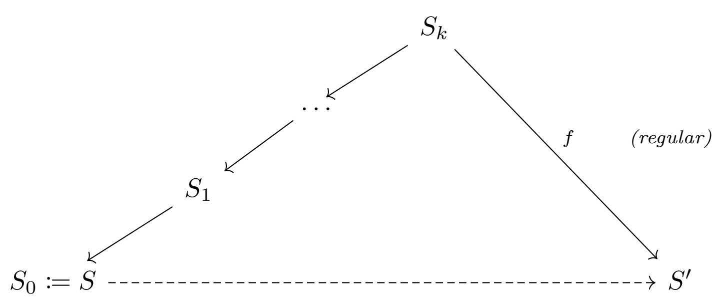
- $\mathrm{Base}(|D|) = \emptyset$, or equivalently $h^0(D-p) = h^0(D) - 1$. These numbers are equal iff every divisor passes through p. This is proved by Riemann-Roch.
- Separating points: $h^0(D-p-q) = h^0(D-p) - 1$, so not all of the divisors that pass through p also pass through q.
- Separating tangents: $h^0(D-2p) = h^0(D-p) - 1$.

Note that on a surface, $|D-p|$ is a just convenient notation for divisors that pass through p, since $D-p$ is not a divisor.

Theorem 6.5. *If X is a smooth variety, then rational maps $X \dashrightarrow \mathbf{P}^n$ are in bijection with linear systems without base components, i.e.* $\mathrm{Base}(L)$ *does not contain a divisor.*

Remark 6.6. One can always remove the indeterminacy in codimension 1, but not necessarily in codimension 2.

Theorem 6.7. *Suppose $\phi : S \to S'$ is a map from a smooth surface to a projective surface. Then there exists a diagram where $S_i = \mathrm{Bl}_{p_i} S_{i-1}$:*



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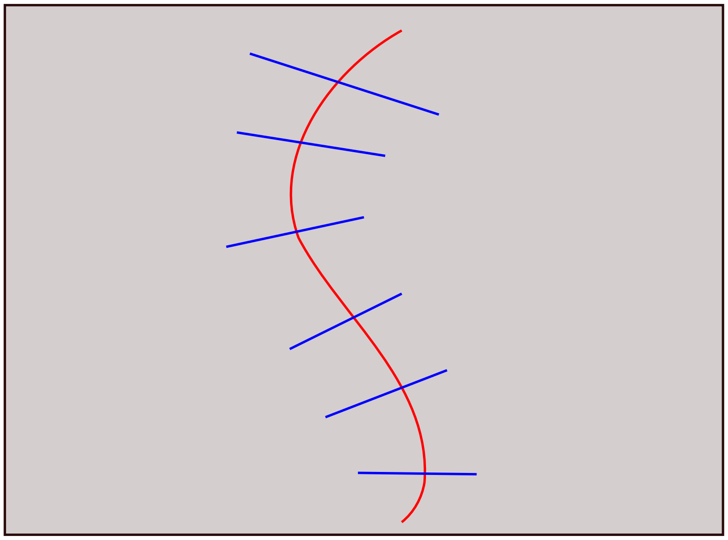
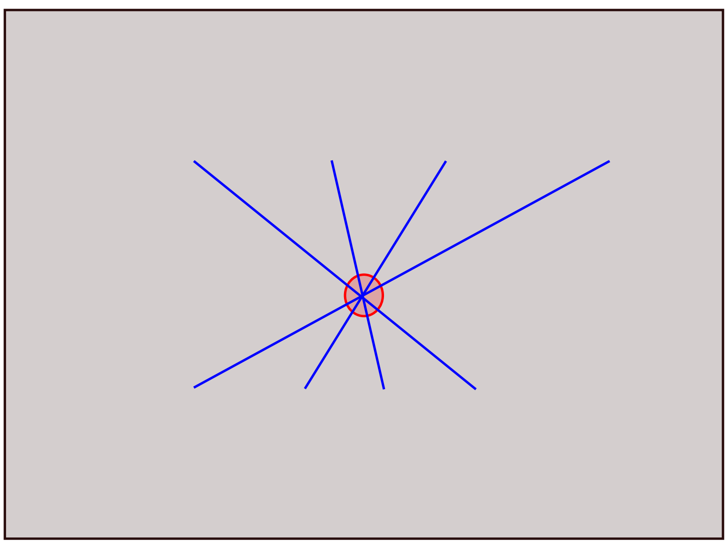
Remark 6.8. Idea:





On the proof: let $D = D_0$ be a divisor in the linear system. Resolve to get $D_1 = f_1^* D - m_1 E_1$ where $m_1 = \operatorname{mult}_p(L) := \min \left\{ \operatorname{mult}_p(D) \;\middle|\; D \in Ł \right\}$. Check that $D_1^2 = D_0^2 - m_1^2$ and $D_i^2 \geq 0$ for all i. But the self-intersection can't decrease forever.

Remark 6.9. Let $L = \mathbf{P}^1$ in $\mathbf{P}^2$ be the set of lines through zero. Blowing up yields a fibration $\operatorname{Bl}_p \mathbf{P}^2 \to \mathbf{P}^1$ which is a ruled surface with fibers the exceptional curves. This surface is $\mathbf{F}_1$.

Theorem 6.10. *Let $f : S' \to S$ be a birational map between smooth surfaces where f^{-1} is undefined at a point $p \in S$. Let $\varepsilon : \widehat{S} \to S$ be the blowup at p, then f factors through ε.*

Corollary 6.11. *Any birational map of smooth projective surfaces $S \to S'$ factors into a sequence of blowups and blowdowns.*

Remark 6.12. Blowup enough to make the rational map regular. Either it's an isomorphism or undefined at a point. In the latter case, blowup and apply the theorem. Why this terminates: $f^{-1}(p)$ contains finitely many divisors.

Toward proving the theorem:

Lemma 6.13. *Let $f : S' \to S$ be birational with $g := f^{-1}$ undefined at p. Then $\exists C \subseteq S'$ a curve with $f(C) = p$.*

Why this implies the theorem:

Proof. Suppose $C \mapsto p$ and consider $\varepsilon : \widehat{S} = \mathrm{Bl}_p S \to S$. Extend $f : S' \to S$ to $g = \varepsilon^{-1} \circ f$. If g is a morphism, we're done. If g is undefined then there exists a curve in $\widehat{S}$ which is contracted by g^{-1}. Note that g^{-1} is only undefined at finitely many points, and thus can't contract a curve. So the exceptional point is contracted to a point q in S.

DZG: Missed parts here!

Conclusion: $\varepsilon^{-1}(z)$ vanishes along E with multiplicity $m \geq 2$. Conclude that $f^{-1}(z)$ also vanishes along C with multiplicity ≥ 2, but this contradicts the construction of the blowup.

□

Remark 6.14. Note that this proof does not use Castelnuovo's criterion.

7. 2024-09-05-12-47-08

Remark 7.1. Last time: $S' \xrightarrow{f} S$ with S smooth and f^{-1} undefined at $p \in S$. Then f contracts a nonempty curve $C \subseteq S'$, i.e. $f(C) = p$.

Remark 7.2. Recall Zariski's main theorem: if $f : X \to Y$ is finite and birational with Y normal, then f is an isomorphism. Recall that Y is normal iff $\mathcal{O}_{Y,p}$ is integrally closed in its fraction field, and smooth implies normal because UFDs are integrally closed. We give two proofs of the theorem from last time.

Proof. We will show that f is finite and affine, i.e. preimages of affines are affines. Recall a morphism $\operatorname{Spec} A \to \operatorname{Spec} B$ is finite if $B \to A$ makes A a finite B-module. Finite is equivalent to quasifinite (finite fibers) and proper. Note that projective morphisms are proper, and any morphism between projective varieties is automatically projective.

□

Proof. Recall S is smooth, so $\mathcal{O}_{S,p}$ is a UFD. Embed $S' \subseteq \mathbf{P}^n$ and consider $f^{-1} : U \to \mathbf{A}^n \subseteq \mathbf{P}^n$ where U is an affine neighborhood of p. Then $f^{-1} = (\varphi_1, \cdots, \varphi_n)$ is given by n rational functions. Write u/v for where ϕ_1 is undefined at p, then $u(p) = v(p) = 0$. Let u/v be coprime. Pullback and consider f^*u, f^*v. Then $f^*u = f^*v \cdot x_1$ where x_i are coordinates on U. They are coprime in the UFD $\mathcal{O}_{S,p}$. So these differ by a regular function. Consider the curve where $f^*v = 0$, then $f^*u = 0$. By the principal ideal theorem, $f^*v = 0$ is codimension one, hence a curve.

□

Remark 7.3. Slogan: non-regular rational maps insert curves. Note that it may factor as a sequence of blowups. Recall Castelnuovo's contractibility criterion: rational (-1)-curves can be blown down.

Lemma 7.4. *If S is smooth projective and contains $E \cong \mathbf{P}^1$ with $E^2 = -1$, then $\exists f : S \to S'$ with S' smooth and $S = \mathrm{Bl}_p S'$, i.e. $f(E) = p$ is a point and $f\big|_{S \setminus E}$ is an isomorphism.*

Proof. Consider $\mathcal{O}_S(1) = \mathcal{O}_{\mathbf{P}^m}(1)\Big|_S$ and let H be a hyperplane. Then H is very ample. Let $A = mH$ for $m \gg 0$; note A is also very ample. I.e. there is some $S \hookrightarrow \mathbf{P}^k$ where $\mathcal{O}_S(m) = \mathcal{O}_{\mathbf{P}^k}(1)\Big|_S$, this follows from pulling back a Veronese embedding $V_m : \mathbf{P}^m \to \mathbf{P}^k$. The degree of the curve under this embedding is $A.E > 0$. Define $L = A + mE$, then $L.E = 0$. Claim: $|L|$ is basepoint-free, contracts E to a point, and the induced morphism $\phi_{|E|}$ is an isomorphism outside of E.

Step one: L is basepoint-free. Consider $H^0(L)$. We know $\mathrm{Base}(L) \subseteq E$ since the only possible zeros are along E. Consider $H^0(L\Big|_E)$ induced by the restriction $\mathcal{O}(L) \twoheadrightarrow \mathcal{O}_E(L)$. Note that $\deg \mathcal{O}_E(L) = 0$, and since $\mathrm{Pic}(\mathbf{P}^n) = \mathbf{Z}$ we have $\mathcal{O}_E(L) \cong \mathcal{O}_{\mathbf{P}^n}$. There is a SES $0 \to \mathcal{O}_S(L - E) \to \mathcal{O}_S(L) \to \mathcal{O}_E(L)$, so by taking the LES, it suffices to show $H^1(\mathcal{O}_S(L - E)) = 0$. Now apply Serre's vanishing theorem: if X is projective and $F \in \mathsf{Coh}(X)$, the twist $F(m)$ for $m \gg 0$ has vanishing higher cohomology and $F(m)$ is globally generated. So $H^1(\mathcal{O}_S(A)) = 0$, and $L = A+mE$, so $L-E = A+(m-1)E$. We want $H^0(A+(m-1)E) = 0$. There are SESs $0 \to \mathcal{O}_S(A + (k-1)E) \to \mathcal{O}_S(A + kE) \to \mathcal{O}_E(A + kE)$. By induction, it suffices to show $H^1(\mathbf{P}^1, \mathcal{O}(m + k)) = 0$. This is dual to $H^0(\mathbf{P}^1, \mathcal{O}(-2 - m - k))$, which is zero if $k \leq m - 1$.

Missed details here, the markers were dying!

Delicate part of the argument: showing S' is smooth. It suffices to see that $\dim_{\mathbf{C}} \mathfrak{m}_p/\mathfrak{m}_p^2 = 2$. For that, we produce two explicit generators. Take a SES $0 \to \mathcal{O}_S(L - 2E) \to \mathcal{O}_S(L - E) \to \mathcal{O}_E(L - E) = \mathcal{O}_{\mathbf{P}^1}(1) \to 0$. Take the LES, note $H^0(\mathcal{O}_{\mathbf{P}^1}(1)) = \langle x, y\rangle$ is 2-dimensional. This is isomorphic to $H^0(L-E)/H^0(L-2E)$, we want to show this is $F \otimes \mathfrak{m}_p/\mathfrak{m}_p^2$ for F some invertible sheaf.

Write $S' \hookrightarrow \mathbf{P}^n$ and $g : S \to \mathbf{P}^n$ for the composition with f. Then $L = g^*\mathcal{O}_{\mathbf{P}^n}(1)$. Check that sections of $L - E$ vanish along E to order at least 1, so are in $\mathfrak{m}_p$. Similarly sections of $L - 2E$ vanish along E to order at least 2 and are thus in $\mathfrak{m}_p^2$. Thus $F = \mathcal{O}_{S'}(1)$.

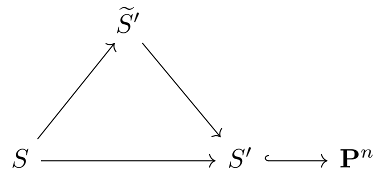
Note $f_*\mathcal{O}_S = \mathcal{O}_{S'}$ and $L = f^*\mathcal{O}_{S'}(1)$, so by the projection formula,

$$f_*L = f_*\left(f^*\mathcal{O}_{S'}(1)\right) = f_*\mathcal{O}_S \otimes \mathcal{O}_{S'}(1) = \mathcal{O}_{S'}(1).$$

Apply f_* to $\mathcal{O}(L-E) \hookrightarrow \mathcal{O}(L) \twoheadrightarrow \mathcal{O}_E(L)$ and show one gets $\mathfrak{m}_p(1) \to \mathcal{O}_{S'}(1) \to \mathbf{C}_p \to \mathbf{R}^1 f_*\mathcal{O}_S(L - E) = 0$. Something similar must be shown for $L - 2E$, but this requires a spectral sequence argument. We instead appeal to Grothendieck's theorem on formal functions.

□

Remark 7.5. Normalization produces a Stein factorization:



> Link to Diagram

Here $\widetilde{S'}$ is the normalization, and can be written as $\widetilde{S'} = \operatorname{Spec} f_* \mathcal{O}_S$.

Remark 7.6. Grothendieck's theorem on formal functions: let $f : X \to Y$ be projective, $F \in \mathsf{Coh}(X)$, and $p \in Y$. Consider the higher direct images $\mathbf{R}^i f_* F$ on Y, which is an $\mathcal{O}_Y$-module. Complete the stalk at p to get $\widehat{\mathbf{R}^i f_* F}_p$ which is a module over $\widehat{\mathcal{O}_{Y,p}}$. This can be computed as a limit $\operatorname{colim}_Z F \Big|_Z$ where Z are thickenings of the fiber $f^{-1}(p)$.

Remark 7.7. We first apply this for $i = 0$; we get $\widehat{f_* \mathcal{O}_S} = \widehat{\mathcal{O}_{S'}}$. For smoothness, we want to show the latter is isomorphic to $\mathbf{C}[\![x, y]\!]$. The theorem says to compute $\operatorname{colim}_m H^0(mE)$. We have $0 \to \mathcal{O}_S(mE) \to \mathcal{O}_S \to \mathcal{O}_{mE}$. The claim is that the colimit is $\mathbf{C}[x, y, x^2, xy, y^2, \cdots, y^m]$, the monomials up to the mth order. Use the fact that $\mathcal{O}_E(-mE) = \mathcal{O}_{\mathbf{P}^1}(m)$.

Remark 7.8. This concludes chapter 2. Next time: ruled surfaces.

8. 2024-09-12-12-46-17

8A. Classification of ruled surfaces.

Remark 8.1. Recalling ruled surfaces: let $f : S \to C$ where all fibers are isomorphic to $\mathbf{P}^1$. We proved it is a locally free $\mathbf{P}^1$ bundle, so $\forall p \in C$ there is some $U \ni p$ such that $f^{-1}(U) \cong U \times \mathbf{P}^1$.

Remark 8.2. Classification of locally free rank r sheaves $\mathcal{E}$ and locally free $\mathbf{A}^1$ bundles (vector bundles). Letting $X = \bigcup U_i$, there is an isomorphism $g_i : \mathcal{E}|_{U_i} \xrightarrow{\sim} \mathcal{O}_{U_i}^{\oplus r}$. Different choices of isomorphisms yield elements $g_i \in \mathrm{GL}_r(\mathcal{O}(U_i)) = \mathrm{Aut}(\mathcal{O}_{U_i}^{\oplus r})$. The coefficients are in the ring of regular functions on U_i. How do these isomorphisms glue? We specify transition functions $g_{ij} \in \mathrm{GL}_r(\mathcal{O}(U_i \cap U_j))$ satisfying the 1-cocycle condition on triple intersections $g_{ij} g_{jk} g_{ki} = 1 \in \mathrm{GL}_r(\mathcal{O}(U_i \cap U_j \cap U_k))$. We mod out by the relation $g'_{ij} = g_j^{-1} g_{ij} g_i$, the 1-coboundary condition. This yields Čech cohomology $\check{H}^1(U, \mathrm{GL}_r(\mathcal{O})) \cong H^1(X, \mathrm{GL}_r(\mathcal{O}))$. This works generally: global versions on locally free objects live in $\check{H}^1(U, \mathrm{Aut}(\mathcal{E}))$.

Remark 8.3. Vector bundles are locally free $\mathbf{A}^r$ bundles. Let $f : Y \to X$ have fibers $\mathbf{A}^1$, so $U_i \subseteq X$ lifts to $f^{-1}(U_i) = U_i \times \mathbf{A}^r$. What are the transition functions? One looks for automorphisms of $(U_i \cap U_j) \times \mathbf{A}^r \ni (x, y_1, \cdots, y_r)$, which correspond to $g_{ij}(x) \in \mathrm{GL}_r(\mathcal{O}(U_i \cap U_j))$. What is the corresponding sheaf? Let $\mathcal{E}(U)$ be sections over U, so morphisms $s : U \to f^{-1}(U)$. Locally this is given by r regular functions, so $\mathcal{E}(U_i) = \mathcal{O}(U_i)^{\oplus r}$.

Remark 8.4. For ruled surfaces, we consider $\check{H}^1(C, \mathrm{PGL}_2(\mathcal{O}))$. More generally, take $\mathrm{PGL}_r(\mathcal{O})$.

Theorem 8.5. *All geometrically ruled surfaces are of the form* $\mathbf{P}_C(\mathcal{E})$ *where* $\mathcal{E}$ *is a locally free sheaf of rank 2. Moreover,* $\mathbf{P}_C(\mathcal{E}) \xrightarrow{\sim} \mathbf{P}_C(\mathcal{E}')$ *over* C *iff* $\mathcal{E}' \cong \mathcal{E} \oplus L$ *for* L *some line bundle.*

Remark 8.6. Thus $\mathcal{E}$ on C is in bijection with an $\mathbf{A}^2$ bundle $\mathbf{A}(\mathcal{E}) \to C$. What is $\mathbf{P}_C(\mathcal{E})$? This is the bundle of lines in the vector bundle. Dually, one could define this as rank 1 quotients of the vector bundle. Passing back and forth: replace $\mathcal{E}$ by $\mathcal{E}^\vee$.

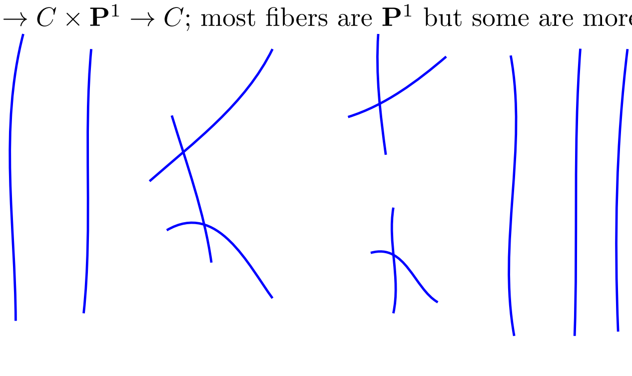
Proof. There is a sequence $1 \to \mathbf{C}^2 \to \mathrm{GL}_2(\mathbf{C}) \to \mathrm{PGL}_2(\mathbf{C}) \to 1$ which globalizes to $1 \to \mathcal{O}^\times \to \mathrm{GL}_2(\mathcal{O}) \to \mathrm{PGL}_2(\mathcal{O}) \to 0$. Taking the LES yields $\mathrm{Pic}(C) \to H^1(\mathrm{GL}_2(\mathcal{O})) \to H^1(\mathrm{PGL}_2(\mathcal{O}))$ where the middle corresponds to rank 2 locally free bundles and the right corresponds to locally free $\mathbf{P}^1$ bundles. It suffices to show this is surjective, so consider $H^2(\mathcal{O}_C^\times)$ – this is trivial by Grothendieck vanishing since $\dim C = 1$.

□

Remark 8.7. Goal: understand locally free sheaves on curves, e.g. $\mathbf{P}^1$ (easy), elliptic curves (harder), and curves of general type (generally hard). Next: relate birationally and geometrically ruled surfaces.

Remark 8.8. A (classically) minimal surface is a smooth projective surface with no -1 curves.

Remark 8.9. Ruled surface: $S \overset{\sim}{\dashrightarrow} C \times \mathbf{P}^1$ for C a curve. Take a sequence of blowups to obtain $\widetilde{S} \to C \times \mathbf{P}^1$ a regular morphism. Consider the composite $\widetilde{S} \to C \times \mathbf{P}^1 \to C$; most fibers are $\mathbf{P}^1$ but some are more complicated:



Claim: either $\widetilde{S} = \mathbf{P}_C(\mathcal{E})$ or there exists a -1 curve in a fiber. In the first case, $\widetilde{S} \to \mathbf{P}_C(\mathcal{E}) \to C$ is a relatively minimal model.

Proof. Consider a reducible fiber $F = \sum m_i D_i$. Then every $D_i^2 < 0$. Use that $F^2 = 0$. Consider $D_{i_0} \cdot F = 0$ one one hand, and is equal to $D_{i_0}\left(m_{i_0} D_{i_0} + \sum_{j\neq i} m_j D_j\right)$. Then $D_{i_0} D_j \geq 0$ for all j, so we must have $D_{i_0}^2 < 0$ for this to equal zero.

Moreover $(K_S + F)\Big|_F = K_F = K_{\mathbf{P}^1}$, and $F\Big|_F = 0$, so $K_S F = -2$. Thus there exists a $D_i \subseteq F$ with $K_S D_i < 0$. If D_i is a connected curve, one can show $p_a(D_i) = \frac{1}{2}(KD_i + D_i^2) \geq 0$. Since $D_i^2 < 0$ and $KD_i < 0$, this forces $D_i \cong \mathbf{P}^1$.

Missed last part of this argument.

□

Remark 8.10. Fact: there are no covers $\mathbf{P}^1 \to C$ if $g(C) > 0$. By Riemann-Hurwitz, $K_{\mathbf{P}^1} = f^*K_C + \sum(m_i - 1)p_i$, and taking degrees yields -2 on the LHS and $2g-2 > 0$ and a positive contribution on the RHS, a contradiction.

Alternatively, consider $\pi : C' \to C$ and consider the induced map $\pi^* : H^0(\Omega_C) \to H^0(\Omega_{C'})$. Note $y = f(x) \implies dy = f'(x)dx$. The claim is that this map is injective, so the genus increases.

8B. **Classification of $\mathbf{P}_C(\mathcal{E})$.**

Theorem 8.11. *(Grothendieck) On $\mathbf{P}^1$, every locally free sheaf is uniquely isomorphic to $\bigoplus_i \mathcal{O}(n_i)$.*

Corollary 8.12. *$L \otimes \mathcal{E} = \mathcal{O} \oplus \mathcal{O}(n)$ for some $n \geq 0$. Thus geometrically ruled surfaces over $\mathbf{P}^1 = C$ are Hirzebruch surfaces $\mathbf{F}_n \coloneqq \mathbf{P}_{\mathbf{P}^1}(\mathcal{O} \oplus \mathcal{O}(n))$. Note that $\mathbf{F}_0 = \mathbf{P}^1 \times \mathbf{P}^1$.*

Theorem 8.13. *(Atiyah) If C is an elliptic curve, then for any rank r there exists a unique indecomposable rank r locally free sheaf $\mathcal{E}$, modulo twisting by a degree zero line bundle $L \in \operatorname{Pic}^0(C) = C$.*

Corollary 8.14. *If $\deg \mathcal{E}$ is even, then there exists an L such that $\deg(\mathcal{E} \otimes L) \deg(\mathcal{E}) + 2\deg(\mathcal{L}) = 0$. Thus either $\mathcal{E} = L_1 \oplus L_2$ or $\mathcal{O} \oplus L_3$. If $\deg \mathcal{E}$ is odd, $\deg(\mathcal{E} \otimes L) = 1$.*

Remark 8.15. Note $\operatorname{Ext}(A, B) = \operatorname{Ext}(\mathcal{O}, B \otimes A^{-1}) = H^1(B \otimes A^{-1})$. Considering an elliptic curve and $0 \to \mathcal{O} \to \mathcal{E} \to \mathcal{O} \to 0$, one has $H^1(\mathcal{O} \otimes \mathcal{O}^{-1}) = H^1(\mathcal{O}) = g(C) = 1$, so there is a nontrivial extension.

Remark 8.16. Next time: $\deg \mathcal{E} = \deg \det \mathcal{E}$ as a line bundle.

9. 2024-09-17-12-46-29

Remark 9.1. We proved that every geometrically ruled surface is of the form $\mathbf{P}_C(\mathcal{E})$ with $\operatorname{rank} \mathcal{E} = 2$. Recall that locally free $\mathcal{O}_C$ modules are in bijection with $\mathbf{A}^n$ bundles over C.

Lemma 9.2. *For all rank 2 locally free sheaves $\mathcal{E}$, there is a SES*

$$0 \to L \to \mathcal{E} \to M \to 0$$

where L, M are line bundles. If $h^0(\mathcal{E}) > 0$, then $L = \mathcal{O}(D)$ for D an effective divisor. Note that $\deg \mathcal{E} \coloneqq \deg \det(\mathcal{E}) = \deg \wedge^2 \mathcal{E}$.

Proof. Sections $s \in H^0(C; \mathcal{E})$ correspond to morphisms $\mathcal{O}_C \xrightarrow{s} \mathcal{E}$ where $1 \mapsto s$. Note that the cokernel of a morphism of locally free sheaves may not be locally free. Taking duals yields $\mathcal{E}^\vee \xrightarrow{s^\vee} \mathcal{O}_C$. Note $\mathcal{E}^\vee \twoheadrightarrow Q = \mathcal{O}(-D)$ with kernel K, which is locally free. This yields $0 \to K \to \mathcal{E}^\vee \to Q \to 0$, so dualize to get the result.

This holds under the hypothesis $h^0(\mathcal{E}) > 0$, so there is a section. By Serre, $\mathcal{E}(n) \coloneqq \mathcal{E} \otimes \mathcal{O}_C(n)$ is generated by global sections for $n \gg 0$. So $H^0(\mathcal{E}(n)) \otimes \mathcal{O}_C \twoheadrightarrow \mathcal{E}(n)$, and the previous argument applies, then untwist.

□

Remark 9.3. Recall Riemann-Roch for line bundles on a curve: $\chi(L) = \deg L + 1 - g$. Note that $\det(\mathcal{E}) \cong L \otimes M$, so $\deg(\mathcal{E}) = \deg(L) + \deg(M)$. Thus

$$\chi(\mathcal{E}) = \chi(L) + \chi(M) = (\deg L + 1 - g) + (\deg M + 1 - g) = \deg(\mathcal{E}) = 2(1 - g).$$

Remark 9.4. When does this extension split? Twist $L \to \mathcal{E} \to M$ by M^{-1} to get $L \otimes M^{-1} \to \mathcal{E} \otimes M^{-1} \to \mathcal{O}$. A section of the former is the same as a map $\mathcal{O} \to \mathcal{E} \otimes M^{-1}$. A section $s \in H^0(\mathcal{E} \otimes M^{-1})$ maps to $1 \in H^0(\mathcal{O})$ which further maps to some $e \in H^1(L \otimes M^{-1})$ in the associated long exact sequence. The extension splits iff the extension class $e = 0$. As a corollary, if $H^1(L \otimes M^{-1}) = 0$ then the extension splits.

Remark 9.5. On $\mathbf{P}^1$, any rank r vector bundle E splits as a sum of $\mathcal{O}(n_i)$.

Proof. Compute $\deg(E \otimes A) = \deg E + 2 \deg A$ when $\operatorname{rank} A = 1$. Replace E by some $E(n)$ so that $\deg E = 0$ or -1. By Riemann-Roch, $\chi(E) = \deg E + 2 = 2$ or 1. Thus $h^0(E) \geq 1$. There is then a SES $\mathcal{O}(n) \to E \to \mathcal{O}(m)$ where $m = \deg E - n$ with $n \geq 0$. The extension class $e \in H^1(\mathcal{O}(2n - \deg E)) = 0$, since $2n - \deg E \geq 0$. This uses that $H^1(\mathcal{O}(d)) = 0$ if $d \geq -1$, since this is dual to $h^0(\mathcal{O}(-2-d))$.

□

Remark 9.6. Why this decomposition is unique: this minimum of n_1, n_2 is unique as a function of E. The $\max(n_1, n_2)$ is the maximal n such that $h^0(E(-n)) > 0$, since twisting by $-n$ yields $\mathcal{O} \oplus \mathcal{O}(n_2 - n)$ where $n_2 - n \leq 0$.

Remark 9.7. For an elliptic curve C, the only indecomposable rank 2 bundles are extensions E in $\mathcal{O} \to E \to \mathcal{O}$ where $e \in H^1(\mathcal{O}) = \mathbf{C}$, or $\mathcal{O} \to E \to \mathcal{O}(p)$ with $e \in H^1(\mathcal{O}(-p)) = H^0(\mathcal{O}(p))^\vee = \mathbf{C}$ for $p \in C$ a point. One can take either of these and tensor by a line bundle.

Remark 9.8. For curves of genus $g \geq 2$, there is a moduli space of indecomposable rank 2 vector bundles of degrees 0 or 1. The dimension is $3g - 3$.

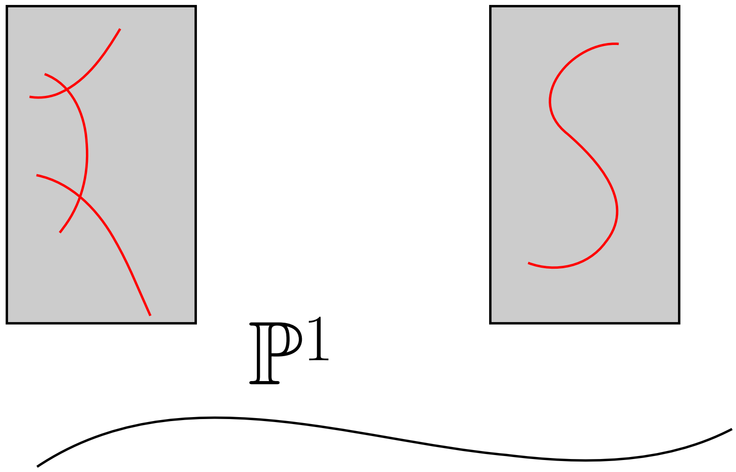
Corollary 9.9. *The geometrically ruled surfaces over* $\mathbf{P}^1$ *are* $\mathbf{P}(\mathcal{O} \oplus \mathcal{O}(n)) = \mathbf{F}_n$, *the Hirzebruch surfaces, for* $n \geq 0$.

9A. **Tautological line bundles on $\mathbf{P}_C(E)$.**

Remark 9.10. On S, there is a natural SES $N \to p^*E \to \mathcal{O}_{\mathbf{P}(E)}(1)$ where $p : S \to C$, p^*E is rank 2, and $N, \mathcal{O}_{\mathbf{P}(E)}(1)$ are line bundles. There is a similar sequence on fibers $\mathcal{O}(-1) \to \mathcal{O} \oplus \mathcal{O} \to \mathcal{O}(1)$. Regard a point in a fiber as an $\mathbf{A}^1 \subseteq \mathbf{A}^2$. We thus call $\mathcal{O}_{\mathbf{P}(E)}(1)$ the tautological bundle.

Remark 9.11. Sections $s : C \to \mathbf{P}_C(E)$ are in bijection with rank 1 locally free quotients of E on C.

Remark 9.12. Intersection theory on a deeper level, toward the enumerative geometry of $\mathbf{P}_C(E)$. Let X be a variety, and F a rank r vector bundle on X. Then there exist Chern classes $c_i(F) \in A^i(X)$ for $0 \leq i \leq \dim X$. Recall that $A^i(X)$ are codimension i cycles $\sum n_k Z_k$ modulo linear equivalence, so $A^1(X) = \operatorname{Pic}(X)$. Recall linear equivalence:



The total Chern class is $c(F) \coloneqq \sum_i c_i(F)$. It is additive in SESs: for $A \to B \to C$, one has $c(B) = c(A) \cdot c(C)$. A map $f : X \to Y$ induces $f^* : A^i(Y) \to A^i(X)$ where $c_i(f^*F) = f^*(c_i(F))$. If L is a line bundle, then $c(L) = 1 + L$.

Remark 9.13. By general principles, $c_2(p^*E) = 0$. Next time $[N] \cdot [\mathbf{P}_C(E)] = 0$.

10. 2024-09-19-12-49-21

Remark 10.1. Let C be a curve of genus g and E a rank 2 vector bundle, and let $S = \mathbf{P}_C(E) \to C$. There is an exact sequence $0 \to N \to p_*E \to \mathcal{O}_S(1) \to 0$ over S where the ranks are 1,2,1 respectively. Let $h = c_1(\mathcal{O}_S(1)) \in \mathrm{NS}(S)$; on any fiber f this restricts to $\mathcal{O}_{\mathbf{P}^1}(1)$. Note that h need not be effective.

Theorem 10.2. *Three equations:*

- $\mathrm{Pic}(S) = p^* \mathrm{Pic}(C) \oplus \mathbf{Z}h$
- $\mathrm{NS}(S) = p^*\mathrm{NS}(C) \oplus \mathbf{Z}h \cong \mathbf{Z} \oplus \mathbf{Z}h$.
- $H^2(S; \mathbf{Z}) = p^* H^2(C; \mathbf{Z}) \oplus \mathbf{Z}h$.

Moreover:

- $f^2 = 0$
- $hf = 1$
- $h^2 = \deg E$

Finally,

$$K_S = -2h + (\deg E + 2g - 2)f.$$

Proof. Let $D \in \mathrm{Pic}(S)$ and consider $d \coloneqq D.f$. Then $D' \coloneqq D - dh$ yields $D'.f = 0$, and we claim that $D' = p^*L$ for some $L \in \mathrm{Pic}(C)$. It suffices to show that $h^0(D' + nf) > 0$ for some n. Note that $h^2(D' + nf) = h^0(K - D' - nf) = 0$, since $(K - D' - nf)f = -2$ which follows from $(K + f)\Big|_f = K_f = K_{\mathbf{P}^1} = -2$. Riemann-Roch yields

$$\chi(D' + nf) = \frac{1}{2}(D' + nf)(D' + nf - K) + \chi(\mathcal{O}_S) = n + c$$

for c some constant. Let $A \sim D' + nf$, then $A.f = 0$. So A is effective and intersects every fiber by zero, so every reducible component is contained in a fiber. So A is a sum of fibers with some multiplicities, which is thus the pullback of some divisor on C.

Note that we have a SES $\mathrm{Pic}^0(S) \to \mathrm{Pic}(S) \to H^2(S;\mathbf{Z}) \to H^2(\mathcal{O}_S)$. Note that $\mathrm{Pic}^0(S) = H^1(\mathcal{O}_S)/H^1(S;\mathbf{Z})$ and $H^2(S;\mathbf{Z}) = \mathrm{NS}(S)$ in this case. This yields the conclusion.

□

Proof. For F a sheaf on X, $c_0(F) = \mathrm{rank}(F)[X]$, $c_1(F)$ is a divisor, and $c_2(F)$ is codimension 2. Recall $c(N) \cdot c(\mathcal{O}_S(1)) = c(p^*E)$. This yields $(1+n)(1+h) = 1 + c_1(p^*E) + c_2(p^*E)$. Note that $c_1(p^*E) = p^*c_1(E) = (\deg E)f$ and $c_2(p^*E) = p^*c_2(E) = p^*0 = 0$, since c_2 vanishes for a curve. Thus

$$(1+N)(1+h) = 1 + (\deg(E))f,$$

and so

- $n + h = (\deg(E))f$,
- $nh = 0$.

Thus $((\deg(E))f - h)\, h = 0$.

□

Remark 10.3. Note that twisting E by a line bundle changes h but does not change $\mathbf{P}_C(E)$.

Proof. Let s be a section, then $s = h + af$ since $s.h = 1$ and $s.f = a$ is unknown. Note that $\deg K_s = 2g-2$, and we can write $K_S = -2h + bf$ for some b. By adjunction, $K_s = (K_S + s)\Big|_s$, and taking degrees yields

$$2g - 2 = (-2h + bf + h + af)(h + af) = (-h + (a+b)f)(h+af) = -\deg(E) + b.$$

Note that b does not depend on the section s, hence the cancellation of a. Solving yield $b = \deg(E) + 2g - 2$.

□

Remark 10.4. Recall that N was defined by varying lines in $\mathbf{A}^2$ along a fiber. Twisting by a line bundle yields $0 \to N \otimes p^*L \to p^*(E \otimes L) \to \mathcal{O}_S(1) \otimes p^*L \to 0$. This yields $h' \coloneqq h + p^*L$ and $(h')^2 = h^2 + 2\deg(L)$.

Remark 10.5. Basic numerical invariants:

- $q(S) = h^1(\mathcal{O}_S) = h^0(\Omega_S)$,
- $p_g(S) = h^2(\mathcal{O}_S) = h^0(\Omega_S^2)$,
- $P_m(S) = h^0(\mathcal{O}_S(mK)) = h^0((\Omega_S^2)^{\otimes m})$.

Note that $P_1 = p_g$ and $P_0 = 1$.

Theorem 10.6. *q, p_g, P_m are birational invariants for smooth projective varieties.*

Proof. Step 1: Global sections of differential forms tend to be birational invariants. Let X be such a variety and $\phi : X \dashrightarrow Y$ a rational map. Then ϕ is undefined in codimension ≥ 2 – there exists $Z \subseteq X$ with $\mathrm{codim}_X(Z) \geq 2$ and $U \coloneqq X \setminus Z$ such that $\varphi|_U = f : U \to Y$ is a regular map. Let $D \subseteq X$ be a divisor cut out by $t = 0$ in a local coordinate. Embed $Y \subseteq \mathbf{P}^N$, and clear denominators and rescale by powers of t. We can thus pull back differential forms to U.

Step 2: Hartog's theorem. A differential form on U with $\mathrm{codim}(X \setminus U) \geq 2$ can be extended to all of X when X is smooth.

□

Corollary 10.7. *If $\phi : X \to Y$ is a dominant rational map then*

- $q(X) \geq q(Y)$
- $p_g(X) \geq p_g(Y)$
- $P_m(X) \geq P_m(Y)$

Remark 10.8. Rational maps thus yield inequalities in both directions.

Remark 10.9. We now consider ruled surfaces $S \overset{\sim}{\dashrightarrow} C \times \mathbf{P}^1$. We obtain

- $q(S) = g$,
- $p_g(S) = 0$,
- $P_m(S) = 0$.

Let $p_i : C \times \mathbf{P}^1 \to C, \mathbf{P}^1$ be the coordinate projections. Then $\mathbf{T}_{C\times\mathbf{P}^1} = p^*\mathbf{T}_C \oplus p^* T_{\mathbf{P}^1}$. Take duals to get $\Omega_S = p^*\Omega_C \oplus \Omega_{\mathbf{P}^1}$ and check $h^0(\Omega_S) = h^0(\Omega_C) + h^0(\Omega_{\mathbf{P}^1}) = g + 0 = g$. Note that $p_g(S) = h^0(K_S)$. Also note that K^2 is not a birational invariant, since blowing up decreases this by 1. Here we have $K_S^2 = 8 - 8g$ by computing $(-2h + (\deg E + 2g - 2))^2 = 4\deg E - 4(\deg E + 2g - 2)$. For $g = 0$, we get $\mathbf{P}^1 \times \mathbf{P}^1$, check $K_S^2 = (2h + 2f)^2 = 8$. For $g = 1$, check $K_S^2 = 0$.

Remark 10.10. Note that a surface S is birationally ruled iff $P_m(S) = 0$ for all m. This is a classical theorem. It is rational iff $q(S) = P_2(S) = 0$. Recall that Kodaira dimension is defined by $P_m \sim m^{\kappa(S)}$. For $\kappa(S) = 0$, there is a full classification in terms of the above invariants. There is a structure theorem for $\kappa(S) = 1$, elliptic surfaces. For $\kappa(S) = 2$, there are finitely many types where $q = p_g = 0$, but there is not a full classification.

11. 2024-09-24-12-46-22

Remark 11.1. New chapter: rational surfaces. Recall that a variety X is rational iff $\exists f : X \overset{\sim}{\dashrightarrow} \mathbf{P}^N$ for some N. Examples include Hirzebruch surfaces $\mathbf{F}_n$, geometrically ruled surfaces over $\mathbf{P}^1$. These are fibrations $S \to \mathbf{P}^1$ where every fiber is $\mathbf{P}^1$. Recall that geometrically ruled surfaces are of the form $\mathbf{P}_{\mathbf{P}^1}(V)$ where V is a vector bundle of rank 2. We showed that any such bundle on $\mathbf{P}^1$ is of the form $\mathcal{O}(a) \oplus \mathcal{O}(b)$, and tensoring by $\mathcal{O}(-a)$ yields $\mathcal{O} \oplus \mathcal{O}(n)$ where $n \in \mathbf{Z}_{\geq 0}$. We write $\mathbf{F}_n = \mathbf{P}_{\mathbf{P}^1}(\mathcal{O} \oplus \mathcal{O}(n))$. Note that $\mathbf{F}_0 = \mathbf{P}^1 \times \mathbf{P}^1$. We can write $\mathrm{Pic}(\mathbf{F}_0) = \mathbf{Z}e + \mathbf{Z}f$ where e, f are fibers of the two projections. These satisfy $e^2 = f^2 = 0$ and $ef = 1$. Any effective curve $C \subseteq \mathbf{F}_0$ can be written as $ae + bf$ for some $a, b \geq 0$. Why this is true: note that $C.f = a \geq 0$ because $|f|$ is basepoint free. Alternatively, check that $p_*(f) = 0$ where p is the projection onto $\mathbf{P}^1$, and $p_*(e) = [\mathbf{P}^1]$. Thus $p_*(C) = [\mathbf{P}^1] \cdot \deg(C \to \mathbf{P}^1) \geq 0$.

Proposition 11.2. *In parts:*

- *In $\mathbf{F}_n$ there exists a unique section s_n such that $s_n^2 = -n$. This is the unique section with negative square.*
- *For all irreducible curves $C \neq s_n$, one has $C^2 \geq 0$.*
- $\mathbf{F}_n \cong \mathbf{F}_m \iff n = m$.

Proof. Write $\mathbf{F}_n = \mathbf{P}_C(E) \xrightarrow{p} C$. We have a sequence $0 \to N \to p_*E \to \mathcal{O}_{\mathbf{P}_C(E)}(1) \to 0$. We write $[\mathcal{O}_{\mathbf{P}_C(E)}] = h$, and we showed $\mathrm{NS}\mathbf{P}_C(E) = \mathbf{Z}h \oplus \mathbf{Z}f$ for $hf = 1$ and $h^2 = \deg(E)$. We have a bijection between sections of p and sequence $p_*E \to F \to 0$. We have $i_*\mathcal{O}_{\mathbf{P}_C(E)}(1) = F$ as sheaves on C. Consider the sequence $\mathcal{O} \oplus \mathcal{O}(n) \to \mathcal{O}(d) \to 0$ on $\mathbf{P}^1$. What values of d can occur? First we analyze for which a, b there is a nontrivial homomorphism $\mathcal{O}(a) \to \mathcal{O}(b)$. Untwisting, this is equivalent to a homomorphism $\mathcal{O} \to \mathcal{O}(a-b)$, which is equivalent to a global section s of $\mathcal{O}(a-b)$. Such an s exists if $a - b \geq 0$. By cases:

- If $d < 0$, there is no homomorphism $\mathcal{O} \oplus \mathcal{O}(n) \to \mathcal{O}(d)$.
- If $d = 0$, there is a unique such homomorphism up to scaling $1 \mapsto \lambda$, which is surjective. This yields the *exceptional section* s_n.
- If $0 < d < n$, $s \in H^0(\mathcal{O}(d))$ has d zeroes and thus fails surjectivity at these points.
- If $d \geq n$, there are many such sections, which can be chosen to have disjoint zeros and thus generate the image.

We now examine s_n. We have $\mathcal{O} \oplus \mathcal{O}(n) \twoheadrightarrow \mathcal{O}$, this yields a section $i : \mathbf{P}^1 \to \mathbf{F}_n$. Since $\mathcal{O} = i^* \mathcal{O}_{\mathbf{P}(E)}(1)$, we have $hs = 0$. We have $h^2 = \deg(E) = n$, $fh = 1$, and $f^2 = 0$. We can write $s = ah + bf$, now check that $1 = sf = a$ so $s = h + bf$. Then $0 = sh = (h+bf)h = n+b$ and thus $b = -n$. Finally $s^2 = (h-nf)^2 = n-2n = -n$.

Any other section s' corresponding to $\mathcal{O} \oplus \mathcal{O}(n) \twoheadrightarrow \mathcal{O}(n)$ yields $s' \sim h + (d-n)f$, and one can check $(s')^2 = n + 2(d-n) \geq n$. Any two such s' intersect at $h^2 = n$ points and are disjoint from s_n. Other sections are sometimes denoted s_∞, and one has $s_\infty \sim h$. One can also write $\operatorname{Pic}(S) = \langle h, f \rangle = \langle s_\infty, f \rangle$.

□

Proof. Write $C = ah + bf$, then $C.s_n \geq 0$ and in fact $C.s_n = b$. One has $C^2 = a^2 n + 2ab \geq 0$.

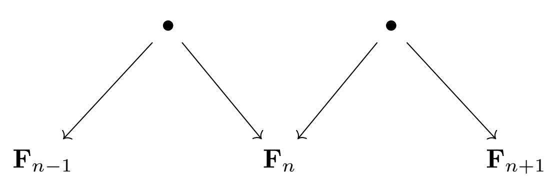
□

Proposition 11.3. $\mathbf{F}_1 \cong \operatorname{Bl}_1 \mathbf{P}^2$.

Proof. Check that $\operatorname{Bl}_1 \mathbf{P}^2 \to \mathbf{P}^2$ is geometrically ruled and has a fiber of square -1.

□

Lemma 11.4. *There is a diagram of blowups and blowdowns:*



> *Link to Diagram*

Proof. Consider the $-n$ section s_n intersecting a fiber f with $s_n.f = 1$. One checks that $(\pi^* s_n - E)(\pi^* f - E) = s_n f + E^2 - \pi^* s_n E - \pi^* f E = 1 - 1 - 0 - 0 = 0$. Because the fiber directions are different than the section directions, s_n intersects E and not the strict preimage of f.

□

Remark 11.5. Note that the classical minimal model is not unique. If $n \neq 1$, there are no -1 curves. If $n = 1$ the -1 curve can be blown down to obtain $\mathbf{P}^2$. Thus $\mathbf{F}_n$ is minimal when $n \geq 1$, but all of the $\mathbf{F}_n$ are birational.

Theorem 11.6. *(Mumford) Suppose $S \to S'$ contracts several curves $\bigcup_{1 \leq i \leq k} E_i$ to a point. Then the $k \times k$ matrix $M_{ij} = E_i E_j$ is negative definite.*

Remark 11.7. If f contracts a curve E, it must be the case that $E^2 < 0$. Why: write $f^*C = C" + mE$, then $E.f^*C = f_*E.C = 0.C = 0$ on one hand, and $E.f^*C = (C' + mE)E = C'.E + mE^2$, and since $C'.E > 0$ and $m > 0$, one must have $E^2 < 0$.

Remark 11.8. Question: conversely, if $\cup E_i \subseteq S$ and $[E_i.E_j]$ is negative definite, does there exist a contracting morphism $S \to S'$? Answer: yes, if S' is a complex analytic surface. This in fact holds for any Moishezon variety. By Artin, the answer is yes if the configuration is *rational*, which is computable. This is proved constructively.

Remark 11.9. Next time: del Pezzo surfaces.

12. 2024-10-03-12-49-19

12A. **Del Pezzo surfaces.**

Definition 12.1. A **del Pezzo surface** is a smooth projective surface over $\mathbf{C}$ such that $-K_S$ is ample.

Remark 12.2. Recall that if S is a surface then $\mathbf{T}_S$ is a rank 2 vector bundle with dual Ω_S. We define the canonical line bundle $\omega_S \coloneqq \det \Omega_S = \wedge^2 \Omega_S$. This is spanned by $dx \wedge dy$ and is thus rank 1. Since this is a line bundle, it is of the form $\mathcal{O}_S(K_S)$ for some divisor K_S, denoted that canonical divisor. The anticanonical bundle is ${\omega_S}^\vee$, the dual of the canonical, with associated divisor $-K_S$, the anticanonical divisor. Note that any toric boundary divisor is anticanonical. This divisor is not unique.

Remark 12.3. Recall that a line bundle L is **very ample** if the associated rational map $\phi : S \to \mathbf{P}H^0(L)^\vee$ (where $x \mapsto \mathbf{P}(s \mapsto s(x))$) is well-defined everywhere. Equivalently, L is basepoint free, i.e. for every $x \in S$ there is a section s where $s(x) \neq 0$. This follows because $[0 : \cdots : 0]$ is not a point in $\mathbf{P}^N$, and this condition guarantees that at least one coordinate is nonzero. In this case, ϕ is an embedding. A divisor L is very ample iff $\mathcal{O}_S(L)$ is very ample. We say L is ample if $L^{\otimes n}$ is very ample for some $n \in \mathbf{Z}_{>0}$.

Remark 12.4. In higher dimensions, any X with $-K_X$ ample is called a **Fano variety**. Thus a Fano surface is by definition a del Pezzo surface. This condition roughly describes having positive curvature – determining precisely when such metrics exist uses K-stability. An interesting research question: can Fano varieties be classified? Some classification results:

- $\dim X = 1$: $X = \mathbf{P}^1$ is the only Fano.
- $\dim X = 2$: del Pezzo surfaces, which we will classify today.
- $\dim X \geq 3$: there are finitely many Fanos up to deformation (theorem, 1960s). In $\dim X = 3$, there are 105. In $\dim X = 4$, the number is unknown.

Remark 12.5. Claim: $\mathbf{P}^2$ is a del Pezzo surface. Check that $-K_{\mathbf{P}^2} = \mathcal{O}_{\mathbf{P}^2}(3)$, which is very ample. Sections intersect the zero section along a cubic, and this has 10 global sections since these biject with degree 3 homogeneous polynomials in 3 variables x, y, z on $\mathbf{P}^2$. Note that $\mathbf{P}^2$ is toric, so choosing this as a polarization yields a moment polytope with 10 integral points: a triangle with side lengths 3. $-K_{\mathbf{P}^2}$ defines an embedding $\mathbf{P}^2 \hookrightarrow \mathbf{P}H^0(\mathcal{O}(3))$ which sends $[x : y : z]$ to $[x^3 :, x^2y :, \cdots ,]$, all monomials of degree 3.

Theorem 12.6. *A surface S is a del Pezzo if it either obtained from $\mathbf{P}^2$ by blowing up k points in general position where $0 \leq k \leq 8$, or $S \cong \mathbf{P}^1 \times \mathbf{P}^1$. Note that* general position *means*

- *No 3 points are on a line, and*
- *No 6 points are on a conic.*

Remark 12.7. If S is a del Pezzo surface, then

- $K_S^2 > 0$, and
- $-K_S.C > 0$ for any algebraic curve C on S.

Note that $(-K_S)^2 = K_S^2$. Generally, if L is ample then $L^2 > 0$. We define the **degree** of a del Pezzo surface to be K_S^2. Note that conversely, any surface satisfying these properties is a del Pezzo by the Nakai-Moishezon criterion. This is proved in Hartshorne chapter 5.

Remark 12.8. Some non-obvious del Pezzo surfaces:

- $(\mathbf{P}^2, \mathcal{O}(3))$
- $(\mathbf{P}^1 \times \mathbf{P}^1, \mathcal{O}(2,2))$.
- $(S = \mathrm{Bl}_k \mathbf{P}^2, \mathcal{O}(-K_S))$ where $-K_S = 3H - \sum E_i$ with $E_i^2 = -1$ the exceptional curves and $H \in \mathrm{Pic}(S)$ is the class of a general line in $\mathbf{P}^2$ (avoiding the blowup points).

The first two are very ample, which is easy to check. Note that $\mathrm{Pic}(S) = H_2(S; \mathbf{Z})$ in the third case, and that the k points must be in general position. Different configurations of points yield deformation-equivalent surfaces. In $\dim X = 2$, there are thus 10 possible del Pezzo surfaces up to deformation equivalence. To compute $(-K_S)^2$, one needs to know

- $H^2 = 1$
- $H.E_i = 0$
- $E_i E_j = 0$ for $i \neq j$
- $E_i^2 = -1$

Distributing and multiplying, one obtains $K_S^2 = 9 - k$ which is positive iff $k \leq 8$.

Remark 12.9. A counterexample where the points are not in general position: let $p_1, p_2, p_3 \in l$ be three points contained in a line in $\mathbf{P}^2$. The strict transform of l is of the form $H - E_1 - E_2 - E_3$. Intersect

$$-K_S.(H - E_1 - E_2 - E_3) = (3H - E_1 - E_2 - E_3)(H - E_1 - E_2 - E_3) = 3 - 1 - 1 - 1 = 0,$$

which violates the second condition in the definition of a del Pezzo. Similarly, taking 6 points on a conic, the strict transform is $2H - E_1 - \cdots - E_6$. Intersecting as above yields $6 - 1 - 1 - 1 - 1 - 1 - 1 = 0$. One could formulate similar conditions on higher numbers of points, but we already require $k \leq 8$ and the next condition would be on 9 points.

Remark 12.10. Denote by S_d a del Pezzo of degree d, obtained by blowing up $k = 9 - d$ points in general position in $\mathbf{P}^2$. How to remember: $d = 9$ yields $k = 0$ points, and $\deg \mathcal{O}_{\mathbf{P}^2}(3) = 3^2 = 9$. By assumption $-K_{S_d}$ is ample, and it will be very ample if $3 \leq d \leq 9$. This corresponds to blowing up $k \leq 6$ points. One can calculate $h^0(-K_{S_d}) = 10 - (9 - d) = d + 1$, so $S_d \hookrightarrow \mathbf{P}^d$ where $3 \leq d \leq 9$. This is referred to as the anticanonical embedding of S_d into $\mathbf{P}^d$.

Example 12.11. For $d = 3$, we have $S_3 = \mathrm{Bl}_6 \mathbf{P}^2 \hookrightarrow \mathbf{P}^3$, and in fact any smooth projective cubic surface can be embedded in $\mathbf{P}^3$.

Theorem 12.12. *Any smooth projective cubic surface is isomorphic to the del Pezzo surface S_3.*

Example 12.13. $S_4 = \mathrm{Bl}_5 \mathbf{P}^2 \hookrightarrow \mathbf{P}^4$, which is true since S_4 is a complete intersection of two quadrics in $\mathbf{P}^4$. Note that a complete intersection is a transverse intersection of hypersurfaces. Note that S_5 is no longer a complete intersection.

Corollary 12.14. *Any del Pezzo surface is birational to $\mathbf{P}^2$.*

Remark 12.15. One could ask if this is true for higher dimensional Fano varieties. It fails in dimension 3 – not every Fano threefold is rational.

13. 2024-10-08-12-51-02

Remark 13.1. Last time: del Pezzo surfaces, where $-K_S$ is ample. Note that for del Pezzos, $-K_S$ is also effective, and is of the form $S = \mathbf{P}^1 \times \mathbf{P}^1$ or $S = \mathrm{Bl}_k \mathbf{P}^2$ where $0 \leq k \leq 8$.

Example 13.2. Let $S = \mathrm{Bl}_8 \mathbf{P}^2$, then there always exists a curve C through these 8 points whose preimage satisfies $[-K_S] = [\widetilde{C}] = 3H - \sum E_i$.

Remark 13.3. Generally, L is an effective line bundle iff L has a nonzero section s. By intersecting s with the zero section of L, one obtains a divisor D. In the 1-dimensional case, this is the correspondence $D \rightleftharpoons L := \mathcal{O}(D)$.

Remark 13.4. Question: is $-K_X$ effective for any (smooth) Fano of any dimension over $\mathbf{C}$?

- Dimension 2: true.
- Dimension 3: true (Sokurov, 1980).
- Dimension 4: true (Kawamata, 2000).
- Dimension $n \geq 5$: open.

Remark 13.5. Generally ample does not imply effective. For example, let C be a genus 2 curve and L a degree 1 line bundle. Any positive degree line bundle on a curve is ample. Consider the moduli space $\mathrm{Pic}^1(C)$ of degree 1 line bundles on C. This is isomorphic to the Jacobian of C, an abelian variety of dimension 2. If L is effective as a line bundle, then $L \cong \mathcal{O}(D)$ with D effective. By definition, D is effective iff $D = \sum a_i D_i$ with $a_i > 0$. Since $\deg L = 1$, without loss of generality we can write $a_1 = 1$ and thus $a_i = 0$ for $i \neq 1$. So $L = \mathcal{O}(a_1 p)$ where $p \in C$ is a point. Thus the moduli space of such bundles is in bijection with the points of C, and is dimension 1. Since $\mathrm{Pic}^1(C)$ includes ample line bundles, there are many ample but non-effective bundles.

Remark 13.6. For any algebraic curve C, there is an Abe-Jacobi map given by $\phi : \mathrm{Sym}^d(C) \to \mathrm{Pic}^d(C)$. Note that $\mathrm{Sym}^d(C) = C^d/S_d$ where S_d is the symmetric group, and is the moduli space of degree d effective divisors on C. On the other hand, $\mathrm{Pic}^d(C)$ is the moduli space of degree d line bundles. This map is given by $D \mapsto \mathcal{O}(D)$, and is not surjective in general.

Remark 13.7. Recall that X is rational if $X \overset{\sim}{\dashrightarrow} \mathbf{P}^n$. In dimension 1, $\mathbf{P}^1$ is the only smooth projective rational curve. In dimension 2, any del Pezzo is rational. Questions:

- Can we classify all rational surfaces?

- Can we find criteria to decide if a surface is rational or not?

Remark 13.8. Recall that Ω_X^k is the vector bundle of algebraic k-forms on a smooth projective variety X of dimension n. These are used to define Hodge numbers $h^{k,0}(X) = \dim H^0(\Omega_X^k)$, which are birational invariants.

Remark 13.9. For $X = \mathbf{P}^n$, one has $h^{k,0} = 0$ for $k \neq 0$ and $h^{0,0} = 1$. For $\mathbf{P}^2$, this yields the following Hodge diamond:

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 0 & & 1 & & 0 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

> Link to Diagram

Thus any rational surface should have a Hodge diamond of the following form:

$$\begin{array}{ccccc} & & h^{2,2} & & \\ & h^{2,1} & & h^{1,2} & \\ 0 & & h^{1,1} & & h^{2,0} \\ & 0 & & h^{1,0} & \\ & & 1 & & \end{array}$$

> Link to Diagram

One can conclude, for example, that K3 surfaces are not rational.

Remark 13.10. Question: is any surface S with $h^{1,0}(S) = h^{2,0}(S) = 0$ necessarily rational? Answer: no, consider Enriques surfaces, which are not rational:

1

0 0

0 10 0

0 0

1

> Link to Diagram

Remark 13.11. We instead introduce more refined birational invariants in terms of $h^0((\Omega^k)^{\otimes m})$ for $m > 0$. For $\mathbf{P}^n$, these vanish for all m. For $\dim X = k$, we define the **plurigenus**

$$P_m(X) := H^0((\Omega_X^k)^{\otimes m}).$$

For $m = 1$, this is referred to as the **geometric genus**.

Theorem 13.12. *(Castelnuovo's rationality criterion) If X is a smooth projective surface, then X is rational iff*

- $h^{1,0}(X) = 0$
- $P_2(X) = 0$

14. 2024-10-10-12-51-51

Remark 14.1. Let S be a smooth projective surface. Then S is rational iff

- $q := h^{1,0} = 0$ (vanishing irregularity)
- $P_2 = h^0(\omega_S^{\otimes 2}) = 0$.

In the forward direction, this is clear since these are birational invariants and $q(\mathbf{P}^2) = h^{1,0}(\mathbf{P}^2) = 0$ and $P_2(\mathbf{P}^2) = 0$. The other direction requires a difficult proof. We'll assume S is minimal, i.e. S does not contain a smooth rational curve C with $C^2 = -1$ (which can be contracted).

Proposition 14.2. *If S is a minimal smooth projective surface with $q = P_2 = 0$, then there exists a curve $C \cong \mathbf{P}^1$ on S such that $C^2 \geq 0$.*

Remark 14.3. General philosophy: C is given by $V(f)$ for some polynomial f, and since $C^2 \geq 0$ it can be deformed to $V(f')$. One can form the pencil $\{uf + vf'\}$.

Lemma 14.4. *Under the same conditions as the last proposition, there exists an irreducible algebraic curve C with*

- $K_S.C < 0$, *and*
- $|K_S + C| = \emptyset$.

Remark 14.5. Consider $S = \mathbf{P}^2$, take C to be any line in $\mathbf{P}^2$. Then $\mathcal{O}(-3).[C] = -3 < 0$, and $L := \mathcal{O}(K_S + C) \cong \mathcal{O}(-2)$ has no sections.

Remark 14.6. Why the lemma implies the proposition: let C be a curve satisfying the conditions of the lemma, then we claim $C \cong \mathbf{P}^1$. The proof uses adjunction and Riemann-Roch: $2g - 2 = C(C + K_S)$ where g is the arithmetic genus, and $\chi(\mathcal{O}(D)) = \frac{1}{2}D(D - K_S) + \chi(\mathcal{O})$. Note that $\chi(\mathcal{O}) = 1 - h^{1,0} + h^{2,0} = 1 - 0 + 0$ by assumption, so $\chi(\mathcal{O}(D)) = \frac{1}{2}D(D - K_S) + 1$. Take $D \coloneqq K_S + C$. Recall that by Serre duality, $h^2(\mathcal{O}(D)) = h^0(\mathcal{O}(K_S - D))$. Thus $h^2(\mathcal{O}(K_S + C)) = h^0(\mathcal{O}(-C)) = 0$ since $-C$ is not effective. As a result, $\chi(K_S + C) = h^0(K_S + C) - h^1(K_S + C)$. We know by assumption that $h^0(K_S + C)$, so $\chi(K_S + C) \leq 0$. Substituting this into Riemann-Roch yields $\frac{1}{2}(K_S + C)C + 1 \leq 0$. Solving for g in the adjunction formula yields $g = \frac{1}{2}(K_S + C)C + 1$, so $g \leq 0$. The arithmetic genus can not be negative for an irreducible curve, hence $g = 0$ and $C \cong \mathbf{P}^1$.

It remains to show that $C^2 \geq 0$. By adjunction, $-2 = 2g - 2 = C(K_S + C)$ since $g = 0$. We assumed $C.K_S < 0$, so $C^2 + C.K_S < C^2$, so $C^2 = -1$ or $C^2 \geq 0$. The former case is ruled out because we assumed S to be minimal.

Remark 14.7. On the proof of the lemma: it is broken into the following three cases,

- $K^2 < 0$,
- $K^2 = 0$,
- $K^2 > 0$.

We'll consider the case $K^2 = 0$. We first claim $-K_S$ is effective, i.e. $h^0(-K_S) \neq 0$. Applying Riemann-Roch to $-K_S$ yields $\chi(-K_S) = \frac{1}{2}(-K)(-K - K) + 1 = -K^2 + 1 = 1$ since $K^2 = 0$. By Serre duality, $h^2(-K_S) = h^0(K_S - (-K_S)) = h^0(2K_S)$, but note $\mathcal{O}(2K_S) = \mathcal{O}(K_S) \otimes \mathcal{O}(K_S) = \omega_S^{\otimes 2}$, so $h^0(2K_S) = P_2 = 0$ by assumption. Thus $h^0(-K_S) = 1 + h^1(-K_S) > 0$.

Claim: let H be a very ample divisor, which is in particular effective. Then $-K_S.H > 0$ since $-K_S$ is effective, so $K_S.H < 0$. We'll show $H + nK_S$ is not effective if n is large enough. Consider $H(H + nK_S) = H^2 + nHK_S$; sine $HK_S < 0$, this is negative for some n since H^2 is a fixed number. If $H + nK_S$ were effective, this would have to be positive. So there exists a unique minimal n_0 such that $H + n_0K_S$ is not effective for any $n \geq n_0$. So take $C \coloneqq H + n_0K_S$. One then checks $K_S.C = K_S.H < 0$ since $H^2 = 0$ and $-K_S$ is effective with H ample. Furthermore $K_S + C = K_S + H + n_0K_S = H + (n_0 + 1)K_S$ which is not effective by definition of n_0, thus $|K_S + C| = \emptyset$.

Remark 14.8. Note that we didn't show C was irreducible. If it is not, then take any irreducible component C' with $C'.K_S < 0$, which must exist since $K_S.C < 0$.

15. 2024-10-15-12-51-37

Remark 15.1. Recall that a *minimal* surface is a smooth projective surface which does not contain any smooth rational (-1)-curves. Castelnuovo's contraction theorem implies that any smooth projective surface is birational to a minimal surface. Goal: classify all smooth projective surfaces up to birational equivalence. A related question: in any birational equivalence class, is there a unique minimal surface? Or can two minimal surfaces be birational? Answer:

- No for rational surfaces,
- Yes for non-ruled surfaces.

Recall that a ruled surface is birational to $C \times \mathbf{P}^1$ for C a smooth projective curve. Note that rational implies ruled, since $\mathbf{P}^2 \overset{\sim}{\dashrightarrow} \mathbf{P}^1 \times \mathbf{P}^1$ which is ruled. Every non-ruled surface admits a unique minimal model up to isomorphism.

Remark 15.2. Let S, S' be non-ruled surfaces and $\phi : S' \overset{\sim}{\dashrightarrow} S$ be a birational map. We want to show that $S' \cong S$. Note that ϕ is a composition of blowups and blowdowns, so there exists some $\widehat{S}$ with $f : \widehat{S} \to S$ and $\varepsilon : \widehat{S} \to S'$ where f, ε are compositions of blowups. Choose a diagram such that ε is composed of a minimal number $n \in \mathbf{Z}_{\geq 0}$ of blowups. By cases: if $n = 0$, then $S' \cong \widehat{S}$, and $f : S' \to S$ is a composition of blowups. The last blowup would introduce an exceptional curve, contradicting the fact that S' is minimal. Thus f is an isomorphism.

Suppose now that $n \neq 0$; we'll show that this can never happen. Let E be the exceptional curve of the blowup ε in $\widehat{S}$. Consider $f(E)$, then either

- $f(E)$ is contracted, so $f(E)$ is a point, or
- $f(E)$ is a curve.

The first case contradicts the minimality of n by removing the blowup/blowdown of E from the diagram. In the second case, we'll show such a curve can not exist, reaching a contradiction.

Lemma 15.3. *Let X be any surface and $f : \widehat{X} \to X$ is the blowup at a point. Let $\widehat{\Gamma} \subseteq \widehat{X}$ be an irreducible curve which is not contracted by f, so $\Gamma := f(\widehat{\Gamma}) \subseteq X$ is a curve. Then*

$$K_{\widehat{X}}.\widehat{\Gamma} \geq K_X.\Gamma.$$

Proof. As long as $\widehat{\Gamma} \not\subseteq E$, it is not contracted to a point when E is blown down. So this yields some curve $\Gamma \subseteq X$, possibly singular. Let $m := E.\widehat{\Gamma} < \infty$, which is finite because $\widehat{\Gamma}$ is not contained in E. Note $K_{\widehat{X}} = f^*K_X + E$. We can write $\widehat{\Gamma} = f^*\Gamma + mE$. Recall that $f^*D_1.f^*D_2 = D_1.f_*f^*D_2$, so $f^*K_X.f^*\Gamma = K_X.f_*f^*\Gamma = K_X.\Gamma$, so

$$\begin{aligned} K_{\widehat{X}}.\widehat{\Gamma} &= (f^*K_X + E)(f^*\Gamma - mE) \\ &= f^*K_X.f^*\Gamma + E.f^*\Gamma - mf^*K_X.E - mE^2 \\ &= K_X.\Gamma + 0 + 0 - 1. \end{aligned}$$

Moreover, $f^*D_1.D = D_1.f_*D$, so we have

$$E.f^*\Gamma = f^*\Gamma.E = \Gamma.f_*E = 0$$

since f_*E is a point and not a divisor. This proves the lemma. Note that equality is attained iff $m = 0$, so $\widehat{\Gamma}$ does not intersect E.

□

Remark 15.4. We now apply this lemma to the previous proof. Recall that $\varepsilon : \widehat{S} \to S$ produces an exceptional curve $E = \widehat{\Gamma}$ where the last blowup ε_n produces E. Let $C = f(E)$. Apply the lemma to each blowup $f_1, \cdots, f_m$ by setting $\widehat{\Gamma} = E$. By the adjunction formula, $2g(E) - 2 = E.(E + K_{\widehat{S}})$, and since $g(E) = 0$ one obtains $E.K_{\widehat{S}} = -1$. Thus $C.K_S \leq -1$ by the lemma. Claim: $C.K_S \neq -1$, so the inequality is strict. If $K_S.C = -1$, then $K_{\widehat{S}}.E = K_S.C$ and this only happens if E does not intersect the other exceptional divisor E. So $f(E) \cong C \cong E$, so $C \cong \mathbf{P}^1$ with $C^2 = -1$, but this contradicts minimality of S. Thus $C.K_S \leq -2$.

By adjunction,

$$2g(C) - 2 = C.(C + K_S) = C^2 + C.K_S \implies C^2 = 2g - 2 - C.K_S \implies C^2 \geq 0$$

since $C.K_S \leq -2$.

Lemma 15.5. *We thus have two properties:*

- $C.K_S \leq -2$,
- $C^2 \geq 0$.

If such a curve exists, all plurigenera $P_m(S)$ vanish.

Proof. Note that $P_m(S) = 0$ iff mK_S is not effective. If $D = mK_S$ is effective, then $D = D' + aC$ where $a \geq 0$ such that D' does not contain C. Then $D.C = D'.C + aC^2 \geq 0$ by assumption, but this implies $mK_S.C \geq 0$, a contradiction. □

Remark 15.6. Hence $P_m(S) = 0$, and in particular $P_2(S) = 0$. Cases:

- $h^{1,0}(S) = 0$, which implies S is rational by Castelnuovo, hence ruled, a contradiction.
- $h^{1,0}(S) \neq 0$, which we claim also can not happen.

Assume the second case. Black box: for any surface S, there is a natural map $\psi : S \to \operatorname{Alb}(S)$ where $\psi(S)$ is a curve of genus $h^{1,0}$. Moreover, ψ is generically finite, and $\operatorname{Alb}(S)$ is generally a genus g abelian variety. Since C is rational, it must be a fiber of ψ. One can show $S \cong \operatorname{Alb}(S) \times \mathbf{P}^1$, showing S is ruled, a contradiction. This forces $n = 0$ and $S \cong S'$.

Remark 15.7. Next time: the Albanese variety.

16. 2024-10-17-12-49-00

Remark 16.1. Let S, S' be non-ruled surfaces, then any birational map $S \overset{\sim}{\dashrightarrow} S'$ is an isomorphism. Factoring as a composition of n blowups and m blowdowns, we saw that if $n = 0$ this is true. For $n \geq 1$, we showed existence of a curve C with $K_S.C \leq -2$ and $C^2 \geq 0$, which implied $P_m(S) = 0$, and in particular $P_2(S) = 0$. There were two cases:

- $h^{1,0}(S) = 0 \implies S$ is rational, hence ruled, a contradiction, or
- $h^{1,0} \neq 0$.

In the latter case, we have a natural map $\psi : S \to \operatorname{Alb}(S)$, which is an abelian variety. The image $\psi(S)$ is a genus g curve where $g = h^{1,0}(S)$. Note that a priori, one could have $\dim \psi(S) = 0, 1, 2$, so we argue that $\dim \psi(S) \neq 0, 2$. If $\psi(S)$ is a point, the universal property of the Albanese yields a contradiction. If $\psi(S)$ is a surface, then ψ is generically finite and $P_2(S) = P_2(\operatorname{Alb}(S)) \neq 0$.

16A. Albanese varieties.

Remark 16.2. Let X be any smooth projective variety, then there is an abelian variety $\operatorname{Alb}(X)$. This can be written as $H^0(\Omega_X)^\vee / \iota(H_1(X; \mathbf{Z}))$ where $\iota : H_1(X; \mathbf{Z}) \to H^0(\Omega_X)^\vee$ is defined by $\gamma \mapsto (\omega \mapsto \int_\gamma \omega)$. It is a theorem that the image is always a lattice Γ, so $\Gamma \otimes \mathbf{C} = H^0(\Omega_X)^\vee$. The proof uses deep facts from Hodge theory.

Remark 16.3. Recall that $h^{1,0} = \dim_{\mathbf{C}} H^0(\Omega_X) := n$, so $H^0(\Omega_X) \cong \mathbf{C}^n \cong \mathbf{R}^{2n}$. Writing $H_1(X; \mathbf{Z}) = \mathbf{Z}^{b_1} + T$ where T is torsion, note that $\iota(T) = 0$. Note also that $b_1 = h^{1,0} + h^{0,1}$. Moreover $\iota(\mathbf{Z}^{b_1}) \cong \mathbf{Z}^{b_1}$. Thus $\operatorname{Alb}(X) \cong \mathbf{R}^{2n}/\mathbf{Z}^{2n} \cong (S^1)^{2n}$. There is a natural complex structure on $\mathrm{Alb}(X)$, and moreover is smooth and projective. The proof again uses Hodge theory. By GAGA, $\mathrm{Alb}(X)$ is the unique variety with this underlying complex manifold structure.

Remark 16.4. Recall $\mathrm{Pic}^0(X)$ is the connected component of $\mathrm{Pic}(X)$ containing the trivial line bundle. A theorem of Grothendieck shows that $\mathrm{Pic}^0(X)$ is an abelian variety of dimension $h^{1,0}$. We also have $\mathrm{Alb}(X) = \mathrm{Pic}^0(X)^\vee$ – more generally, for any abelian variety A, one defines $A^\vee := \mathrm{Pic}^0(A)$ since $\mathrm{Pic}^0(\mathrm{Pic}^0(A)) = A$. We thus regard $\mathrm{Alb}(X)$ as the "smallest" abelian variety associated to X. The map $\psi : X \to \mathrm{Alb}(X)$ can be given by $x \mapsto (\omega \mapsto \int_x^p \omega)$ where p is a fixed choice of point on X. The universal property is that for any abelian variety A, any map $X \to A$ factors as $X \to \mathrm{Alb}(X) \to A$.

17. 2024-10-22-12-46-33

Remark 17.1. Recall Castelnuovo's rationality criterion: let S be a smooth projective surface, then S is rational iff $q := h^1(\mathcal{O}_X) = h^0(\Omega_X) = 0$ and $P_2 := h^0(2K_S) = 0$. As a corollary, S is rational iff S is *unirational*, i.e. there is a dominant map $\mathbf{P}^2 \dashrightarrow S$ of some degree d. We discussed minimal surfaces, for which there are two cases:

- S is ruled, so $S \overset{\sim}{\dashrightarrow} C \times \mathbf{P}^1$ for some curve C. This implies that a generic point is contained in a rational curve.
- S is not ruled. In this case, there are few rational curves.

We showed that if S is not ruled, there exists a unique minimal model in every birational equivalence class. That is, if $f : S' \overset{\sim}{\dashrightarrow} S$ with S, S' minimal surfaces, then f is an isomorphism. If S is ruled, minimal models are not unique – blow up any point in a fiber of $S \to C$ to transform one (0)-curve to a union of two (-1)-curves, and blowdown the other curve to get another ruled surface.

Theorem 17.2. *If S is a rational minimal surface, then $S \cong \mathbf{P}^2$ or $\mathbf{F}_n$ for some n.*

Theorem 17.3. *If S is ruled and not rational, then $S \cong \mathbf{P}_C(\mathcal{E})$ is a geometrically ruled surface where C is a curve of genus $g \geq 1$.*

Remark 17.4. Important trick: suppose C is an effective irreducible curve where $|K_S + C| = \emptyset$. Suppose also that $\chi(\mathcal{O}_S) = 1$, noting that $\chi(\mathcal{O}_S) = 1 - q + p_2$ which vanishes for rational surfaces. Then $C \cong \mathbf{P}^1$.

Recall that $\chi(\mathcal{O}(D)) = \chi(\mathcal{O}_S) + \frac{1}{2}D(D - K_S)$. Applying this to $D = K + C$ yields $\chi(K + C) = 1 + \frac{1}{2}(K + C)C = p_a(C)$. On the other hand, $\chi(K + C) = h^0(K+C) - h^1(K+C) + h^0(-C)$. Here $h^0(K+C) = h^0(-C) = 0$, so $\chi(K+C) \leq 0$, and thus $p_a(C) \leq 0$, forcing $p_a(C) = 0$ and $C \cong \mathbf{P}^1$.

Recall that if $\nu : \widetilde{C} \to C$ is the normalization, then $p_a(C) \geq p_a(\widetilde{C}) = g(\widetilde{C})$.

Corollary 17.5. *Suppose $\chi(\mathcal{O}_S) = 1$ and $C = \sum a_i C_i$ is an effective curve, and suppose $|K + C| = \emptyset$. Then each $C_i \cong \mathbf{P}^1$ since $|K + C_i| = \emptyset$.*

Remark 17.6. Consider $S = \mathbf{P}^2$; the key is to look at minimal rational curves on S. Note that this generalizes to higher dimensions, viz. Mori theory. For $\mathbf{P}^2$, these minimal curves will be lines. If C is a line, then $KC = -3 < 0$ and $|K + C| = |-3H + H| = \emptyset$.

For $S = \mathbf{F}_n$, note that this is a $\mathbf{P}^1$ fibration over $\mathbf{P}^1$ and the minimal rational curves are its fibers. Letting C be a fiber, we have $C^2 = 0$ and by the genus formula, $KC = -2$. Moreover $|K + C| = \emptyset$, using the following observation:

Remark 17.7. Suppose A is an irreducible effective curve with $A^2 \geq 0$ with $D.A < 0$. Then D is not effective and $|D| = \emptyset$. This follows from $D = nA + \sum a_i B_i$ with $AB_i \geq 0$ and $nA^2 \geq 0$.

Remark 17.8. One can then check that if $K + C$ is effective, then its restriction to a fiber is still effective. However, $K_{\mathbf{P}^1} = \mathcal{O}_{\mathbf{P}^1}(-2)$ is not effective.

Remark 17.9. We now prove that if S is rational and minimal, then $S \cong \mathbf{P}^2, \mathbf{F}_n$. Consider the set of irreducible curves $\left\{ C \cong \mathbf{P}^1 \subseteq S \;\middle|\; C^2 \geq 0 \right\} \neq \emptyset$. This is nonempty by a previous result on existence of special curves on such surfaces. Consider the subset of such curves with minimal square $C^2 = m$. Fix a hyperplane section H, so $C.H = \deg(C)$, and consider the further subset with minimal $C.H$.

Proposition 17.10. *Step 1: in $|C|$, every curve is irreducible and isomorphic to $\mathbf{P}^1$. Thus in this minimal covering set, the curves do not split into multiple components.*

Proof. Suppose $D \in |C|$ and write $D = \sum a_i C_i$, then all $C_i \cong \mathbf{P}^1$. Then $|K + D| = |K + C| = \emptyset$. This follows from the "useful observation", since $(K + C)C = 2p_a(C) - 2 = -2$. Note that $H.C_i < H.C$.

We'll return to this proof later.

□

Proposition 17.11. *Step 2:* $\dim |C| \leq 2$. *Note that if one considers the divisors in $|C|$ passing through a fixed point p, the dimension either stays the same or decreases by one. This is a* codim ≤ 1 *condition. Those passing through p with multiplicity ≥ 2 is a* codim ≤ 3 *condition. Given $\mathcal{O}_S \twoheadrightarrow \mathcal{O}_S/\mathfrak{m}^2$, we have $H^0(\mathcal{O}(C)) \to H^0(\mathcal{O}(C) \otimes \mathcal{O}_S/\mathfrak{m}^2)$ and $\mathcal{O}_S/\mathfrak{m}^2$ has dimension 3, so the kernel has dimension at most 3. Supposing* $\dim |C| \geq 3$*, there exists a curve $D \in |C|$ passing through p with multiplicity ≥ 2, which is thus a singular curve – but this contradicts step 1.*

Proposition 17.12. *Step 3: there is a SES $0 \to \mathcal{O}_S \to \mathcal{O}_S(C) \to \mathcal{O}_C(C) \to 0$ gotten by twisting $\mathcal{O}_S(-C) \hookrightarrow \mathcal{O}_S \twoheadrightarrow \mathcal{O}_C$ by C. Note that $\mathcal{O}_C(C) = \mathcal{O}_{\mathbf{P}^1}(m)$. Taking the LES, note that $H^1(\mathcal{O}_S) = 0$ by rationality, so we have*

$$0 \to H^0(\mathcal{O}_S) = \mathbf{C} \to H^0(\mathcal{O}_S(C)) \to H^0(\mathcal{O}_{\mathbf{P}^1}(m)) \to 0.$$

We claim $|C|$ has no base points. Note that $h^0(\mathcal{O}_S(C)) = m$, and by the LES, $\dim |C| = m + 1$ *since $h^0(\mathcal{O}_{\mathbf{P}^1}(m)) = m + 1$. Since $m + 1 \leq 2$, we have $m \leq 1$ and thus $m = 0, 1$. If $m = 0$, then* $\dim |C| = 1$ *yielding a morphism $S \to \mathbf{P}^1$ with C a fiber, making S ruled by a previous theorem. Since S is minimal, this yields $\mathbf{F}_n$, since it is a geometrically ruled surface over $\mathbf{P}^1$. If $m = 1$,* $\dim |C| = 2$ *and this yields a morphism $S \to \mathbf{P}^2$. Since the degree is $C^2 = 1$, this is an isomorphism.*

Remark 17.13. Next time: finish step 1, understand the minimal surfaces which are rational and not ruled.

18. 2024-10-24-12-40-57

18A. Chapter 5 Recap.

Remark 18.1. For minimal surfaces, there is a division:

- Non-ruled surfaces: $\exists!$ minimal model in every birational class. We showed there is a sequence of contractions $S \to\to\to S_{\min}$ where $S_{\min}$ has no (-1)-curves. Therefore our next step will be classifying such minimal surfaces.
- Ruled surfaces: minimal models are never unique. There are two cases:
  - $q \coloneqq h^1(\mathcal{O}_S) = 0$: regular surfaces. Minimal surfaces are $\mathbf{P}^2$ and $\mathbf{F}_n$ for $n \neq 1$, thus rational.
  - $q \neq 0$: non-regular surfaces. Minimal surfaces are $\mathbf{P}_C(\mathcal{E})$ with $\operatorname{rank} \mathcal{E} = 2$ and $\mathcal{E}$ a vector bundle, i.e. geometrically ruled surfaces (every fiber is $\mathbf{P}^1$). Here $g(C) = q$ and $p : S \to C$ is the Albanese map.

Last time: suppose S is a rational surface, we considered minimal rational curves C on S, i.e.

$$\left\{ C \cong \mathbf{P}^1 \subseteq S \;\middle|\; C^2 \geq 0 \right\} \supseteq \left\{ C \;\middle|\; C^2 = m \text{ is minimal} \right\} \supseteq \left\{ C \;\middle|\; \deg(C) \coloneqq C.H \text{ is minimal} \right\}.$$

Then every $D \in |C|$ is smooth, irreducible, and $D \cong \mathbf{P}^1$, i.e. curves in $|C|$ do not split. For example, conics in $\mathbf{P}^2$ degenerate into unions of lines and thus do split.

Proof. Suppose minimal just means $C.H$ is minimal, dropping the condition that $C^2 = m$ is minimal. Write $D \in |C|$ as $D = \sum n_i C_i$. By the genus formula, $0 = g(C) = \frac{C^2 + CK}{2} + 1$, and so $C^2 + CK = 2g - 2 = -2$ and thus $CK = -2 - C^2 < 0$ since $C^2 \geq 0$. Moreover $DK = CK < 0$. There exists an i such that $C_i K < 0$, and if $C_i^2 \geq 0$ then C is not minimal since $\deg C_i \leq \deg C$. If $C_i^2 < 0$, combine this with $C_i K < 0$ and substitute into the genus formula to conclude $C_i^2 = -1$ and $g(C) = 0$, this C_i is a rational (-1)-curve, contradicting minimality of S.

□

Remark 18.2. This argument generalizes to higher dimensions. Note that $|C|$ is a covering family of S of rational curves with minimal degree. See e.g. Mori's Annals paper proving Hartshorne's conjecture, which started the Mori program, or papers that use the bend-and-break technique. E.g. if X is a smooth projective Fano variety, so $-K_X$ is ample, then X is covered by rational curves and such covering families exist.

Remark 18.3. Recall that ruled non-regular surfaces ($q \neq 0$) are birational to $C \times \mathbf{P}^1$ where $g(C) = q$. First factor $S \to C \times \mathbf{P}^1$ as $S \to\to\to S' \to\to\to C \times \mathbf{P}^1$ as a composition of blowdowns and blowups. Note that blowing up points in fibers of $S \to C \times \mathbf{P}^1$ yields unions of $\mathbf{P}^1$s. The claim that E_i must be contained in a fiber, and thus not horizontal. This is because $g(E_i) = 0$ but $g(C) = q > 0$, and $\mathbf{P}^1$ can not map to a curve of higher genus. One can show this using Riemann-Hurwitz, since $\deg K_C = 2g(C) - 2$ and $\deg K_{\mathbf{P}^1} = -2 < 0$. Alternatively, $f^*\Omega_C \subset \Omega_E$ which, by taking dimensions, yields $g(C) \leq g(E)$. Thus the exceptional curves are vertical. Since there is a fibration $S' \to C$, there are thus fibrations to C after each blowdown and $S \to C$ fibers over C.

Remark 18.4. Note that the E_i are not equivalent to fibers, because the intersection form of the exceptional curves is negative semi-definite. Next chapter: ruled, non-regular surfaces, then we move on to non-ruled surfaces.

18B. **Chapter 6:** $p_g = 0, q > 0$.

Remark 18.5. Recall $p_g := h^0(K_S)$, so if $p_g = 0$ then there is not effective $D \sim K_S$. For ruled surfaces, all $P_m := h^0(mK_S) = 0$, so no multiple of K_S is effective. Use the useful observation: if $C^2 \geq 0$ with $mK.C < 0$ then mK is not effective. Take C to be a fiber F, then $mK_S.F + F^2 = 2g(F) - 2$ and so $mK_S.F = -2 < 0$.

Remark 18.6. Noether's formula: $\chi(\mathcal{O}_S) = \frac{K^2 + \chi(S)}{12} = \frac{c_1^2 + c_2}{12}$, which was derived from Hirzebruch-Riemann-Roch,

$$\chi(D) = \int_X e^D \operatorname{Td}(\mathbf{T}_X),$$

where you multiply the corresponding power series and take the top degree terms. We have $\chi(\mathcal{O}_S) = 1 - q + p_g$ in general, and note that $p_g = 0$ in this case. Note that $\chi(S) = \sum (-1)^i b_i = 2 - 2b_1 + b_2$ by Poincare duality, and $b_1 = h^{1,0} + h^{0,1} = 2q$ by the Hodge decomposition. Substituting these facts into Noether's formula yields

$$12(1-q) = K^2 - 4q + b_2 \implies K^2 = 10 - 8q - b_2.$$

Lemma 18.7. *In parts:*

a. *If $p_g = 0$ and $q > 0$ then $K^2 \leq 0$ and $K^2 < 0$ only if $q = 1, b_2 = 2$.*
b. *If S is minimal with $K^2 < 0$ then S is ruled.*

Proof. We know $q \geq 1$ and $b_2 \geq 1$ since there is always a hyperplane section. If $q \geq 2$, we're done, since $K^2 < 0$ by the above formula. If $q = 1$ then $b_2 \geq 2$: this follows from producing two linearly independent divisors in $H^2(S; \mathbf{Z})$. Consider the Albanese map $p : S \to \operatorname{Alb}(S)$. If S is a curve C, recall that $\operatorname{Alb}(C) = \operatorname{Jac}(C) = \operatorname{Pic}^0(C)$. This map contracts rational curves. Two cases:

- $\dim p(X) = 1$
- $\dim p(X) = 2$

It's nonzero since $p(X)$ generates $\operatorname{Alb}(X)$ and thus can not be a point if $q = \dim p(X) > 0$. In the first case, the generic fiber F is a curve with $F^2 = 0$ which is not a multiple of a hyperplane section H since $H^2 > 0$; then $\langle H, F \rangle \subseteq H^2(S; \mathbf{Z})$. In the second case, one can pull back differential forms on $p(X) \cong \mathbf{C}^g / \mathbf{Z}^{2g}$, contradicting $p_g = 0$.

□

19. 2024-10-29-12-44-54

Remark 19.1. Surfaces with $p_g = 0$ and $q \geq 1$, i.e. ruled irregular surfaces (plus a little more). The next main goal: minimal non-ruled surfaces.

Lemma 19.2. *In parts:*

a. *(A little more) $p_g = 0, q \geq 1 \implies K^2 \leq 0$, or $K^2 = 0$ and $q = 1, b_2 = 2$.*
b. *(Ruled irregular) If S is minimal and $K^2 < 0$ then $p_g = 0$ and $q \geq 1$.*

Proposition 19.3. *If S is minimal and $K^2 < 0$ then S is ruled.*

Proof. Of lemma part (a): by Noether's formula, $K^2 = 10 - 8q - b_2$. Use the Albanese map to show $b_2 \geq 2$.

Of lemma part (b): if $p_g := h^0(K_S) \neq 0$, then K is effective. Write $K = \sum n_i C_i$, then $K^2 < 0 \implies \exists C_j$ such that $KC_j < 0$ and $C_j^2 < 0$. By the genus formula, $p_g(C_j) = \frac{KC_j + C_j^2}{2} + 1$, forcing C_j to be a negative curve, contradicting minimality of S. This argument shows that in fact $P_m = 0$ for all m. Suppose that $q \neq 0$, we will then argue S is rational, and hence either $\mathbf{P}^2$ or $\mathbf{F}_n$ by minimality. But $K_{\mathbf{P}^2}^2 = 9$ and $K_{\mathbf{F}_n}^2 = n > 0$, contradicting $K^2 < 0$. By Castelnuovo's criterion, S is rational iff $2K_S$ is effective, which it is not (by repeating the above argument). □

Remark 19.4. Examples where K is not effective but $2K$ is effective: Enriques surfaces.

Remark 19.5. Proving the proposition, the main tool: the Albanese map $p : S \to \mathrm{Alb}(S)$ which is an abelian variety of dimension q. Note that $p(S)$ generates $\mathrm{Alb}(S)$, and p has connected fibers. If $q \geq 1$ then $\dim p(S) = 1, 2$. If $p_g = 0$, then $\dim p(S) = 1$. Any abelian variety is of the form $A = \mathbf{C}^g/\mathbf{Z}^{2g}$, and the differential forms are those with constant coefficients, e.g. $dz_1, \cdots, dz_n$, $dz_1 \wedge dz_2 \cdots$, etc. These are translation invariant and thus descend to the quotient. One then produces a differential form in $\mathrm{Alb}(S)$ which pulls back to S, yielding a nontrivial section of $\Omega^2(S)$, forcing $\dim p(S) = 1$. In this case, $p : S \to B$ where B is a smooth curve of genus q. Note that if a single fiber is $\mathbf{P}^1$ then S is ruled. We're trying to rule out fibers with higher genus.

Proof. Step 1: suppose $C \subseteq S$ is a curve with $KC < 0$ and $|K + C| = \emptyset$. Then $p|_C : C \to B$ is etale (unramified), and if $q \geq 2$ then $p|_C : C \xrightarrow{\sim} B$. Suppose C is contained in a fiber, so C collapses to a point. We first claim that this fiber F must be irreducible. Toward a contradiction, write $F = C + \sum n_i C_i$. Since the intersection form on fibers is negative definite, we have $C^2 < 0$. But then C is forced to be a (-1)-curve, contradicting minimality. Suppose $F = nC$, then $C^2 = 0$. The genus formula yields $p_a(C) = \frac{KC + C^2}{2} + 1 = 0 \implies KC = -2$, then $C \cong \mathbf{P}^1$ and S is ruled, a contradiction. So C is necessarily a horizontal curve.

Step 1.5: Riemann-Roch. Write $0 = h^0(K + C) \geq \chi(K + C)$ since $h^2(K + C) = h^0(-C) = 0$ since C is effective. We have

$$\chi(K + C) = \chi(\mathcal{O}_S) + \frac{(K + C)C}{2} = 1 - q + (p_a(C) - 1) = p_a(C) - q \leq 0,$$

and thus $p_a(C) \leq q$. But this contradicts the fact that the genus decreases if $q \geq 2$, by Hurwitz: $2g(C) - 2 = d(2g(B) - 2) + \varepsilon$. The only way this can happen is if $B \cong C$, in which case $g(C) = g(B) = 1$ (they are elliptic curves) and $\varepsilon = 0$ since the map is unramified. □

Proof. Step 2: we'll show there exists an irreducible curve $C \subseteq S$ such that $KC \leq -2$ and $|K + C| = \emptyset$. Recall that we proved the following: if S is minimal and $K^2 < 0$, then there exists a D' such that $KD' < -n$ is arbitrarily negative and $|K + D| = \emptyset$. Write $D = \sum_{1 \leq i \leq r} n_i C_i \leq D' = \sum n_j C_j$ where all $KC_i \leq -1$, i.e. keep just those curves that intersect K negatively. We have $|K + D| = \emptyset$. Thus we are done unless

a. there exists some $n_i \geq 2$ or
b. $r \geq 2$,

i.e. we want to show that D is a single curve with coefficient 1.

In case (a), $|K+2C_i| = \emptyset$ and thus

$$\begin{aligned} 0 = h^0(K+2C_i) &\geq \chi(K+2C_i) = \chi(\mathcal{O}_S) + \frac{(K+2C_i)2C_i}{2} \\ &= (1-q+p_g) + (KC_i + 2C_i^2) \\ &= (1-q+p_g) + (2KC_i + 2C_i^2 - KC_i) \\ &= (1-q+p_g) + (2(2p_a(C_i)-2) - KC_i) \\ &= (1-q+p_g) + (2(2q-2) - KC_i) \\ &= (1-q) + (4(q-1) - KC_i) \\ &> 3(q-1) \\ &\geq 0 \end{aligned}$$

noting that $h^2(K+2C_i) = 0$. This yields $0 > 0$, a contradiction.

□

Proof. Step 3: get a contradiction. Take C as in step 2.

Case 1: $p : C \xrightarrow{\sim} B$. By Riemann-Roch,

$$h^0(C) \geq \chi(\mathcal{O}_S) + \frac{C(C-K)}{2} = (1-q) + (q-1-KC) \geq 2,$$

noting that $\frac{C(C+K)}{2} + 1 = q$. Note that this implies that C, as a section of p, moves. Thus intersecting C with the general fiber F yields a point that moves, thus defining a map $F \cong \mathbf{P}^1$.

Case 2: $g(B) = g(C) = 1$ (elliptic curves) and p is étale. In this case, make a base change. Write $S' = S \underset{B}{\times} C$, which changes a multisection into an honest section. Note that $q(S') \geq q(S) \geq 1$, and $p_g(S') = h^0(\omega_{S'}^2) = 0$.

□

Remark 19.6. What remains to do: look at the "a little more" case. Two cases:

- The genus of the general fiber satisfies $g(F) \geq 2$, then p is a smooth map yielding a family of elliptic curves over an elliptic curve
- $g(F) = 1$, there may be multiple fibers. These appear in the classification of minimal models with $\kappa = 0$. Proofs here involve the topological Euler characteristic.

20. 2024-10-31-12-45-31

Remark 20.1. Recall that we are considering minimal surfaces S with $q \geq 1, p_g = 0$. We had $K^2 = 10 - 8q - b_2$, and concluded that if $K^2 < 0$ then S is ruled. We consider now the extreme case where $K^2 = 0$, so $q = 1, p_g = 0, b_2 = 2$, and S is not ruled. Consider the Albanese morphism $p : S \to B \subseteq \mathrm{Alb}(S)$ with $g(B) = q$; we'd like to show every fiber is irreducible. If any fiber is reducible, this forces $b_2 \geq 3$ by taking two components along with a horizontal part. If any fiber is $\mathbf{P}^1$, then S is ruled and birational to $B \times \mathbf{P}^1$. Since S is reduced and p is nonconstant onto a smooth curve, p is flat.

Theorem 20.2. *If $g \geq 2$ then p is smooth, so every fiber is smooth and reduced (multiplicity one). If $g = 1$ then every fiber F is of the form $F = nC$ for C a smooth elliptic curve. In any case, p is an isotrivial family, i.e. any two fibers are isomorphic.*

Remark 20.3. In the $g = 1$ case, the fibration p is almost smooth. Note that isotrivial families aren't necessarily cartesian products. Moreover, we claim that p is never the trivial family $S = B \times F$. If it were, $K_S = \pi_1^* K_B + \pi_2^* K_F$ is effective, contradicting $p_g = 0$.

Remark 20.4. To study this, we consider the topological Euler characteristic, defined as $\chi_{\mathsf{Top}}(X) = \sum_{i=0}^{\dim_{\mathbf{R}} X} (-1)^i \dim H_c^i(X)$ for X any algebraic variety over $\mathbf{C}$, not necessarily compact. Fact: if $X \supseteq Z$ a closed set with open complement $U := X \setminus Z$, one has $\chi_{\mathsf{Top}}(X) = \chi_{\mathsf{Top}}(Z) + \chi_{\mathsf{Top}}(U)$. As a corollary, for a fibration like p, one has

$$\chi_{\mathsf{Top}}(S) = \chi_{\mathsf{Top}}(B) \cdot \chi_{\mathsf{Top}}(F_{\text{gen}}) + \sum_{\text{finite}} \chi_{\mathsf{Top}}(F_s) - \chi_{\mathsf{Top}}(F_{\text{gen}})$$

where F_s is the special fiber. Note that there is a LES $H_c^i(U) \to H_c^i(X) \to H_c^i(Z)$ given by including cycles along $Z \hookrightarrow X$ and restricting cycles to the open subset $U \subseteq X$.

Note that $b_2 = 2q$ by Hodge conjugate symmetry, and so in our case

$$\chi_{\mathsf{Top}}(S) = 1 - b_1 + b_2 - b_3 + 1 = 2 - 2b_1 + b_3 = 4 - 4q = 0.$$

If C is a smooth curve of genus g,

$$\chi_{\mathsf{Top}}(C) = 1 - b_1 + 1 = 2 - 2g.$$

Note that when C is smooth, $\chi(\mathcal{O}_C) = h^0(\mathcal{O}_C) - h^1(\mathcal{O}_C) = 1 - g$, and so $\chi_{\mathsf{Top}}(C) = 2\chi(\mathcal{O}_C)$.

Remark 20.5. Claim: $\chi_{\mathsf{Top}}(F_s) > \chi_{\mathsf{Top}}(F_{\text{gen}})$, so the finite sum in $\chi_{\mathsf{Top}}(S)$ is nonzero. Equality only holds in the $g = 1$ case of the above theorem. This will prove the first part of the theorem. Let F be a fiber, then there is a SES $\mathcal{O}_S(-F) \hookrightarrow \mathcal{O}_S \twoheadrightarrow \mathcal{O}_F$ given by restriction to a fiber. Thus

$$\chi(\mathcal{O}_F) = \chi(\mathcal{O}_S) - \chi(\mathcal{O}_S(-F)) \quad = \chi(\mathcal{O}_S) - \left(\chi(\mathcal{O}_S) + \frac{F(F+K)}{2}\right) = \frac{-FK}{2}.$$

By the same argument, $\chi(\mathcal{O}_C) = \frac{-CK}{2} = \frac{-FK}{2n}$ since $C^2 = 0$. Consider the case $g(F_{\text{gen}}) \geq 2$, then $\chi(\mathcal{O}_C) - \chi(\mathcal{O}_{F_{\text{gen}}}) > 0$. Similarly when $g = 1$. It remains to show smoothness.

Lemma 20.6. *If C is singular, then*

$$\chi_{\mathsf{Top}}(C) > 2\chi(\mathcal{O}_C).$$

Example 20.7. Let C be a cuspidal rational curve, i.e. a cubic curve with a cusp. Let $\eta : \widetilde{C} \to C$ be its normalization, so $\widetilde{C} = \mathbf{P}^1$. Note that η is not an algebraic isomorphism, but is a homeomorphism. Thus $\chi_{\mathsf{Top}}(C) = 2$. There is an embedding $C \hookrightarrow \mathbf{P}^2$ as a plane cubic, and thus a SES $\mathcal{O}_{\mathbf{P}^2}(-3) \to \mathcal{O}_{\mathbf{P}^2} \to \mathcal{O}_C$ and thus $\chi(\mathcal{O}_C) = 0$.

Example 20.8. Let C be a rational cubic with a node, then again $\widetilde{C} = \mathbf{P}^1$. Note that $\chi_{\mathsf{Top}}(\text{pt}) = 1$, and by an argument deleting points in C and $\widetilde{C}$ one has $\chi_{\mathsf{Top}}(C) = 1$. Again one can argue $\chi(\mathcal{O}_C) = 0$.

Example 20.9. The general case is a combination of the two above cases, unibranch (where η is a homeomorphism) and two branches crossing. Consider a curve with a node with $b = 3$ independent tangent directions. Take the seminormalization and then normalize, which separates the node into 3 points. In the first step, the seminormalization is a homeomorphism, so χ_{Top} doesn't change. In the second step, it increases by $b - 1$. Let $\pi : C^{\setminus} \to C$ be the seminormalization, then there is a SES

$$\mathcal{O}_C \to \pi_* \mathcal{O}_{C^{\setminus}} \to Q$$

where Q is a skyscraper sheaf, which is nonzero if π is not an isomorphism. Thus $\chi(\mathcal{O}_C)$ increases in the first step. Similarly, for the normalization $\eta : C^{\setminus} \to \widetilde{C}$, there is again a SES

$$\mathcal{O}_{C^{\setminus}} \to \eta_* \mathcal{O}_{\widetilde{C}} \to Q'$$

where Q' is a skyscraper sheaf of length $b - 1$.

Remark 20.10. Fact: any proper non-isotrivial family of curves of genus $g \geq 1$ must degenerate, i.e. there must be a singular fiber. A similar statement holds for abelian varieties. For $g \geq 2$, consider $\mathcal{A}_g$ as a coarse space. Adding a level structure for $m \geq 3$ yields a fine space $\mathcal{A}_g(m)$, where a level structure is an isomorphism $A[m] \xrightarrow{\sim} (\mathbf{Z}/m\mathbf{Z})^{2g}$. This kills automorphisms. Note that for $g = 1$, this is easier since $M_1 = \mathbf{A}^1$ and any map from a projective to an affine variety is constant.

21. 2024-11-05-12-50-39

Remark 21.1. Recall: let S be a minimal non-ruled surface with $p_g = 0$ and $q \geq 1$, which implies $K^2 = 0$ and $b_2 = 2$. Then $\pi : S \to B$ is isotrivial, where B is a curve with $g(B) = q$. Either

- All fibers are smooth with multiplicity 1 and $g(B) \geq 2$, or
- $g(B) = 1$ and all fibers are either smooth or a multiple of a smooth curve.

Theorem 21.2. *Let G be a finite group acting freely on $B' \times F$. Then there is an etale cover $B' \times F \to \frac{B' \times F}{G} = S$ where $\frac{B' \times F}{G} \to B'/G = B$. Thus any such surface is a quotient of a cartesian product. Here $g(B'), g(F) \geq 1$.*

Remark 21.3. Thus there is an etale base change such that S is a Cartesian product and the group action is free.

Lemma 21.4. *If $g(B') = g(F) = 1$, then $\exists N$ such that $NK_S \sim 0$. Otherwise, if either $g(B) \geq 2$ or $g(F) \geq 2$, there exists a sequence of integers $n_1 \leq n_2 \leq \cdots$ such that $P_{n_i} \to \infty$.*

Remark 21.5. In the first case, $\kappa(S) = 0$, and in the second, $\kappa(S) \geq 1$. Note that if $G \curvearrowright B'$ by translations and B' is elliptic, then $B = B'/G$ is elliptic as well.

Proof. Let $S' \coloneqq B' \times F$ and $f : S' \to S$. Then by Riemann-Hurwitz, $K_{S'} = f^* K_S + \varepsilon$, and since f is étale, $\varepsilon = 0$. Moreover $f_* K_{S'} = f_* f^* K_S = dK_S$ where $d \coloneqq \deg f = \sharp G$. Thus $f_*(nK_S) = ndK_S$. Note that $K_{B' \times F} = \pi_1^* K_{B'} + \pi_2^* K_F$, so if either $g(B'), g(F) \geq 2$, then the plurigenera go to infinity. □

Example 21.6. Let $S' = E_1 \times E_2$ be a product of elliptic curves and let $G = \langle \iota \rangle$ be the group generated by an involution. Define $\iota(x, y) = (x + a, -y)$ where $a \in E_1[2]$ is 2-torsion, so the action on the first coordinate is free. Note $S' \to E_1' \coloneqq E_1/\iota_1$ with fibers E_2, and $B \coloneqq E_1'$ is again an elliptic curve. There is also a projection $S' \to E_2' \coloneqq E_2/\iota_2$ with fibers either E_1 or $2E_1'$. Analyzing $E_2 \to E_2'$, the ramification points are y where $2y = 0$, so this is ramified over 4 torsion points. By Riemann-Hurwitz, the genus decreases, so $E_2' \cong \mathbf{P}^1$. Locally this can be written $y^2 = x(x-1)(x-\lambda)$ and the involution is $(x, y) \mapsto (x, -y)$ which ramifies over $0, 1, \lambda, \infty \in \mathbf{P}^1$.

Claim: $p_g = 0, q = 1$. Write $p_g(S') = h^0(\Omega^2_{E_1 \times E_2})$. Note that $h^0(\Omega_{E_1}) = 1$, generated by dx which is translation invariant on $\mathbf{C}$. We have $H^0(\Omega^2_{E_1 \times E_2}) = \langle dx \wedge dy \rangle_{\mathbf{C}}$, and $H^0(\Omega^2_{E_1 \times E_2/G}) = H^0(\Omega^2_{E_1 \times E_2})^G$, the G-invariant differential forms on S'. One now checks that under $(x, y) \mapsto (x + a, -y)$, we have

- $dx \mapsto dx$
- $dy \mapsto -dy$
- $dx \wedge dy \mapsto -dx \wedge dy$.

Thus there are no invariant forms and $H^0(\Omega^2_{E_1 \times E_2})^G = 0$. This also shows $H^0(\Omega_{S'/G}) = H^0(\Omega_{S'})^G = \langle dx \rangle_{\mathbf{C}}$. Note also that $K_S \not\sim 0$ but $2K_S \sim 0$.

Remark 21.7. Other examples: let $G \curvearrowright E_1 \times E_2$ by a finite group action where $G \curvearrowright E_1$ by translation and $E_2/G \cong \mathbf{P}^1$. These are referred to as **bielliptic surfaces**. Consider $\mathrm{Aut}(E)$ for E an elliptic curve. This always contain $\mathrm{Aut}^0(E)$ generated by translations $x \mapsto x + a$ for $a \in E$. One can write $\mathrm{Aut}(E) = H \rtimes \mathrm{Aut}^0(E)$ where H is a finite group isomorphic to $\mathrm{Aut}(E, 0)$, the automorphisms that fix the origin. Generically $H = C_2$ generated by the elliptic involution. Let $E_\tau := \frac{\mathbf{C}}{\mathbf{Z} \oplus \mathbf{Z}\tau}$, then

$$H = \begin{cases} C_2 & y \mapsto -y \\ C_4 & y \mapsto iy (E_i) \\ C_6 & y \mapsto \zeta_3 y (E_{\zeta_3}) \end{cases}.$$

Remark 21.8. There is a finite list of possible actions $G \curvearrowright E_1 \times E_2$ for bielliptic surfaces:

1. C_2
2. $C_2 \times C_2$
3. C_4
4. $C_4 \times C_2$
5. C_3
6. $C_3 \times C_3$
7. C_6

We have

- $2K_S \sim 0$ in cases 1,2,
- $4K_S \sim 0$ in 3,4,
- $3K_S \sim 0$ in 5,6,
- $6K_S \sim 0$ in 7.

Note that as a corollary, we always have $12K_S \sim 0$ in this case.

Remark 21.9. Claim: there is a base change of $S \to B$ to $S' \to B'$ such that S' is a Cartesian product. This is case 1 from earlier. An idea of the proof: there is a constant map $B \to \mathcal{M}_g \to \mathcal{A}_g$ at the level of coarse spaces. Pass to fine spaces $\mathcal{M}_g(N) \to \mathcal{M}_g$, and $\mathcal{A}_g(N) \to \mathcal{A}_g$, which are quotients by finite groups. The latter is comprised of pairs $[C, \mathrm{Jac}(C)_N \xrightarrow{\sim} C_N^{2g}]$. One then considers trivializing the local system $R^1\pi_* C_N$.

Remark 21.10. The $\kappa(S) = 0$ case is particularly important case for us.

22. 2024-11-12-12-46-21

Remark 22.1. Goal: classify surfaces with $\kappa(S) = 0$. Let S be a minimal non-ruled surface, so $P_m = 0, 1$ for all m and $P_m = 1$ for at least one m. We know K_S is nef and thus $K_S^2 = 0$.

Theorem 22.2. *There are four possibilities for S:*

1. $p_g = 0, q = 0$ *and* $\chi(\mathcal{O}_S) = 1 - q + p_g = 1$ *($K \neq 0, 2K = 0$: Enriques),*
2. $p_g = 0, q = 1$ *and* $\chi(\mathcal{O}_S) = 0$ *(Bielliptic),*
3. $p_g = 1, q = 0$ *and* $\chi(\mathcal{O}_S) = 2$ *($K = 0$: K3 surfaces),*
4. $p_g = 1, q = 1$ *and* $\chi(\mathcal{O}_S) = 1$ *(not possible),*
5. $p_g = 1, q = 2$ *and* $\chi(\mathcal{O}_S) = 0$ *($K = 0$: abelian surfaces).*

Corollary 22.3. $4K_S = 0$ *or* $6K_S = 0$*, so* $12K_S = 0$ *in any case.*

Remark 22.4. Note that if X is a projective variety of dimension n and D is a nef divisor, then $D^n \geq 0$. If A is an ample divisor, then Kleiman's criterion $D + \varepsilon A$ is ample $\forall \varepsilon > 0$. Ampleness implies $(D + \varepsilon A)^n > 0$ since this is the degree of X. Then take $\varepsilon \to 0$. Note that torsion in $\mathrm{Pic}(X)$ is numerically zero.

Kleiman's criterion: consider $N_1(X) = \{\sum n_i C_i\} / \sim$, sums of curves modulo numerical equivalence. Similarly $N^1(X) = \{\sum n_j D_j\} / \sim$, sums of divisors modulo numerical equivalence. There is an intersection product $N_1(X) \times N^1(X) \to \mathbf{Z}$, and numerical equivalence means they intersect the same way. This is a perfect pairing, and $\mathrm{rank}_{\mathbf{R}} N_1(X) \otimes \mathbf{R} = \mathrm{rank}\, \mathrm{NS}(X)$, so $N^1(X) = \mathrm{NS}(X)/\mathrm{tors}$. There is a cone $\mathrm{NE}_1(X) \subseteq N_1(X) \otimes \mathbf{R}$, the Mori-Kleiman cone (or Mori cone). Any divisor E gives a linear function f_E on $\mathrm{NE}_1(X)$. The Kleiman criterion states that a divisor E is ample if $f_E > 0$ on $\overline{\mathrm{NE}(X)} \setminus \{0\}$. Thus nef divisors are limits of ample divisors.

Mori proved that if one partitions $\mathrm{NE}_1(X)$ by the hyperplane K_X, the negative part is polyhedral, generated by extremal rays. For Fano varieties, the entire cone is polyhedral, while for abelian surfaces it is the round cone. More cones of K3 surfaces are partly polyhedral, partly round.

Lemma 22.5. $\chi(\mathcal{O}_S) \geq 0$.

Remark 22.6. Note $\chi(\mathcal{O}_S) = 1 - q + p_g$, and by Noether's formula, $\chi(\mathcal{O}_S) = \frac{c_1^2 + c_2}{12} = \frac{K^2 + \chi(S)}{12}$. If $K^2 = 0$, then $12\chi(\mathcal{O}_S) = \chi(S) = \sum b_i = 1 - 2q + b_2 - 2q + 1 = 2 - 4q + b_2$. Note $b_2 = h^{0,2} + h^{1,1} + h^{2,0} = 2p_g + h^{1,1}$, thus $b_2 \geq 2p_g$ and

$$12\chi(\mathcal{O}_S) = 2 - 4q + b_2 \geq 2 - 4q + 2p_g.$$

There is a problem if q is large. Note that $-4\chi(\mathcal{O}_S) = -4 + 4q - 4p_g$, and adding this to the above equation yields

$$8\chi(\mathcal{O}_S) = -2 - 4p_g + b_2 \geq -2 - 2p_g.$$

Note that $-4p_g = 0, -4$, but if $-4p_g = -4$ then $\chi(\mathcal{O}_S) < 0$ would contradict this inequality.

Remark 22.7. Thus the possibilities for $\chi(\mathcal{O}_S) = 1 - q + p_g$ are

- $p_g = 0 \implies \chi(\mathcal{O}_S) = 1 - q \geq 0$,
- $p_g = 1 \implies \chi(\mathcal{O}_S) = 2 - q \geq 0$.

These yield the four possibilities in the theorem.

Lemma 22.8. *Suppose* $\kappa(S) = 1$*. Let* $m, n > 0$ *and* $d = \gcd(m, n)$*. If* $P_m = P_n = 1$ *then* $P_d = 1$*.*

Remark 22.9. Consider $\mathcal{S} \coloneqq \left\{ m \;\middle|\; |mK_S| \neq \emptyset \right\} \subseteq \mathbf{Z}$; this forms a semigroup. We will show that $\mathcal{S}$ is in fact a group.

Proof. Case 1: $mK \sim 0$, which happens iff $nK \sim 0$. This happens if $K.H = 0$ for H a hyperplane. Then for all $a, b \in \mathbf{Z}$, we have $(am + bn)K = 0$, so $dK = 0$.

Case 2: $mK = C_m \neq 0$ is a nonzero effective curve iff $nK = C_n \neq 0$ is as well. This happens if $K.H > 0$. Write $m = m'd, n = n'd$ where $\gcd(m', n') = 1$. Write $mK = \sum n_i C_i$ and $nK = \sum b_j D_j$. Note that $dm'n'K \sim \sum n_i n' C_i \sim \sum b_j m' D_j$, but in fact these are equal divisors (not just linearly equivalent) since $P_m = 0, 1$ forces uniqueness of effective divisors in $|mK|$ when they exist. Note that $\sum \frac{a_i}{n'} C_i$ and $\sum \frac{b_j}{n'} D_j$ are integral divisors, and we claim these divisors are in $|dK|$.

□

Remark 22.10. We now discuss the proof of the theorem. Case 1: $p_g = 0, q = 0, \chi(\mathcal{O}_S) = 1$. By Riemann-Roch, $\chi(D) = \chi(\mathcal{O}_S) + \frac{D(D-K)}{2} = h^0(D) - h^1(D) + h^0(K - D)$, and so $h^0(D) + h^0(K - D) \geq \chi(D)$. Applying this to $D = mK$ yields $h^0(mK) + h^0((1 - m)K) \geq \chi(\mathcal{O}_S)$. We'll show a positive multiple and a negative multiple of K is effective, which can only happen if the multiple if zero. If $\chi(\mathcal{O}_S) = 1$, we have $h^0(3K) + h^0(-2K) \geq 1$. If $h^0(-2K) = 1$, then $-2K \sim 0$. Consider the case where $h^0(3K) \geq 1$. This implies $p_3 = 1$. In this case we have $h^0(2K) \geq 1$ since $q = 0$ and S is not rational. This implies $p_2 = 1$, which implies $p_1 = p_g = 1$, contradicting $p_g = 0$.

Remark 22.11. Case 2: we have already considered these in a previous chapter.

Remark 22.12. Case 3: $p_g = 1, q = 0, \chi(\mathcal{O}_S) = 2$, and we want to conclude $K \sim 0$. By Riemann-Roch, $h^0(2K) + h^0(-K) \geq 2$. We know $h^0(2K) = 1$, thus $h^0(-K) \geq 1$ and $-K$ is effective. Since K is effective, we have $K \sim 0$.

Remark 22.13. Case 4: $p_g = 1, q = 1, \chi(\mathcal{O}_S) = 1$, we want to show this is impossible. Recall the SES $\mathrm{Pic}^0(S) \hookrightarrow \mathrm{Pic}(S) \twoheadrightarrow \mathrm{NS}(S)$, and note $\mathrm{Pic}^0(S) \cong \mathbf{C}^q/\mathbf{Z}^{2q} = \mathbf{C}/\mathbf{Z}^2$ in this case. This is a complex torus, which has a 2-torsion point $\varepsilon \in \mathrm{Pic}^0(S)$. By Riemann-Roch, $h^0(\varepsilon) + h^0(K - \varepsilon) \geq 1$. Since ε can not be effective since $2\varepsilon = 0$ (intersect with a hyperplane), we have $h^0(\varepsilon) = 0$. So $h^0(K - \varepsilon) \neq 0$. We thus have $\exists D \in |K|$ and $\exists E \in |K - \varepsilon|$. Note that $2D = 2E \in |2K|$ since there is a unique divisor in this linear system. Either this is zero, or effective and nonzero. We want to conclude $D = E$, and thus $\varepsilon = D - E = 0$, a contradiction.

Called case 3.5 in class.

Remark 22.14. Case 5: the hardest case. Use the Albanese map $\alpha : S \to \mathrm{Alb}(S)$, an abelian variety of dimension $q = 2$. Claim: α is surjective and étale. Fact: if $X \to Y$ is an etale map between projective varieties and Y is an abelian variety, then X is an abelian variety. I.e. any etale cover of an abelian variety is again an abelian variety. We have to prove the image of α is not just a curve, and that α is unramified.

Called case 4 in class.

23. 2024-11-19-12-46-23: K3 Surfaces

Remark 23.1. Let S be a minimal surface with $p_g = 1, q = 0, \chi(\mathcal{O}_S) = 2$ with $\kappa = 0$. We proved $K_S \sim 0$. By the Noether formula, $\chi(\mathcal{O}_S) = \frac{K^2 + \chi(S)}{12}$, so $\chi(S) = 24$. We have the following Hodge diamond:

$$h^{2,2} = 1$$

$$h^{1,2} = 0 \qquad h^{2,1} = 0$$

$$h^{0,2} = h^0(\omega_S) = 1 \qquad h^{1,1} = 20 \qquad h^{2,0} = 1$$

$$h^{0,1} = q = 0 \qquad h^{1,0} = 0$$

$$h^{0,0} = 1$$

> Link to Diagram

Example 23.2. Let $S = X_4 \subseteq \mathbf{P}^3$ be a quartic, then $K_S = K_{\mathbf{P}^3} + S\Big|_S = -4H + 4H = 0$. Take the SES $\mathcal{O}_{\mathbf{P}^3}(-4) \to \mathcal{O}_{\mathbf{P}^3} \to \mathcal{O}_S$ given by restriction of functions. Note that $H^i(\mathcal{O}_{\mathbf{P}^n}(d)) = 0$ unless $d = 0, n$, so by the LES, $H^1(\mathcal{O}_S) = 0$.

Example 23.3. Let $X_{3,2} \subseteq \mathbf{P}^4$ be a complete intersection, then $K_X = K_{\mathbf{P}^4} + D_3 + D_2\Big|_X = -5H + 3H + 2H = 0$. Computing $H^1(\mathcal{O}_X)$ proceeds similarly. A similar example is $X_{2,2,2} \subseteq \mathbf{P}^4$.

Remark 23.4. Each of these defines a K3 with an ample line bundle $L \coloneqq \mathcal{O}_X(-1)$, the restriction of $\mathcal{O}_{\mathbf{P}^n}(-1)$ to X. In these cases:

- $L^2 = H^2 X = 4$
- $L^2 = 2 \cdot 3 = 6$
- $L^2 = 2 \cdot 2 \cdot 2 = 8$.

All of these yield a pair (X, L), and we define the degree of the pair as $d \coloneqq L^2$. These are the only complete intersections which are K3s.

Lemma 23.5. *On a K3 surface, for any divisor D, one has D^2 even.*

Remark 23.6. By Riemann-Roch, $\chi(D) = \frac{D^2 - DK}{2}$, and $DK = 0$. This in fact holds for all minimal surfaces with $\kappa = 0$, since some multiple of K is zero.

Remark 23.7. Suppose $d = 2$. Then L can not be ample. There is a double cover $f : X \to \mathbf{P}^2$ branched over $B = \{f_6 = 0\}$, a smooth curve given by an equation of degree 6. By Riemann-Hurwitz, $K_X \sim_{\mathbf{Q}} f^*(K_{\mathbf{P}^2} + \frac{1}{2}B) \sim 0$, which forces B to be a sextic. Clearing denominators yields $2K_X \sim f^*(2K_{\mathbf{P}^2} + B)$. We generally have $f_* \mathcal{O}_X = \mathcal{O}_{\mathbf{P}^2} \oplus F^{-1}$ where $F^2 = \mathcal{O}(B)$. This is the decomposition into the ± 1 eigenspaces of $f_* \mathcal{O}_X$ respectively under an involution. There is a grading, this is given by a multiplication morphism $F^{-1} \otimes F^{-1} \to \mathcal{O}_{\mathbf{P}^2}$, or equivalently a map $\mathcal{O} \to F^2$ where $1 \mapsto s$. The cover has local coordinates $z^2 = s(x, y)$, and $(s) = B$ cuts out the ramification locus, so $s \in \mathcal{O}_{\mathbf{P}^2}(B)$ and $F^2 = \mathcal{O}_{\mathbf{P}^2}(B)$ is necessary. We have $f_* \omega_X = \omega_{\mathbf{P}^2} \oplus \omega_{\mathbf{P}^2}(F)$. Moreover $H^1(\mathcal{O}_X) = H^1(\mathcal{O}_{\mathbf{P}^2}) \oplus H^1(\mathcal{O}_{\mathbf{P}^2}(-3)) = 0$, so $q = 0$. Similarly, $H^0(\omega_X) = H^0(\mathcal{O}_{\mathbf{P}^2}(-3)) \oplus H^0(\mathcal{O}_{\mathbf{P}^2}(-3 + 3)) \cong 0 \oplus \mathbf{C}$, so ω_X has a section and this forces $K_X \sim 0$.

Example 23.8. Let $f : X \to \mathbf{P}^1 \times \mathbf{P}^1$ with $B \subseteq |\mathcal{O}_{\mathbf{P}^1\times\mathbf{P}^1}(2,2)|$, noting that $K_{\mathbf{P}^1\times\mathbf{P}^1} = \mathcal{O}(2,2)$. A similar calculation shows that this is a K3 surface. One sets $L = f^*\mathcal{O}(1,1)$, where $\mathcal{O}(1,1)$ is very ample and embeds $\mathbf{P}^1 \times \mathbf{P}^1 \hookrightarrow \mathbf{P}^3$ as a conic. Then $L^2 = 2 \cdot 2 = 4$, yielding a degree 4 K3. Note that $X \to \mathbf{P}^3$ is not an embedding in this case; it covers a degree 2 surface.

Remark 23.9. One could generalize this to consider double covers of other del Pezzo surfaces branched over $B \in |-2K|$.

Example 23.10. Elliptic K3s: by analogy, these are the $d = 0$ case. These come with a line bundle which is not ample. They are elliptic surfaces: fibrations $f : S \to C$ with generic fiber F_g of genus 1. In the K3 case, $C = \mathbf{P}^1$ since there are no differential forms. We have $\chi(S) = \chi(C)\chi(F_g) + \sum_F(\chi(F) - \chi(F_g))$, and since $\chi(F) = 2 - 2g = 0$, all contributions come from the excess term. The simplest singular fibers are nodal curves, which normalize to $\mathbf{CP}^1 \cong S^2$. So there are generically 24 singular fibers, and there may or may not be a section.

Remark 23.11. Minimal surfaces with $\kappa = 1$ are elliptic, but not every elliptic surface satisfies $\kappa = 1$. If the fibration admits a section, the fibers are genus 1 curves with a point and the fibration is called Jacobian. The fibers have Weierstrass form $y^2 = x^3 + Ax + B$ where $A(t_0, t_1), B(t_0, t_1)$ are homogeneous in the coordinates t_i on $\mathbf{P}^1$. One checks $\Delta_{24} = 4A_8^3 + 27B_{12}^2$.

Remark 23.12. Note $L = f^*\mathcal{O}_{\mathbf{P}^1}(1)$ yields $L^2 = 0$. One can choose other polarizations, e.g. $L = s + df$. Then $L^2 = s^2 + 2dsf + f^2 = -2 + 2d + 0 = 2d - 2$, where $s^2 = -2$ follows from the genus formula $g(s) = \frac{s^2+sK}{2} + 1$ with $sK = 0$. For generic choices, L will be ample, and $L^2 = 2d - 2$ achieves every even integer.

Remark 23.13. Let's count moduli for the examples given above.

- X_4: there are $\binom{7}{4}$ monomials, minus $\dim \mathrm{PGL}_4 =?$, minus 1 for projectivizing equations. This yields $35 - 16 = 19$.
- $X_{2,3}$: a more delicate count yields 19.
- $X_{2,2,2}$: one gets 19 by computing the dimension of a Grassmannian.
- $f : X \to \mathbf{P}^2$: the number of sextics in $\mathbf{P}^2$, minus $\dim \mathrm{PGL}_2$, yields $\binom{8}{2} - 9 = 19$.
- $f : X \to \mathbf{P}^1 \times \mathbf{P}^1$: counting yields 18.
- Elliptic: count possibilities for A_8 and B_{12}, modulo PGL_2, yields 18.

Why the difference: $\operatorname{rank} \operatorname{Pic}(X_4) = 1$ generically, while $\operatorname{rank} \operatorname{Pic}(X) = 2$ for e.g. $X \to \mathbf{P}^1 \times \mathbf{P}^1$. So larger Picard rank yields fewer moduli; in general one has $20 - \operatorname{rank} \operatorname{Pic}(X)$ moduli.

Remark 23.14. In every even degree $d \geq 4$, there exist K3s with a very ample line bundle. This comes from studying things like X_4 which also admit an elliptic fibration, and setting $\widetilde{L} := L + kF$ for F a fiber class. Then $\widetilde{L}^2 = L^2 + 2kLF$, and since L is very ample and F is basepoint free, $\widetilde{L}$ is again very ample.

Remark 23.15. Suppose X is a K3 and $C \subseteq X$ is a smooth curve with $g(C) \geq 2$. Note $C^2 = 2g - 2$ by the genus formula, since $K = 0$. Then $|C|$ is basepoint free, and $\phi_{|L|} : X \to \operatorname{im}(X) \subseteq \mathbf{P}^3$ is degree either 1 or 2. In this case we say X is a hyperelliptic K3.

Proof. Consider the SES $\mathcal{O}_X \to \mathcal{O}_X(C) \to \mathcal{O}_C(C)$. By adjunction $K_C = K_X + C\Big|_C = \mathcal{O}_C(C)$, so $\mathcal{O}_C(C) = \mathcal{O}_C(K_C)$. The result follows from the theory of curves. □

Remark 23.16. Next time: Hodge theory, deformations, Enriques surfaces. Last class: elliptic surfaces.

Remark 23.17. Note $\rho \leq 20$ in characteristic zero, but possibly $\rho = 22$ in positive characteristic.

24. 2024-11-21-12-48-29

Remark 24.1. K3 surfaces: X with $p_g = 1, q = 0, K_X \sim 0, \chi(X) = 24, \chi(\mathcal{O}_X) = 2$. Polarized: pairs (X, L) with L an ample line bundle satisfying $L^2 = 2d$. Note that any two K3 surfaces over $\mathbf{C}$ are homeomorphic, and $\pi_1(X) = 1$ and thus $H^1(X;\mathbf{Z}) = 0$. By UCT, $H^2(X;\mathbf{Z}) = \mathrm{Hom}(H_1(X;\mathbf{Z}), \mathbf{Z}) \oplus T$, where T is the torsion in H_1, is torsionfree is isomorphic to $\mathbf{Z}^{22}$. There is an integral bilinear form $H^2(X;\mathbf{Z}) \times H^2(X;\mathbf{Z}) \to H^4(X;\mathbf{Z}) \cong \mathbf{Z}$. Note that (x,x) is even for all x, symmetric, and unimodular. The form has signature $(3, 19)$. Such indefinite unimodular even forms are unique, so this lattice is $II_{3,19} \cong U^3 \oplus E_8^2$. Note that $U = II_{1,1}$ and $E_8(-1) = II_{0,8}$.

Note that $\mathrm{Pic}(X) = H^1(\mathcal{O}_X^\times)$. Take the LES for the exponential SES $\mathbf{Z} \to \mathcal{O}_X \to \mathcal{O}_X^\times$ to get

$$0 \to \mathrm{Pic}^0(X) \to \mathrm{Pic}(X) \to \mathrm{NS}(X) = \ker\left(H^2(X;\mathbf{Z}) \xrightarrow{f} H^2(\mathcal{O}_X)\right).$$

We have $\mathrm{Pic}^0(X) = H^1(\mathcal{O}_X)/H^1(X;\mathbf{Z}) = \mathbf{C}^q/\mathbf{Z}^{2q}$, and $\mathrm{Pic}(X) \cong \mathrm{NS}(X) \subseteq H^2(X;\mathbf{Z}) = II_{3,19}$. By the Hodge index theorem, $\mathrm{NS}(X)$ has signature $(1, r-1)$ for some r. The inclusion $\mathrm{NS}(X) \subseteq H^2(X;\mathbf{Z})$ is a primitive embedding of lattices, i.e. $nv \in \mathrm{NS}(X) \implies v \in \mathrm{NS}(X)$, since $\mathrm{NS}(X)$ is isomorphic to a kernel. Nikulin classified which hyperbolic lattices can be primitively embedded into $II_{3,19}$, and moreover shows they are all attained by some K3 surface.

Remark 24.2. Understanding $\ker f$ from above: it is the composition

$$H^2(X;\mathbf{Z}) \to H^2(X;\mathbf{C}) = H^{2,0} \oplus H^{1,1} \oplus H^{0,2} \twoheadrightarrow H^{2,0} = H^2(\mathcal{O}_X).$$

Write $H^{0,2} = H^0(\Omega^2) = H^0(K_X) = \mathbf{C}\omega_X$.

Lemma 24.3.

$$\mathrm{NS}(X) = H^2(X;\mathbf{Z}) \cap H^{1,1} = \left\{\alpha \in H^2(X;\mathbf{Z}) \;\middle|\; \alpha \cdot \omega_X = 0\right\}.$$

Proof. Write $\alpha = \alpha^{2,0} + \alpha^{1,1} + \alpha^{0,2}$ where $\overline{\alpha}^{2,0} = \alpha^{0,2}$ and $\overline{\alpha}^{1,1} = \alpha^{1,1}$. If $\alpha \in \ker f$ then $\alpha^{2,0} = 0$ and so $\alpha = \alpha^{1,1}$. The second identification comes from the fact that $\alpha \cdot \omega_X \in H^{2,2} + H^{1,3} + H^{0,4} = H^{2,2}$ and the pairing $H^{2,0} \times H^{0,2} \to H^{2,2}$ is nondegenerate. □

Remark 24.4. Thus varying ω_X amounts to varying $\mathrm{NS}(X)$, and one has $0 \leq r \leq 20$ for $r := \mathrm{rank}\,\mathrm{NS}(X)$. We define a period domain

$$D := \left\{[\omega] \;\middle|\; \omega \in II_{3,19} \otimes \mathbf{C}, \omega \cdot \omega = 0, \omega \cdot \overline{\omega} > 0\right\} \subseteq \mathbf{CP}^{21}.$$

The Torelli theorem states that for any $\omega \in D$ there exists a complex analytic K3 surface X with $\omega_X = \omega$, together with a marking $H^2(X;\mathbf{Z}) \xrightarrow{\sim} II_{3,19}$. Further, two marked K3s are isomorphic iff $\omega_X = \omega_{X'}$.

25. 2024-11-26-12-46-18: Elliptic Surfaces

25A. Examples of Elliptic Surfaces.

Definition 25.1. An **elliptic surface** X is a (smooth) fibration $\pi : X \to C$ over a curve C whose generic fiber is a genus 1 curve, not necessarily with an origin fixed. We assume there are no (-1)-curves in a fiber, so π is relatively minimal.

Remark 25.2. Multiple fibers are allowed, as well as non-reduced and non-irreducible curves.

Example 25.3. Begin with $\mathbf{P}^2$ and take two general polynomials $f_3(x_0, x_1, x_2), g_3(x_0, x_1, x_2)$ and consider the pencil $sf_3 + tg_3 = 0$ where $[s : t] \in \mathbf{P}^1$. Generically, $f_3 \cap g_3$ is 9 points, so blow these up. These yields a pencil without basepoints with a map $\mathrm{Bl}_9 \mathbf{P}^2 \to \mathbf{P}^1$ with coordinates $[s : t]$. The generic fiber is an elliptic curve, and this surface has 9 sections $E_1, \cdots, E_9$.

More generally, it's enough to assume this pencil contains an irreducible curve, and the points of blowup are allowed to be infinitely near (e.g. tangency of the curves). In this case, at least E_9 is a section, so this always yields an elliptic surface with a section. Thus the points need not be in general position. The most degenerate case is when all other 8 curves are contained in a fiber. In this case, $\kappa = -\infty$, so X is a RES (rational elliptic surface). Note that there are many (-1)-curves, but they are all sections and not contained in fibers.

Remark 25.4. Such surfaces (X, E) with a section biject with del Pezzo surfaces of degree 1. Start with such a $X = \mathrm{dP}_1$ and consider $|-K_X|$. Since $h^0(-K_X) = 2$, this linear system is a $\mathbf{P}^1$ and any two elements intersect at one point. This yields a pencil with a basepoint. If $C \in |-K_X|$, then $K_C = K_X + C\Big|_C = 0$, so this is an elliptic curve. Conversely, taking an RES and blowing down E yields a dP_1.

Remark 25.5. Such a family of curves can be regarded as a single curve over the function field of $\mathbf{P}^1$. There is still a section, so Mordell's theorem guarantees that the sections form a finitely generated group. This produces a surface with infinitely many (-1)-curves.

Example 25.6. Some K3 surfaces are elliptic; there is an 18-dimensional family of them. Produce these using Weierstrass equations. Here $\kappa = 0$.

Example 25.7. All Enriques surfaces are elliptic (without a section). There are always two double fibers in a fibration over $\mathbf{P}^1$. Note that if you have a section, you can not have a multiple fiber, by considering intersection multiplicities. Again, here $\kappa = 0$.

Remark 25.8. All surfaces S with $\kappa(S) = 1$ are elliptic. The fibration is induced by mK_X for $m \gg 1$. The rational map it induces maps to a curve since $\kappa = 1$. The nontrivial fact is that this map is basepoint free and is thus regular. Adjunction shows that the generic fiber is an elliptic curve.

Lemma 25.9. *Surfaces with $\kappa(S) = 2$ are not elliptic.*

Remark 25.10. $K_X.C \geq 0$ for all curves C. However, the restriction of K_X to a fiber must be zero, so $K_X.F.F = 0$ for every fiber F. So every multiple mK_X must contract every fiber, along with all curves it intersects by zero. Thus the image must be a curve, but $\kappa = 2$ forces the image to be a surface.

Remark 25.11. A generic K3 has $\rho = 1$, but any elliptic fibration has a fiber and a multisection, so $\rho \geq 2$. So the generic K3 is not elliptic.

Remark 25.12. Halphen pencils: start with a cubic f_3 and a sextic g_6 and take the pencil $sf_3 + tg_6$. The intersection $f_3 \cap g_6$ is 9 points, and we can arrange for the multiplicity is 2 at each point. Taking $\mathrm{Bl}_9 \mathbf{P}^2$, this has a fibration to $\mathbf{P}^1$. This is a RES without a section with a multiple fiber.

There is a variation $sf_{3d} + tg_{3d}$, and blowing up again yields a RES with multiple fiber. We recover the previous case by taking $f_{3d} = f_3^d$ for $d = 2$.

Remark 25.13. There is a construction with takes elliptic surfaces $\pi : X \to C$ to Jacobian elliptic surfaces : $\pi : JX \to C$, i.e. elliptic surfaces with a section. On smooth fibers, this sends X_t to $(JX_t, 0)$, since the Jacobian has a distinguished origin. This construction removes multiple fibers, and on singular fibers one takes a compactified Jacobian. Multiple curves nE are mapped to E, any other curves X_t are sent to $JX_t \cong X_t$ (which has a section). It follows that $\chi(X_t) = \chi(JX_t)$.

Remark 25.14. What happens to an Enriques surface under this construction? Considering the first example: since $\chi(\mathbf{P}^2) = 3$ and $\chi(\mathbf{P}^1) = 1$, we have $\chi(\mathrm{Bl}_9 \mathbf{P}^2) = 12$ since points are replaced with $\mathbf{P}^1$s, increasing χ by one each time. Note that $\chi(\mathcal{O}_X) = 1$ for this RES, for K3s $\chi(\mathcal{O}_X) = 2, \chi = 24$, and for Enriques surfaces $\chi(\mathcal{O}_X) = 1, \chi = 12$. Here we've used that $K^2 = 0$ and thus $\chi = 12\chi(\mathcal{O}_X)$ by Noether's formula.

Lemma 25.15. *Let X be relatively minimal. Then $K_X \sim cF$ where $c \geq 1$ and F is a fiber.*

Proof. Write $K_X = \sum n_i C_i$ with C_i in fibers. Pick a fiber F, then for some index set J we have $\sum m_j C_j = F$. Compute $p_a(F) = 1$ since this is constant in a flat family. On the other hand, we have $p_a(F) = \frac{(K_X + F)F}{2} + 1$, and since $F^2 = 0$ we have $K_X.F = 0$. Then either

1. $K_X.C_j = 0$ for all j, or
2. $\exists j$ such that $K_x C_j < 0$.

This follows from the fact that $K_X.F = \sum m_j K_X.C_j = 0$. If $C_j^2 < 0$, then note that $p_a(C_j) = \frac{K_X.C_j + C_j^2}{2} + 1$. Then either

- $F = m_j C_j$ is irreducible, or
- $p_a(C_j) = 0, C_j \cong \mathbf{P}^1, C_j^2 = -2$, and $K_X.C_j = 0$.

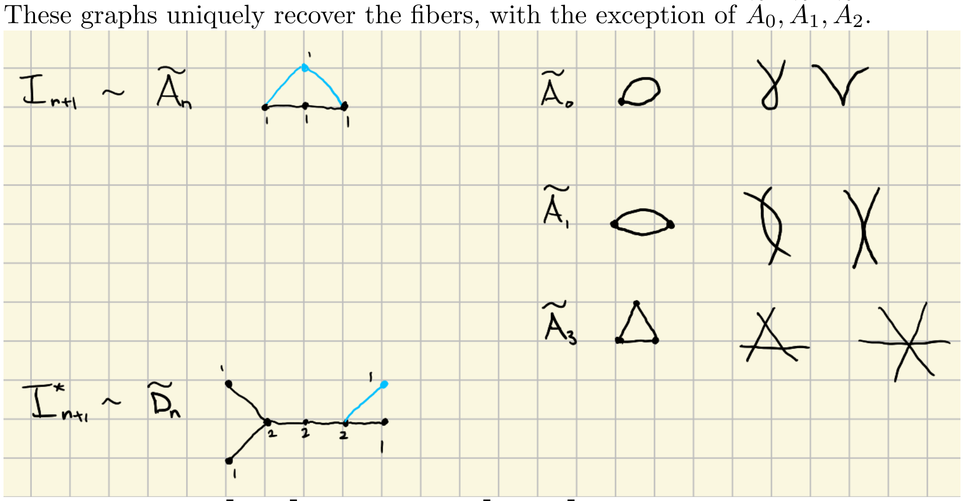
This follows from the fact that the numerator is either $(-1) + (-1)$, which is ruled out by relative minimality, or $(0) + (-2)$. The latter is the only possibility. In any case, $K_X.C_j = 0$, and $\sum n_i C_i = \sum d_i F_i$ for some collection of fibers F_i. □

25B. **Singular fibers.**

Remark 25.16. Since $g(E) = 1$, the singular fibers are either of the for mE, or a cuspidal or nodal curve. If the singular fiber is a collection of (-1)-curves, take the dual graph. This has an associated matrix $M_{ij} = C_i.C_j$ which is negative semidefinite. The possible graphs that can arise are $\widetilde{A}_n, \widetilde{D}_n, \widetilde{E}_m$ for $m = 6, 7, 8$. Taking the determinants of these matrices yields

- $\det A_n = n + 1$,
- $\det D_4 = 4$,
- $\det E_6 = 3$,
- $\det E_7 = 2$,
- $\det E_8 = 1$.

These graphs uniquely recover the fibers, with the exception of $\widetilde{A}_0, \widetilde{A}_1, \widetilde{A}_2$.



One has $M(I_n) = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$ and $M(I_n^*) = \begin{bmatrix} 1 & -n \\ 0 & 1 \end{bmatrix}$ for monodromy around these fibers.

Remark 25.17. The canonical class formula:

$$K_X \sim_{\mathbf{Q}} \pi^* K_C + \sum \frac{m_i - 1}{m_i} F_i + M.$$

where the sum is over multiple fibers and called the *divisor part*, and M is called the *moduli part*. One sets $M = j^* \mathcal{O}_{\mathbf{P}^1_j}(1) + \sum c_i F_i$ where the c_i are prescribed based on the Dynkin type. Here $j : C \to \mathbf{P}^1$ is the map sending a fiber X_t for $t \in C$ to its j-invariant. This puts significant restrictions on the fiber types.

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