DISSERTATION

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Abstract. Rough draft of my dissertation

Contents

Date: January 21, 2025.

1. Introduction

Enriques surfaces Y are minimal algebraic surfaces of Kodaira dimension zero satisfying $h^1(\mathcal{O}_Y) = h^2(\mathcal{O}_Y) = 0$ and $K_Y \neq 0$ but $2K_Y = 0$. A fundamental property of an Enriques surfaces Y is that its universal cover X is isomorphic to a K3 surface[1]. For both K3 and Enriques surfaces, the theory of compactifications is very rich: once a polarization L is fixed, there is a Hodge-theoretic period domain parametrizing isomorphism classes of polarized K3 or Enriques surfaces. These are bounded Hermitian symmetric domains, and thus for appropriate choices of arithmetic subgroups, the resulting arithmetic quotients admit Baily–Borel [BB66], toroidal [Ash+75], and Looijenga semitoroidal compactifications [Loo03].

It is then natural to ask about geometric compactifications such as stable pair compactifications and how they relate to these Hodge theoretic compactifications. In a series of recent papers, Alexeev–Engel–Thompson have made breakthroughs for K3 surfaces (see [ABE22; AE21; AE22; AEH22; AET23]). In particular, there are explicit and effective answers to this question for K3 surfaces equipped with a non-symplectic involution whose fixed locus is a curve.

Stable pair compactifications of the moduli space of Enriques surfaces are less well-studied. In [Sch22] the second author studied the stable pair compactification of the moduli space of Enriques surfaces with a degree 6 polarization (Enriques'

[1] A K3 surface is a smooth projective surface X with $K_X = 0$ and $h^1(\mathcal{O}_X) = 0$

original construction) and give a full description for a 4-dimensional subfamily of the moduli space. One of the obstructions to extending similar results to the entire 10-dimensional family is the high degree d of the polarization needed. It is thus natural to consider instead the lowest value possible, which is $d = 2$. In this situation, X is naturally equipped with a non-symplectic involution, the **Enriques involution**, whose fixed locus is a curve, and thus the theory developed in [AE22] is applicable.

However, the theory in [AE22] does not immediately apply in this situation – one must account for the fact that there are certain natural geometric automorphisms. Horikawa gives a construction of a K3 surface X as a degree 2 cover $\rho : X \to \mathbf{P}^1 \times \mathbf{P}^1$ branched over a divisor $D \in -2K_{\mathbf{P}^1 \times \mathbf{P}^1}$ of bidegree $(4, 4)$. It can be shown that ρ is symplectic and that ρ commutes with the Enriques involution, and thus the theory in [AE22] must be modified to keep track of this extra symmetry.

2. Preliminaries

We follow closely the exposition in [AE22, § 2].

2A. Lattices.

Remark 2.1. By a *lattice* we mean a finitely generated free abelian group L of finite rank equipped with a nondegenerate symmetric bilinear form $b\colon L \times L \to \mathbf{Z}$. In particular, two lattices are *isometric* there exists an isomorphism of the underlying abelian groups which preserves the bilinear forms. Given a set of generators $e_1, \ldots, e_r$ of L, we can associate a *Gram* matrix given by $(b(e_i, e_j))_{i,j}$. The lattice L is called *unimodular* provided the determinant of a Gram matrix is ± 1. The lattice L is called *even* provided $b(v, v) \in 2\mathbf{Z}$ for all $v \in L$. Given a lattice L we denote by L^* its dual $\mathrm{Hom}_{\mathbf{Z}}(L, \mathbf{Z})$. As the bilinear form is nondegenerate, we have an inclusion $L \hookrightarrow L^*$ and the quotient $A_L = L^*/L$ is a finite abelian group called the *discriminant group* of L. The discriminant group A_L comes equipped with a quadratic form $q_L\colon A_L \to \mathbf{Q}/\mathbf{Z}$ by sending $v + L \mapsto b(v, v) \bmod \mathbf{Z}$. If $S \subseteq L$ is a sublattice and T is its orthogonal complement, we have that $A_L \cong A_T$ and $q_S = -q_T$ under this correspondence. The lattices for which $A_L \cong \mathbf{Z}_2^a$ for some positive integer a are called **2-elementary**.

2B. K3 surfaces, and nonsymplectic involutions.

Remark 2.2. A lot of the geometry and moduli theory of K3 surfaces is regulated by lattice theory. For a K3 surface X it is well-known that $H^2(X, \mathbf{Z})$, endowed with the cup product, is an even, unimodular lattice of signature $(3, 19)$. It follows that $H^2(X, \mathbf{Z})$ is isometric to the so-called *K3 lattice* $\mathrm{II}_{3,19} := U^{\oplus 3} \oplus E_8^{\oplus 2}$, where U is the hyperbolic plane $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ and E_8 is the negative definite root lattice associated to the corresponding Dynkin diagram. In particular, symmetries of the surface X translate into symmetries of the K3 lattice $\mathrm{II}_{3,19}$.

Remark 2.3. A particularly rich setting is provided by *nonsymplectic involutions*, i.e. order 2 automorphisms $\iota\colon X \to X$ such that the induced map $\iota^*\colon H^{2,0}(X) \to H^{2,0}(X)$ satisfies $\iota^*\omega_X = -\omega_X$. Then we can look at the action of ι^* on $H^2(X, \mathbf{Z})$ and we denote by S its $(+1)$-eigenspace. It turns out that S is a hyperbolic 2-elementary lattice, and all the possibilities for S up to isometries were classified by Nikuln. More precisely, there are 75 cases which correspond bijectively to the triples of invariants (r, a, δ), where r is the rank of S, $A_S \cong \mathbf{Z}_2^a$, and δ is the so-called *coparity* of L: $\delta = 0$ provided $q_L(v) \equiv 0 \bmod \mathbf{Z}$, and $\delta = 1$ otherwise.

3. Moduli via period domains

3A. A general construction.

Remark 3.1. We describe here a construction common to the construction of many Hodge-theoretic moduli spaces. Let Λ be an ambient lattice, $S \leq \Lambda$ a primitive sublattice, and $T := S^{\perp\Lambda}$ its orthogonal complement in Λ. Define the **period domain associated to** S to be

$$\Omega_S^{\pm} := \left\{ [v] \in \mathbf{P}(S \otimes_{\mathbf{Z}} \mathbf{C}) \;\middle|\; v^2 = 0 \text{ and } v\overline{v} = 0 \right\}.$$

As a matter of notation, we also set

$$\Omega^S := \Omega_{S^\perp} := \Omega_T.$$

In cases of interest, we have a decomposition $\Omega_S^{\pm} = \Omega_S^{+} \amalg \Omega_S^{-}$ into irreducible components, both of which are type IV bounded Hermitian symmetric domains which are permuted by $\mathrm{Gal}(\mathbf{C}/\mathbf{R})$. We fix a choice of component Ω_S^+, and let $\mathrm{O}(S)^+ \leq \mathrm{O}(S)$ be the subgroup fixing this component. We then form a locally symmetric space and a corresponding Baily-Borel compactification

$$F(S) := \mathrm{O}^+(S)\diagdown\Omega_S^+ \qquad \overline{F(S)}^{\mathrm{BB}} := \overline{\mathrm{O}^+(S)\diagdown\Omega_S^+}^{\mathrm{BB}}.$$

More generally, one can let Γ be any neat arithmetic group that acts properly discontinuously on $S\Omega_S^+$. One can then similarly form

$$F(S, \Gamma) := \Gamma\diagdown\Omega_S^+, \qquad \overline{F(S, \Gamma)}^{\mathrm{BB}} := \overline{\Gamma\diagdown\Omega_S^+}^{\mathrm{BB}}.$$

Specific choices of S are used throughout our work to construct various coarse moduli spaces. In some instances, we must remove a hyperplane arrangement to form the correct moduli space. Let

$$\mathcal{H}_{-2} := \left(\bigcup_{\delta \in \Phi_N} \delta^\perp \right) \cap \Omega_N^+ = \bigcup_{\substack{\delta \in N, \\ \delta^2 = -2}} \left\{ [v] \in \Omega_N^+ \;\middle|\; v \cdot \delta = 0 \right\}.$$

and define

$$F(S, \Gamma, \mathcal{H}_{-2}) := \Gamma\diagdown\left(\Omega_S^+ \setminus \mathcal{H}_{-2}\right), \qquad \overline{F(S, \Gamma, \mathcal{H}_{-2})}^{\mathrm{BB}} := \overline{\Gamma\diagdown\left(\Omega_S^+ \setminus \mathcal{H}_{-2}\right)}^{\mathrm{BB}}$$

3B. Generally finding cusps.

Remark 3.2. We now discuss how $\partial F(S, \Gamma, \mathcal{H}_{-2})$ can be described lattice-theoretically. Let $\mathrm{Gr}^{\mathrm{iso}}(S)$ be the isotropic Grassmannian of the lattice S, and write $\partial F(S, \Gamma, \mathcal{H}_{-2}) = \bigcup_{i \geq 0} \partial F(S, \Gamma, \mathcal{H}_{-2})_i$ for a stratification of the boundary by i-dimensional components. One can show that there are bijections

$$\mathrm{Gr}_1^{\mathrm{iso}}(L)/\Gamma \cong \partial F(S, \Gamma, \mathcal{H}_{-2})_0, \qquad \mathrm{Gr}_2^{\mathrm{iso}}(L)/\Gamma \cong \partial F(S, \Gamma, \mathcal{H}_{-2})_1,$$

and so 0-cusps correspond to Γ-orbits of primitive isotropic lines and 1-cusps to orbits of isotropic planes.

3C. **Moduli of K3 surfaces with nonsymplectic involution.**

Remark 3.3 (Constructing moduli of quasi-polarized K3 surfaces lattice-theoretically)**.** The coarse moduli space F_{2d} of polarized K3 surfaces (X, L) can be realized using the construction described in subsection 3A. Recall that $H^2(X;\mathbf{Z}) \cong \mathrm{II}_{3,19}$. Fix a marking $\varphi : H^2(X;\mathbf{Z}) \to \mathrm{II}_{3,19}$ and a polarization L of degree $2d$, and let $h \coloneqq \varphi([L]) \in \mathrm{II}_{3,19}$. One can then show that $h^\perp \cong L_{2d}$. Let

$$\mathrm{Stab}_{\mathrm{O}(\mathrm{II}_{3,19})}(h) \coloneqq \left\{ \gamma \in \mathrm{O}(\mathrm{II}_{3,19}) \;\middle|\; \gamma(h) = h \right\}$$

be the stabilizer of h in $\mathrm{II}_{3,19}$ and define

$$\Gamma_h \coloneqq \mathrm{Stab}_{\mathrm{O}(\mathrm{II}_{3,19})}(h)^+$$

to be the finite index subgroup fixing $\Omega^+_{\Lambda_{2d}}$. Letting $\mathcal{F}^{\mathrm{qp}}_{2d}$ be the moduli stack of quasi-polarized K3 surfaces of degree $2d$, there is an analytic isomorphism at the level of coarse spaces

$$F^{\mathrm{qp}}_{2d} \cong \Gamma_h \diagdown \Omega^+_{\Lambda_{2d}}.$$

However, $\mathcal{F}^{\mathrm{qp}}_{2d}$ is generally not a separated stack. We can instead use the stack $\mathcal{F}^{\mathrm{ADE}}_{2d}$ of polarized K3s with ADE singularities, since there is an isomorphism $F^{\mathrm{ADE}}_{2d} \cong F^{\mathrm{qp}}_{2d}$ at the level of coarse spaces.

Definition 3.4. The theory of moduli of pairs (X, ι) with X a K3 surface and ι a nonsymplectic involution can be approached using the construction in subsection 3A as well. Let $S \subseteq \mathrm{II}_{3,19}$ be a primitive hyperbolic 2-elementary sublattice which is the $(+1)$-eigenspace of an involution ρ of $\mathrm{II}_{3,19}$. A *ρ-marking* of (X, ι) is an isometry $\varphi\colon H^2(X,\mathbf{Z}) \to \mathrm{II}_{3,19}$ such that $\iota^* = \varphi^{-1} \circ \rho \circ \varphi$. Fix such a marking ρ. We have a **period domain** Ω^+_S associated to S, and we define the change-of-marking group associated to ρ to be

$$\Gamma_\rho = \left\{ \gamma \in \mathrm{O}(\mathrm{II}_{3,19}) \;\middle|\; \gamma \circ \rho = \rho \circ \gamma \right\}.$$

One can then show that the coarse moduli space of ρ-markable K3 surfaces is analytically isomorphic to the locally symmetric space

$$F_S \coloneqq F(S^\perp, \Gamma_\rho, \mathcal{H}_{-2}) \coloneqq \Gamma_\rho \diagdown (\Omega_{S^\perp} \setminus \mathcal{H}_{-2}).$$

In particular, the point corresponding to (X, ι) is $[\varphi(\mathbb{C}\omega_X)]$.

3D. **Hodge theoretic compactifications.**

Remark 3.5. Hodge theory provides different ways to compactify $\Omega_{S^\perp}/\Gamma$ for any finite index subgroup $\Gamma \subseteq \mathrm{O}(S^\perp)$. A standard way that involves no choices is provided by the *Baily–Borel compactification* $\overline{\Omega_{S^\perp}/\Gamma}^{\mathrm{bb}}$. This is a projective normal compactification whose boundary is stratified into 0-cusps and 1-cusps which correspond to Γ-orbits of isotropic vectors $I \subseteq T$ and isotropic planes $J \subseteq T$. *Toroidal compactifications* $\overline{\Omega_{S^\perp}/\Gamma}^{\mathfrak{F}}$ are blow-ups of $\overline{\Omega_{S^\perp}/\Gamma}^{\mathrm{bb}}$ which depend on the choice of a compatible system of admissible fans $\mathfrak{F} = \{\mathfrak{F}_K\}$ for each isotropic vector I or plane J. The fan $\mathfrak{F}_K$ is a rational polyhedral decomposition of the rational closure $C_{K,\mathbf{Q}}$ of the positive cone $C_K \subseteq K^\perp/K \otimes \mathbb{R}$. It is required to satisfy the usual fan axioms, and additionally be Γ-invariant with only finitely many orbits of cones. As this datum is trivial for isotropic planes, it is sufficient to provide the fan only for the isotropic vectors I, hence $\mathfrak{F} = \{\mathfrak{F}_I\}$. Lastly, *Semitoroidal compactifications* are due to Looijenga and simultaneously generalize the Baily–Borel and toroidal compactifications by allowing the fans $\mathfrak{F}_I$ to be not necessarily finitely generated.

4. Hyperelliptic K3s

Definition 4.1. Let X be a K3 surface and let $L \in \operatorname{Pic}(X)$ be a line bundle with $L^2 > 0$ where the linear system $|L|$ has no fixed components. We say that $|L|$ is a **hyperelliptic linear system on** X and X is a **hyperelliptic** K3 **surface** if $|L|$ contains a hyperelliptic curve.

Remark 4.2. The induced morphism $\varphi_{|L|} : X \to \mathbf{P}^g$ where $L^2 = 2g-2$ in this case is a generally 2-to-1 morphism onto a surface F of degree $g-1$ in $\mathbf{P}^g$. By the classification of surfaces, either $F \cong \mathbf{P}^2$ or $\mathbf{F}_n$[2] with $n \in \{0,1,2,3,4\}$ ramified over a curve $C \in |-2K_F|$.

Remark 4.3. The open locus of $\overline{F_{4,\text{h.e.}}}^{\text{BB}}$ can be realized using the standard construction of L-polarized K3 surfaces, taking $L = U^{\oplus 2} \oplus D_{16}$. More generally, degree n hyperelliptic K3 surfaces can be constructed by taking $L = U^{\oplus 2} \oplus D_{n-2}$.

4A. Hyperelliptic quartic K3s.

Remark 4.4. We now focus back on our main case of interest: hyperelliptic quartic K3s, i.e. hyperelliptic K3 surfaces of degree 4. In this case, the hyperbolic 2-elementary even lattice S is given by $U(2)$, which corresponds to the invariants $(r, a, \delta) = (2, 2, 0)$.

The Baily–Borel compactification $\overline{\Omega_{S^\perp}/\Gamma}^{\text{bb}}$ … for which Γ? Was studied by Laza–O'Grady.

Now relate $\overline{\mathbf{K}}_{\text{h}}$ with $\overline{\Omega_{S^\perp}/\Gamma}$ and an appropriate Looijenga semitoroidal. Where is this in Valery and Phil's work? Give appropriate references.

Remark 4.5. Following [LO21], consider the period domain construction described in subsection 3A using the lattice $\Lambda_N \coloneqq U^{\oplus 2} \oplus D_{N-2}$ and $\Gamma = \mathrm{O}(\Lambda_N)^+$.[3] We then obtain a sequence of locally symmetric spaces

$$\mathcal{F}(N) \coloneqq F(\Lambda_N, \mathrm{O}(\Lambda_N)^+) \coloneqq \mathrm{O}(\Lambda_N)^+ \diagdown \Omega^+_{\Lambda_N}.$$

In particular, taking $N = 19$ yields the F_4, the coarse moduli space of standard polarized K3 surfaces of degree 4, and taking $N = 18$ yields a coarse moduli space $F_{4,\text{h.e.}}$ of quartic (i.e. degree 4) hyperelliptic K3 surfaces. The lattice embedding $\Lambda_{18} \hookrightarrow \Lambda_{19}$ induced by $D_{16} \hookrightarrow D_{17}$ produces an inclusion $F_{4,\text{h.e.}} \subseteq F_4$ realizing $F_{4,\text{h.e.}}$ as a normal Heegner divisor in F_4. This in turn induces a morphism $\overline{F_{4,\text{h.e.}}}^{\text{BB}} \to \overline{F_4}^{\text{BB}}$. The Baily-Borel compactification $\overline{F_{4,\text{h.e.}}}^{\text{BB}}$ was studied in LO16 and [LO21], where in the latter they show

$$\overline{F_{4,\text{h.e.}}}^{\text{BB}} \cong \mathrm{Chow}_{2,4} /\!\!/ \mathrm{SL}_4,$$

a GIT quotient of the Chow variety of $(2,4)$ curves in $\mathbf{P}^3$.

See section 2.1 here https://arxiv.org/pdf/1801.04845.pdf#page=6&zoom=auto,-87,319

[2] A Hirzebruch surface $\mathbf{F}_n \coloneqq \operatorname{Proj}_{\mathbf{P}^1}(\mathcal{O}_{\mathbf{P}^1}(-n) \oplus \mathcal{O}_{\mathbf{P}^1})$.

[3] For moduli-theoretic purposes, if $N \equiv 6 \bmod 8$, one then instead passes to a finite index subgroup as detailed in [LO21].

Theorem 4.6 ([LO21, Theorem 2.3])**.** *The Baily–Borel compactification*

$$\overline{F_{4,\mathrm{h.e.}}}^{\mathrm{BB}} \cong \overline{\mathrm{O}(\Lambda_{18})^+ \backslash \Omega^+_{\Lambda_{18}}}^{\mathrm{BB}}$$

has two 0*-cusps (type III boundary components) and eight* 1*-cusps (type II boundary components). The incidences between* 0*-cusps and* 1*-cusps are represented in Figure 1.*

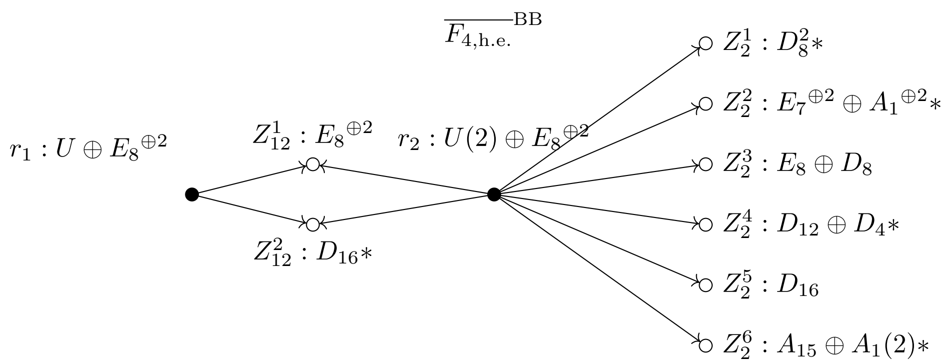


FIGURE 1. Cusp diagram for degree 4 hyperelliptic K3 surfaces $\overline{F_{4,\mathrm{h.e.}}}^{\mathrm{BB}}$.

Remark 4.7. See https://arxiv.org/pdf/2006.06816.pdf#page=1&zoom=100,-274,431

If $C \subseteq \mathbf{P}^1 \times \mathbf{P}^1$ is a smooth curve of bidegree $(4,4)$ and $\pi : X_C :\to \mathbf{P}^1 \times \mathbf{P}^1$ is the double cover branched along C, then X_C is a smooth hyperelliptic polarized K3 surface of degree 4 and thus $X_C \in \overline{F_{4,\mathrm{h.e.}}}^{\mathrm{BB}}$. Letting $M := |\mathcal{O}_{\mathbf{P}^1\times\mathbf{P}^1}(4,4)| /\!\!/ \operatorname{Aut}(\mathbf{P}^1 \times \mathbf{P}^1)$ be the GIT quotient, LO21 describes a birational period map $M \dashrightarrow \overline{F_{4,\mathrm{h.e.}}}^{\mathrm{BB}}$.

Remark 4.8. The K3 surfaces parameterized by $\overline{\mathbf{K}}$ are double covers of $\mathbf{P}^1 \times \mathbf{P}^1$ branched along curves of class $(4,4)$ in the monomials listed in (1). More in general, the double covers of $\mathbf{P}^1 \times \mathbf{P}^1$ branched along general curves of class $(4,4)$ give rise to K3 surfaces known as *hyperelliptic* K3 surfaces. Let us construct their family and the KSBA compactification.

Let $\mathbf{P}^{24}$ be the space of coefficients, up to scaling, for a bidegree $(4,4)$ polynomial in $\mathbf{P}^1 \times \mathbf{P}^1$. In this case, a monomial $X_0^i X_1^j Y_0^k Y_1^\ell$ is indexed by

$$M_{\mathrm{h}} := \{(i,j,k,\ell) \in \mathbf{Z}^4_{\geq 0} \mid i+j = k+\ell = 4\}.$$

Let $\mathbf{U}_{\mathrm{h}} \subseteq \mathbf{P}^{24}$ be the dense open subset of coefficients $[\ldots : c_{ijk\ell} : \ldots]$ such that the corresponding $(4,4)$ curve is smooth. We can define a KSBA-stable family

$$\left(\mathcal{X}_{\mathrm{h}} := \mathbf{U}_{\mathrm{h}} \times (\mathbf{P}^1 \times \mathbf{P}^1), \frac{1+\epsilon}{2}\mathcal{B}_{\mathrm{hyp}}\right) \to \mathbf{U}_{\mathrm{hyp}},$$

where $\mathcal{B}_{\mathrm{h}}$ is the relative divisor given by

$$\sum_{(i,j,k,\ell)\in M_{\mathrm{h}}} c_{ijk\ell} X_0^i X_1^j Y_0^k Y_1^\ell = 0.$$

We can consider the fiberwise double cover $(\mathcal{T}_{\mathrm{h}}, \epsilon\mathcal{R}_{\mathrm{h}}) \to \left(\mathcal{X}_{\mathrm{h}}, \frac{1+\epsilon}{2}\mathcal{B}_{\mathrm{h}}\right)$, which gives rise to the family of hyperelliptic K3 surfaces. The automorphism group of $\mathbf{P}^1 \times \mathbf{P}^1$ acts on $\mathbf{U}_{\mathrm{h}}$ identifying isomorphic fibers. In particular, $\mathbf{U}_{\mathrm{h}}/\mathrm{Aut}(\mathbf{P}^1 \times \mathbf{P}^1)$ is the moduli space of smooth hyperelliptic K3 surfaces. To compactify it, we can consider the stack $\overline{\mathcal{P}}'_{\mathrm{h}}$ given by the closure of the image of the morphism $\mathbf{U}_{\mathrm{h}} \to \mathcal{SP}\left(\frac{1+\epsilon}{2}, 2, 8\epsilon^2\right)$. Let $\overline{\mathbf{P}}'_{\mathrm{h}}$ be the corresponding coarse moduli space and denote by $\overline{\mathbf{P}}_{\mathrm{h}}$ its normalization, which gives rise to a compactification of the 18-dimensional moduli space $\mathbf{U}_{\mathrm{h}}/\mathrm{Aut}(\mathbf{P}^1 \times \mathbf{P}^1)$. Alternatively, by using the family $(\mathcal{T}_{\mathrm{h}}, \epsilon\mathcal{R}_{\mathrm{h}}) \to \mathbf{U}_{\mathrm{h}}$ and the moduli functor $\mathcal{SP}\left(\epsilon, 2, 16\epsilon^2\right)$ instead, we obtain the compactifications $\overline{\mathbf{K}}_{\mathrm{h}}$, which instead parameterize generically the hyperelliptic K3 surfaces. We have that $\overline{\mathbf{K}}_{\mathrm{h}} \cong \overline{\mathbf{U}}_{\mathrm{h}}$.

Remark 4.9. The inclusion $\mathbf{U} \hookrightarrow \mathbf{U}_{\mathrm{h}}$ induces an inclusion of the stacks $\overline{\mathcal{P}}' \hookrightarrow \overline{\mathcal{P}}'_{\mathrm{h}}$, and hence an inclusion of the corresponding coarse moduli spaces $\overline{\mathbf{P}}' \hookrightarrow \overline{\mathbf{P}}'_{\mathrm{h}}$. Therefore, we have an induced morphism $\overline{\mathbf{P}} \to \overline{\mathbf{P}}_{\mathrm{h}}$ which is finite and birational onto its image. {Luca: The reason why this morphism exists is nontrivial! The normalization is not functorial, so one has to really prove this.} {The above is also missing the following. Do we have an embedding of $\mathbf{U}/G$ into $\mathbf{U}_{\mathrm{h}}/\mathrm{Aut}(\mathbf{P}^1 \times \mathbf{P}^1)$? Recall $G = \mathbb{G}_m^2 \rtimes (\mathbf{Z}/2\mathbf{Z})$.}

Remark 4.10. The compactification $\overline{\mathbf{P}}_{\mathrm{h}}$ should be fully understood from the work in [AE22]. Moreover, the GIT and Baily–Borel should be understood by [LO21].

5. Enriques surfaces

5A. The unpolarized case.

Remark 5.1. If Y is an Enriques surface, it is well known that the universal cover $\pi\colon X \to Y$ is a μ_2 Galois cover where X is a K3 surface and $Y \cong X/\iota$ for ι the basepoint-free involution swapping the sheets of the cover. We write $V_{+1}(\iota^*), V_{-1}(\iota^*) \subseteq H^2(X;\mathbf{Z})$ for the $(+1)$ and (-1)-eigenspaces respectively of the induced involution in cohomology $\iota^*\colon H^2(X;\mathbf{Z}) \to H^2(X;\mathbf{Z})$. It is well-known that $V_{+1}(\iota)^{\perp H^2(X;\mathbf{Z})} = V_{-1}(\iota)$. The covering map π induces an embedding of lattices

$$\pi^*\colon H^2(Y;\mathbf{Z}) \hookrightarrow H^2(X;\mathbf{Z}),$$

whose image is $V_{+1}(\iota^*)$. It is well known that

- $H^2(X;\mathbf{Z}) \cong \mathrm{II}_{3,19} \cong U^{\oplus 3} \oplus E_8^{\oplus 2}$ is the K3 *lattice*;
- $M := H^2(Y,\mathbf{Z})/\mathrm{tors} \cong \mathrm{II}_{1,9} \cong U \oplus E_8$ is the *Enriques lattice*;
- $V_{+1}(\iota^*) \cong U(2) \oplus E_8(2) \cong \mathrm{II}_{1,9}(2)$;
- $V_{-1}(\iota^*) \cong U \oplus U(2) \oplus E_8(2) = U \oplus \mathrm{II}_{1,9}(2)$.

Remark 5.2. We will use the decomposition of the K3 lattice into summands involving the Enriques lattice

$$\mathrm{II}_{3,19} = \mathrm{II}_{1,9} \oplus \mathrm{II}_{1,9} \oplus U,$$

and describe a vector in the K3 lattice $\mathrm{II}_{3,19}$ with three coordinates (x, y, z) accordingly. Let $U = \mathbf{Z}e \oplus \mathbf{Z}f$ with $\{e, f\}$ the standard hyperbolic basis satisfying $e^2 = f^2 = e \cdot f - 1 = 0$.

Remark 5.3 (Period domain for unpolarized Enriques surfaces)**.** Again following the period domain construction described subsection 3A, now with the lattice

$$N \coloneqq U \oplus \mathrm{II}_{1,9}(2).$$

The period domain for unpolarized Enriques surfaces is Ω_N^+, and the correct associated locally symmetric space is

$$\mathcal{E}_\emptyset \coloneqq F(N, \mathrm{O}(N)^+, \mathcal{H}_{-2}) \coloneqq \mathrm{O}(N)^+ \diagdown \left(\Omega_N^+ \setminus \mathcal{H}_{-2}\right).$$

Lemma 5.4 (Torelli for Enriques surfaces, Horikawa)**.** *Points in $\mathcal{E}_\emptyset$ correspond to isomorphism classes of unpolarized Enriques surfaces.*

Theorem 5.5 ([Ste91, Propositions 4.5 and 4.6])**.** *The Baily–Borel compactification*

$$\overline{\mathcal{E}_\emptyset}^{\mathrm{BB}} \coloneqq \overline{\mathrm{O}^+(N) \diagdown (\Omega_N^+ \setminus \mathcal{H}_{-2})}^{\mathrm{BB}}$$

has two 0*-cusps and two* 1*-cusps. The incidences between* 0*-cusps and* 1*-cusps are represented in Figure 2.*

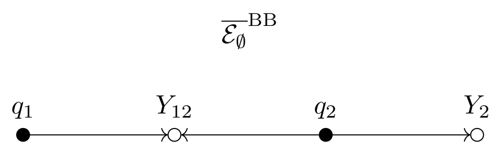


FIGURE 2. Cusp diagram for the moduli space of unpolarized Enriques surfaces $\overline{\mathcal{E}_\emptyset}^{\mathrm{BB}}$.

5B. **Degree 2 polarized Enriques surfaces.** For degree 2 polarized Enriques surfaces, we consider the same period domain, but we change the arithmetic group acting on it.

Definition 5.6. A **polarization** on an Enriques surface Y is a pseudo-ample (i.e. big and nef) line bundle L on Y; we call this an **ample polarization** if L is ample.

Definition 5.7. A **numerical (resp. ample numerical) polarization** on Y is a choice L of a numerical equivalence class of pseudo-ample (resp. ample) line bundle L.

Definition 5.8. A **numerically polarized Enriques surface** is a pair (Y, L) where Y is an Enriques surface and L is a numerical polarization on Y.[4]

Remark 5.9. Let $L \coloneqq U \oplus {\mathrm{II}_{1,9}}^{\oplus 2}$, noting that $N \leq L$, and define the following involution:

$$I\colon L \to L,$$
$$(x, y, z) \mapsto (y, x, -z).$$

[4]Why introduce **numerical** polarizations? Recall that A is a polarized abelian variety if it is equipped with an isogeny $\lambda : A \to A^\vee$. If L is a numerical polarization on A, it induces a unique isogeny λ_L, and every such isogeny comes from such an L, so numerical polarization strictly generalizes this notion to other varieties.

A result of Horikawa shows that there is an isometry $\mu : H^2(X; \mathbf{Z}) \to L$ such that $I \circ \mu = \mu \circ I^*$ and produces an embedding

$$\begin{aligned} M &\to L \\ m &\mapsto (m, m, 0). \end{aligned}$$

Reference: https://file.notion.so/f/s/b5171ee5-610c-489f-b347-5839cc0005f0/Sterk.pdf?id=2a417bca-589b-4639-86e6-6901fe36ff30&table=block&spaceId=7cb2f7c7-7373-4d11-91ab-284625335dc8&expirationTimestamp=1686073894799&signature=AIqn4Lp1sajAsu5XQEbx0 155,749

Define

$$\Gamma' := \{ g \in \mathrm{O}(L) \mid g \circ I = I \circ g \text{ and } g(e+f, e+f, 0) = (e+f, e+f, 0) \},$$

automorphisms in the centralizer of I in $\mathrm{O}(L)$ fixing the point $(e+f, e+f, 0)$. If $g \in \Gamma'$ then $g|_N \in \mathrm{O}(N)$. So define

this is what Sterk claims, needs proof.

$$\Gamma := \{ g|_N \mid g \in \Gamma' \} \leq O(N),$$

which is the image of Γ' in $\mathrm{O}(L_-)$. Again using the construction in subsection 3A, the moduli space for Enriques surfaces with a polarization of degree 2 is given by the locally symmetric space

$$\mathcal{E}_2 \cong F(N, \Gamma) = \Gamma \diagdown \Omega_N^+.$$

Theorem 5.10 ([Ste91, §4.3]). *The Baily–Borel compactification*

$$\overline{\mathcal{E}_2}^{\mathrm{BB}} := \overline{\Gamma \diagdown \Omega_N^+}^{\mathrm{BB}}$$

has five 0*-cusps and nine* 1*-cusps. The incidences between* 0*-cusps and* 1*-cusps are represented in Figure 3.*

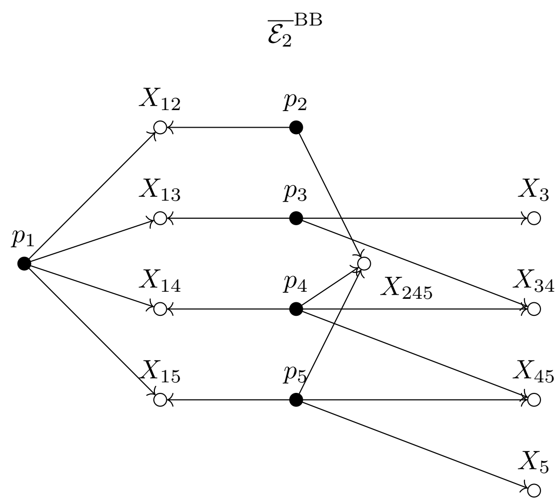


FIGURE 3. Cusp diagram for degree 2 polarized Enriques surfaces $\overline{\mathcal{E}_2}^{\mathrm{BB}}$.

5C. **Numerically polarized.** [GH16]

5D. **The family of degree 2 polarized Enriques surfaces.**

Remark 5.11. We review the construction of degree 2 polarized Enriques surfaces following [Bar+04, Chapter V, § 23]. Let us consider the involution on $\mathbf{P}^1 \times \mathbf{P}^1$ given by
$$\iota\colon ([X_0 : X_1], [Y_0 : Y_1]) \mapsto ([X_0 : -X_1], [Y_0 : -Y_1]).$$
We have that ι has precisely four isolated fixed points, namely
$$([0 : 1], [0 : 1]),\ ([0 : 1], [1 : 0]),\ ([1 : 0], [0 : 1]),\ ([1 : 0], [1 : 0]).$$
Let $B \subseteq \mathbf{P}^1 \times \mathbf{P}^1$ be a general ι-invariant curve of class $(4, 4)$ not passing through the fixed points of ι. Then, the bi-homogeneous polynomial giving B consists of the following monomials:
$$\begin{gathered} X_0^4Y_0^4,\ X_0^4Y_0^2Y_1^2,\ X_0^4Y_1^4,\ X_0^3X_1Y_0^3Y_1,\ X_0^3X_1Y_0Y_1^3,\ X_0^2X_1^2Y_0^4, \\ X_0^2X_1^2Y_0^2Y_1^2,\ X_0^2X_1^2Y_0^4,\ X_0X_1^3Y_0^3Y_1,\ X_0X_1^3Y_0Y_1^3,\ X_1^4Y_0^4,\ X_1^4Y_0^2Y_1^2,\ X_1^4Y_1^4. \end{gathered} \tag{1}$$
The coefficients of $X_0^4Y_0^4, X_0^4Y_1^4, X_1^4Y_0^4, X_1^4Y_1^4$ must be nonzero to guarantee that B does not pass through the torus fixed points of ι.

Remark 5.12. The double cover $\pi\colon T \to \mathbf{P}^1\times\mathbf{P}^1$ branched along B is a well known to be a K3 surface: T is smooth and minimal, $K_T \sim \pi^*\left(K_{\mathbf{P}^1\times\mathbf{P}^1} + \frac{1}{2}B\right) \sim 0$, and $\pi_*\mathcal{O}_T = \mathcal{O}_{\mathbf{P}^1\times\mathbf{P}^1} \oplus \mathcal{O}_{\mathbf{P}^1\times\mathbf{P}^1}\left(-\frac{1}{2}B\right)$, which gives $h^1(\mathcal{O}_T) = 0$.

Remark 5.13. Let $\mathcal{L}^{\otimes 2} = \mathcal{O}_{\mathbf{P}^1\times\mathbf{P}^1}(4, 4)$ and let $p\colon L \to \mathbf{P}^1 \times \mathbf{P}^1$ be the total space of the line bundle $\mathcal{L}$. Then the double cover T of $\mathbf{P}^1 \times \mathbf{P}^1$ branched along B can be viewed inside L as the vanishing locus of $t^2 - p^*s = 0$, where $B = V(s)$ and $t \in \Gamma(L, p^*\mathcal{L})$ is the tautological section. We have that then ι lifts to an involution $\widetilde{\iota}$ of T with exactly eight fixed points: two over each fixed point of ι. If τ denotes the deck transformation of the cover, i.e. $t \mapsto -t$, then we have that $\widetilde{\iota}$ commutes with τ and the composition $\sigma = \tau \circ \widetilde{\iota}$ is a fixed-point free involution of T. The quotient $q\colon T \to T/\sigma = S$ is then an Enriques surface called *Horikawa model*, and comes equipped with a degree 2 polarization induced by $\mathcal{O}_{\mathbf{P}^1\times\mathbf{P}^1}(1, 1)$.

Let $R \subseteq T$ be the ramification locus, so that $2R = \pi^*B$, define $\overline{R} = q(R)$, and let $0 < \epsilon \ll 1$ rational. Then we have the two following covering equalities:
$$K_T + \epsilon R \sim_{\mathbf{Q}} \pi^*\left(K_{\mathbf{P}^1\times\mathbf{P}^1} + \frac{1+\epsilon}{2}B\right),$$
$$K_T + \epsilon R \sim_{\mathbf{Q}} q^*\left(K_S + \frac{\epsilon}{2}\overline{R}\right).$$

Lemma 5.14. *With the notation introduced above, we have the following self-intersection numbers:*
- $\left(K_{\mathbf{P}^1\times\mathbf{P}^1} + \frac{1+\epsilon}{2}B\right)^2 = 8\epsilon^2$;
- $(K_T + \epsilon R)^2 = 16\epsilon^2$;
- $\left(K_S + \frac{\epsilon}{2}\overline{R}\right)^2 = 8\epsilon^2$.

Remark 5.15. We now relativize the above construction. Let $\mathbf{P}^{12}$ be the space of coefficients, up to scaling, for a bidegree $(4, 4)$ polynomial in the monomials in (1). So, if $c_{ijk\ell}$ denotes the coefficient of $X_0^iX_1^jY_0^kY_1^\ell$, then $[\ldots : c_{ijk\ell} : \ldots] \in \mathbf{P}^{12}$ with (i, j, k, ℓ) within the following set:
$$M \coloneqq \left\{(i, j, k, \ell) \in \mathbf{Z}^4_{\geq 0} \;\middle|\; i + j = k + \ell = 4,\ i + k \equiv j + \ell \equiv 0 \bmod 2\right\}.$$

Let $\mathbf{U} \subseteq \mathbf{P}^{12}$ be the dense open subset of coefficients such that the corresponding ι-invariant $(4,4)$ curve $B \subseteq \mathbf{P}^1 \times \mathbf{P}^1$ is smooth and does not pass through the torus fixed points of $\mathbf{P}^1 \times \mathbf{P}^1$. Define $\mathcal{X} \coloneqq \mathbf{U} \times (\mathbf{P}^1 \times \mathbf{P}^1)$ and let $\mathcal{X} \to \mathbf{U}$ be the projection. Let

$$\mathcal{B} \coloneqq V\left(\sum_{(i,j,k,\ell)\in M} c_{ijk\ell} X_0^i X_1^j Y_0^k Y_1^\ell\right) \subseteq \mathcal{X}.$$

Then $\left(\mathcal{X}, \frac{1+\epsilon}{2}\mathcal{B}\right) \to \mathbf{U}$ is a family of stable pairs with fibers given by $\left(\mathbf{P}^1 \times \mathbf{P}^1, \frac{1+\epsilon}{2}B\right)$ as described above. Additionally, we observe that $\left(\mathcal{X}, \frac{1+\epsilon}{2}\mathcal{B}\right) \to \mathbf{U}$ is a KSBA-stable as defined in [Kol23, p. 8.7].

Remark 5.16. The family $\left(\mathcal{X}, \frac{1+\epsilon}{2}\mathcal{B}\right) \to \mathbf{U}$ has isomorphic fibers. To eliminate this redundancy, we consider the action of $\operatorname{Aut}(\mathbf{P}^1 \times \mathbf{P}^1) \cong (\mathrm{PGL}_2 \times \mathrm{PGL}_2) \rtimes \mathbf{Z}/2\mathbf{Z}$ (see [Dol12]) on $H^0(\mathcal{O}(4,4))$. More precisely, we want to look at the subgroup G which preserves ι-invariant $(4,4)$-curves not passing through the torus fixed points. Note that the $\mathbf{Z}_2$-action preserves the set of monomials M as $(i,j,k,\ell) \in M$ if and only if $(k,\ell,i,j) \in M$. Now consider a generic $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \mathrm{PGL}_2$ acting on $[X_0 : X_1]$. One can check directly that the action of this matrix preserves the monomials in M if and only if $b = c = 0$, and the same holds if we consider the action of the second copy of PGL_2 which acts on $[Y_0 : Y_1]$. In particular, we have that $G \cong \mathbb{G}_m^2 \rtimes (\mathbf{Z}/2\mathbf{Z})$. Therefore, we have an action $G \curvearrowright \mathbf{U}$ which identifies the isomorphic fibers of $\left(\mathcal{X}, \frac{1+\epsilon}{2}\mathcal{B}\right) \to \mathbf{U}$.

Over $\mathbf{U}$, we can also consider the cover $(\mathcal{T}, \epsilon\mathcal{R}) \to \left(\mathcal{X}, \frac{1+\epsilon}{2}\mathcal{B}\right)$ which gives the family of isomorphism classes of pairs $(T, \epsilon R)$ and the fiberwise quotient by the Enriques involution $(\mathcal{T}, \epsilon\mathcal{R}) \to \left(\mathcal{S}, \frac{\epsilon}{2}\overline{\mathcal{R}}\right)$ which gives the family of isomorphism classes of Enriques surfaces $\left(S, \frac{\epsilon}{2}\overline{R}\right)$. Summarizing, we have the following commutative diagram:

$$\begin{array}{ccc} (\mathcal{T}, \epsilon\mathcal{R}) & \longrightarrow & \left(\mathcal{X}, \frac{1+\epsilon}{2}\mathcal{B}\right) \\ \downarrow & & \downarrow \\ \left(\mathcal{S}, \frac{\epsilon}{2}\overline{\mathcal{R}}\right) & \longrightarrow & \mathbf{U} \end{array}$$

Definition 5.17. Following the notation in [Kol23, Theorem 8.1], consider the moduli functors $\mathcal{SP}(\mathbf{a}, d, \nu)$ for

$$(\mathbf{a}, d, \nu) = \left(\frac{1+\epsilon}{2}, 2, 8\epsilon^2\right),\ \left(\epsilon, 2, 16\epsilon^2\right),\ \left(\frac{\epsilon}{2}, 2, 8\epsilon^2\right).$$

and the corresponding coarse moduli spaces $\mathrm{SP}(\mathbf{a}, d, \nu)$. We now define the following stacks:

Consider the KSBA-stable family $\left(\mathcal{X}, \frac{1+\epsilon}{2}\mathcal{B}\right) \to \mathbf{U}$. Therefore there is an induced morphism $\mathbf{U} \to \mathcal{SP}\left(\frac{1+\epsilon}{2}, 2, 8\epsilon^2\right)$ and denote by $\mathcal{P}'$ the closure of its image. Let $\overline{\mathbf{P}}'$ be the coarse moduli space corresponding to $\overline{\mathcal{P}}'$, and denote by $\overline{\mathbf{P}}$ its normalization. We have that $\overline{\mathbf{P}}$ provides a projective compactification of $\mathbf{U}/G$. By using the families $(\mathcal{T}, \epsilon\mathcal{R}) \to \mathbf{U}$ and $\left(\mathcal{S}, \frac{\epsilon}{2}\overline{\mathcal{R}}\right) \to \mathbf{U}$ instead, we obtain the compactifications $\overline{\mathbf{K}}$ and $\overline{\mathbf{E}}$ of $\mathbf{U}/G$ respectively, which instead parameterize generically the K3 and Enriques surfaces.

Remark 5.18. It is a standard observation that the compactifications $\overline{\mathbf{P}}, \overline{\mathbf{K}}, \overline{\mathbf{E}}$ are isomorphic to each other (see [MS21, § 3] for an analogous situation). We will mostly focus on $\overline{\mathbf{P}}$ as it parameterized the simplest objects.

6. Morphisms of moduli and cusps

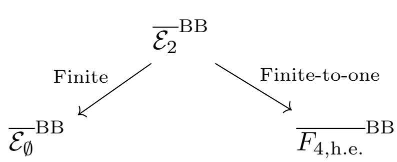
6A. Mapping the boundaries: matching cusp diagrams.

Remark 6.1. Let $\mathcal{F}_{2d}$ be the moduli space of polarized K3 surfaces of degree $2d$. How do we match the cusp diagram for the Baily–Borel compactification for the moduli of degree 2 Enriques surfaces with the Baily–Borel compactification for degree 4 hyperelliptic K3 surfaces? This is actually quite subtle, and it works as follows:

- [Ste91, § 5] matches cusps for degree 2 Enriques surfaces and degree 4 K3 surfaces,
$$\overline{\mathcal{E}_2}^{\mathrm{BB}} \rightleftharpoons \overline{\mathcal{F}_4}^{\mathrm{BB}}.$$
- Using Scattone's method in [Sca87a, § 6], we can match the cusps for degree 4 hyperelliptic K3 surfaces and degree 4 K3 surfaces,
$$\overline{F_{4,\mathrm{h.e.}}}^{\mathrm{BB}} \rightleftharpoons \overline{\mathcal{F}_4}^{\mathrm{BB}}.$$
- The above two points imply the matching we need.

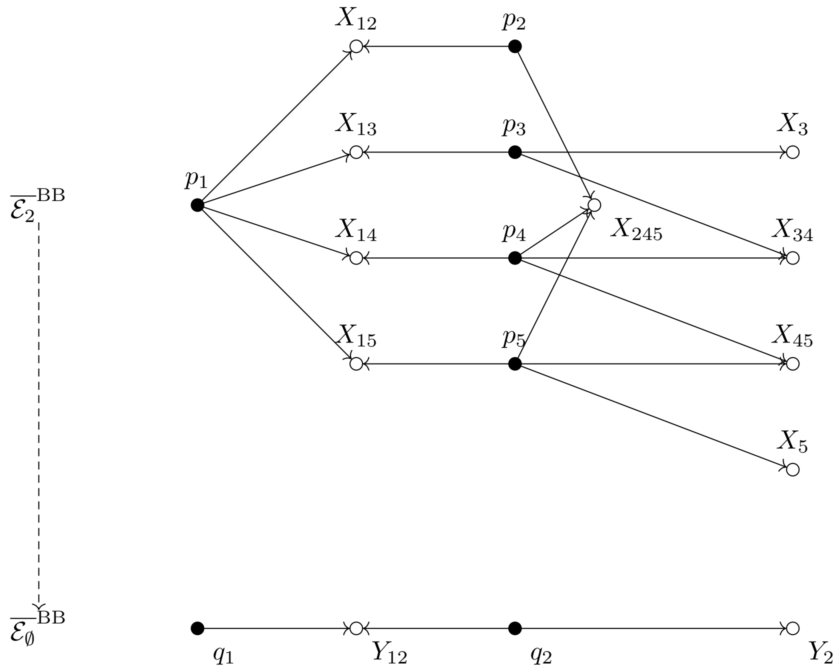
Remark 6.2. What is Scattone's method uses the following observations:

- 1-cusps of $\overline{\mathcal{F}_4}^{\mathrm{BB}}$ are in one-to-one correspondence with the orthogonal complements of D_8 in the Niemeier lattices, and
- The 1-cusps of $\overline{F_{4,\mathrm{h.e.}}}^{\mathrm{BB}}$ are in one-to-one correspondence with the orthogonal complements of D_7 in the Niemeier lattices.

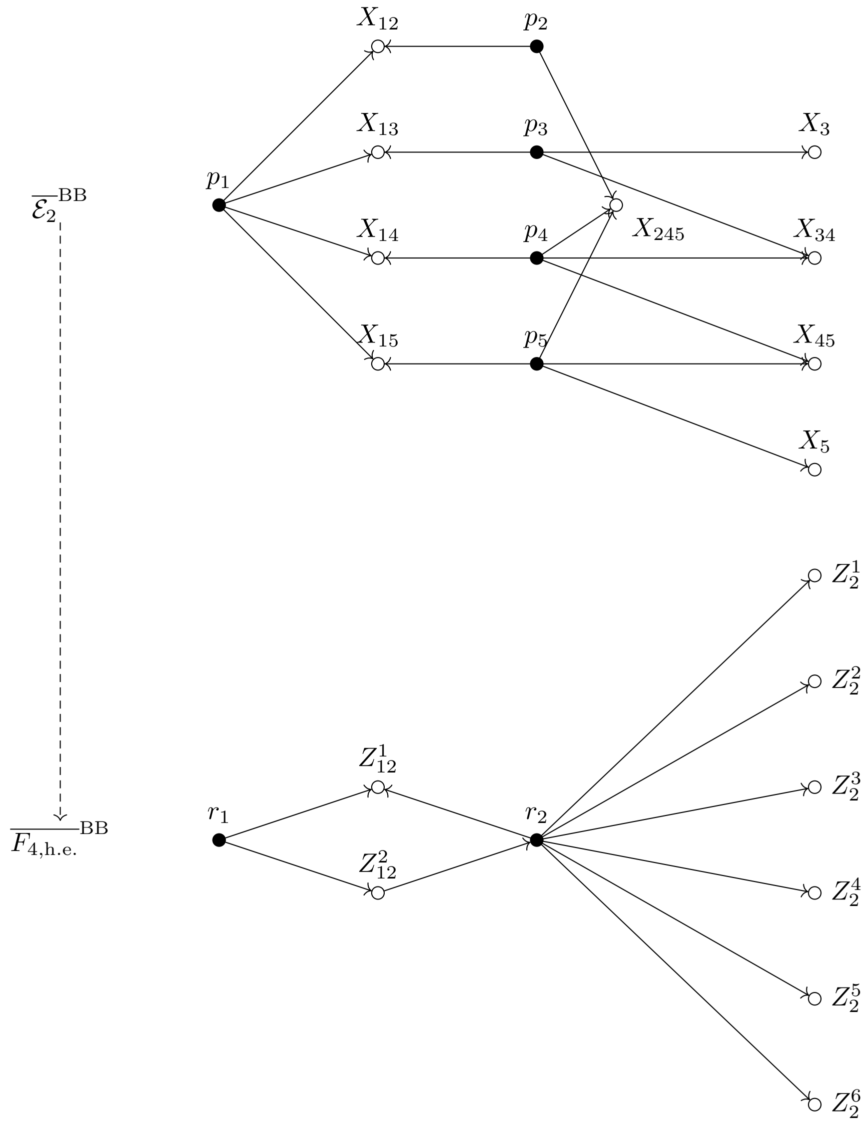


As a result, there are two cusp incidence diagrams to match:

- Polarized $\{X_i, p_i\}$ to unpolarized $\{Y_i, q_i\}$:



- Polarized $\{X_i, p_i\}$ to hyperelliptic $\{Z_i, r_i\}$:



6B. **Type III KPP model at the (18,2,0) odd 0-cusp.**

Remark 6.3. At the (18, 2, 0) odd 0-cusp the Type III stable models are of pumpkin type

Stable models vs KPP models, write down the definition and the differences.

Let X be a K3 surface with a nonsymplectic involution ι with induced involution i^* on $H^2(X;\mathbf{Z})$. We define S to be the $(+1)$-eigenspace ι^*; it is a hyperbolic lattice 2-elementary lattice, and all the possibilities for such lattices were classified by Nikulin. We denote by T the orthogonal complement of S in $H^2(X;\mathbf{Z})$.

Definition 6.4. What makes an **odd** 0-cusp different from an **even** 0-cusp?

Definition 6.5. We define two types of stable models $\overline{X}_0 = \cup \overline{V}_i$:

(1) *Pumpkin.* Each surface $\overline{V}_i$ has two sides $\overline{D}_i = \overline{D}_{i,\text{left}} + \overline{D}_{i,\text{right}}$, they are glued in a circle, all of D_i meeting at the north and south poles.
(2) *Smashed pumpkins.* Starting with a surface of the pumpkin type, one short side is contracted to a point, so that the north and south poles are identified.

If the surface V_i, say to the left, is $(\mathbb{F}_1, D_1 + D_2)$, where $D_1 \sim f$ is the short side being contracted, $D_2 \sim 2s + 2f$ is the other side, and $C_g \sim f$ on V contract V_i by the $\mathbf{P}^1$-fibration $V_i \to \mathbf{P}^1$. Then on the next surface V_{i-1} to the left the long side will fold $2:1$ to itself, creating a non-normal singularity along that side.

If on V_i the divisor C_g has degree $C_g^2 \geq 2$, then only the short side is contracted and the resulting surface $\overline{V}_i$ is normal in codimension 1, with only two points in the normalization glued together (the poles).

Theorem 6.6 ([AE22, Theorem 9.9])**.** *Let $(\overline{X}_0, \cup \overline{V}_i, \epsilon \overline{C}_g)$ be the stable model of a pair $(X_0 = \cup V_i, \epsilon C_g)$, where X_0 is the KPP model of a Type III Kulikov surface and C_g is the component of genus $g \geq 2$ in the ramification divisor R. Then the normalization of each $\overline{V}_i$ is an ADE surface with an involution from [AT21, Table 2]. Moreover,*

- *If $\overline{T}$ is an odd 0-cusp of F_S, then $\overline{X}_0$ is of pumpkin type.*
- *If $\overline{T}$ is an even 0-cusp of F_S, then $\overline{X}_0$ is of smashed pumpkin type. The surfaces V_i of the last type in definition Definition˜9.8, on which $V_i \to \overline{V}_i$ contracts one side are surfaces of [AT21, Table 2] for which one of the sides has length 0, i.e. those with a double prime or a "+".*

6C. ADE surfaces.

Definition 6.7. An **ADE surface** is a pair (Y, C), where Y is a normal surface. (Y, C) has log canonical singularities and the divisor $-2(K_Y + C)$ is Cartier and ample. $L \coloneqq -2(K_Y + C)$ is referred to as the **polarization** of the ADE surface (Y, C).

Remark 6.8. Let $B \in |L|$ effective divisor such that $(Y, C + \frac{1+\epsilon}{2}B)$ is log canonical for $0 < \epsilon \ll 1$, then $(Y, C + \frac{1+\epsilon}{2})$ is called an ADE pair. We can take the double cover $X \to Y$ branched along B and I guess possibly along C. It can happen that Y is toric and C is part of the toric boundary.

ADE surfaces admit a combinatorial classification. The classes of ADE surfaces are called shapes. A shape can be *pure* or *primed*. Surfaces of pure shape are fundamental. Surfaces of primed shape are secondary and can be obtained from surfaces of pure shape using an operation called **priming**.

The ADE surfaces of pure shape are all toric. To construct these we start from a polarized toric surface (Y, L), where $L = -2(K_Y + C)$. This corresponds to a lattice polytope P in $M \otimes_{\mathbf{Z}} \mathbf{R}$. Given a surface (Y, C) of pure shape, the irreducible components of C are called **sides**. There are two sides with a point in common called left or right. They decompose $C = C_1 + C_2$. A side can be **long** or **short** depending on whether a side C' satisfies $C' \cdot L = 2, 4$ or $C' \cdot L = 1, 3$ respectively. The ADE surfaces of pure shape are listed in [AT21, Table 1] (see Figures$_{1}$,2,˜3 therein). Here are some basic examples:

- The ADE surface (Y, C) corresponding to D_4 is $Y = \mathbf{P}^1 \times \mathbf{P}^1$ and C is the sum of two incident torus fixed curves.
- The ADE surface (Y, C) corresponding to A_1 is $Y = \mathbf{P}^2$ and C is the sum of two torus fixed curves. The polarization is $\mathcal{O}(2)$.

Remark 6.9. The superscripts minus signs on the left or right denote the location of the short side. Note both sides can be long or short. Do they correspond to the visible length? Not at all! The ADE surface A_3 has two long sides, but one edge is shorter than the other. By the way, in this case, $Y = \mathbb{F}_2^0$. A_2^- has a long side on the left and a short side on the right.

Primed shapes. Priming is an operation that produces a new del Pezzo surface $(\overline{Y}', \overline{C}')$ from an old one (Y, C). The priming operation is basically a weighted blow-up given by the composition of two ordinary blow-ups and the contraction of a (-2)-curve making an A_1 singularity. Weighted blow-ups of this form are the basis of the priming operation. Weighted blow-up with respect to the idea (y, x^2). Priming has the meaning of disconnecting a curve from another. Given an ADE pair $(Y, C + \frac{1+\epsilon}{2})$, then the priming operation is performed on the points of intersection between C and B, which intersect transversely by [AT21, Remark 3.3]. Priming may not exist, and there are some necessary and sufficient conditions for priming to exist.

For an ADE shape, we add a prime symbol when priming on a long side. When priming a short side, we change the minus into a plus. All the ADE surfaces, pure or primed are in [AT21, Table 2].

7. Our new results

7A. Enriques strategy.

Remark 7.1. This story suggests the following approach to Enriques surfaces:

- Fully understand the cusps of the Enriques moduli space, possibly in terms of what has been done for K3s already.
- For each cusp, find the Coxeter diagram.
- For each Coxeter diagram, cook up the right IAS pair of a manifold and a divisor R_{IAS}. For us, instead of an IAS^2 it may be an IARP^2, and may come from some fusion of known IAS^2s for K3s, maybe as simple as quotienting the IAS^2 by the antipodal map.
- Reverse-engineer the IARP^2 so that it carries two commuting involutions, and probably take R_{IAS} to be the intersection of the two ramification divisors on the IARP^2.
- Describe all of the ways the IARP^2 can degenerate, a la Valery's pumpkin-type models.

7B. Lemmas/theorems.

Lemma 7.2. *If* $\operatorname{sig}(L) = (p, q)$ *and* $e \in L$ *is isotropic, then* $\operatorname{sig}(\mathbf{Z}e) = (?, ?)$ *and* $\operatorname{sig}(\mathbf{Z}e^\perp) = (?, ?)$.

Lemma 7.3. *Let* L *be a lattice of signature* (p, q) *and let* $e \in L$ *be an isotropic vector. Then*

$$\operatorname{sig}(e^\perp / e) = (p-1, q-1).$$

Proposition 7.4. *Let* $\text{II}_{3,19}$ *be the K3 lattice and let* h *be an ample class of degree* d. *Then*

$$L_{2d} := h^{\perp \text{II}_{3,19}} \cong \langle -2d \rangle \oplus U^{\oplus 2} \oplus {E_8}^{\oplus 2}$$

and $\operatorname{sig} L_{2d} = (2, 19)$. *Thus* F_{2d} *arises as the Hodge-theoretic moduli space associated with the period domain* D_L *for the lattice* $L := L_{2d}$.

Remark 7.5. The theorem below needs the following notations and conventions (these will all be introduced before as needed).

$\mathrm{II}_{3,19} = U^{\oplus 3} \oplus E_8^{\oplus 2} = (U \oplus E_8)^{\oplus 2} \oplus U$ and

$$I(m, m', h) = (m, m', -h).$$

$$L_- = U \oplus U(2) \oplus E_8(2)$$

$$\Omega_- = \left\{[v] \in \mathbf{P}(L_- \otimes \mathbb{C}) \;\middle|\; v^2 = 0,\ v \cdot \overline{v} > 0\right\}$$

Definition 7.6. Consider the following subgroup of $O(\mathrm{II}_{3,19})$:

$$\Gamma' = \left\{g \in O(\mathrm{II}_{3,19}) \;\middle|\; g \circ I = I \circ g,\ g(e+f, e+f, 0) = (e+f, e+f, 0)\right\}$$

Note that we have a natural group homomorphism $\Gamma' \to O(L_-)$ given by $g \mapsto g|_{L_-}$. To prove that $g|_{L_-} \in O(L_-)$ it is enough to observe that $g(L_-) = L_-$. Let $x \in L_-$. We have that $g(x) \in L_-$ if $I(g(x)) = -g(x)$. This holds because

$$I(g(x)) = g(I(x)) = g(-x) = -g(x).$$

We denote by Γ the image of $\Gamma' \to O(L_-)$.

Remark 7.7.

$$E_2 = \Omega_-/\Gamma$$

$$\Omega_{4,\mathrm{h}} = \left\{[v] \in \mathbf{P}(\Lambda_{18} \otimes \mathbb{C}) \;\middle|\; v^2 = 0,\ v \cdot \overline{v} > 0\right\}$$

$\Lambda_{18} = U^{\oplus 2} \oplus D_{16}$, $\Gamma_{4,\mathrm{h}} = O(\Lambda_{18})$.

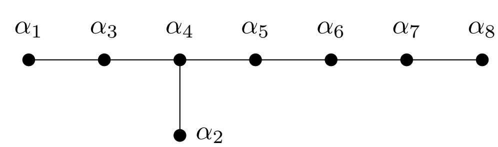


Figure 4. The E_8 lattice (Sterk's convention).

Theorem 7.8. *There exists an injective morphism*

$$E_2 \to F_{4,\mathrm{h}}$$

which extends to a morphism of the Baily–Borel compactifications

$$\overline{E}_2^{\mathrm{bb}} \to \overline{F}_{4,\mathrm{h}}^{\mathrm{bb}}.$$

Proof. Consider the inclusion of $U(2)$ into $U(2) \oplus E_8(2)$ as direct summand. By considering the orthogonal complements in $\mathrm{II}_{3,19}$ we obtain that

$$L_- \subseteq \Lambda_{18}.$$

From this follows from the definitions of Ω_- and $\Omega_{4,\mathrm{h}}$ that we have an inclusion

$$\Omega_- \hookrightarrow \Omega_{4,\mathrm{h}}.$$

Let us show that this descends to a morphism

$$\Omega_-/\Gamma_2 \to \Omega_{4,\mathrm{h}}/\Gamma_{4,\mathrm{h}}.$$

Let $[v], [w] \in \Omega_2$ and assume there exists $g \in \Gamma$ such that $g([v]) = [w]$. We show that there exists $h \in \Gamma_{4,\mathrm{h}}$ such that $h([v]) = [w]$. By the definition of Γ, $g = \widetilde{g}|_{L_-}$, there exists $\widetilde{g} \in O(\mathrm{II}_{3,19})$ such that $\widetilde{g} \circ I = I \circ \widetilde{g}$ and $f(e+f, e+f, 0) = (e+f, e+f, 0)$. Then, by the proof of [Ste91, Proposition 2.7], we have that $\widetilde{g}$ preserves $e + f$ and $e - f$ in $L_+ = U(2) \oplus E_8(2)$. In particular, $\widetilde{g}$ preserves the summand $U(2) \subseteq U(2) \oplus E_8$. This implies that $\widetilde{g}$ preserves $U(2)^\perp = \Lambda_{18}$. In particular, by setting $h = \widetilde{g}|_{\Lambda_{18}}$ we obtain what we needed.

We now prove that the morphism $\varphi\colon \Omega_-/\Gamma \to \Omega_{4,\mathrm{h}}/\Gamma_{4,\mathrm{h}}$ is injective. Let $x_1, x_2 \in \Omega_-/\Gamma$ and assume that $\varphi(x_1) = \varphi(x_2)$. Let S_i be the Enriques surface corresponding to x_i. Then S_i is the quotient of a K3 surface T_i which is the double cover $\pi_i\colon T_i \to \mathbf{P}^1 \times \mathbf{P}^1$ branched along a $(4, 4)$ curve B_i which is invariant with respect to the involution $\iota\colon (x, y) \mapsto (-x, -y)$. Because of the assumption that $\varphi(x_1) = \varphi(x_2)$, we must have that

$$\begin{array}{ccc} T_1 & \xrightarrow{\cong} & T_2 \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \pi_2} \\ \mathbb{P}^1 \times \mathbb{P}^1 & \xrightarrow{\cong} & \mathbb{P}^1 \times \mathbb{P}^1 \end{array}$$

where the bottom isomorphism commutes with ι and the top map commutes with $\widetilde{\iota}$. Let τ_i be the deck transformation of the cover π_i, so that we have the two Enriques involutions $\sigma_i = \tau_i \circ \widetilde{\iota}$. Then we have an isomorphism between $S_1 = T_1/\sigma_1 \cong T_2/\sigma_2$, which implies that the period points x_1, x_2 are equal.

The morphism $\Omega_-/\Gamma \to \Omega_{4,\mathrm{h}}/\Gamma_{4,\mathrm{h}}$ extends to a morphism of the Baily–Borel compactifications by [KK72, Theorem 2], and sends boundary components to boundary components.

Next, we describe the cusp correspondence. Recall, $\overline{E}_2^{\mathrm{bb}}$ has five 0-cusps $p_1, \ldots, p_5$ corresponding to the following isotropic vectors in

$$L_- = U \oplus U(2) \oplus E_8(2) = \langle e, f\rangle \oplus \langle e', f'\rangle \oplus \langle \alpha_1, \ldots, \alpha_8\rangle.$$

1. $\delta_1 = e$;
2. $\delta_2 = e'$;
3. $\delta_3 = e' + f' + \overline{\alpha}_8$;
4. $\delta_4 = 2e' + f' + \overline{\alpha}_1$;
5. $\delta_5 = 2e + 2f + \overline{\alpha}_1$.

Note that $e' \cdot f' = 2$ and $\overline{\alpha}_i \cdot \alpha_j = \delta_{ij}$. We have that $\delta_1^\perp/\delta_1 \cong U(2) \oplus E_8(2)$ and $\delta_i^\perp/\delta_i \cong U \oplus E_8(2)$ for $i = 2, \ldots, 5$.

On the other hand, $\overline{F}_{4,\mathrm{h}}^{\mathrm{bb}}$ has two 0-cusps q_1, q_2 for which the corresponding isotropic vectors $\eta_1, \eta_2 \in \Lambda_{18}$ satisfy $\eta_1^\perp/\eta_1 \cong U \oplus E_8^{\oplus 2}$ and $\eta_2^\perp/\eta_2 \cong U(2) \oplus E_8^{\oplus 2}$.

To understand whether $p_1 \mapsto q_1$ or $p_1 \mapsto q_2$, it is enough to compute

$$\delta_1^{\perp_{\Lambda_{18}}}/\delta_1.$$

But this is clear after realizing that $\Lambda_{18} \cong U \oplus U(2) \oplus E_8^{\oplus 2}$, and there is the explicit embedding

$$\begin{aligned} L_- = U \oplus U(2) \oplus E_8(2) &\subseteq U \oplus U(2) \oplus E_8^{\oplus 2} \\ (u, v, w) &\mapsto (u, v, w, w). \end{aligned}$$

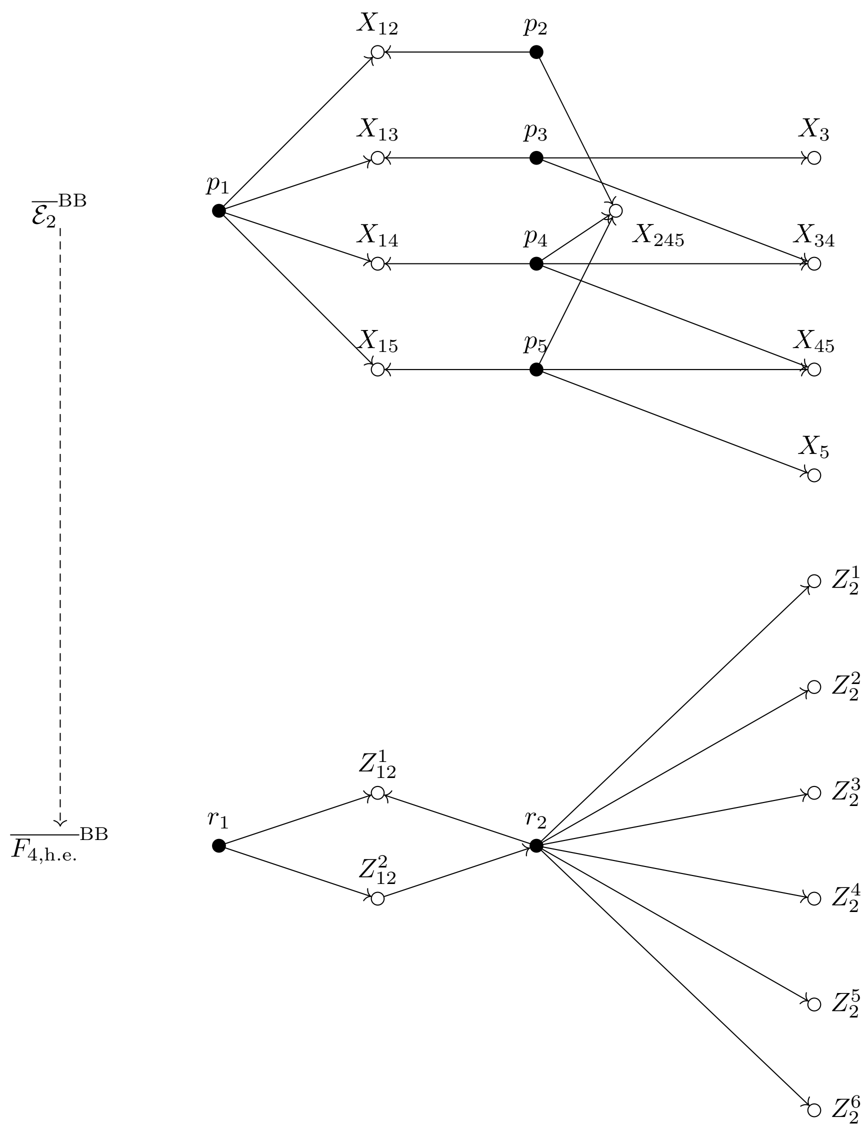
So that it is clear that

$$\delta_1^{\perp \Lambda_{18}}/\delta_1 = U(2) \oplus E_8(2).$$

□

Lemma 7.9. *Cusp correspondence 1*

We have a cusp correspondence from polarized $\{X_i, p_i\}$ *in* $\partial \overline{\mathcal{E}_2}^{\mathrm{BB}}$ *to hyperelliptic* $\{Z_i, r_i\}$ *in* $\partial \overline{F_{4,\mathrm{h.e.}}}^{\mathrm{BB}}$*:*



8. The Baily-Borel compactification

Remark 8.1. The Baily-Borel and toroidal compactifications are defined for quotients of Hermitian symmetric spaces by actions of arithmetic subgroups of their automorphism groups, i.e. those that can be written as $\Gamma\diagdown\Omega$. BB compactifications are generally small, e.g. $\dim F_2 = 19$ but $\operatorname{codim} \partial\overline{F_2}^{\mathrm{BB}} = 18$, and this often precludes having a satisfactory modular interpretation of its boundary points. In particular, given an arc in this compactification with endpoint in the boundary, one can not generally construct a birationally unique limit. Toroidal compactifications $\overline{\Gamma\diagdown\Omega}^{\mathrm{Tor}}$ are obtained as certain blowups of $\overline{\Gamma\diagdown\Omega}^{\mathrm{BB}}$, and e.g. for F_2 some boundary components become divisors (codimension 1). However these are highly non-unique and depend on choices of fans. One might hope there are canonical such choices. The semitoric compactifications of Looijenga interpolate between $\overline{\Gamma\diagdown\Omega}^{\mathrm{BB}}$ and $\overline{\Gamma\diagdown\Omega}^{\mathrm{Tor}}$.

Definition 8.2. The group G acts transitively on the set of boundary components $F \subseteq \partial\mathcal{D}_L \coloneqq \widetilde{\mathcal{D}}_L \setminus \mathcal{D}_L$, and $\operatorname{Stab}_G(F) \leq G$ is a maximal parabolic subgroup. Taking stabilizers establishes a bijection

$$\begin{aligned}\{\text{Boundary components } F \subseteq \partial\mathcal{D}_L\} &\to \{\text{Maximal parabolic subgroups } P \leq G\} \\ F &\mapsto P_F \coloneqq \operatorname{Stab}_G(F)\end{aligned}$$

For $G \coloneqq \mathrm{SO}(V)$, parabolic subgroups P are stabilizers of flags of isotropic subspaces in V, and since $\operatorname{sig}(V) = (2, n)$, a flag has length at most 3 and a maximal flag is of the form $p \subseteq I \subseteq J$ where p is a point, I is an isotropic line, and J is an isotropic plane. The only flags that define maximal parabolic subgroups of $\mathrm{SO}(V)$ are of length 1, consisting of either a single line or a single plane. Thus we have bijections

$$\begin{aligned}\{\text{Rational boundary components of } (\mathrm{SO}_{2,n}(\mathbf{R}), \mathrm{SO}_2(\mathbf{R}) \times \mathrm{SO}_n(\mathbf{R}))\} \\ \downarrow\!\uparrow \\ \{\text{Maximal parabolic subgroups of } \mathrm{SO}(V)\} \\ \downarrow\!\uparrow \\ \{\text{Isotropic lines } I \subset V\} \cup \{\text{Isotropic planes } J \subset V\}\end{aligned}$$

where a boundary component F is rational if $\operatorname{Stab}_G(F)$ is defined over $\mathbf{Q}$. For an arithmetic subgroup $\Gamma \leq G$, letting $\partial(\mathcal{D}_L)_{\mathbf{Q}}$ be the set of all rational boundary components of $\mathcal{D}_L \subset \widetilde{\mathcal{D}}_L$, we produce a compactification

$$\overline{\Gamma\diagdown\mathcal{D}_L} = \Gamma\diagdown\mathcal{D}_L \bigcup_{F \in \partial(\mathcal{D}_L)_{\mathbf{Q}}} (G_F(\mathbf{Q}) \cap \Gamma)\diagdown F$$

Definition 8.3. Let L be a lattice of signature $(2, n)$ for $n \geq 1$, let Ω_L be the associated period domain, let $\mathrm{O}^+(L) \leq \mathrm{O}(L)$ be the subgroup preserving Ω_L and let $\widetilde{\Omega}_L$ be the affine cone over Ω_L. Let $n \geq 3$, let $k \in \mathbf{Z}$, and let $\Gamma \leq \mathrm{O}^+(L)$ be a finite-index subgroup with $\chi : \Gamma \to \mathbf{C}^*$ a character. A holomorphic functional $f : \Omega_L \to \mathbf{C}$ is called a **modular form of weight** k **and character** χ **for** Γ if

- Factor of automorphy: $f(\lambda z) = \lambda^{-k} f(z)$ for any $\lambda \in \mathbf{C}^*$.
- Equivariance: $f(\gamma z) = \chi(\gamma) f(z)$ for all $\gamma \in \Gamma$.

Definition 8.4. Let Ω_L as above and let $M_k(\Gamma, \chi)$ be the $\mathbf{C}$-vector space of such modular forms of weight k for Γ with character χ. The **Baily-Borel compactification** can be defined as

$$\overline{\Gamma \diagdown \Omega_L}^{\mathrm{BB}} := \operatorname{Proj} \bigoplus_{k \geq 1} M_k(\Gamma, \chi_{\mathrm{triv}})$$

where χ_{triv} is the trivial character.

Remark 8.5. $\partial \overline{\Gamma \diagdown \Omega_L}^{\mathrm{BB}}$ decomposes into points p_i and curves C_j, which are in bijection with Γ-orbits of isotropic lines i and isotropic planes j in $L_{\mathbf{Q}}$. Moreover $p_i \in \overline{C_j} \iff$ one can choose representatives lines i and planes j such that $i \subseteq j$.

Remark 8.6. A theorem of Baily-Borel gives the existence of an ample automorphic line bundle $\mathcal{L}$ on $\overline{\mathcal{D}_L}$ giving it the structure of a normal projective variety isomorphic to a canonical model $\operatorname{Proj} \bigoplus_{k \geq 0} H^0(L^k)$??. We denote this compactification $\overline{\Gamma \diagdown \mathcal{D}_L}^{\mathrm{BB}}$.

I don't quite remember what this graded ring is.

8A. **Toroidal and semitoroidal compactifications.**

Definition 8.7. A **toroidal compactification** $\overline{\Gamma \diagdown \mathcal{D}_L}^{\mathrm{Tor}}$ is a certain blowup of $\overline{\Gamma \diagdown \mathcal{D}_L}^{\mathrm{BB}}$, so there is a birational map $\overline{\Gamma \diagdown \mathcal{D}_L}^{\mathrm{Tor}} \dashrightarrow \overline{\Gamma \diagdown \mathcal{D}_L}^{\mathrm{BB}}$. It is defined by a collection of admissible fans $\{F_i\}_{i \in I}$ where I ranges over an index set for all cusps.

Definition 8.8. A **semitoroidal compactification** is a generalization due to Looijenga for which the cones of F_i are not required to be finitely generated.

Remark 8.9. [AE21] shows that semitoroidal compactifications are characterized as exactly the normal compactifications $\overline{M}^{\mathrm{SemiTor}}$ fitting into a tower

$$\begin{array}{c} \overline{M}^{\mathrm{Tor}} \\ \uparrow \\ \overline{M}^{\mathrm{SemiTor}} \\ \uparrow \\ \overline{M}^{\mathrm{BB}} \end{array}$$

where $\overline{M}^{\mathrm{Tor}}$ is some toroidal compactification of M.

Promising stuff here: https://dept.math.lsa.umich.edu/~idolga/EnriquesOne.pdf#page=567&zoom=160
136,765

Remark 8.10. On the toroidal compactification associated with $\Gamma \diagdown \Omega_L$: for a cusp C_i of the BB compactification, let $F \subset L_{\mathbf{Q}}$ be the corresponding Γ-orbit of an isotropic line or plane. Consider its stabilizer $S(F) := \operatorname{Stab}_{\mathrm{O}^+(L_{\mathbf{R}})}(F)$, and its unipotent radical $U(F)$. Then $U(F)$ is a vector space containing a lattice $U(F) \cap \Gamma$ and an open convex cone $C(F)$. Let $C(F)^{\mathrm{rc}}$ be the rational closure of the cone, so the union of $C(F)$ and rational rays in its closure. We then choose a fan $\Sigma(F)$ with $\operatorname{supp}(\Sigma(F)) = C(F)^{\mathrm{rc}}$ which is invariant under $S(F) \cap \Gamma$ and produce an associated toric variety $X_{\Sigma(F)}$. If one does this for every F to produce a Γ-admissible collection of polyhedra Σ, their quotients by Γ glue to give a toroidal compactification $\overline{\Gamma \backslash \Omega_L}^{\mathrm{Tor}}$, which has the structure of a (complex) algebraic space. There is a surjection $\overline{\Gamma \backslash \Omega_L}^{\mathrm{Tor}} \twoheadrightarrow \overline{\Gamma \backslash \Omega_L}^{\mathrm{BB}}$. Why $e^\perp/e$ shows up: if e is an isotropic line in L corresponding to a cusp F, there is an isomorphism of lattices $U(F) \cap \widetilde{\mathrm{O}}^+(L) \cong e^\perp/e$ where $\widetilde{\mathrm{O}}^+ \coloneqq \ker\left(\mathrm{O}^+(L) \to \mathrm{O}(A_L)\right)$.

8B. Misc.

Definition 8.11 (Log CY pairs)**.** A **log Calabi-Yau (CY) pair** is a pair (X, D) with X a proper variety and D an effective $\mathbf{Q}$-Cartier divisor such that the pair is log canonical and $K_X + D \sim_{\mathbf{Q}} 0$.

Definition 8.12. A degeneration $\pi : \mathcal{X} \to \Delta$ is a **CY degeneration** if π is proper, $K_{\mathcal{X}} \sim_{\mathbf{Q}} 0$, and $(\mathcal{X}, \mathcal{X}_0)$ is dlt. This implies that $\mathcal{X}_t$ is a Calabi-Yau variety for $t \neq 0$ and $\mathcal{X}_0$ is a union of log CY pairs (V_i, D_i). If $\mathcal{X}_t$ is a strict CY of dimension n, so $\pi_1 \mathcal{X}_t = 0$ and $h^i(\mathcal{X}_t, \mathcal{O}_{\mathcal{X}_t}) = 0$ for $1 \leq i \leq n-1$, and $\dim \Gamma(\mathcal{X}_0) = n$, we say $\mathcal{X}$ is a large complex structure limit or equivalently a maximal unipotent or MUM degeneration.

Remark 8.13. If $n = 2$, Kulikov shows that $\Gamma(\mathcal{X}_0)$ is always isomorphic to a 2-sphere S^2. Whether $\Gamma(\mathcal{X}_0) \cong S^n$ or a quotient thereof for $n \geq 3$ more generally is an open question, posed by Kontsevich-Soibelman. It has recently been shown by Kollár-Xu that in the case of degenerations of Calabi-Yau or hyperkähler manifolds, the dual complex is always a rational homology sphere.

Remark 8.14. Why this is useful to us: one formulation of mirror symmetry is the formulation due to Strominger-Yau-Zaslow, aptly called SYZ mirror symmetry. Conjecturally, the general fiber $\mathcal{X}_t$ of a punctured family of CYs $\mathcal{X}^\circ \to \Delta^\circ$ can be given the structure of a special Lagrangian torus fibration $\mathcal{X}_t \to B$, one can "dualize" the fibration to obtain a mirror CY $\widehat{\mathcal{X}_t} \to B$ over the same base. The common base B of these two fibrations is conjecturally of the form $\Gamma(\mathcal{X}_0)$, the dual complex of a degeneration $\mathcal{X} \to \Delta$ extending $\mathcal{X}^\circ$.

This might have something to do with the discrete Legendre transform Phil mentions.

8C. Baily-Borel cusps and incidence diagrams.

8D. Other compactifications.

Remark 8.15. See https://arxiv.org/pdf/2010.06922.pdf#page=1&zoom=auto,-17,32

Write $F_{2d} \coloneqq \Gamma_{2d} \backslash D_{2d}$. A cusp p_i of $\overline{F_{2d}}^{\mathrm{BB}}$ determines a cone C_i. Toroidal and semitoroidal compactifications are then determined by a collection of Γ_{2d}-invariant fans supported on C_i for i ranging over an index set for all cusps. If $d = 1$ (or more generally if $2d$ is squarefree), there is a single 0-cusp p_{2d} whose cone C_{2d} has a description as the positive light cone in the rational closure of C_{2d} with respect to a certain lattice M_{2d}. This can be written $C_{2d}^{\mathrm{rc}} \coloneqq \operatorname{Conv}(\overline{C}_{2d} \cap M_{2d} \otimes_{\mathbf{Z}} \mathbf{R})$. A semitoroidal compactification of F_{2d} is then determined by a semitoric fan in $M_{2d,\mathbf{R}}$ supported on C_{2d}^{rc} which is invariant for a particular subgroup $\Gamma_{2d}^+ \leq \mathrm{O}(M_{2d})$. In this case, one can make a canonical choice for such a semitoric fan: the Coxeter fan

Todo: can spell out what M_{2d}, Γ, Γ^+ are.

Σ_{2d}^{Cox} whose cones are precisely the fundamental domains for a Weyl group action on C_{2d}^{rc}, see AET19.

9. Scattone's Baily Borel compactifications

9A. Degree $2d$ compactifications.

Remark 9.1. The main reference for this section is [Sca87b], which describes the Baily-Borel compactifications of F_{2d}. The main result of this work is to describe $\partial \overline{F_{2d}}^{\text{BB}}$ using lattice-theoretic techniques, giving partial cusp diagrams for certain arithmetically constrained values of d. In particular, it shows that the number of 1-cusps is asymptotic to d^8, and the number of 0-cusps is 1 when d is squarefree, and otherwise is given by the function ???. Complete details are given for the cases $d = 1, 2$.

Remark 9.2. One first notes that by the global Torelli theorem for algebraic K3 surfaces [PS71b], F_{2d} admits a coarse space of the form D_{L_d}/Γ_{L_d} for a certain choice of lattice L_d. Note that D_{L_d} is a 19-dimensional bounded symmetric domain of type IV and Γ_{L_d} is an arithmetic group acting upon it. Recall that the theory of automorphic forms realizes $\overline{D_{L_d}/\Gamma_{L_d}}^{\text{BB}}$ as a projective variety. If L_d is the primitive cohomology of a polarized K3 surface, there is a correspondence between n-dimensional boundary cusps and Γ_{L_d}-orbits of $n+1$-dimensional isotropic subspaces in L_d. The boundary of the Baily-Borel compactification is "small" in the sense that it has very high codimension, and thus the geometric information it contains is insufficient to reconstruct a birationally unique family from a family over the punctured disc.

Remark 9.3. A polarization of degree $2d$ on a K3 surface is a primitive divisor H with $H^2 = 2d > 0$ which is pseudoample, i.e. $HD \geq 0$ for any effective $D \in \operatorname{Div}(X)$. A primitively polarized K3 surface is a pair (X, H). Choose a marking $\phi : H^2(X; \mathbf{Z}) \to \mathrm{II}_{3,19}$ such that $\phi(H) = h$ where h is a fixed primitive vector satisfying $h^2 = 2d$. So $\mathrm{O}(\mathrm{II}_{3,19})$ acts transitively on the set of primitive vectors of a fixed square, such an isometry can always be found and the choice of h is irrelevant. Note that $(H, \omega_X) = 0$, and thus the period of (X, H) lies in $\Omega_{2d} \coloneqq \Omega_S$ for $S \coloneqq h^{\perp \mathrm{II}_{3,19}}$. Set $\Gamma_{2d} \coloneqq \operatorname{Stab}_{\mathrm{O}(\mathrm{II}_{3,19})}(h)$; then $\Gamma_{2d} \curvearrowright \Omega_{2d}$ discontinuously and Ω_{2d}/Γ_{2d} is a normal complex analytic space which serves as a coarse space for F_{2d} by [PS71b; Fri84]. Because any two choices of h are equivalent modulo $\mathrm{O}(\mathrm{II}_{3,19})$, the isomorphism class of $h^{\perp \mathrm{II}_{3,19}}$ depends only on d. Making an appropriate choice of h, one can identify

$$L_{2d} \coloneqq h^{\perp \mathrm{II}_{3,19}} \cong \langle -2d \rangle \oplus U^{\oplus 2} \oplus E_8^{\oplus 2}.$$

One can now define the period domain as $\Omega_{2d} \coloneqq \Omega_{L_{2d}}$. This consists of two connected components interchanged by conjugation, so $\Omega_{2d} = D_{2d} \cup \widetilde{D}_{2d}$, and we fix once and for all a choice of one component which we will denote D_{2d}. It is well known that

$$D_{2d} \cong \frac{\mathrm{SO}_{2,19}^0}{\mathrm{SO}_2 \times \mathrm{SO}_{19}}.$$

We set $\widetilde{\mathrm{O}}(L_{2d})$ to be the image of Γ_{2d} under the injection $\Gamma_{2d} \hookrightarrow \mathrm{O}(L_{2d})$ induced by restriction – note that this coincides with the general definition $\widetilde{\mathrm{O}}(L) \coloneqq \ker(\mathrm{O}(L) \to$

$\mathrm{O}(q_L)$). We set $\mathrm{O}_-(L_{2d})$ to be the index 2 subgroup that preserves the component D_{2d}, and $\Gamma_{2d} \coloneqq \widetilde{\mathrm{O}}(L_{2d}) \cap \mathrm{O}_-(L_{2d})$, we obtain identifications

$$F_{2d} \cong \Omega_{2d}/\widetilde{\mathrm{O}}(L_{2d}) = D_{2d}/\Gamma_{2d}.$$

We focus our attention on the latter definition, $F_{2d} \coloneqq D_{2d}/\Gamma_{2d}$, and more generally on compactifications of general D/Γ.

9B. **General theory.**

Remark 9.4. Let D be a symmetric bounded domain and $\Gamma \leq \mathrm{Aut}(D)$ a discrete arithmetic subgroup of automorphisms. Equivalently, we can write $D = G(\mathbf{R})/K$ for G a connected linear algebraic group defined over $\mathbf{Q}$ and K a maximal compact subgroup of $G(\mathbf{R})$. We then require that $\Gamma \leq G$ is arithmetic, i.e. $\Gamma \subseteq G(\mathbf{Q})$ and is commensurable with $G(\mathbf{Z})$. We will generally define $\overline{D/\Gamma}$ as D^*/Γ where $D \subseteq D^* \subseteq D^\vee$ is a subset of the compact dual via the Borel embedding, comprised of D and rational boundary components.

Remark 9.5. Regarding $D \subseteq D^\vee$, we have $\partial D = \coprod F_i$ where each F_i is a boundary component, i.e. a maximal connected complex analytic set. We set

$$N_F \coloneqq \left\{ g \in G(\mathbf{R}) \;\middle|\; gF = F \right\} \coloneqq \mathrm{Stab}_{G(\mathbf{R})}(F)$$

to be the stabilizer of a boundary component and note that the maximal parabolic subgroups of $G(\mathbf{R})$ are precisely those of the form N_F. A boundary component is rational when $N_F(\mathbf{C})$ is defined over $\mathbf{Q}$. Let $B(D)$ be the set of proper rational boundary components of D. Then there is a bijection

$$\begin{aligned} B(D) &\rightleftharpoons \{\text{Proper maximal parabolic } \mathbf{Q}\text{-subgroups of } G(\mathbf{C})\} \\ F &\rightleftharpoons N_F(\mathbf{C}) \end{aligned}$$

We can write

$$D^* = D \cup \amalg_{F \in B(D)} F.$$

Then $\overline{D/\Gamma} \coloneqq D^*/\Gamma$ can be written as

$$D/\Gamma \cup \coprod_{[F] \in B(D)/\Gamma} V_F$$

where V_F are varieties and we index over orbits of rational boundary components modulo Γ.

Remark 9.6. We can identify the V_F explicitly: write $G_F \coloneqq \mathrm{Stab}_{G(\mathbf{R})}(F)/\mathrm{Fix}_{G(\mathbf{R})}(F)$ and $N_{\Gamma,F} \coloneqq \mathrm{Stab}_\Gamma(F)/\mathrm{Fix}_\Gamma(F)$, then $V_F = F/N_\Gamma(F)$. Note that in applications to K3 surfaces, we have $G(\mathbf{R}) = \mathrm{SO}^0_{2,19}$. In this situation, we have a correspondence

$$\begin{aligned} \partial D_L &\rightleftharpoons \mathrm{OGr}(L_{\mathbf{R}}) \\ F &\rightleftharpoons E \end{aligned}$$

where E corresponds to F iff $\mathrm{Stab}_{\mathrm{O}(L_{\mathbf{R}})}(E) = \mathrm{Stab}_{G(\mathbf{R})}(F) \coloneqq N_F$. Restricting to rational boundary components corresponds to $\mathrm{OGr}(L_{\mathbf{Q}})$, which are further identified with $\mathrm{OGr}(L)$, the primitive isotropic sublattices of L. For any subgroup $\Gamma_L \leq G(\mathbf{Z})$ we obtain a bijection

$$B(D_L)/\Gamma_L \rightleftharpoons \mathrm{OGr}(L)/\Gamma_L$$

which preserves incidence relations.

Example 9.7. As an example, one can take the symplectic form on $\mathbf{Z}^{2g}$, which yields $G(\mathbf{R}) = \mathbf{P}\mathsf{Sp}_g(\mathbf{R})$ and D is the Siegel upper half space $\mathcal{H}^g$. Let $\Gamma = G(\mathbf{Z}) = \mathbf{P}\mathsf{Sp}_g(\mathbf{Z})$ be the full Siegel modular group. Then maximal parabolic $\mathbf{Q}$-subgroups of $G_\mathbf{R}$ correspond to stabilizers of rational isotropic subspaces of $L_\mathbf{R}$. Let $H^n \coloneqq \mathcal{H}^n/\mathsf{Sp}_n(\mathbf{Z})$, then $\overline{D/\Gamma} = H_g \cup H_{g-1} \cup \cdots \cup H_1 \cup H_0$ coincides with the Satake compactification.

Consider now the example of F_{2d}. Recall that D is a component of $\Omega \coloneqq \left\{ z \in \mathbf{P}L_\mathbf{C} \;\middle|\; z^2 = 0,\, z\overline{z} > 0 \right\}$ and $D^\vee = \left\{ z \in \mathbf{P}L_\mathbf{C} \;\middle|\; z^2 = 0 \right\}$ is a quadric. Stratify $\partial D = \partial_1 D \coprod \partial_0 D$, noting that $\partial_0 D = D^\vee \cap \mathbf{P}L_\mathbf{R}$. Then all points of $\partial_0 D$ are of the form $\mathbf{P}(\langle v \rangle_\mathbf{C})$ where $v \in L_\mathbf{R}, v^2 = 0$ is isotropic. All components in $\partial_1 D$ are of the form $\mathbf{P}(\langle v, w \rangle_\mathbf{C}) \cap \partial_1 D$ where $\langle v, w \rangle_\mathbf{R}$ varies in $\mathrm{OGr}_2(L_\mathbf{R})$. Restricting to rational components, one considers $\mathrm{OGr}_2(L)$ instead, i.e. subspaces $E_\mathbf{R}$ arising from sublattices $E \leq L$. We thus obtain bijections

$$\begin{aligned} B_0(D) &\rightleftharpoons \mathrm{OGr}_1(L) \\ \mathbf{P}\langle v \rangle_\mathbf{C} &\rightleftharpoons \langle v \rangle_\mathbf{Z} \end{aligned}$$

and

$$\begin{aligned} B_1(D) &\rightleftharpoons \mathrm{OGr}_2(L) \\ \mathbf{P}\langle v, w \rangle_\mathbf{C} \cap \partial_1 D &\rightleftharpoons \langle v, w \rangle_\mathbf{Z} \end{aligned}$$

One can then write

$$\overline{D/\Gamma} = D/\Gamma \cup \coprod_{[\langle v \rangle_\mathbf{Z}] \in \mathrm{OGr}_1(L)/\Gamma} p_v \cup \coprod_{[\langle v, w \rangle_\mathbf{Z}] \in \mathrm{OGr}_2(L)/\Gamma} \frac{\mathbf{P}\langle v, w \rangle_\mathbf{C} \cap \partial_1 D}{N_\Gamma(\langle v, w \rangle_\mathbf{Z})}$$

where $N_\Gamma(E) = \mathrm{Stab}_\Gamma(E)/\mathrm{Fix}_\Gamma(E)$ can be identified with the image of $\mathrm{Stab}_\Gamma(E)$ in $\mathrm{SL}(E) \cong \mathrm{SL}_2(\mathbf{Z})$.

10. Moduli of polarized K3 surfaces of degree $2d$

Remark 10.1 (K3 surfaces)**.** A **K3 surface** is a smooth projective surface with trivial canonical bundle $\omega_X \cong \mathcal{O}_X$ and $h^1(\mathcal{O}_X) = 0$. Prototypical examples include double branched covers of sextic curves in $\mathbf{P}^2$ and smooth quartic hypersurfaces in $\mathbf{P}^3$. All K3 surfaces are diffeomorphic, and thus have the Hodge diamond shown in fig. 5.

$$\begin{matrix} & & h^{2,2} & & \\ & h^{2,1} & & h^{1,2} & \\ h^{2,0} & & h^{1,1} & & h^{0,2} \\ & h^{1,0} & & h^{0,1} & \\ & & h^{0,0} & & \end{matrix} \quad = \quad \begin{matrix} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{matrix}$$

Figure 5. The Hodge diamond of a K3 surface.

Remark 10.2 (Lattice theory in moduli)**.** The cup product endows the singular cohomology $H^2(X, \mathbf{Z})$ with the structure of a lattice, where by a **lattice** we mean a finitely generated free **Z**-module with a nondegenerate **Z**-valued symmetric bilinear form. It is isometric to the K3 lattice:

$$H^2(X; \mathbf{Z}) \cong \mathrm{II}_{3,19} \coloneqq U \oplus U \oplus U \oplus E_8 \oplus E_8,$$

where U is the hyperbolic lattice, the unique even unimodular lattice of rank 2 with Gram matrix $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ and E_8 is the negative-definite lattice associated to the E_8 Dynkin diagram.

By the weak Torelli theorem for K3 surfaces [Bar+04, Cor. 8.1.1.2], the moduli theory of K3 surfaces is regulated by this lattice structure. Of particular importance is the **Néron-Severi** lattice $\mathrm{NS}(X) \coloneqq H^{1,1}(X) \cap H^2(X, \mathbf{Z})$ of integral $(1,1)$ forms, and its orthogonal complement in $H^2(X, \mathbf{Z})$, the **transcendental lattice**. We refer to these as S_X and T_X respectively. By the Lefcshetz $(1,1)$ theorem [Bar+04, Thm. 4.2.13], the first Chern class $c_1 : \mathrm{Pic}(X) \to \mathrm{NS}(X)$ induces an isometry. Because the naive construction of a coarse moduli space of projective K3 surfaces yields a non-Hausdorff space, we restrict our attention to **polarized K3 surfaces of degree** $2d$ – pairs (X, L) where X is a K3 surface and L is an ample line bundle on X satisfying $L^2 = 2d > 0$.

Remark 10.3 (Lattice polarized K3 surfaces)**.** Let S be a non-degenerate lattice of signature $(1, n)$ which admits a primitive embedding into $\mathrm{II}_{3,19}$ (which is of signature $(3, 19)$) and $T \coloneqq S^{\perp \mathrm{II}_{3,19}}$ which is of signature $(2, 19-n)$. We then define the period domain associated to S as a connected component D_S of

$$\Omega_S \coloneqq \left\{ [\sigma] \in \mathbf{P}\left(T \otimes \mathbf{C}\right) \;\middle|\; \sigma^2 = 0, \sigma\overline{\sigma} > 0 \right\},$$

yielding a Hermitian symmetric domain of Type IV. Let $\Gamma_S \coloneqq \widetilde{\mathrm{O}}(T)$ where

$$\widetilde{\mathrm{O}}(T) \coloneqq \ker(\mathrm{O}(T) \to \mathrm{O}(A_T))$$

and $A_T \coloneqq T^\vee / T$ is the discriminant group of T. It can be shown that

$$F_S \coloneqq \Gamma_S \backslash D_S$$

is a coarse moduli space of S-polarized K3 surfaces. Taking $S \coloneqq \langle h \rangle$ the sublattice generated by an ample class h satisfying $h^2 = 2d$ recovers $F_{2d} \coloneqq F_{\langle h \rangle}$, noting that $\langle h \rangle^{\perp \mathrm{II}_{3,19}} \cong \langle -2d \rangle \oplus U^{\oplus 2} \oplus E_8^{\oplus 2}$. By [PS71a], F_{2d} constructed in this way is a coarse moduli space of degree $2d$ primitively polarized K3 surfaces.

To such surfaces, by [BB66] there is a canonically defined quasiprojective compactification $\Gamma_S \backslash D_S \hookrightarrow \overline{\Gamma_S \backslash D_S}^{\mathrm{BB}}$ whose boundary consists of 0-cusps (points) and 1-cusps (curves). The 0-cusps are in bijection with Γ_S-orbits of isotropic lines $I \subseteq T$, and the 1-cusps are in bijection with orbits of isotropic planes $J \subseteq T$. By [Ash+75], there exists a class of compactifications $\Gamma_S \backslash D_S \hookrightarrow \overline{\Gamma_S \backslash D_S}^{\mathcal{F}}$ defined by the combinatorial data of a fan $\mathcal{F} \coloneqq \{\mathcal{F}_I\}$ where I ranges over Γ-orbits of isotropic lines in T, equivalently as I ranges over the 0-cusps of $\partial\overline{\Gamma_S \backslash D_S}^{\mathrm{BB}}$. These are referred to as toroidal compactifications. Noting that $\mathrm{sig}(I^{\perp T}/I) = (1, 18)$, we obtain a hyperbolic lattice and can construct a model of hyperbolic space $\mathbb{H}^{18}$ as a projectivization of the positive cone

$$C^+ \coloneqq \left\{ v \in I^\perp / I \otimes \mathbf{R} \;\middle|\; v^2 > 0 \right\}.$$

Letting $\Gamma_{S,I} \coloneqq \mathrm{Stab}_{\Gamma_S}(I)$, the data of $\mathcal{F}_I$ is specified by a $\Gamma_{S,I}$-invariant rational polyhedral tiling of $\mathbb{H}^{18}$. The combinatorics of such a tiling determines a union of toric varieties which are adjoined as the boundary strata of $\Gamma_S \backslash D_S$ at the 0-cusp corresponding to I. This produces a divisorial boundary with mild singularities. The work of [Loo86] introduces semitoroidal compactifications, allowing for the tiling to be *locally* rationally polyhedral, which simultaneously generalizes the Baily-Borel and toroidal compactifications described above.

Remark 10.4. A modular alternative to these compactifications was introduced in [KS88], denoted the space of stable slc pairs, which is proper. A pair (X, R) of a projective variety X with a $\mathbf{Q}$-divisor D is stable if $K_X + R$ is ample and $\mathbf{Q}$-Cartier and (X, R) has slc singularities. These strictly generalize the stable curves that appear in $\overline{\mathcal{M}_{g,n}}$, and naturally generalize these notions to all dimensions. For an appropriately universal choice of polarizing divisor R on the generic K3 surface in F_{2d}, one can define a compactification $\overline{F_{2d}}^R$ as the closure of the space of pairs $(X, \varepsilon R)$ in a Zariski open subset of F_{2d} in the space of stable slc pairs. For example, consider a degree 2 polarized K3 surface (X, L). The linear system $|L|$ induces a branched 2-to-1 cover $X \to \mathbf{P}^2$, and one can choose R to be the ramification divisor of the covering involution. By ???, when R is a *recognizable divisor*, there is a unique semifan $\mathcal{F}_R$ such that the normalization of $\overline{F_{2d}}^R$ is isomorphic to the semitoroidal compactification $\overline{F_{2d}}^{\mathcal{F}_R}$.

Remark 10.5 (Kulikov models)**.** Any degeneration of K3 surfaces is birational to a **Kulikov model**: after a birational modification and a ramified base change, it may be put in the form of a degeneration $\pi : \mathcal{X} \to \mathbf{D}$ over the complex disc such that π is semistable with trivial canonical $\omega_{\mathcal{X}} \cong \mathcal{O}_{\mathcal{X}}$.

By ?, the central fiber of a type III Kulikov model is encoded in an integral affine 2-sphere, abbreviated IAS^2.

11. Boundary

Goal for this section: describe how Coxeter-Vinberg diagrams are used to get models at BB cusps.

11A. The case of abelian varieties.

Let us consider the setup first for moduli of principally polarized abelian varieties. We have the following:

Theorem 11.1. *There is an isomorphism*

$$\eta : \overline{\mathcal{A}_g}^{\mathfrak{F}} \xrightarrow{\sim} (\overline{\mathcal{A}_g}^{\mathrm{KSBA}})^{\nu}$$

for F the second Voronoi fan.

Is there always a natural morphism from toric compactifications to KSBA? Not in general! We can talk about it when we meet.

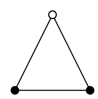
Corollary 11.2. *As a result, any punctured 1-parameter family $\mathcal{X}^\circ \to \Delta^\circ$ has a unique limit $\mathcal{X}_0$ which can combinatorially be described as a tropically polarized abelian variety $(X_{\mathrm{trop}}, \Theta_{\mathrm{trop}})$ with a tropical Θ divisor Θ_{trop}.*

More is true: the fan F is itself a moduli space for such tropical abelian varieties.

Remark 11.3 (Motivation from abelian varieties)**.** To see how this works, consider a 1-parameter family of abelian varieties. These are tori of the form

$$\mathcal{X}_t \coloneqq \mathrm{coker}(\phi_t : \mathbf{Z}^g \hookrightarrow (\mathbf{C}^*)^g),$$

where ϕ_t are embeddings that vary in the family. Write this embedding as a matrix M; this is a matrix of periods. Then for $t \approx 0$ one exponentiates M_{ij} to get a symmetric positive-definite $g \times g$ matrix B. There is a cone $C \subseteq \{B = B^t > 0\}$ in $\mathrm{GL}_g(\mathbf{Z})$ and a Coxeter fan F supported on its rational closure C^{rc} which corresponds to affine Dynkin diagram $\widetilde{A}_2$:

$\widetilde{A}_2$:


This corresponds to a triangular fundamental domain for a reflection group that acts on C. For a cartoon picture, think of a hyperbolic disc $\mathbf{D}$ and let the fundamental domain be a triangle with ideal vertices:

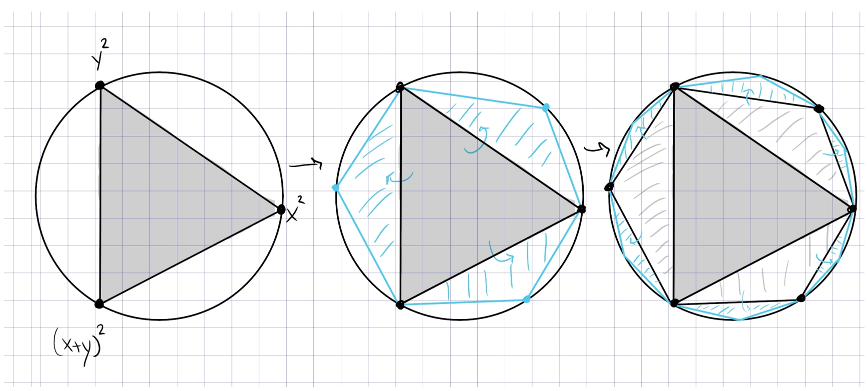


Figure 6. Tessellating a hyperbolic disc by triangles

Note that the straight lines forming the edges should "really" be curved hyperbolic geodesics. One continues reflecting in order to tessellate the hyperbolic disc, then puts this disc in $\mathbf{R}^3$ at height one and cones it to the origin to get an infinite-type fan F:

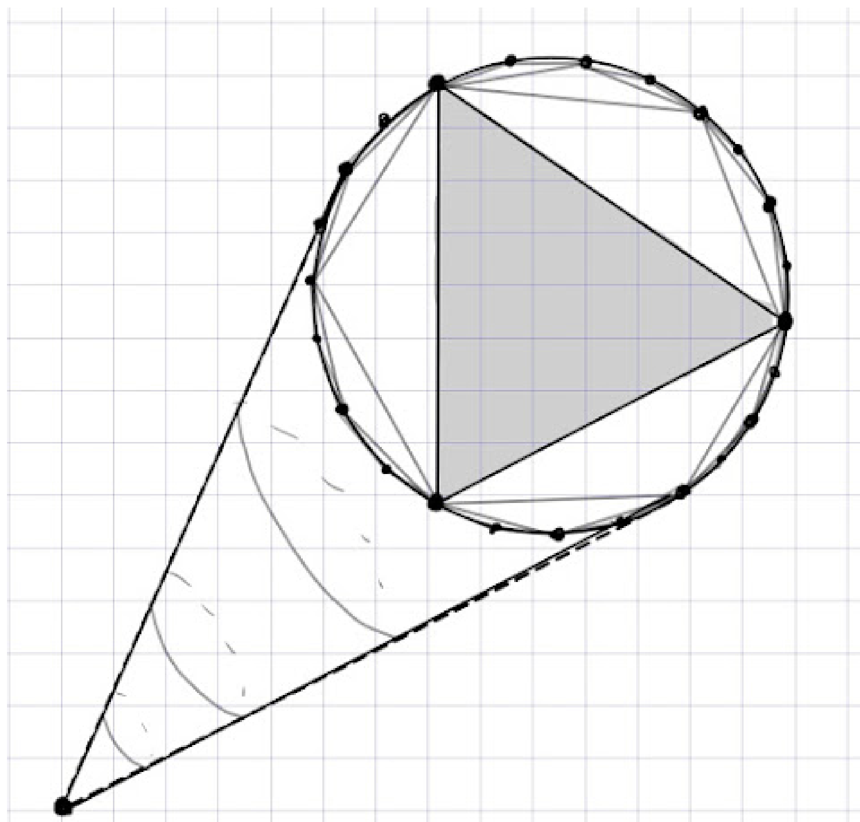


FIGURE 7. Coning off the hyperbolic disc to form a fan.

Note that the entire fan F admits an $\mathrm{SL}_2(\mathbf{Z})$ action. Now if one rewrites B as a form $B(x,y) = ax^2 + by^2 + c(x+y)^2$ with $a,b,c \in \mathbf{Z}_{\geq 0}$, the coordinate vector $\vec{v} \coloneqq (a,b,c)$ defines a point in some chamber of C^{rc}. In turn, $\vec{v}$ defines a 1-parameter degeneration of abelian varieties, and thus a pair $(X_{\mathrm{trop}}, \Theta_{\mathrm{trop}})$. When B is integral it defines an embedding $B : \Lambda \hookrightarrow \Lambda^\vee$ and thus one can construct a torus $T \coloneqq \Lambda^\vee_{\mathbf{R}}/\Lambda \cong \mathbf{R}^2/\mathbf{Z}^2$ which has finitely many integral points defined by Λ.

What is Λ?

Recall that for any lattice L there is an associated Voronoi tessellation by polytopes P_i, one such P_i centered around each lattice point ℓ_i. Let Vor_B be the Voronoi tessellation of Λ; this can be identified with a hexagonal honeycomb tessellation of $\mathbf{R}^2$:

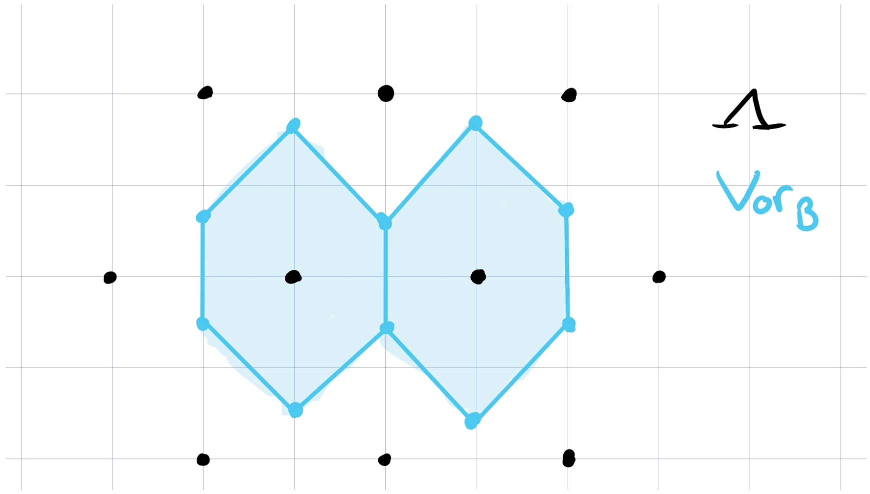


FIGURE 8. The Voronoi tesselation associated to $\widetilde{A}_2$, the triangular lattice.

One then defines $\Theta_{\mathrm{trop}} := B(\mathrm{Vor}_B)/\Lambda$, a quotient of the image of the hexagonal tessellation. Although the blue vertices in Vor_B generally have vertices with fractional coordinates, the vertices in the image have integral coordinate vertices with respect to $\Lambda^\vee$. The image of a regular hexagon is now a hexagon with side lengths a, b, c, and since we've quotiented by Λ, Θ_{trop} is determined by two hexagons with side lengths determined by $\vec{v} = (a, b, c)$ which are glued together:

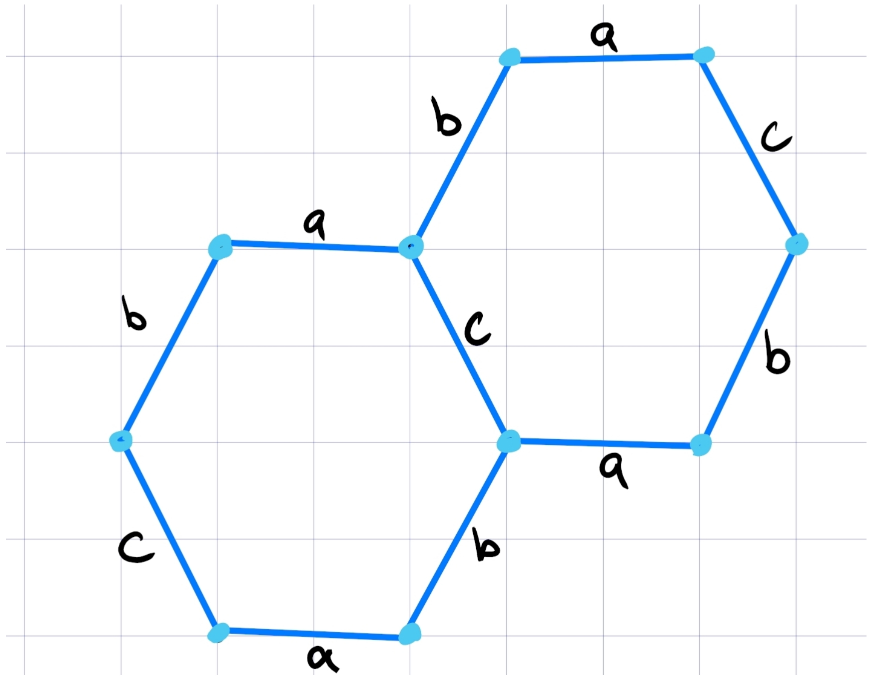


FIGURE 9. A picture of two relevant polytopes in the image of the Voronoi tessellation, which tessellates the entire dual lattice. After quotienting, these will be the only two relevant polygons.

The claim is that this picture describes an entire degeneration $\mathcal{X}$ of abelian varieties. To see the central fiber: every vertex w_i in this new tessellation defines an honest fan via $\mathrm{Star}(w_i)$; here there are 2 vertices of valence 3 and 3 edges in the quotient, so the central fiber $\mathcal{X}_0$ is two copies of $\mathbf{P}^2$ corresponding to w_1, w_2 glued together along three curves corresponding to a, b, c. To see the entire family: put this entire picture at height 1, cone to the origin to get a fan, and quotient that fan by a $\mathbf{Z}^2$ action to get $\mathcal{X}$.

Note that in the K3 case, things are harder because the combinatorics only describes $\mathcal{X}_0$ and not the entire family $\mathcal{X}$, so one has to appeal to abstract smoothing results to obtain the existence of a family $\mathcal{X}$ extending $\mathcal{X}_0$.

Moreover, the original fan F is a moduli of these polyhedral pictures. One can degenerate Θ_{trop} by sending some coordinates a, b, c to zero. This degenerates the honeycomb 6-gons into 4-gons if just one side goes to zero. For example, if $a \to 0$, this corresponds to being on a wall in Figure 7. If two coordinates degenerate, say $a, b \to 0$, this corresponds to being on a ray. This can be read off by recalling $B(x, y) = ax^2 + by^2 + c(x + y)^2$ and labeling the ideal vertices with monomials $x^2, y^2, (x+y)^2$ as in Figure 6. Thus varying $\vec{v} = (a, b, c)$ corresponds to varying the side lengths of hexagons and correspondingly moving through C^{rc}. Staying in the fundamental chamber doesn't change the overall combinatorial type of Figure 9, but passing through a wall will flip the hexagonal tiling in various ways.

11B. **To the K3 case.** The claim is that a similar story more or less goes through for K3s: the Coxeter diagram is much more complicated, and the relevant combinatorial device is an IAS^2 with 24 singularities instead of a tropical variety. We have the following:

Theorem 11.4. *There is a morphism*

$$\eta : \overline{F_2}^{\mathfrak{F}} \to (\overline{F_2}^{\mathrm{KSBA}})^{\nu}$$

where F is fan of a Coxeter diagram associated to a cusp of F_2, and the Stein factorization of η is through a semitoroidal compactification.

Corollary 11.5. *Any punctured 1-parameter family $\mathcal{X}^\circ \to \Delta^\circ$ has a unique limit $\mathcal{X}_0$ which can be combinatorially described as a singular integral-affine sphere with an integral-affine divisor* $(\mathrm{IAS}^2, R_{\mathrm{IAS}})$.

Remark 11.6. This is a much harder theorem than the $\mathcal{A}_g$ case: periods of K3 surfaces are highly transcendental, and the period map is not well-understood. Also note that the relevant Coxeter diagram for $\mathcal{A}_g$ was relatively simple, while the diagram for F_2 is the following:

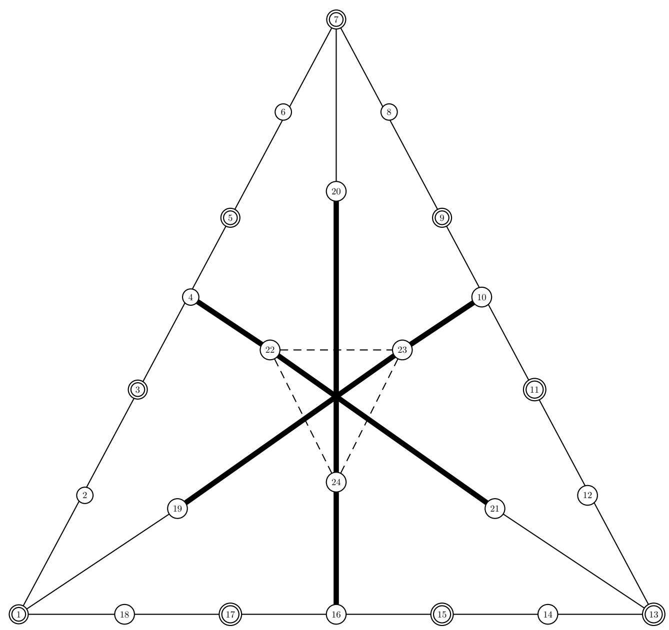


FIGURE 10. The Coxeter diagram for type $(19, 1, 1)$.

Nodes in Figure 10 correspond to roots spanning a hyperbolic lattice

$$N \coloneqq U \oplus E_8(-1)^{\oplus 2} \oplus A_1(-1), \qquad \operatorname{sig}(N) = (1, 18)$$

which is the Picard lattice of the Dolgachev-Nikulin mirror K3. Decorated nodes v_i record self-intersection numbers v_i^2, and edges between v_i and v_j record the intersection numbers $v_i.v_j$. Note that the Coxeter diagram also captures the data of all (-2) curves on the mirror K3 surface and their intersections.

This diagram again describes the fundamental chamber of a reflection group, and the cone in this case $C = \left\{ v^2 > 0 \right\}$. Toroidal compactifications of F_2 correspond to fans whose support is C^{rc} (i.e. the interior, plus rational rays on the boundary). There is a natural fundamental chamber defined by $\left\{ v \;\middle|\; v.r_i \geq 0 \right\}$ where $\{r_i\}$ are roots, the difference is that now some vertices of the fundamental chamber may be ideal vertices:

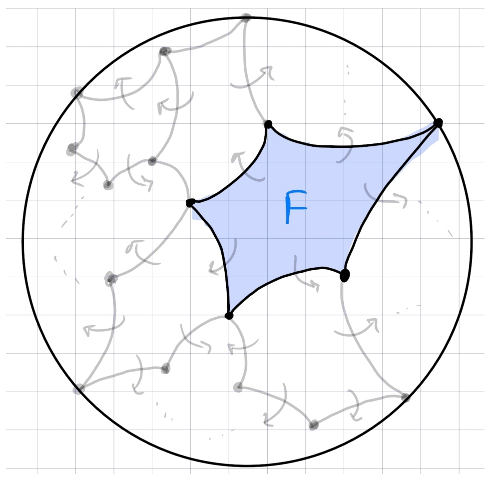


FIGURE 11. A fundamental chamber F for a reflection group. Reflecting over walls of F successively generates a tiling of the hyperbolic disc by copies of F. Note that one vertex is an ideal vertex, i.e. it is in $\partial \overline{\mathbb{H}^n}$.

Proceeding similarly to take the cone over this picture and allow rational boundary points yields the cone C^{rc} and a corresponding infinite-type fan $\mathcal{F}$ – this is a fan since the faces are rationally generated, F is a fundamental chamber for the reflection group $W(N)$, the fan is W-invariant by construction and moreover invariant under $\mathrm{O}(N)$. Since there is a short exact sequence

$$0 \to W(N) \to \mathrm{O}(N) \to S_3 \to 0$$

the index of $W(N)$ is finite and thus F is finite volume.

Points in this fan can naturally be interpreted as period points, so a choice of a point in the fan yields a degenerating family of K3 surfaces by the Torelli theorem. Let $v \in F$ be a point in the fundamental chamber, we will next consider how this corresponds to a combinatorial object, the same way $\vec{v} = (a, b, c)$ did in the case of $\mathcal{A}_g$.

First consider a fan with 18 rays, corresponding to a toric surface Σ with 18 curves. Note that the rays alternative between long and short vectors:

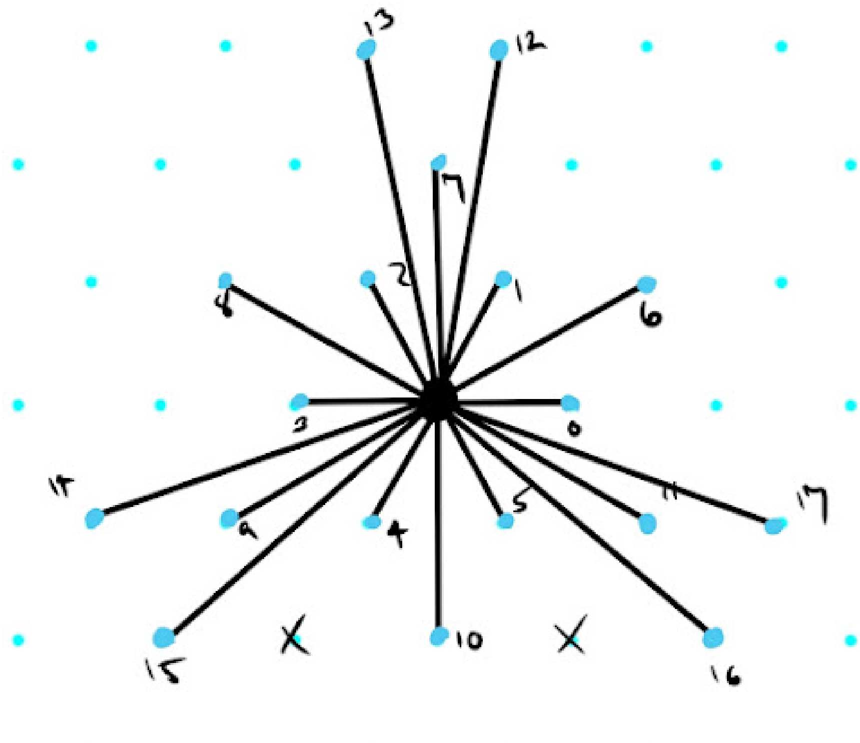


Figure 12. The starting point: a toric surface with 18 rays.

This corresponds to a polytope P_Σ which is an 18-gon (not necessarily regular) which is the moment polytope for X_Σ where Σ is the fan in Figure 12 and has edge lengths $\ell_0, \cdots, \ell_{17} \in \mathbf{R}$, which determines a polarization L for X_Σ. Although not shown in the picture here, we can call each edge "long" if it was dual to a long vector, and similarly "short" if dual to a short vector.

Note also that each edge can be written as $\ell_i v_i$ for v_i some unit vectors, and it is a nontrivial condition on $\vec{\ell}$ that this polygon closes. In particular, one needs $\sum_{i=0}^{17} \ell_i v_i = 0$.

Now cut triangles out of sides $0, 6, 12$ and call the resulting polygon non-convex polygon P. Each triangle cut corresponds to a non-toric blowup of X_Σ, i.e. a blowup at a point p which is not T-invariant. This introduces three new length parameters $\ell_{18}, \ell_{20}, \ell_{20}$ corresponding to the heights of these three triangles. Each will introduce an I_1 singularity to the moment polytope.

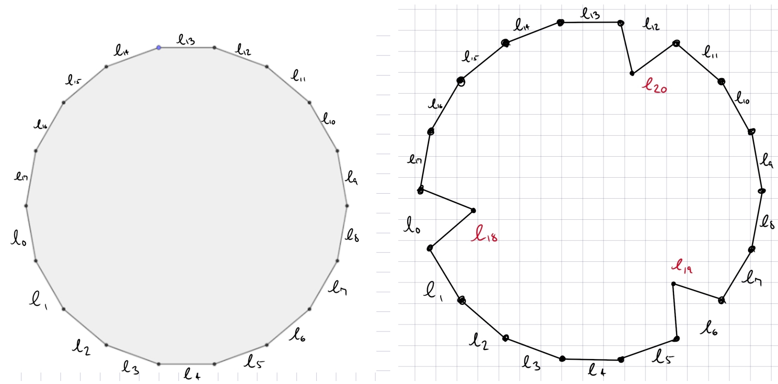


FIGURE 13. The Symington polytope: an 18-gon, before and after a nontoric blowup corresponding to cutting out triangles.

Regarding such polytopes as the Symington polytopes, which are bases of Lagrangian torus fibrations, these are in particular elliptic fibrations and these singularities precisely correspond to introducing singular type I_1 fibers in Kodaira's classification.

Take two copies of P, say P and P^{op}, and glue them together along the outer edges and call the result B. This is topolologically the gluing of two discs, and thus B is homeomorphic to S^2. Each gluing along the outer edges introduces a new I_1 singularity, yielding $3+3=6$ singularities in the hemispheres and 18 singularities along the equator for a total of 24 singularities of type I_1 and thus an IAS^2 with charge 24.

Note that there are now 24 length parameters: $\ell_0, \cdots, \ell_{17}$ along the equator, $\ell_{18}, \ell_{19}, \ell_{20}$ in the northern hemisphere, and $\ell_{21}, \ell_{22}, \ell_{23}$ in the southern hemisphere. The tuple $\vec{\ell} = (\ell_1, \cdots, \ell_{23})$ turns out to correspond to 24 vectors in a 19 dimensional space, and there are enough conditions to ensure the polygons actually close.

This produces the tropical sphere IAS^2, so one also needs to describe its tropical divisor R_{IAS}. The above construction works for any K3 with a nonsymplectic involution, e.g. an elliptic K3, and the IAS^2 is naturally equipped with an involution ι that swaps P and P^{op}. The ramification divisor of ι is the equator, highlighted in blue in the following cartoon picture of B, and one takes R_{IAS} to be the sum of the blue edges with coefficient 2 for even (short?) sides and coefficient 1 for odd (long?) sides:

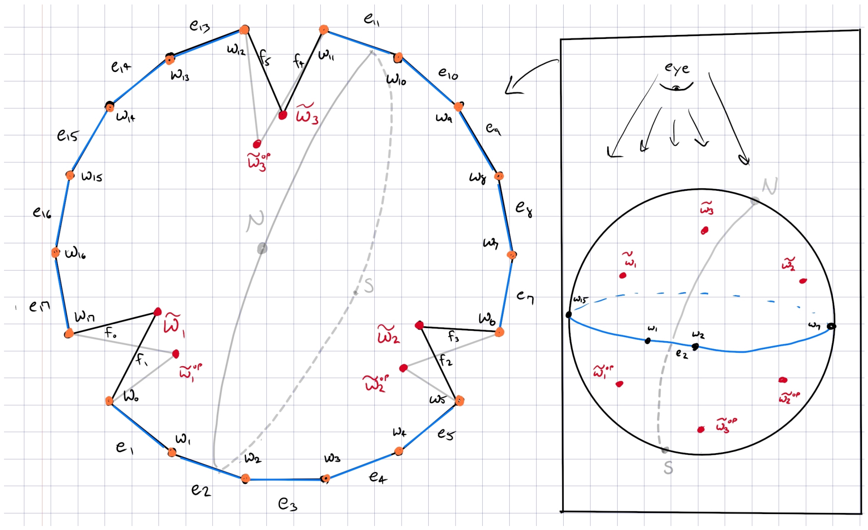


FIGURE 14. Caption

We now describe how one obtains a degeneration $\mathcal{X}$ of K3 surfaces from this combinatorial picture. One must first extend this IAS^2 to a complete triangulation by basis triangles. This triangulation should be done on P first, before the doubling construction, so that the vertices and edges in the northern hemisphere are perfectly matched with those in the southern. Here is a cartoon of what this might look like on one copy of P, before gluing:

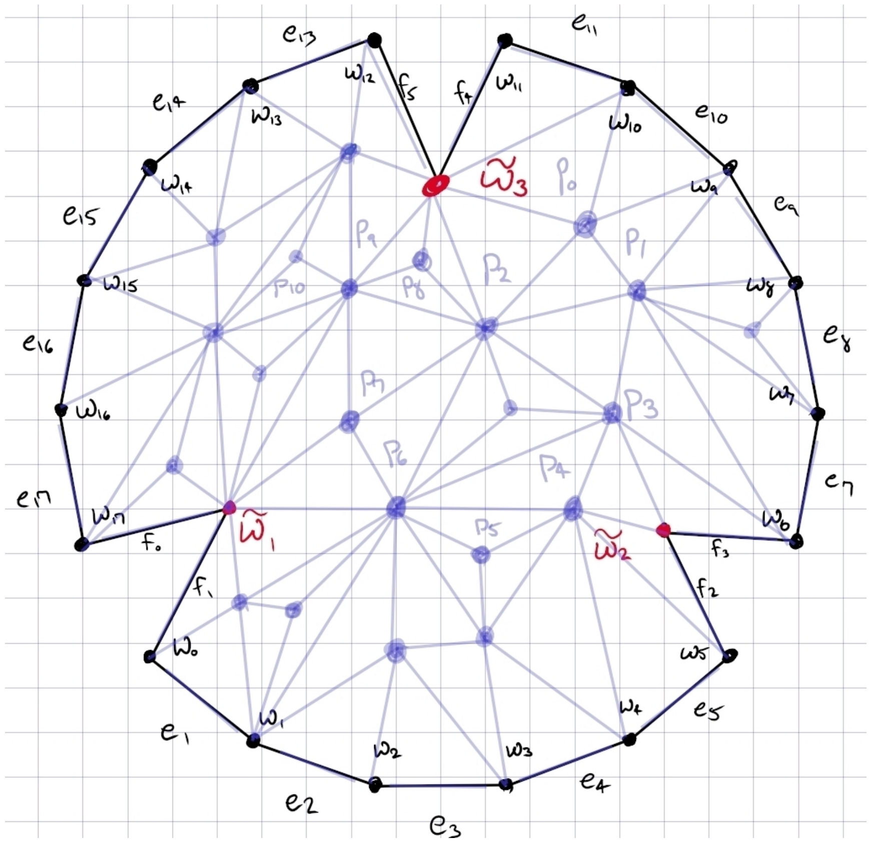


FIGURE 15. The IAS on P extended to a complete triangulation by basis triangles.

This is again a cartoon picture, meant to show how vertices and triangles in the hemispheres should match in pairs exchanged by the involution ι. Here e.g. the blue triangles are meant to match, as well as $\mathrm{Star}(\widetilde{w}_1)$ and $\mathrm{Star}(\widetilde{w}_1^{\mathrm{op}})$:

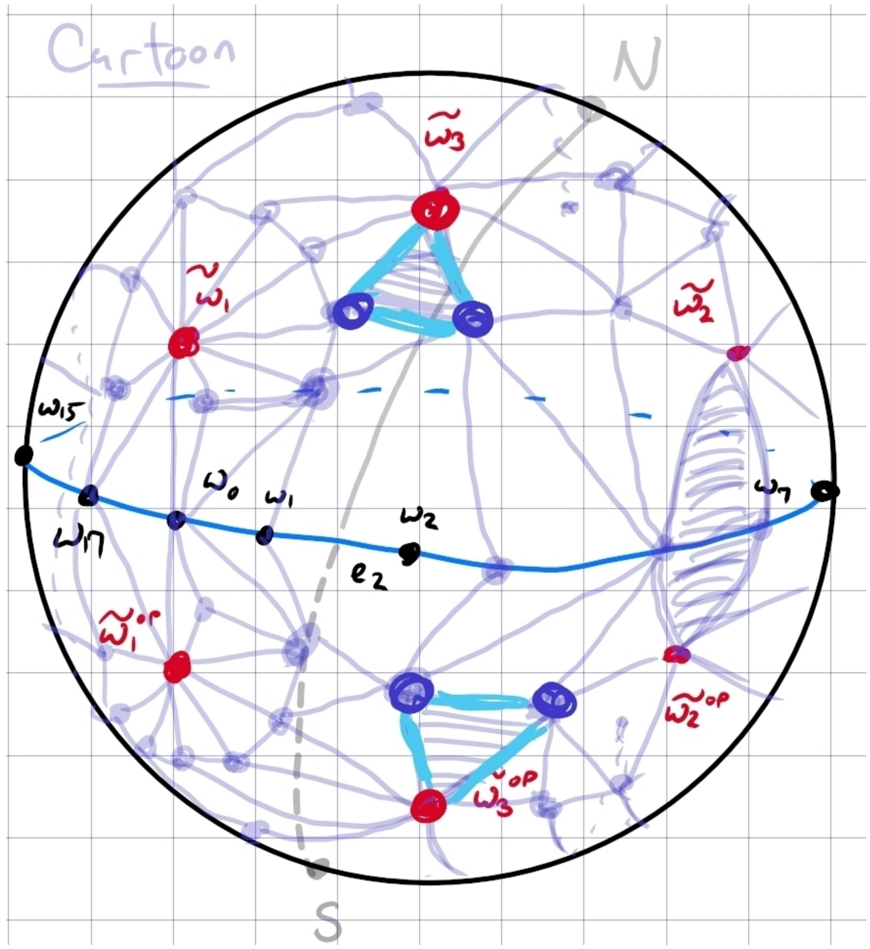


FIGURE 16. A completely triangulated IAS^2 defined by $B := P \cup P^{\mathrm{op}}$.

This final picture describes the central fiber $\mathcal{X}_0$ of a Kulikov degeneration of K3 surfaces in the following way: there are many non-singular vertices p_i, and exactly 24 singular vertices w_i and $\widetilde{w}_i, \widetilde{w}_i^{\mathrm{op}}$. For the p_i, there is a fan defined by $\mathrm{Star}(p_i)$ which defines a toric surface V_i. For the 24 singular vertices, there is a modified recipe to cook up a semi-toric surface – since the singularity is type I_1, this will be a charge 1 surface, and thus realizable as a toric surface with a single non-toric blowup, a *semitoric* surface. How to make this blowup is uniquely determined by an additional omitted decoration called the *monodromy ray* at the singular vertex. Roughly, this is a preferred ray cooked up from the Picard-Lefschetz monodromy operator around the singular vertex. One can think of this as a “singular fan”.

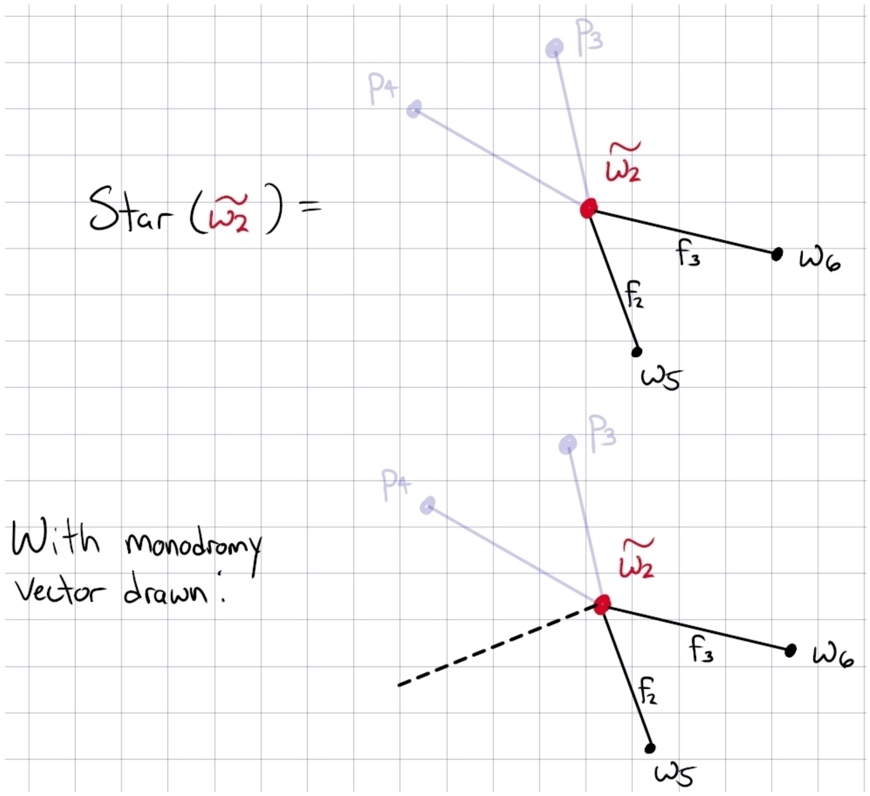


FIGURE 17. Star($\widetilde{w}_2$) in Figure 15 with the extra data of a monodromy vector.

So

$$\mathcal{X}_0 = \bigcup_i V_i \cup \bigcup_{j=1}^{24} W_j$$

where the V_i are all toric surfaces and the W_j are all semitoric surfaces of charge 1, and the triangulation determines how they are all glued together. To see how this gluing is done, consider the following local picture in the triangulation:

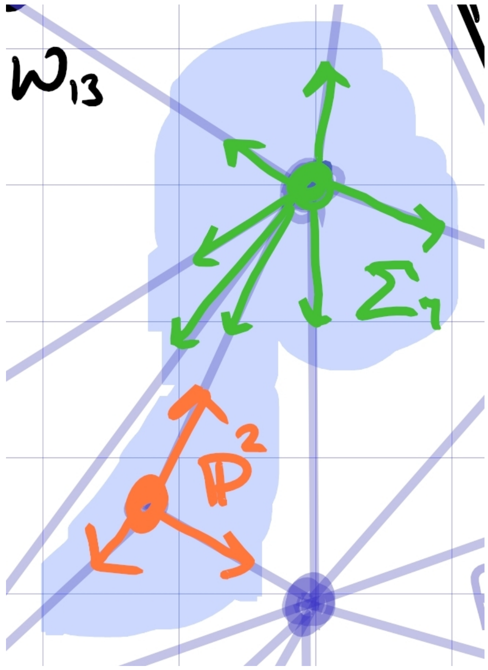


FIGURE 18. Local gluing in the IAS^2 of two toric surfaces Σ_7 and $\mathbf{P}^2$

At the orange vertex, taking the star we see three rays and thus a copy of $\mathbf{P}^2$. At the green vertex, we see 7 rays, and thus some toric surface Σ_7 which is probably something like a Hirzebruch surface $\mathbf{F}_n$ with 3 toric blowups. Since the orange and green vertices are adjacent by exactly one edge, this means we glue $\mathbf{P}^2$ to Σ_7 along the the curves determined by rays pointing along that edge. Moreover, whenever there is a triangle, this corresponds to three surfaces glued together along a triple point.

The general case is that the IAS^2 has 24 copies of I_1 singularities; these singularities can collide to produce semitoric surfaces with multiple nontoric blowups.

Remark 11.7. Note that this *only* describes $\mathcal{X}_0$ and not an entire family $\mathcal{X}$. Friedman solved this problem: there is a technical condition called ***d*-semistability**, and if this is satisfied then $\mathcal{X}_0$ is smoothable. Moreover the smoothing will have the correct period and/or monodromy vector λ.

Haven't discussed λ here yet!

To obtain all degenerations, one considers all of the ways this combinatorial object can degenerate. Sending some $\ell_i \to 0$ causes the 18-gon to collapse into a small polygon, or causes some hemispherical singularities to descend into the equator. This corresponds to moving an interior point of original fundamental chamber F onto a wall, and wall-crossing mutates the IAS^2 in some other ways.

Remark 11.8. Some miscellaneous remarks:

- The Kulikov models are highly non-unique, differing by flops. Adding the divisor R_{IAS} fixes this and pins down $\mathcal{X}$ uniquely.
- It seems one can read off the stable model from the IAS^2. For the honeycombs in the $\mathcal{A}_g$ case, everything was contracted down to two $\mathbf{P}^2$s glued

along their 3 boundary curves in a Θ-graph. In the F_2 case, one contracts everything in the IAS^2 except for the equator, i.e. the interiors of the hemispheres are contracted. The most general degeneration is 18 copies of $\mathbf{P}^2$ glued in a cycle; one can then send some $\ell_i \to 0$ to collide the vertices and get fewer than 18 surfaces.

- It seems one can also read off Type II degenerations from the IAS^2. Here there are 4 Type II cusps, 3 correspond to collapsing the 18-gon in the IAS^2 in the equatorial plane to an interval. The 4th involves collapsing the 18-gon to a point with bits sticking out. Type I degenerations correspond to collapsing everything to a point.
- Why everything works simply here: there is only one relevant cusp in F_2, and the involution propagates to everything including the IAS^2. The Coxeter diagram is also highly symmetric, hinting at how to make the right toric and IAS construction.

11C. **Notes from Phil's talk.**

Remark 11.9. For Σ_g a compact complex curve of genus g, choose a symplectic basis $\{\alpha_i, \beta_i\}_{i\leq g}$ of $H_1(\Sigma_G; \mathbf{Z})$, then there is a unique basis $(\omega_1, \cdots, \omega_g)$ of $H^0(\Omega_{\Sigma_g})$ such that $\int_{\alpha_i} \omega_j = \delta_{ij}$. In this basis, form the **period matrix** $\tau = (\int_{\beta_i} \omega_j)_{i,j=1}^g$. This satisfies $\tau^t = \tau$ and $\Im(\tau) > 0$ is positive-definite, and is thus an element in The **Siegel upper half-space**

$$\mathcal{H}_g := \left\{ \tau \in \mathrm{Sym}_{g\times g}(\mathbf{C}) \;\middle|\; \Im(\tau) > 0 \right\}.$$

The **Jacobian** of Σ is defined as $\mathrm{Jac}(\Sigma) := \mathbf{C}^g / (\mathbf{Z}^g \oplus \tau \mathbf{Z}^g)$. Note that we made a choice of "marking" by choosing the symplectic basis $\{\alpha_i, \beta_i\}$, and any two such choices are related by $\begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathsf{Sp}_{2g}(\mathbf{Z})$, the isometry group of $\mathbf{Z}^{2g}$ with the standard symplectic form, where the action is $\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} I_g \\ \tau \end{bmatrix} = \begin{bmatrix} A + B\tau \\ C + D\tau \end{bmatrix}$, the analogue of a linear fractional transformation. To renormalize the 1-forms, we change basis to get a similar matrix $\begin{bmatrix} I_g \\ (A + B\tau)^{-1}(C + D\tau) \end{bmatrix}$. Thus to get an invariant, we consider

$$[\tau] \in \mathsf{Sp}_{2g}(\mathbf{Z}) \backslash \mathcal{H}_g := \mathcal{A}_g,$$

the moduli space of PPAVs. Here one can realize the polarization on A as a symplectic form on $H_1(A; \mathbf{Z})$ which is represented by a holomorphic line bundle $L \in \mathrm{Pic}(A)$, i.e. identifying a symplectic form on $H_1(A; \mathbf{Z})$ as an element of $H^2(A; \mathbf{Z})$ which we want to be a $(1, 1)$ form.

Remark 11.10. Now $\mathcal{A}_g$ is not compact, so we consider degenerations over $\mathcal{X}^\circ \to \Delta^\circ$ and let $\Delta^\circ \to \mathcal{A}_g$ be the associated period mapping – how does the period map degenerate as $t \to 0$? The answer is that a certain isotropic subspace $I \leq (\mathbf{Z}^{2g}, \omega_{\mathrm{std}})$ becomes distinguished by the fact that periods against $I^\perp$ remain finite.

Example 11.11. Let $y^2 = x^3 + x^2 + t$ be a family of elliptic curves over $\mathbf{A}^1 \setminus 0$. At $t = 0$ this degenerates to a nodal cubic. There is a vanishing cycle α, and the distinguished isotropic subspace is precisely $I := \mathbf{Z}\alpha$. One shows $I^\perp = \mathbf{Z}\alpha$ as well. In this case, to be in $I^\perp$ means to be a curve that does not pass through the thinning neck of the torus that degenerates; any curve that does pass through should intuitively have a period that blows up. We normalize by picking a c_t such that $\int_\alpha c_t \frac{dx}{y} = 1$, then $\int_\beta \omega_t = \tau_t \in \mathbf{C}$. However, this isn't well-defined: one can parallel-transport β around $t = 0$ and the monodromy action will be a Dehn twist, so integrals against β are only well-defined up to $\mathbf{Z}p$ where p are periods against α, here p is normalized to 1. So $\int_\beta \omega_t \in \tau_h + \mathbf{Z} \in \mathbf{C}/\mathbf{Z}$. As $t \to 0$, one was $\tau_t \to +i\infty$ if α, β are oriented properly. We can fix this ambiguity by exponentiation, getting a well-defined invariant $\exp(2\pi i \int_\beta \omega_t) \in \mathbf{C}^*$.

Remark 11.12. How this works for $g \geq 1$: assume I is Lagrangian, so $I^\perp = I$, corresponding to a maximally unipotent degeneration. If this were a genus g curve, we could pinch $\leq g$ disjoint cycles simultaneously, and a maximal degeneration will pinch exactly g. Since $\mathsf{Sp}_{2g}(\mathbf{Z})$ acts transitively on Lagrangian subspaces in $\mathrm{LGr}(V)$, we can assume $I = \bigoplus_i \mathbf{Z}\alpha_i$ is generated by the α curves. Generalizing the $\mathbf{C}^*$ embedding in the previous case, we obtain a torus embedding

$$\begin{aligned} E : \mathcal{H}_g &\hookrightarrow (\mathbf{C}^*)^{\binom{g}{2}} \\ \tau &\mapsto \begin{bmatrix} \exp(2\pi i \tau_{11}) & \cdots & \\ \vdots & \ddots & \vdots \\ & \cdots & \exp(2\pi i \tau_{gg}) \end{bmatrix} \end{aligned}$$

Since τ is symmetric, the image $E(\tau)$ is again symmetric. Note that the β_i cycles are well-defined up to translation in I, but because the 1-form was normalized so that integrals of α_j along β_i were 1 or 0, the entries in this matrix are well-defined up to integers. Thus we can exponentiate every entry in the period matrix to get a well-defined symmetric matrix. The unipotent orbit theorem of Schmid gives an asymptotic estimate

$$E(\tau_t) \sim_{t \to 0} \begin{bmatrix} c_{11} t^{n_{11}} & c_{12} t^{n_{12}} & \cdots & \\ \vdots & \ddots & & \vdots \\ & \cdots & & c_{gg} t^{n_{gg}} \end{bmatrix} \in \mathrm{Mat}_{n \times n}(\mathbf{C}^*)$$

which is a cocharacter of $(\mathbf{C}^*)^{\binom{g}{2}}$, i.e. an inclusion $\mathbf{C}^* \hookrightarrow (\mathbf{C}^*)^{\binom{g}{2}}$ which is a composition of a group morphism and a translation. Here the c_{ij} are the translation parts, and if $c_{ij} = 1$ for all i, j this yields an honest group morphism. Such a cocharacter is called a unipotent orbit. This asymptotic estimate is quantified, so there is a precise speed at which the period matrix approaches the cocharacter.

Setting $N := (n_{ij})$, we have $N \in \mathrm{Sym}_{g \times g}(\mathbf{Z})$ and $N > 0$. These entries capture the relative speeds at which the various cycles are collapsing. Since the c_{ij} are ultimately just translations, we'll omit them from here onward.

Remark 11.13. Define a cone

$$P_g := \left\{ N \in \mathrm{Sym}_{g \times g}(\mathbf{Z}) \;\middle|\; N > 0 \right\} \subseteq \mathbf{Z}^{\binom{g}{2}}$$

and consider the family

$$\frac{(\mathbf{C}^*)^g}{\langle (t^{n_{11}}, t^{n_{12}}, \cdots, t^{n_{1g}}), (t^{n_{21}}, t^{n_{22}}, \cdots, t^{n_{2g}}), \cdots, (t^{n_{g1}}, t^{n_{g2}}, \cdots, t^{n_{gg}}) \rangle}$$

over Δ°. How does one extend this family over $t = 0$? If N has full rank, this entire expression is isomorphic to $(\mathbf{C}^g/\mathbf{Z}^g)/\tau\mathbf{Z}^g$. There are two answers, one by fans and one by polytopes.

Remark 11.14. The following is the fan construction due to Mumford, which most easily generalizes to K3 surfaces.

Consider the example

$$N = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} \rightsquigarrow \frac{(\mathbf{C}^*)^2}{\langle (t^2, t), (t, t^3) \rangle}$$

Note $N > 0$, since $\det N > 0, \mathrm{Trace} N > 0$, and $N(\mathbf{Z}^g)$ is generated by the vector $(2, 1)$ and $(1, 3)$. First quotient $\mathbf{R}^2$ by this lattice to get a flat real 2-torus, then take a polyhedral tiling whose vertices are integer points. Here we take a tiling of the fundamental domain and translate it everywhere. This gives a tiling $\mathcal{F}_0$ on the universal cover $\mathbf{R}^g$. Now put this picture at height 1 in $\mathbf{R}^{g+1}$ to get a tiling $\widetilde{\mathcal{F}}_0$ of $\mathbf{R}^g \times \{1\} \subseteq \mathbf{R}^{g+1}$, and let $\widetilde{\mathcal{F}} := \mathrm{Cone}(\widetilde{\mathcal{F}}_0) \subset \mathbf{R}^{g+1}$ be its cone. Taking the toric variety $X(\widetilde{\mathcal{F}})$, and define $X(\mathcal{F}) := X(\widetilde{\mathcal{F}})/N(\mathbf{Z}^g)$, where the quotient makes sense precisely because $N(\mathbf{Z}^g)$ acts on $\mathbf{R}^g \times \{1\}$ by translation, and this extends to a linear action on $\mathbf{R}^g$, which moreover preserves $\widetilde{\mathcal{F}}$ and thus acts on the toric variety. There is a morphism $\phi : X(\mathcal{F}) \to \mathbf{A}^1$ induced by the morphism of fans given by the height function: projection in $\mathbf{R}^{g+1}$ onto the last coordinate, whose image is in $\mathbf{R}_{\geq 0}$. This map descends to the quotient since the linear action preserves the height function.

This produces a degenerating fan of abelian varieties. A fiber $\phi^{-1}(t)$ of $X(\widetilde{F})$ for $t \neq 0$ yields $(\mathbf{C}^*)^g$, and the action $N(\mathbf{Z}^g)$ acts by translations of the form $(t^{n_{i1}}, t^{n_{i2}}, \cdots, t^{n_{ig}})$ in the original family. Thus we recover the original family as an infinite quotient of a toric variety. But the toric variety has a toric boundary, encoded in the tiling. The fiber $\phi^{-1}(0)$ has dual complex $\Gamma(X_0(\mathcal{F})) = \mathcal{F}_0$ equal to the original tiling, and $X_0(\mathcal{F})$ is a union of toric varieties.

In the original lattice, in the quotient there are precisely 3 0-cells, and we interpret the star of each 0-cell as the fan of a toric surface. They are glued according to the tiling.

Remark 11.15. The polytope construction, which builds the projective coordinate ring instead. One defines Q to be the hull of certain points, constructs a theta function, and takes Proj of a certain graded algebra generated by such functions with an explicit multiplication rule and structure constants. These define a certain PL function with a "bending locus" which gives a polyhedral decomposition of $\mathbf{R}^g/\mathbf{Z}^g$. For any $N \in P_g$ one can define the Delaunay decomposition $\mathrm{Del}(N)$, and the central fiber $\mathcal{X}_0$ of the family will have **intersection** complex $\mathrm{Del}(N)$ – the loci where the PL function is linear will be polytopes which are the cells of the Delaunay decomposition. The second Voronoi fan F^{Vor} is a decomposition of P_g into loci where $\mathrm{Del}(N)$ is constant. One then takes $\mathrm{Sym}_{g \times g}(\mathbf{Z}) \diagdown \mathcal{H}_g \hookrightarrow (\mathbf{C}^*)^{\binom{g}{2}} \to X(F^{\mathrm{Vor}})$. One the quotients by conjugation in $\mathrm{GL}_g(\mathbf{Z})$ to get $X(F^{\mathrm{Vor}})/\mathrm{GL}_g(\mathbf{Z}) \hookrightarrow \overline{\mathcal{A}_g}^{\mathrm{Vor}}$. Correspondingly, for any $\mathcal{X}^\circ$ in $\mathcal{A}_g$, tracing through this construction gives a proper family $\mathcal{X}$ in $\overline{\mathcal{A}_g}^{\mathrm{Vor}}$ – note that we've only described what toric compactification to take for the maximally unipotent degenerations, but one can carry out similar constructions for the other cusps of $\overline{\mathcal{A}_g}^{\mathrm{BB}}$.

Remark 11.16. One should ask if $\overline{\mathcal{A}_g}^{\mathrm{Vor}}$ actually solves a moduli problem, and the answer is yes (up to normalization) by a theorem of Alexeev. The moduli problem is the moduli of semi-abelic pairs. Define $\overline{\mathcal{A}_g}^{\Theta}$ to be the closure of pairs $(X, \varepsilon R)$

where R is their theta divisors, then Alexeev shows

$$\overline{\mathcal{A}_g}^{\mathrm{Vor}} = (\overline{\mathcal{A}_g}^{\Theta})^{\nu}$$

Remark 11.17. How do we do something similar for K3 surfaces? Fix $v \in \mathrm{II}_{3,19}$ primitive with $v^2 = 2d$ and define

$$\begin{aligned}
\Omega_L &\coloneqq \left\{ \mathbf{C}x \in \mathrm{Gr}_1(\mathrm{II}_{3,19_{\mathbf{C}}}) \;\middle|\; (x,x)=0, (x,\overline{x})>0 \right\} \\
\Omega_{2d} &\coloneqq v^{\perp_{\Omega_{\mathrm{II}_{3,19}}}} = \left\{ x \in \mathrm{II}_{3,19_{\mathbf{C}}} \;\middle|\; (x,v)=0 \right\} \\
\Gamma_{2d} &\coloneqq \mathrm{Stab}_{\mathrm{O}(\mathrm{II}_{3,19})}(v) = \left\{ \gamma \in \mathrm{O}(\mathrm{II}_{3,19}) \;\middle|\; \gamma(v)=v \right\} \\
F_{2d} &\coloneqq \Gamma_{2d} \backslash \Omega_{2d}
\end{aligned}$$

Here Ω_{2d} plays the role of $\mathcal{H}^g$ in the abelian variety case, and is a Hermitian symmetric domain of type IV or $\mathrm{SO}_{2,n}$, and F_{2d} is an arithmetic quotient. Fixing a marking $\phi : H^2(X;\mathbf{Z}) \to \mathrm{II}_{3,19}$, the period map for a family $\mathcal{X}^\circ \to \Delta^\circ$ is given by taking $H^{2,0}(X) = \mathbf{C}\omega$ and looking at $[\omega] \coloneqq \phi(\omega) \in F_{2d}$, since $[\omega] \in \Omega_{2d}$ but is ambiguous up to change of marking (elements of Γ). This is a map $\Delta^\circ \to F_{2d}$.

Given a degenerating family, there is a distinguished isotropic lattice $I \leq v^\perp$ where $\operatorname{sig} v^\perp = (2,19)$. Note I can only have rank 1 or 2. The rank 1 case (Type III degenerations) is a maximally unipotent degeneration; the central fiber is as singular as possible, and $\mathcal{X}_0$ will always have 0-strata. In contrast, in the rank 2 case (Type II degenerations) there are models of the degeneration with no 0-strata.

In the rank 1/Type III case, there is a vanishing cycle δ associated to a 0-stratum in $\mathcal{X}_0$ which is topologically a 2-torus. It turns out that δ is an isotropic vector that spans the isotropic lattice, so we can write $I = \mathbf{Z}\delta \subseteq v^\perp$. In the degeneration, the 2-torus collapses to a point.

In the rank 2/Type II case, there are two linearly independent isotropic vectors δ and λ in $v^\perp$ corresponding to 2-tori collapsing simultaneously not to isolated points as in the previous case, but rather to circles in $\mathcal{X}_0$. They are in the singular locus of $\mathcal{X}_0$, which is an elliptic curve.

Remark 11.18. We henceforth assume $\operatorname{rank}_{\mathbf{Z}} I = 1$ and write $I = \mathbf{Z}\delta$ for δ the isotropic vanishing cycle. Normalize ω_t so that $\int_\delta \omega_t = 1$ for $t \neq 0$. Let $\{\gamma_i\}_{i=1}^{19}$ be a basis of $\delta^\perp/\delta$. Since $\delta \in v^\perp$ was isotropic of signature $(2,19)$, we have $\operatorname{sig}(\delta^\perp/\delta) = (1,18)$ and this gives us a hyperbolic lattice of rank 19. Consider the integral $\int_{\gamma_i} \omega_t \in \mathbf{C}$. For this to make sense, one needs to lift the δ_i from $\delta^\perp/\delta$ to $\delta^\perp$, and the choice of lift is ambiguous up to a multiple of δ. By the normalization of the integral, we get a well-defined period

$$\int_{\gamma_i} \omega_t \in \mathbf{C}/\mathbf{Z}$$

As in the PPAV case, we use the exponential to get rid of the quotient by $\mathbf{Z}$. Letting $U_\delta \leq \Gamma_{2d}$ be the unipotent subgroup stabilizing δ, we get the following torus embedding

$$\begin{aligned}
U_\delta \backslash \Omega_{2d} &\xrightarrow{\psi} (\mathbf{C}^*)^{19} \\
\mathbf{C}[\omega_t] &\mapsto \left(\exp\left(2\pi i \int_{\gamma_1} \omega_t\right), \cdots, \exp\left(2\pi i \int_{\gamma_{19}} \omega_t\right) \right)
\end{aligned}$$

and a nilpotent orbit theory yielding an asymptotic estimate

$$\psi_t \sim (c_1 t^{\lambda_1}, \cdots, c_{19} t^{\lambda_{19}})$$

with $c_i \in \mathbf{C}^*$, so the periods are approximated by a cocharacter where the λ_i measure how fast the periods degenerate.

Remark 11.19. Degenerations: a theorem of KPP shows that after a finite base change and birational modifications, any degeneration of K3s has a model where

- X is smooth
- X_0 is RNC
- $K_X = \mathcal{O}_X$

The most famous degeneration of K3s is the Fermat degeneration is a non-example, since smoothness fails:

$$V(x_0 x_1 x_2 x_3 = t(x_0^4 + x_1^4 + x_2^4 + x_3^4))$$

This threefold has precisely 24 conifold singular points. The central fiber at $t = 0$ is a tetrahedron, 4 planes $\mathbf{P}^2$ in $\mathbf{P}^3$, and the singular points come from intersecting each edge of the tetrahedron with the residual quartic. One can get a smooth threefold by taking a small resolution of the singular points. There are choices for the resolutions, differing by flops, so here is a heuristic of a symmetric choice where along each edge there are two resolutions extending into each component:

image

The result has four components V_i which are isomorphic to $\mathrm{Bl}_6 \mathbf{P}^2$, 2 points on each of 3 lines in $\mathbf{P}^2$.

An observation originally due to GHK: there is an IAS on $\Gamma(X_0)$, i.e. there are charts to $\mathbf{R}^2$ up to post-composition with $\mathrm{SL}_2(\mathbf{Z}) \rtimes \mathbf{R}^2$.

Here is an example of $\mathcal{X}_0 = \cup V_i$ for a Kulikov degeneration (written as a decomposition into irreducible components). Each unlabeled edge has an implicit label of -1:

image

This forms a tiling of the sphere. Each tile corresponds to an irreducible component V_i of $\mathcal{X}_0$. The edges correspond to components V_i, V_j glued along an anti-canonical cycle of rational curves V_{ij}. The edge numbers record the self-intersection numbers of the cycles V_{ij} regarded as a cycle in V_i and $V_{ji} = V_{ij}$ regarded as a cycle in j.

A general fact about degenerations of CYs: $\mathcal{X}_0$ is generally a union of log CY varieties, i.e. there are meromorphic 2-forms on components and they are glued along their poles so that the residues agree. The red lines in the image denotes the pole locus of these forms. Each triple point is where 3 surfaces are glued. Since the overall variety is a SNC surface, there are only double curves and triple points. Note that this picture is the **intersection complex** of $\mathcal{X}_0$, and not the dual complex $\Gamma(\mathcal{X}_0)$. To obtain the dual complex, take the dual tiling, regard each integral 0-cell in the result as a fan, and glue the fans.

Here are the fans:

image

Here is how this interacts with the original diagram:

image

This works fine at most vertices, but at most 24 components are non-toric. Note that from toric geometry, if (V, D) is a toric pair then $-D_i^2 v_i = v_{i-1} + v_{i+1}$ and so one can enforce this formula on such components. For example, the following pair has all -1 curves since $v_2 = v_1 + v_3$:

image

Enforcing this formula locally, non-toric points force some $\mathrm{SL}_2(\mathbf{Z})$ monodromy in the IAS.

Remark 11.20. This is the analogue of the Mumford fan construction. Note that in the PPAV case, the lattice didn't specify a Kulikov degeneration since it was not a complete triangulation. But completing this to a complete triangulation of the corresponding real 2-torus does yield a Kulikov model. For K3s, instead of a complete triangulation on T^2, we're taking a complete triangulation of an IAS^2. Note that unlike the PPAV case, a triangulated IAS^2 only gives $\mathcal{X}_0$ (glued from ACPs) and not the entire family $\mathcal{X}$. An abstract theorem of Friedman says it smooth to a K3, but one does not get an explicit construction of the smoothing $\mathcal{X}_t$.

There is also no polytope construction here whatsoever, only the fan construction for the central fiber. GHK and Siebert have been working on the polytope side. It's hard: it's not clear what the multiplication rule for theta functions should be. We represent an IAS^2 with the following data:

Missing, see video.

This recovers $\mathcal{X}_0$ by taking fans at vertices.

Remark 11.21. Joint work with Valery: a polarizing divisor is a divisor R in the generic K3 surface in $F_{2d}(\mathbf{C})$. This corresponds to a choice of ample divisor on a Zariski open subset of $F_{2d}(\mathbf{C})$. For such a choice, we define $\overline{F_{2d}}^R$ to be the closure of K3 pairs $(X, \varepsilon R)$ in the space of KSBA stable pairs. A generic K3 has Picard rank 1, and it's in the ample class, so any divisor on the generic K3 is automatically ample. Thus $K_X + \varepsilon R > 0$ since $K_X = 0$. The pair also has slc singularities. Note that we've allowed all K3s to have ADE singularities, these are examples of slc, and taking ε small enough resolves any problems. One needs R not to pass through log canonical centers, and there are no log canonical centers on an ADE K3. Their theorem gives an explicit description of such a moduli space.

Remark 11.22. We say R is recognizable if it extends to a unique divisor R_0 on any Kulikov surface. Idea: for $\mathcal{X}_0$ there are many different smoothing families $\mathcal{X}_i$ and choices of divisors R_i. For any 1-parameter family, taking the Zariski closure of R_i yields a flat limit $R_{i,0}$ on $\mathcal{X}_0$. If R is recognizable, these flat limits do not vary, so the choice of divisor can be made on *any* K3, even a smooth K3. If R is a recognizable polarizing divisor, there is a unique semifan F_R such that

$$\overline{F_{2d}}^{F_R} = (\overline{F_{2d}}^R)^\nu$$

This relates a Hodge-theoretic compactification on the left with a geometric compactification on the right.

Remark 11.23. A semitoroidal compactification simultaneously generalizes toroidal and BB compactifications.

Recall that assocaited to a degeneration of K3s we had $\vec{\lambda} := (\lambda_1, \cdots, \lambda_{19}) \in \delta^\perp/\delta$, a signature $(1, 18)$ lattice. Friedman-Scattone show that $\vec{\lambda}^2$ is the number of triple points in $\mathcal{X}_0$. The semifan F_R is a locally polyhedral $\Gamma_\delta := \mathrm{Stab}_\Gamma(\delta)$ invariant decomposition of the positive cone $C^+ \subset \delta^\perp/\delta$. This is the future light cone in the corresponding hyperbolic space. Roughly $\overline{F_{2d}}^{F_R}$ is $X(F_R)/\Gamma_\delta$.

Why this? We had a torus embedding of the first partial quotient $U_\delta \setminus \mathbf{D} \to (\mathbf{C}^*)^{19}$ and the latter is canonically identified with $\delta^\perp/\delta \otimes \mathbf{C}^*$. The monodromy invariant $\vec{\lambda}$ was approximated by the cocharacter $\lambda \otimes \mathbf{C}^*$. We extend that torus by a toric variety whose fan has support in $\delta^\perp/\delta$.

Note that here semitoroidal corresponds to *locally* polyhedral. A globally polyhedral tiling condition would just yield a usual fan. For instance, the cones here might have infinitely many rational polyhedral walls. On the other hand, the BB compactification corresponds to the trivial compactification of C^+ which is just the entirety of C^+.

Remark 11.24. AE prove that recognizable divisors $R^{\mathrm{rc}} \coloneqq \sum_{C \in |L|, C^\nu \cong \mathbf{P}^1} C$ exist. The rational curve divisor is always recognizable for any degree $2d$, so this exhibits some semitoroidal compactifications with geometric meaning.

AET give some explicit examples for F_2. Degree 2 K3s are generically 2-to-1 covers $\pi : X \to \mathbf{P}^1$ branched over a sextic, take $L \coloneqq \pi^* \mathcal{O}_{\mathbf{P}^1}(1)$. One takes the R to be the ramification divisor $R \in |3L|$; it is a recognizable divisor. They construct a semifan F_R which is a coarsening of the Coxeter fan for the root system in $\delta^\perp/\delta$; one takes a subset of the root mirrors.

The construction of the singular K3 surface: start with the heart IAS^2, triangulate completely, double this construction, replace each vertex with the surface defined by the star. Note the cuts introducing shears along the boundary. The cuts introduce 3 singularities in each hemisphere, and angular defects of the polygonal gluing introduce 18 singularities along the equator. This IAS^2 has an involution, and this $\mathcal{X}_0$ naturally has an involution. The ramification divisor of the IAS^2 is in blue, it's a tropical ramification divisor. It is the dual complex of the limit of ramification divisors.

Why is this recognizable? $\mathcal{X}_0$ admits an involution ι_0. From this we can determine the limit of $\mathrm{Fix}(\iota)$. Note that $\mathcal{X}_0$ alone determines R_0, the limit of R_t, and $R_0 = \mathrm{Fix}(\iota_0)$. This implies recognizability since the choice of divisor R can be made on any Kulikov surface.

Remark 11.25. On joint work with ABE for elliptic K3s. Take $X \to \mathbf{P}^1$ an elliptic fibration with fiber f and section s. This is not of the form F_{2d}, since here one takes $H = \mathbf{Z}s \oplus \mathbf{Z}f$ for the polarization. Generically the fibration has 24 singular fibers. They show $R \coloneqq s + m \sum f_i$ for f_i the singular fibers is recognizable for any multiple m. The lattice $\mathrm{II}_{1,17}$ is reflective, and F_R here refines the Coxeter chamber into 9 subchambers. This is a fan which is strictly not a semifan. There is a corresponding tropical elliptic K3 given by the following IAS^2.

image

Here one glues the top too the bottom, identifying the segments by a vertical shear. Note it has an S^1 fibration which tropicalizes the elliptic fibration. The blue vertical lines are limits of singular fibers, the blue horizontal is the limit of the section.

12. Coxeter Theory

From https://math.ucr.edu/home/baez/twf_dynkin.pdf#page=1&zoom=160,-141,556

12A. Coxeter groups and diagrams.

Remark 12.1. Main ideas:

- Elliptic subdiagrams of rank r correspond to codimension r faces of a polytope P
- Parabolic subdiagrams (of rank $n-1$) correspond to cusps of P

Remark 12.2 (A summary of hyperbolic Coxeter diagram conventions)**.** Regarding this as a group of reflections in hyperplanes, we have the following interpretations:

Description	Diagram	Notation	m_{ij}	$\angle(H_i, H_j)$	w_{ij}
Labeled simple edge	H_1 ○—m_{ij}—○ H_2	$H_i \pitchfork H_j$	m_{ij}	π/m_{ij}	$\cos\left(\frac{\pi}{m_{ij}}\right)$
No Edge	H_1 ○ ○ H_2	$H_i \perp H_j$	2	$\pi/2$	0
Simple Edge	H_1 ○—○ H_2	$H_i \pitchfork H_j$	3	$\pi/3$	$\frac{1}{2}$
Double Edge	H_1 ○═○ H_2	$H_i \pitchfork H_j$	4	$\pi/4$	$\frac{\sqrt{2}}{2}$
Triple Edge	H_1 ○≡○ H_2	$H_i \pitchfork H_j$	5	$\pi/5$	$\frac{1+\sqrt{5}}{4}$
Thick/bold edge	H_1 ○━○ H_2	$H_i \parallel H_j$	∞	0	1
Dotted Edge	H_1 ○- - w_{ij} - -○ H_2	$H_i \,)(\, H_j$	0	∞	$\cosh(\rho(H_i, H_j))$
Simple vertex	○	$h_i^2 = -1$			1
Black vertex	●	$h_i^2 = -2$			2
Double-circled vertex	◎	$h_i^2 = -4$			4

TABLE 1. A summary of conventions for Coxeter-Vinberg diagrams

Note that generally

- $\cos(\angle(H_i, H_j)) = -(h_i, h_j)$ when $|(h_i, h_j)| < 1$ and
- $\cosh(\rho(H_i, H_j)) = -(h_i, h_j)$ when $|(h_i, h_j)| > 1$.

Here $\rho(H_i, H_j)$ is the length of a common perpendicular to H_i and H_j. Moreover,

- $H_i \pitchfork H_j \iff |w_{ij}| = |(h_i, h_j)| < 1$,
- $H_i \parallel H_j \iff |w_{ij}| = |(h_i, h_j)| = 1$,
- $H_i \,)(\, H_j \iff |w_{ij}| = |(h_i, h_j)| > 1$.

For a hyperbolic Coxeter polytope P bounded by hyperplanes $\{H_1, \cdots, H_n\}$, one constructs the Gram matrix $G(P) = (g_{ij}) \in \mathrm{Mat}_{1 \le i,j \le n}$ defined by

$$g_{ij} = \begin{cases} 1 & i = j \\ -\cos(\pi/m_{ij}) & H_i \pitchfork H_j, \angle(H_i, H_j) = \pi/m_{ij} \\ -\cosh(\rho(H_i, H_j)) & H_i \,)(\, H_j \\ -1 & H_i \parallel H_j \end{cases}.$$

Note that $g_{ij} = (h_i, h_j)$ is the Gram matrix of the corresponding intersection form.

Definition 12.3 (Coxeter groups)**.** A group W is a **Coxeter group** if it has a presentation of the following form:

$$W = \left\langle r_1, \cdots, r_n \;\middle|\; (r_i r_j)^{m_{ij}} \ \forall 1 \leq i, j \leq n \right\rangle \qquad m_{ij} \in \mathbf{Z}_{\geq 1} \cup \{\infty\}$$

where

- $m_{ii} = 1$ for all i,
- $m_{ij} \geq 2$ for $i \neq j$, and
- $m_{i,j} = \infty$ means there is no relation imposed.

If $S = \{r_1, \cdots, r_n\}$ is a fixed generating set, we call the pair (W, S) a **Coxeter system**.

Definition 12.4 (Coxeter diagrams)**.** Given a Coxeter system (W, S), the **the pre-Coxeter diagram** of (W, S) is weighted undirected graph with a single vertex v_i for each $r_i \in S$, and for each pair $i \neq j$, an edge e_{ij} of weight $w_{ij} := m_{ij}$ connecting v_i to v_j. Note that this yields a complete[5] graph on $|S|$ vertices. The **Coxeter diagram** $D(W)$ of (W, S) is the partially weighted graph obtained from the pre-Coxeter diagram by the following modifications:

- Edges e_{ij} of weight $w_{ij} = 2$ are deleted.
- Edges e_{ij} of weight $w_{ij} \geq 7$ are labeled with their weights.
- Edges e_{ij} of $w_{ij} = 3, 4, 5, 6$ follow one of two conventions: they are either replaced with an $(w_{ij} - 2)$-fold multi-edge, or are unmodified and retain their label of w_{ij}.
- Edges e_{ij} of weight $w_{ij} = \infty$ are replaced by bold/thick edges.

Remark 12.5 (Facts about Coxeter diagrams)**.** We summarize several facts about the full Coxeter diagram:

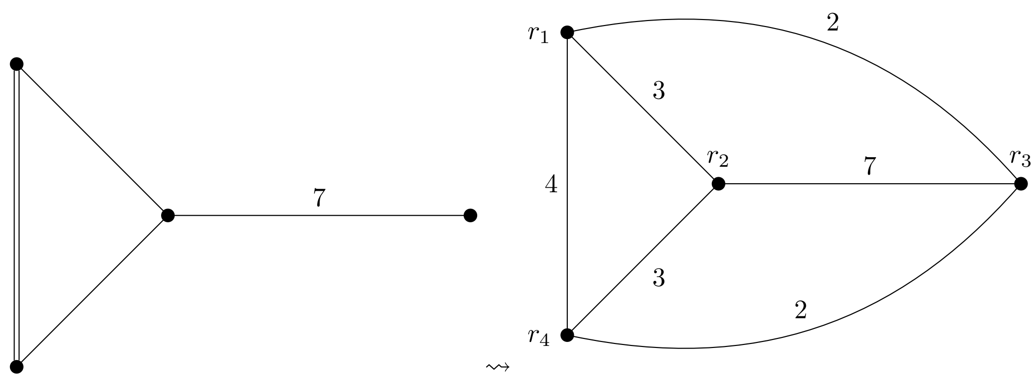
- Vertices v_i and v_j are non-adjacent if and only if $w_{ij} = 2$,
- Vertices v_i and v_j are adjacent if and only if $w_{ij} \geq 3$,
- Edge weights are suppressed for small weights $w_{ij} \leq 6$, and explicitly included for every $w_{ij} \geq 7$.

Remark 12.6 (How to read a group presentation from a Coxeter diagram)**.** One can recover the presentation of a Coxeter group from any Coxeter diagram. Explicitly, given a diagram D, one constructs a group W such that $D = D(W)$ in the following way: first one transforms the Coxeter diagram into a pre-Coxeter diagram by adding weight 2 edges between every pair of non-adjacent vertices, forming a complete graph. One then replaces double/triple/quadruple edges with weight 4/5/6 edges respectively. Finally, reads the group presentation off of the weighted adjacency matrix of the resulting graph. Explicitly, the group W will have a generator for every vertex and a relation $(r_i r_j)^{w_{ij}}$ for each edge e_{ij} of weight w_{ij}.d

Example 12.7 (Passing between Coxeter diagrams and Coxeter groups)**.** Every Coxeter diagram is naturally associated with a weighted graph whose edge weights are all integers $m_{ij} \geq 2$, and from this presentation, one can immediately read off the group presentation. For example, consider the following diagram and the

[5] Recall that a graph is **complete** if every vertex is adjacent to every other vertex.

associated weighted graph:



Reading generators and relations off of this graph, we obtain a group freely generated by r_1, r_2, r_3, r_4 subject to the following relations:

$$W \coloneqq \left\langle r_1, r_2, r_3, r_4 \;\middle|\; \begin{array}{l} r_1^2 = r_2^2 = r_3^2 = r_4^2 = 1 \\ (r_1 r_2)^3 = (r_2 r_3)^7 = (r_1 r_4)^4 = (r_2 r_4)^3 = 1 \\ (r_1 r_3)^2 = (r_3 r_4)^2 = 1 \end{array} \right\rangle.$$

Letting A be the weighted adjacency matrix of this weighted graph, we can read this group presentation directly off of the following symmetric matrix:

$$A = \begin{bmatrix} 1 & 3 & 2 & 4 \\ 3 & 1 & 7 & 3 \\ 2 & 7 & 1 & 2 \\ 4 & 3 & 2 & 1 \end{bmatrix}$$

This matrix defines an exact sequence of $\mathbf{Z}$-modules

$$0 \to \mathbf{Z}^4 \xrightarrow{A} \mathbf{Z}^4 \to W \to 0,$$

realizing $W \cong \operatorname{coker} A$ as a presentation of W by generators and relations.

12B. **Coxeter polytopes.**

Remark 12.8. Recall the cosine formula for Euclidean inner product spaces: in $\mathbb{E}^n$, the norm is $\|x\| \coloneqq \sqrt{x^2} \coloneqq \sqrt{x.x}$, and we have

$$vw = \|v\|\|w\| \cos(\angle(v,w)) = \sqrt{v^2}\sqrt{w^2} \cos(\angle(v,w)) = \sqrt{v^2 w^2} \cos(\angle(v,w))$$

For a general bilinear form, we can define

$$\angle(v,w) \coloneqq \cos^{-1}\left(\frac{vw}{\sqrt{v^2 w^2}}\right).$$

We can thus interpret the pairing as measuring angles in the following way:

$$vw = \frac{\cos(\angle(v,w))}{\sqrt{v^2 w^2}},$$

which moreover allows one to compute intersections vw from knowledge of v^2, w^2, and angles $\angle(v,w)$, which is precisely the data that is encoded in a Coxeter diagram.

Definition 12.9 (Dihedral angles between hyperplanes)**.** If H_i, H_j are intersecting hyperplanes in $\mathbb{E}^n$, we write $H_i \pitchfork H_j$. We write $h_i \coloneqq H_i^\perp$ and $h_j \coloneqq H_j^\perp$ for unit normal vectors spanning their orthogonal complements, and define the **dihedral angle** between H_i and H_j as

$$\angle(H_i, H_j) \coloneqq \angle(h_i, h_j).$$

If H_i is parallel to H_j, we write $H_i \parallel H_j$ and define $\angle(H_i, H_j) = 0$. We similarly write $H_i \perp H_j$ if $\angle(H_i, H_j) = \pi/2$.

Remark 12.10. Note that there is a common trick to get rid of the square root in these formulas: one writes

$$(vw)^2 = v^2 w^2 \cos^2(\angle(v, w))$$

For $\angle(v, w) = \pi/m_{ij}$, this gives a way to recover m_{ij} from the bilinear form.

Definition 12.11 (Coxeter polytopes)**.** Let $X \coloneqq \mathbb{E}^n, \mathbb{S}^n, \mathbb{H}^n$ be a Euclidean, spherical, or hyperbolic geometry. A polytope $P \subseteq X$ is **Coxeter polytope** if all dihedral angles between pairs of intersecting facets H_i and H_j are of the form π/m_{ij} for $m_{ij} \in \mathbf{Z}_{\geq 2}$, and any two non-intersecting facets are parallel.

Remark 12.12 (Coxeter group G_P of a Coxeter polytope P)**.** Every Coxeter polytope P defines a Coxeter group $G_P \leq \operatorname{Isom}(\mathbf{X})$ generated by reflections through the supporting hyperplanes H_i of facets of P and a corresponding Coxeter diagram D_P. For $\mathbf{X} = \mathbb{E}^n$, one constructs G_P in the following way:

- A generator r_i for each facet H_i of P with relation $r_i^2 = 1$, representing reflection through the hyperplane H_i,
- For any facets H_i, H_j where $H_i \pitchfork H_j$, there is a relation $(r_i r_j)^{m_{ij}} = 1$ where m_{ij} is defined by $\angle(H_i, H_j) = \pi/m_{ij}$.
- For non-intersecting facets $H_i \parallel H_j$, we set $m_{ij} = \infty$ and take a relation $(r_i r_j)^\infty = 1$, i.e. no relation is imposed at all.

Remark 12.13. Note that P is a fundamental domain for the action of G_P on $\mathbf{X}$. Moreover, if $G \leq \operatorname{Isom}(\mathbf{X})$ is any discrete finitely generated reflection group, then its fundamental domain is always a Coxeter polytope. If $\mathbf{X} = \mathbb{S}^n$ or $\mathbb{E}^n$, Coxeter polytopes are classified and are either simplices or products of simplices respectively, and full lists can be found. For $\mathbf{X} = \mathbb{H}^n$, the general classification is an open problem. Poincaré classified them in $\mathbb{H}^2$. Vinberg showed that no compact Coxeter polytopes exist in $\mathbb{H}^n$ for $n \geq 30$, and no non-compact but finite volume polytopes exist for $n \geq 996$. These bounds are not sharp. Finding explicit examples of high-dimensional compact Coxeter polytopes is interesting because these can be used to explicitly construct high-dimensional hyperbolic manifolds.

Definition 12.14 (Volumes and covolumes of Coxeter groups/polytopes/diagrams)**.** We define the **covolume** of G_P as the volume of $P \cong \mathbf{X}/G_P$, where the metric on the quotient is induced from the metric defining the geometry on $\mathbf{X}$.

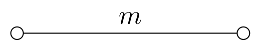
Remark 12.15. We collect some facts about the corresponding Coxeter diagram $D(P)$:

- $D(P)$ has vertices v_i corresponding to H_i, where v_i, v_j are non-adjacent if and only if $H_i \perp H_j$[6],

[6]Recalling that edges with $m_{ij} = 2$ are deleted by convention.

- Edges e_{ij} are plain if $m_{ij} < \infty$ and $m_{ij} \neq 0$, so $H_i \pitchfork H_j$,
- Edges e_{ij} are bold if $m_{ij} = \infty$, so $H_i \parallel H_j$ and $\angle(H_i, H_j) = \pi/\infty = 0$.

Example 12.16 (Euclidean Coxeter polytopes)**.** Consider the following Coxeter diagram:



This corresponds to a non-compact polytope in $\mathbb{E}^2$ bounded by two hyperplanes H_1, H_2 through the origin (i.e. lines), one corresponding to each node, intersecting at an angle of π/m. Without loss of generality, we can take H_1 to be the x-axis and H_2 to be a line of slope π/m:

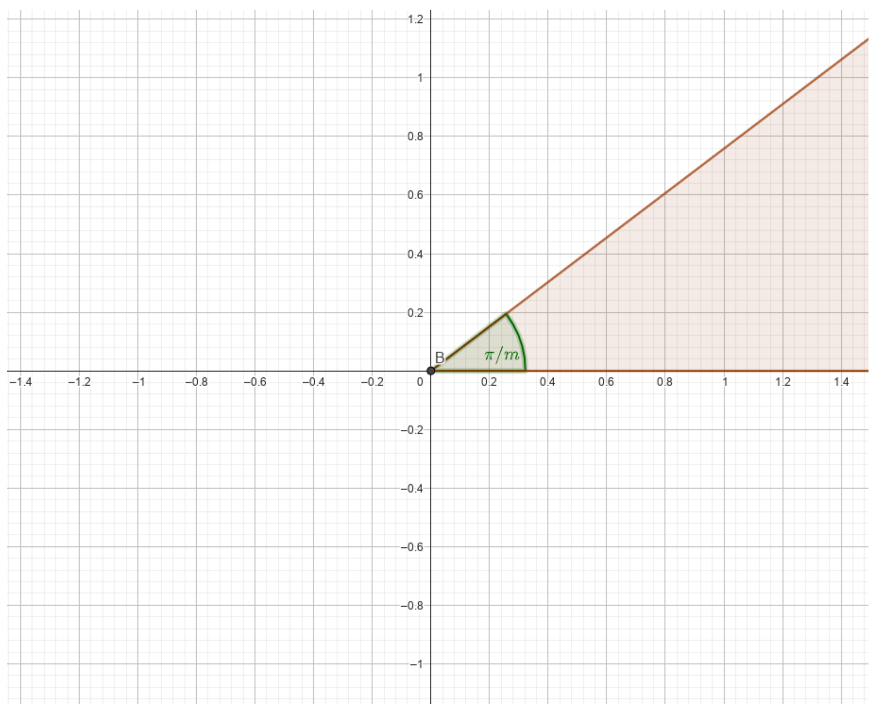
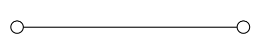


Figure 19. Caption

One can note that if $m = 2$, then one deletes the edge by convention to get the Coxeter diagram

This is the Dynkin diagram of $A^1 \times A^1$, which indeed has fundamental chamber the first quadrant. Similarly, if one takes $m = 3$ on recovers the standard Dynkin diagram for A_2:



We get a fundamental chamber with two walls at a dihedral angle of $\pi/3$, corresponding to the dual hyperplanes of the two standard short rootsα and β with $\angle(\alpha, \beta) = 2\pi/3$ in Lie theory:

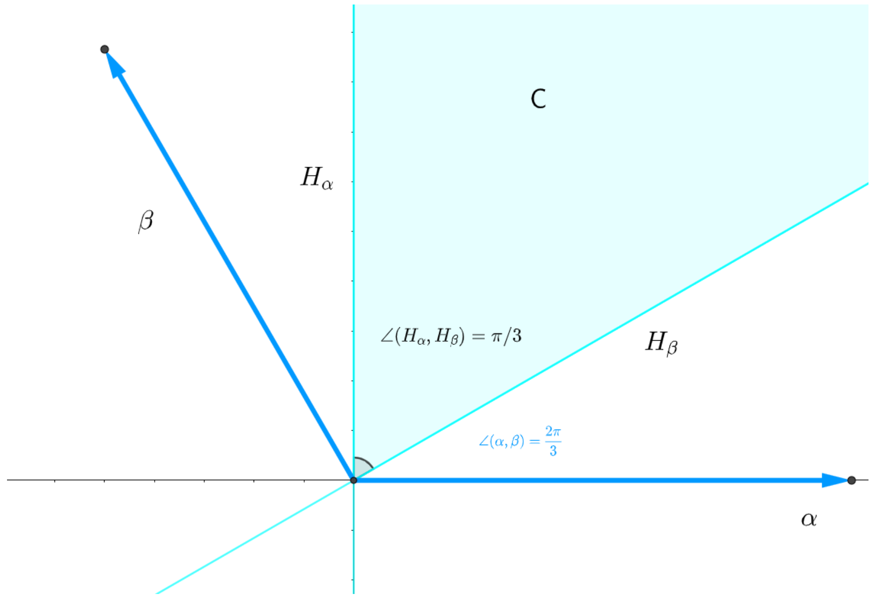


FIGURE 20. Caption

Todo: weighted $\widetilde{A}_2$ as a simplex.

Example 12.17 (Affine examples)**.**

Remark 12.18. Note that taking reflections of the fundamental domain C by the Weyl group generates a **tiling** of the hyperbolic disc in these cases.

Remark 12.19 (Importance of tilings)**.** Why this is important: given *any* tiling of $\mathbb{E}^2$ or $\mathbb{H}$ the hyperbolic disc, we can place it at height one and take a cone to get an infinite-type toric variety. Alternatively, given any tiling we can construct a surface that is a union of toric pairs by interpreting every vertex of the tiling as a fan and the edges of tiles as gluing instructions.

Finally, we can interpret an IAS^2 has an irregular spherical tiling, i.e. a tiling $\mathbb{S}^2$ which is not necessarily generated by reflections, but one which has finitely many tiles. We then regard the tiling as a union of toric surfaces as described above.

Definition 12.20 (The Gram matrix of a Euclidean Coxeter polytope)**.** Let $P \subset \mathbb{E}^n$ be a Euclidean Coxeter polytope, not necessarily compact. One defines the **Gram matrix** $G(P)$ of P as

$$G(P)_{ij} = \begin{cases} 1 & i = j \\ -\cos\left(\frac{\pi}{m_{ij}}\right) & H_i \pitchfork H_j, \quad \angle(H_i, H_j) = \pi/m_{ij} \\ -1 & H_i \parallel H_j, \quad \angle(H_i, H_j) = \pi/\infty = 0 \end{cases}.$$

12C. **Hyperbolic Coxeter polytopes.** %https://hal.science/hal-03345221/file/Survey_-Discrete_Cox_Gp_V3_arxiv.pdf#page=5&zoom=auto,-147,687

Remark 12.21. See the section on hyperbolic geometry for a description of $\mathbb{H}^n := \left\{ x \in \mathbb{E}^{n,1} \;\middle|\; x^2 = -1, x_0 > 0 \right\}$ and terminology (space/time/light-like vectors). As a convention, $\mathbb{H}^n$ means the interior of $\overline{\mathbb{H}^n} := \mathbb{H}^n \cup \partial \mathbb{H}^n$ where $\partial \mathbb{H}^n$ is the boundary at infinity consisting of ideal points.

Some unsorted notes:

- The distance ρ on $\mathbb{H}^n$ is defined such that $\rho(v, w) := \operatorname{arccosh}(vw)$.
- In $\mathbb{H}^n$ Vinberg defines the dihedral angle as $\angle(f_i, f_j) := \pi - \angle(f_i^{\perp}, f_i^{\perp})$.
- The diagram E_{10} describes a polytope in $\mathbb{H}^9$.

Remark 12.22 (Hyperplane incidence relations in hyperbolic spaces)**.** In hyperbolic geometry ($\mathbb{H}^2$ to simplify), there are two types of parallelism: asymptotically parallel (converging) lines, or ultraparallel (diverging) lines. Both are characterized by sharing a common orthogonal line, however, asymptotically parallel lines have a common perpendicular in $\partial\overline{\mathbb{H}^2}$ going through their ideal point of intersection, while ultraparallel lines share a common perpendicular at a point in the interior $\mathbb{H}^2$. By the ultraparallel theorem, H_i, H_j are ultraparallel if and only if $H_i \cap H_j = \emptyset$ in $\overline{\mathbb{H}^2}$.

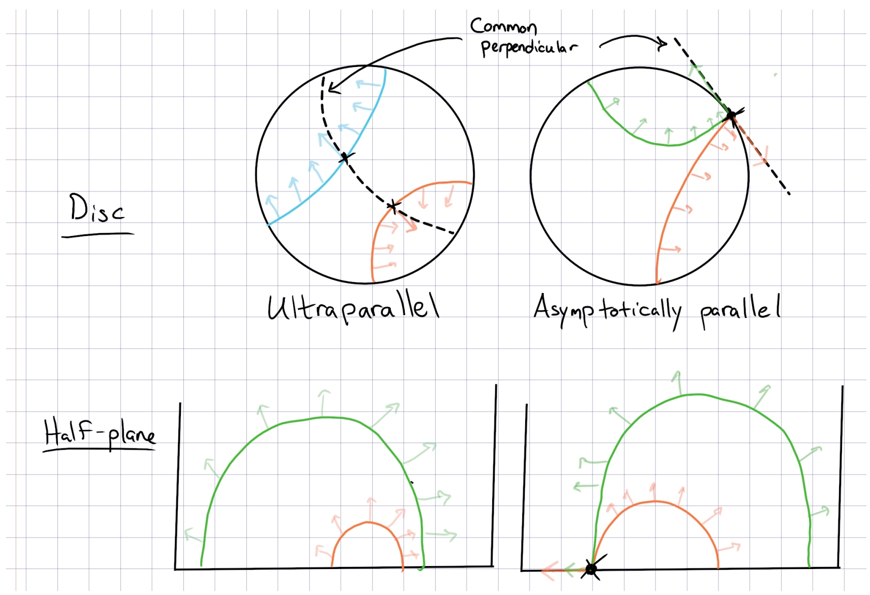


FIGURE 21. The two types of parallelism in hyperbolic space, visualized in the ball model and half-plane model respectively.

Thus given a pair of hyperplanes H_i and H_j, there are thus three possibilities for their incidence relations:

- H_i, H_j are not parallel and thus intersect in $\mathbb{H}^n$. We write $H_i \pitchfork H_j$ and define $\angle(H_i, H_j)$ as the usual dihedral angle.
- H_i, H_j are asymptotically parallel/converging and thus intersect in an ideal point in $\partial\mathbb{H}^n$. We write $H_i \parallel H_j$ and define $\angle(H_i, H_j) = \frac{\pi}{\infty} = 0$.
- H_i, H_j are ultraparallel/diverging and do not intersect in $\overline{\mathbb{H}^n}$. We write $H_i)(H_j$.

Remark 12.23 (Hyperbolic distance between hyperplanes)**.** Note that in the last case above, $\angle(H_i, H_j)$ is undefined but there is a minimal distance $\rho(H_i, H_j)$ between the two hyperplanes. By geometric axioms, if $H_i \cap H_j = \emptyset$ then there is a unique geodesic L_{ij} that is simultaneously orthogonal to both H_i and H_j, intersecting them at points p_i and p_j. One then defines $\rho(H_i, H_j)$ as the length of a geodesic segment along L_{ij} with endpoints at p_i and p_j.

Remark 12.24 (Extending Coxeter diagrams for hyperbolic polytopes)**.** Following Vinberg, one can extend the notion of a Coxeter diagram to a weighted graph with positive weights $w_{ij} > 0$ where all $w_{ij} \in (0,1)$ can be written in the form $w_{ij} = \cos\left(\frac{\pi}{m_{ij}}\right)$ for some $m_{ij} \in \mathbf{Z}_{\geq 2}$ and $w_{ij} \in [1, \infty]$ can be arbitrary real (possibly infinite) numbers. In this convention,

- $w_{ij} = \cos\left(\frac{\pi}{m_{ij}}\right) \in (0,1)$ get simple edges of labeled weight m_{ij} (or multi-edges) corresponding to $H_i \pitchfork H_j$ and $\angle(H_i, H_j) = \left(\frac{\pi}{m_{ij}}\right)$
- $w_{ij} = 1$ get **bold** unlabeled edges of weight 1 corresponding to $H_i \parallel H_j$ and $\angle(H_i, H_j) = \frac{\pi}{\infty} = 0$.
- $w_{ij} \in (1, \infty)$ get **dotted** labeled edges of weight w_{ij} (or unlabeled) corresponding to $H_i)(H_j$ and w_{ij} corresponds to $\rho(H_i, H_j)$

More generally, given a Coxeter-Vinberg diagram set

$$g_{ij} = \frac{h_i h_j}{\sqrt{h_i^2 h_j^2}},$$

then one interprets

- $g_{ij} < 1 \implies g_{ij} = \cos(\angle(h_i, h_j))$ and $H_i \pitchfork H_j$ with $\angle(h_i, h_j) = \pi/m_{ij}$,
- $g_{ij} = 1 \implies H_i \parallel H_j$ with $\angle(h_i, h_j) = 0$,
- $g_{ij} > 1 \implies H_i)(H_j$.

Todo: (∞, ∞, ∞).

Example 12.25 (Hyperbolic Coxeter polytopes)**.**

Remark 12.26. As in the Euclidean case that taking reflections of the fundamental domain C by the corresponding Weyl group naturally constructs a **tiling** of $\mathbb{E}^2$ in all of these cases:

- $A_1 \times A_1$ tiles $\mathbb{E}^2$ with 4 non-compact quadrants,
- A_2 tiles $\mathbb{E}^2$ with 6 non-compact sectors of angle $\pi/3$,
- In general, taking $\circ \to^m \circ$ with $m \in \mathbf{Z}_{\geq 1}$ tiles $\mathbb{E}^2$ with $2m$ non-compact sectors of angle π/m,
- $\widetilde{A}_2$ tiles $\mathbb{E}^2$ with infinitely many compact equilateral triangles of with internal angles $\pi/3$.

Definition 12.27 (The Gram matrix of a hyperbolic polytope)**.** Let $P \subseteq \overline{\mathbb{H}^n}$ be a Coxeter polytope, possibly with ideal points. The **Gram matrix** of P is the matrix

$$G(P)_{ij} = \begin{cases} 1 & i = j \\ -\cos\left(\frac{\pi}{m_{ij}}\right) & H_i \pitchfork H_j, \quad \angle(H_i, H_j) = \pi/m_{ij}, \\ -1 & H_i \parallel H_j, \quad \angle(H_i, H_j) = \pi/\infty = 0, \\ -\cosh(\rho(H_i, H_j)) & H_i)(H_j, \quad \angle(H_i, H_j) = \pi/0 = \infty, \end{cases}.$$

Remark 12.28. When labeling the Coxeter graph, one often puts m_{ij} or $\cosh(\rho(H_i, H_j))$ as the labels, mixing conventions slightly. Edges of weight 2 are deleted, edges of weight 3 are unlabeled simple edges.

Remark 12.29. If $P \subseteq \mathbb{H}^n$ is a compact hyperbolic Coxeter polytope, the quotients $\mathbb{H}^n/G_P$ are hyperbolic orbifolds. The simplest examples of such polytopes are the hyperbolic n-gons defined by integers $p_1, \cdots, p_k \geq 2$ satisfying $\sum p_i^{-1} < k-2$.

Definition 12.30 (Simple systems)**.** We say $\Delta = \{r_i\}$ is a **simple system** of generators for a polytope P if $r_i r_j \geq 0$ for all i and j, and P has a facet presentation by the mirrors H_{r_i}. This allows one to write

$$P = \left\{ v \in L_{\mathbf{R}} \,\middle|\, v^2 = 0,\, r_i v \geq 0 \right\}.$$

We call P a **Weyl chamber**[7]. The closure $\overline{P}$ is a fundamental domain for the action of $W(L)$ and the Weyl group acts simply transitively on the set of chambers.

Remark 12.31 (Decomposing the future orthogonal group into a Weyl and symmetry group)**.** Let $W(L)$ be the reflections in all negative norm vectors. There is an identification

$$\mathrm{O}^+(L) \cong W(L) \rtimes S(C), \qquad S(C) \coloneqq \mathrm{Stab}_{\mathrm{O}^+(L)}(C)$$

where $C \subset \mathbb{B}^n$ be a fundamental chamber of $W(L)$ with respect to some choice of a simple set of generators.

The semidirect might be in the wrong direction here, which one is normal?

Definition 12.32 (Reflective lattices)**.** We say L is **reflective** if $W(L) \leq \mathrm{O}^+(L)$ is finite-index. More generally, if we define $\mathrm{O}^+(L)_k$ as the subgroup generated by all k-reflections, i.e. reflections in roots v with $v^2 = k$, we say L is k-reflective if $W(L)$ is finite index in $\mathrm{O}^+(L)$.

Remark 12.33. If L as above is reflective, it is well-known C is a hyperbolic Coxeter polytope of finite volume.

Definition 12.34 (Vinberg-Coxeter diagrams)**.** A **Vinberg-Coxeter diagram** is an extension of a Coxeter diagram with adds the following decorations:

- Black edges
- Double-circled edges
- Dotted edges
- Thick edges

It is a weighted graph with positive edge weights $w_{ij} > 0$ where we require that any $w_{ij} \in (0,1)$ is of form $w_{ij} = \cos\left(\frac{\pi}{m_{ij}}\right)$ for some $m_{ij} \in \mathbf{Z}_{\geq 2}$, but we explicitly allow some $w_{ij} \in [1, \infty]$ to be real (possibly infinite) numbers. We additionally specify vertex weights r_i for each vertex v_i. In this convention,

- $w_{ij} = \cos\left(\frac{\pi}{m_{ij}}\right) \in (0,1)$ get **simple edges** of labeled weight m_{ij} (or unlabeled multi-edges of multiplicity $m_{ij} - 2$ for $m_{ij} = 3, 4, 5, 6$),
- $w_{ij} = 1$ get **bold unlabeled edges** of weight 1
- $w_{ij} \in (1, \infty)$ get **dotted labeled edges** of weight w_{ij}.

12D. Elliptic and Parabolic subdiagrams.

Remark 12.35. Given these weights, one can construct the weighted adjacency matrix A with $a_{ij} = w_{ij}$ if v_i, v_j are adjacent and zero otherwise.

A matrix A is a **direct sum of matrices** A_i if A is similar via permutations of rows and columns to the block diagonal matrix whose blocks are the A_i. If A can not

[7] This is also sometimes notated C.

be written as a direct sum of two matrices, we say A is **indecomposable**. Every matrix has a unique representation as a sum of indecomposable components. We say a Coxeter polytope is indecomposable if its Gram matrix G_P is indecomposable. Any matrix G_P arising from an irreducible Coxeter polytope is either positive-definite, positive-semidefinite, or indefinite. We say a diagram D_P is elliptic if G_P is PD, parabolic if every subdiagram is elliptic and it has at least one degenerate irreducible component.

Connected components of the diagram correspond to indecomposable sub-block matrices of A. A diagram is elliptic of A is positive-definite, and is parabolic if any indecomposable component of A is degenerate and positive-semidefinite. There are finitely many indecomposable elliptic and parabolic diagrams. If a Coxeter diagram describes a Coxeter polytope P, elliptic subdiagrams of codimension 1 correspond to facets of P. Moreover, P has finite volume iff every such elliptic subdiagram can be extended in exactly 2 ways to either an elliptic subdiagram of rank n or a parabolic subdiagram of rank $n-1$, corresponding to every facet of the polytope meeting each of its adjacent facets at either an interior point or an ideal point of $\mathbb{H}^n$ respectively.

Remark 12.36. Idea: a subdiagram is elliptic if the Gram matrix is negative definite of full rank, and parabolic if negative semidefinite of corank equal to the number of components of the diagram. Elliptic diagrams of rank r biject with codimension r faces of C. Parabolic diagrams of corank 1 correspond to ideal points of C. Vinberg's algorithm produces a simple system Δ of generators for $W(L)$ which determines a hyperbolic polytope C via the corresponding Weyl chamber. If the algorithm terminates, C is of finite volume.

Definition 12.37 (Ranks of subdiagrams)**.** The **rank** of a subdiagram is its number of vertices minus its number of connected components.

Definition 12.38 (Elliptic and parabolic Coxeter subdiagrams)**.** A Coxeter diagram G is called **elliptic** (resp. **parabolic**) if every connected component of G is a Coxeter diagram underlying classical (resp. affine) Dynkin diagram. This is summarized in the following table; note that the classical diagrams B_n and C_n become identified when the arrow is omitted:

A possible reference that mostly agrees with this: http://webdoc.sub.gwdg.de/ebook/serien/e/mpi_-mathematik/2005/8.pdf#page=3&zoom=180,-91,721

Another reference, although they seem to include a mysterious G_-2^m... % https://www.maths.dur.ac.uk/users/anna.felikson/talks/HypCoxPoly17.pdf#page=8&zoom=

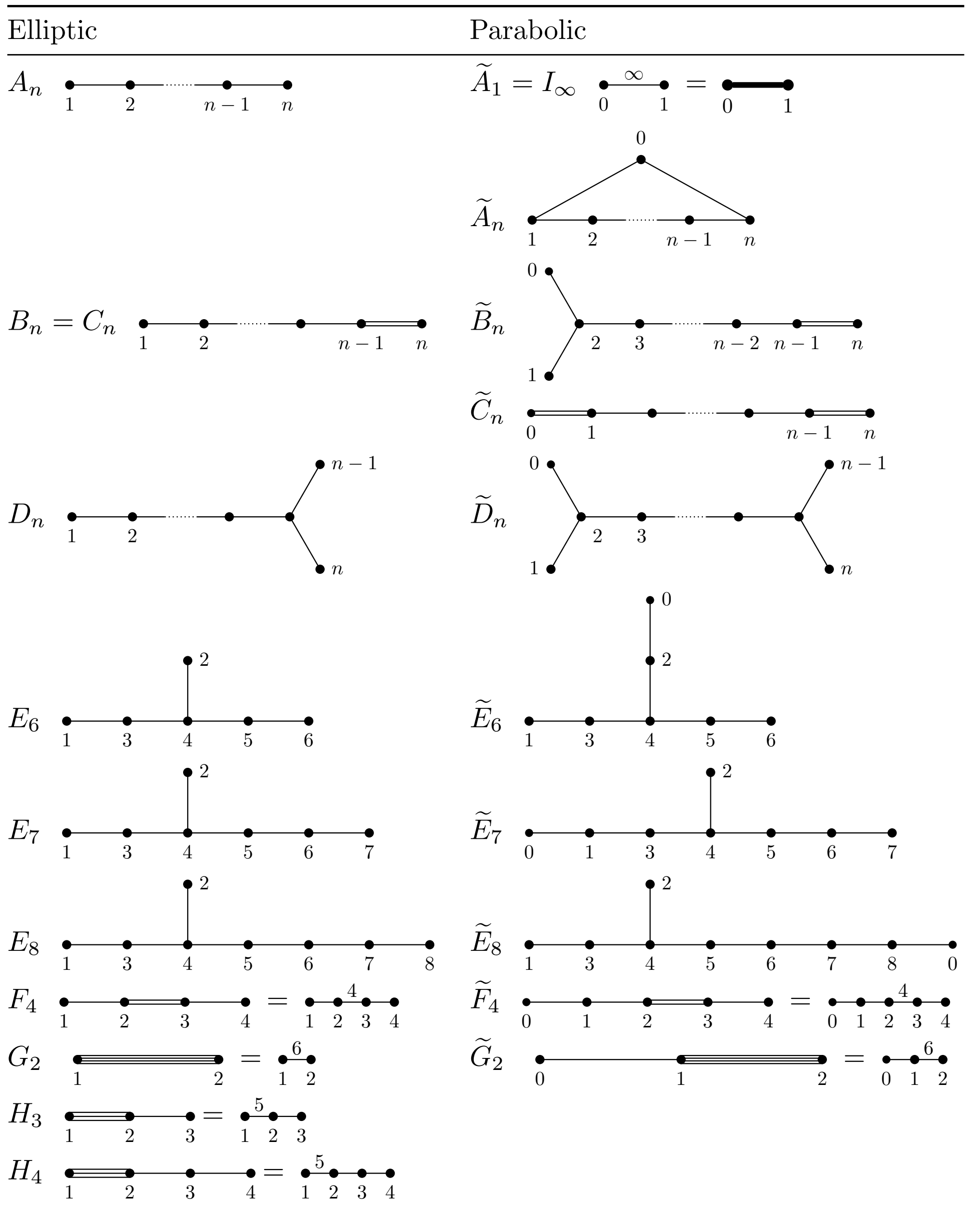


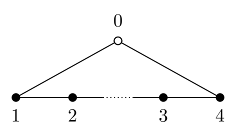
TABLE 2. Classification of elliptic and parabolic subdiagrams of a Coxeter diagram

12E. **Some discrepancies.** Note the following discrepancies when comparing the classification of diagrams of Coxeter diagrams to the usual notions of Dynkin diagrams:

- These are not Dynkin diagrams: we forget the arrows on double, triple, etc edges.

- **Warning**: in a **Coxeter** diagram, an edge of label m always corresponds to an $(m-2)$-fold edge. **In a Dynkin diagram, a 3-fold edge corresponds to $m = 6$. We do not use this convention in the table above!** Compare $G_2, \widetilde{G}_2$ in the table, which have 4-fold edges corresponding to $m = 6$ to the following classical diagrams for G_2 and $\widetilde{G}_2$ which still correspond to $m = 6$:

The reason for this discrepancy: in a **Dynkin** diagram, the edge labels m must satisfy a crystallographic condition and thus $m = 2, 3, 4, 6$. Since $m = 5$ is not possible, this makes the interpretation in that special case unambiguous.
- This discrepancy also occurs for H_i; here a triple edge truly corresponds to $m = 5$.
- In the affine case, we do not distinguish the "new" node, usually denoted by a white dot labeled 0. Compare to the usual diagram e.g. for $\widetilde{A}_n$:



Remark 12.39. Elliptic subdiagrams are a disjoint union of classical Dynkin diagrams, while parabolic subdiagrams are a disjoint union of *affine* Dynkin diagrams.

Why these matter: we are working with Coxeter polytopes P in a hyperbolic space, i.e. hyperbolic Coxeter polytopes. Vinberg has a general theory which says the Coxeter diagram D records the combinatorics of P:

- Facets of $P \rightleftharpoons$ nodes of D,
- Dihedral angles between two facets of $P \rightleftharpoons$ edges of D,
- k-faces of $P \rightleftharpoons$ elliptic subdiagrams of P of co-rank k,
- Ideal vertices of $P \rightleftharpoons$ parabolic subdiagrams of rank k.

Idea: in F_2, Type II strata are classified by maximal parabolic subdiagrams of the single Coxeter diagram, and Type III strata by elliptic subdiagrams. Dimensions of strata correspond to number of vertices in these subdiagrams, and inclusion of diagrams corresponds to degenerations (smaller diagrams correspond to "more degenerate").

12F. **Edge conventions for Coxeter diagrams.** The interpretation of these Coxeter diagrams in terms of root systems:

Needs some notation from [AN06]:

$V(M)$	the light cone $V(M) = \left\{ x \in M \otimes \mathbb{R} \,\middle|\, x^2 > 0 \right\}$ of a hyperbolic lattice M
$V^+(X)$	the half containing polarization of the light cone $V(S_X)$
$\mathcal{L}(S) = V^+(S_X)/\mathbb{R}^+$	the hyperbolic space of a surface S
$W^{(2)}(M)$	the group generated by reflections in all $f \in M$ with $f^2 = -2$
$W^{(4)}(M)$	the group generated by reflections in all (-4) roots of M
$W^{(2,4)}(M)$	the group generated by reflections in all (-2) and (-4) roots of M
$\mathcal{M}^{(2)}$	a fundamental chamber of $W^{(2)}(S)$ in $\mathcal{L}(S)$
$\mathcal{M}^{(2,4)}$	a fundamental chamber of $W^{(2,4)}(S)$ in $\mathcal{L}(S)$
$P^{(2)}\left(\mathcal{M}^{(2,4)}\right)$	all (-2)-roots orthogonal to $\mathcal{M}^{(2,4)}$
$P^{(4)}\left(\mathcal{M}^{(2,4)}\right)$	all (-4)-roots orthogonal to $\mathcal{M}^{(2,4)}$
(X, θ)	a K3 with involution θ
X^θ	the fixed locus of an involution
$P(X)_{+I}$	the subset of exceptional classes of (X, θ) of type I

Description	Symbol
Black vertices: $f \in P^{(4)}\left(\mathcal{M}^{(2,4)}\right)$, i.e. $f^2 = -4$	f ●
White vertices: $f \in P^{(2)}\left(\mathcal{M}^{(2,4)}\right)$, i.e. $f^2 = -2$	f ○
Double-circled vertices: $f \in P(X)_{+I}$, i.e. the class of a rational component of X^θ.	f ◎
No edge: $f_1 \neq f_2 \in P\left(\mathcal{M}^{(2,4)}\right)$ with $f_1 \cdot f_2 = 0$, so $\angle(f_1 f_2) = \pi/2$	f_1 ○ ○ f_2
Simple edges of weight m, or $m-2$ simple edges when m is small: $\frac{2 f_1 f_2}{\sqrt{f_1^2 f_2^2}} = 2\cos\frac{\pi}{m}$, so $\angle(f_1 f_2) = \pi/m$	f_1 ○——m——○ f_2
Thick edges: $\frac{2 f_1 \cdot f_2}{\sqrt{f_1^2 f_2^2}} = 2$	f_1 ○━━━○ f_2
Broken edges of weight t: ?	f_1 ○- - -t- - -○ f_2

Table 3. Edge conventions for Coxeter diagrams

Edge conventions for Coxeter polytopes: nodes correspond to facets f_i, f_j of P and edges record relations in G_P.

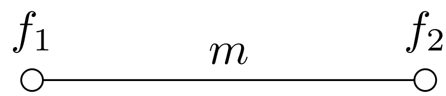
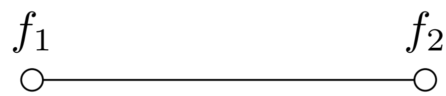
Description	Diagram
$\angle(f_i f_j) = \pi/2$	
$\angle(f_i f_j) = \pi/m$	
$\angle(f_i f_j) = \pi/3$	
$\angle(f_i f_j) = \pi/4$	
$\angle(f_i f_j) = \pi/5$	
f_i, f_j do not intersect	
f_i, f_j intersect in $\partial\overline{\mathbb{H}^n}$	

TABLE 4

12G. **Surfaces associated with Coxeter diagrams.**

Remark 12.40. As described in [AE22, Prop. 4.6], % https://arxiv.org/pdf/2208.10383.pdf#page=18&zoom=145,169 for the Halphen case $S := (10, 10, 1)$, there exists a K3 surface with $S_X^+ = S$ with $\pi : X \to Y := X/\iota$ where $\mathrm{Nef}(Y)$ can be identified with the Coxeter chamber for the full reflection group W_r.

Moreover, [AE23, Cor. 4.8] % https://arxiv.org/pdf/2208.10383.pdf#page=18&zoom=160,134,179 shows that the Coxeter diagram of S can be used to write the dual graph of exceptional curves on Y under the following modifications:

Description	Symbol	Description
A single-circled white vertex	$\overset{F}{\circ}$	$F \cong \mathbf{P}^1$ with $F^2 = -1$
A double-circled white vertex	$\overset{F}{\circledcirc}$	$F \cong \mathbf{P}^1$ with $F^2 = -4$
A black vertex	$\overset{F}{\bullet}$	$F \cong \mathbf{P}^1$ with $F^2 = -2$.
Any single, plain edge	$\overset{F_i}{\circ}$ —— $\overset{F_j}{\circ}$	$F_i, F_j \cong \mathbf{P}^1$ with $F_i F_j = 1$
Any bold edge	$\overset{F_i}{\circ}$ **━━** $\overset{F_j}{\circ}$	$F_i, F_j \cong \mathbf{P}^1$ with $F_i F_j = 2$
Any double edge	$\overset{F_i}{\circ}$ ══ $\overset{F_j}{\circ}$	??

TABLE 5. How to read a surface off of a Coxeter diagram

Unclear what double edges are, need to read further.

12H. **Incidence diagrams/dual complexes.**

12I. **Dual complexes.**

Definition 12.41 (Dual complex)**.** Let D be an RSNC divisor. The **dual complex** $\Gamma(D)$ of D is the PL-homeomorphism type of the simplicial complex whose d-cells correspond with codimension d strata in D, i.e. irreducible components of d-fold intersections $V_{i_0} \cap \cdots \cap V_{i_d}$.

Example 12.42. Let $\mathcal{X} \to \Delta$ be a semistable degeneration and let $\mathcal{X}_0 = V_1 \cup \cdots \cup V_n$ be the smooth surfaces forming the irreducible components of the central fiber. Writing $C_{ij} \coloneqq V_i \cap V_j$ and $p_{ijk} \coloneqq V_i \cap V_j \cap V_k$ for their intersections along curves and points, we call each irreducible component of C_{ij} a **double curve** and the points p_{ijk} **triple points**.

Semistability ensures that the dual complex has dimension at most 3, i.e. there are at worst triple points. Thus concretely the dual complex has

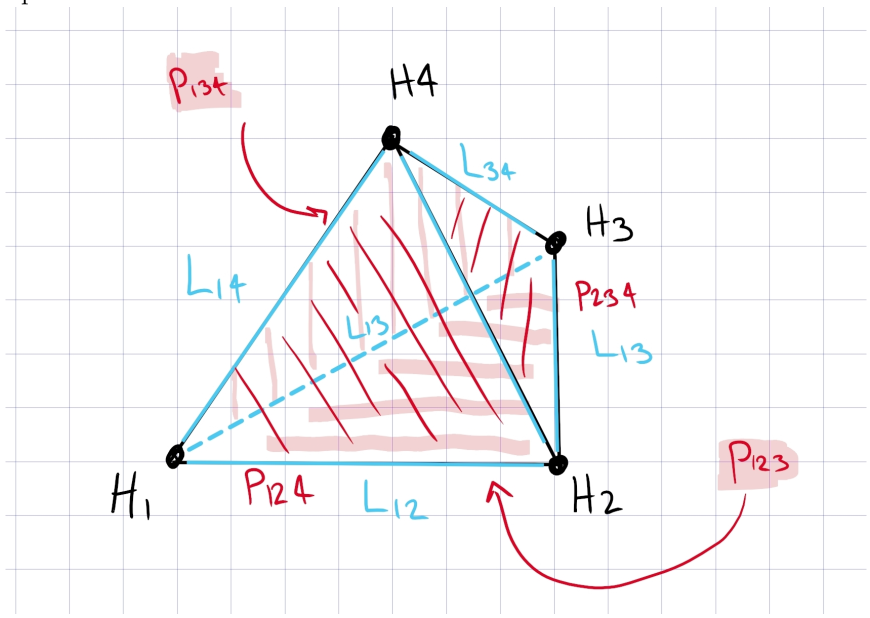
- a vertex for each component V_i,
- an edge from V_i to V_j for each double curve C_{ij}, and
- a 2-simplex spanning V_i, V_j, V_k for each triple point p_{ijk}.

Remark 12.43. For a double curve $C = C_{ij} = C_{ji}$ regarded as a curve in V_i and V_j respectively, Persson's triple point formula holds:

$$C_{ij}^2 + C_{ji}^2 = -T_C$$

where T_C is the number of triple points on C.

Example 12.44. Let $H_i \subset \mathbf{P}^3$ for $0 \leq i \leq 3$ be the four standard coordinate hyperplanes, i.e. $H_i = \left\{ [z_1 : z_2 : z_3 : z_4] \;\middle|\; z_i = 0 \right\}$ and let $D = \sum H_i$. Any 2 planes intersect in a line and any 3 planes intersect in a point, so there are $\binom{4}{2} = 6$ double curves $C_{ij} \coloneqq H_i \cap H_j$ and $\binom{4}{3} = 4$ triple points $p_{ijk} \coloneqq H_i \cap H_j \cap H_k$. The dual complex is the standard tetrahedron:



Definition 12.45 (Incidence complex)**.** Let (X, D) be a RNC compactification. The **incidence complex** $I(D)$ **of** D is the simplicial complex built in the following way: let $D = \sum_i D_i$ be a decomposition into prime divisors, and take a complex $I(D)$ whose k-dimensional cells are in bijection with irreducible components of k-fold intersections of the D_i. The **colored incidence complex** is $I(D)$ with an integer weight (or a coloring) attached to each 0-cell indicating the dimension of the corresponding stratum.

Remark 12.46. In our case of interest, (X, D) will be a Baily-Borel compactification of a moduli space where $D \coloneqq \partial \overline{X}$ is a union of boundary strata of various dimensions. Because we primarily work with hyperbolic lattices, D will only contain strata of dimensions 0 and 1, i.e. points and curves. Thus $I(D)$ will reduce to a graph whose vertices are in bijection with points and curves in D and whose edges record when a point p_j is contained in the closure of a curve C_i. We can thus form the colored incidence complex $I(D)$ with two colors, taking points to be black and curves to be white.

Definition 12.47 (Cusp incidence diagrams)**.** Let Ω_N be the period domain associated with a lattice N and let $\Gamma \subseteq \mathrm{O}(N)$ be a finite-index subgroup. The Baily-Borel compactification $\overline{\Omega_S/\Gamma}^{\mathrm{BB}}$ is a projective variety with a boundary stratification

$$\overline{\Omega_S/\Gamma}^{\mathrm{BB}} = (\Omega_S/\Gamma) \cup \mathcal{I} \cup \mathcal{J}, \qquad \partial\overline{\Omega_S/\Gamma}^{\mathrm{BB}} = \mathcal{I} \cup \mathcal{J}$$

where

- $\mathcal{I}$ is a set of points referred to as 0-*cusps*, which are in bijective correspondence with Γ-orbits of primitive isotropic 2-dimensional sublattices of N, and
- $\mathcal{J}$ is a set of modular curves referred to as 1-*cusps*, which are in bijective correspondence with Γ-orbits of primitive isotropic 1-dimensional sublattices of N.

We summarize below what information the colored incidence complex $I(\partial\overline{\Omega_S/\Gamma}^{\mathrm{BB}})$ captures:

Cusp Type	Type II, $\mathcal{J}$	Type III, $\mathcal{I}$
Boundary Strata	1-cusps/curves C_i	0-cusps/points p_j
Vertex type	C_i ○	p_j ●
Sublattice Type	Isotropic lines $[\mathbf{Z}e] \in \mathrm{Gr}_1^{\mathrm{iso}}(L)/\Gamma$	Isotropic planes $[\mathbf{Z}e \oplus \mathbf{Z}f] \in \mathrm{Gr}_2^{\mathrm{iso}}(L)/\Gamma$
Subdiagram Type	Maximal parabolic	Elliptic

TABLE 6. Cusp types

Moreover, we draw an edge between a black and white node to denote a point p_i contained in the closure of a curve C_j:

Example 12.48. Consider the following colored incidence diagram:

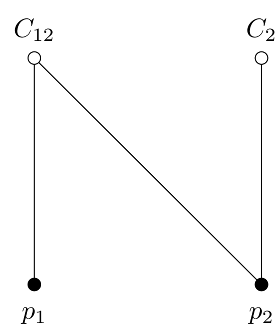


FIGURE 22. A colored incidence diagram $I(D)$ for $D = \mathcal{I} \cup \mathcal{J}$.

This represents the boundary stratification of a Baily-Borel compactification for which $\mathcal{I} = \{p_1, p_2\}$ consists of two points, $\mathcal{J} = \{C_{12}, C_2\}$ is two curves, where $p_1, p_2 \in \overline{C_{12}}$, $p_2 \in \overline{C_2}$, and $p_1 \notin \overline{C_2}$. This can be represented by the following configuration of curves and points:

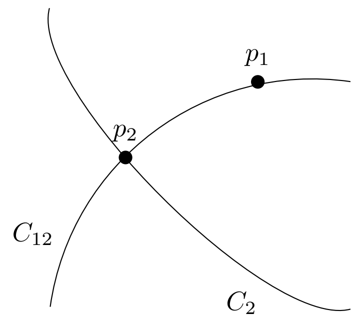


FIGURE 23. A configuration of curves and points representing $I(D)$ in Figure 23.

Spell out which root system this is attached to? Yes, that would be a good idea.

Remark 12.49. Each 0-cusp p_i has an associated Vinberg diagram $\mathcal{D}(p_i)$ whose maximal parabolic subdiagrams enumerate the 1-cusps C_{ij} adjacent to p_i in the incidence diagram.

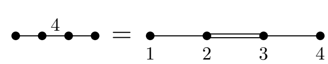
Example 12.50. The following figure shows the Vinberg diagram for the 0-cusp ???? associated to the lattice $N \coloneqq (18, 0, 0)$:

This has the following two maximal parabolic subdiagrams:

Vinberg1Parabolic1 *Vinberg1Parabolic2*

That there are exactly 2 such subdiagrams is reflected in the fact that the vertex ??? in the incidence diagram has valence 2.

\end{document}



Conventions on Coxeter diagrams: https://file.notion.so/f/s/962e72a3-fa3a-45ec-84a3-2bb0e7699a7b/Alexeev-Nikulin_Del_Pezzo_and_K3_-Surfaces.pdf?id=7e4996b4-08e6-4bd2-ac49-8d23080f7577&table=block&spaceId=7cb2f7c7-7373-4d11-91ab-284625335dc8&expirationTimestamp=1683905929597&signature=fMZC5iiiI0MbbvXkv
155,146

The interpretation of these Coxeter diagrams in terms of root systems:
Needs some notation from [AN06]:

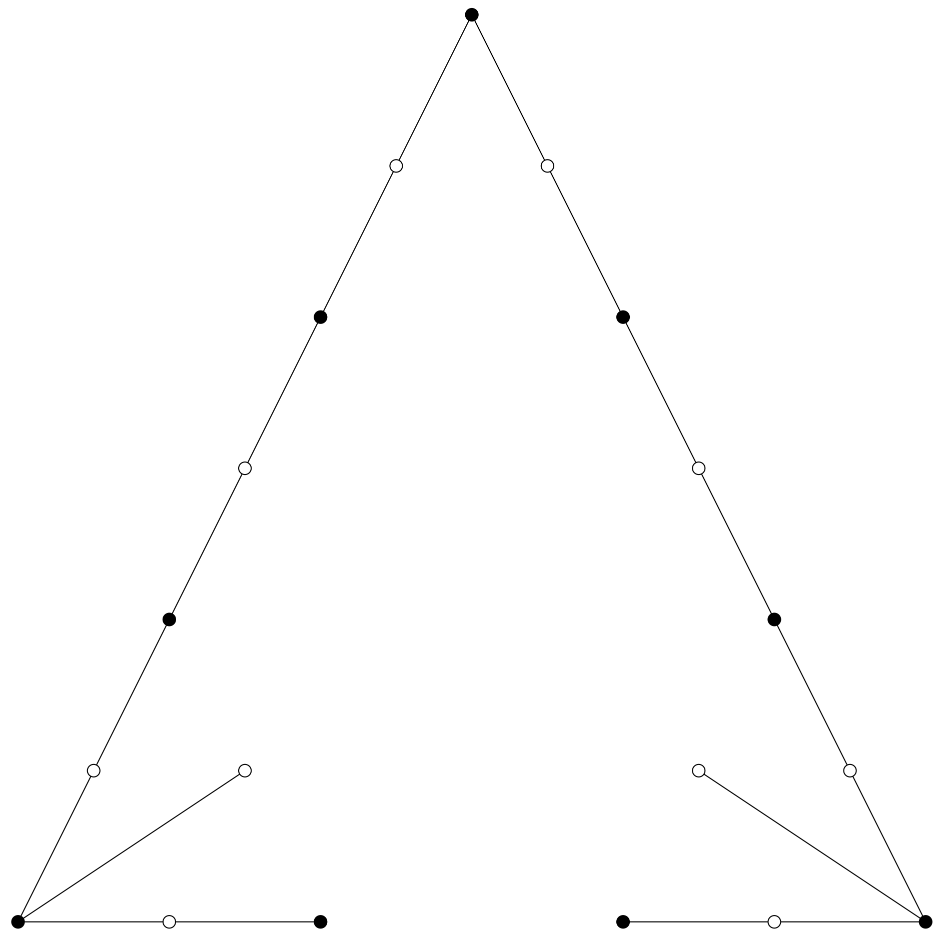


FIGURE 24. Coxeter-Vinberg diagram

$V(M)$	the light cone $V(M)=\left\{x\in M\otimes\mathbb{R}\,\middle|\,x^2>0\right\}$ of a hyperbolic lattice M
$V^+(X)$	the half containing polarization of the light cone $V(S_X)$
$\mathcal{L}(S)=V^+(S_X)/\mathbb{R}^+$	the hyperbolic space of a surface S
$W^{(2)}(M)$	the group generated by reflections in all $f\in M$ with $f^2=-2$
$W^{(4)}(M)$	the group generated by reflections in all (-4) roots of M
$W^{(2,4)}(M)$	the group generated by reflections in all (-2) and (-4) roots of M
$\mathcal{M}^{(2)}$	a fundamental chamber of $W^{(2)}(S)$ in $\mathcal{L}(S)$
$\mathcal{M}^{(2,4)}$	a fundamental chamber of $W^{(2,4)}(S)$ in $\mathcal{L}(S)$
$P^{(2)}\left(\mathcal{M}^{(2,4)}\right)$	all (-2)-roots orthogonal to $\mathcal{M}^{(2,4)}$
$P^{(4)}\left(\mathcal{M}^{(2,4)}\right)$	all (-4)-roots orthogonal to $\mathcal{M}^{(2,4)}$
(X,θ)	a K3 with involution θ
X^θ	the fixed locus of an involution
$P(X)_{+I}$	the subset of exceptional classes of (X,θ) of type I

12J. **Edge notation.**

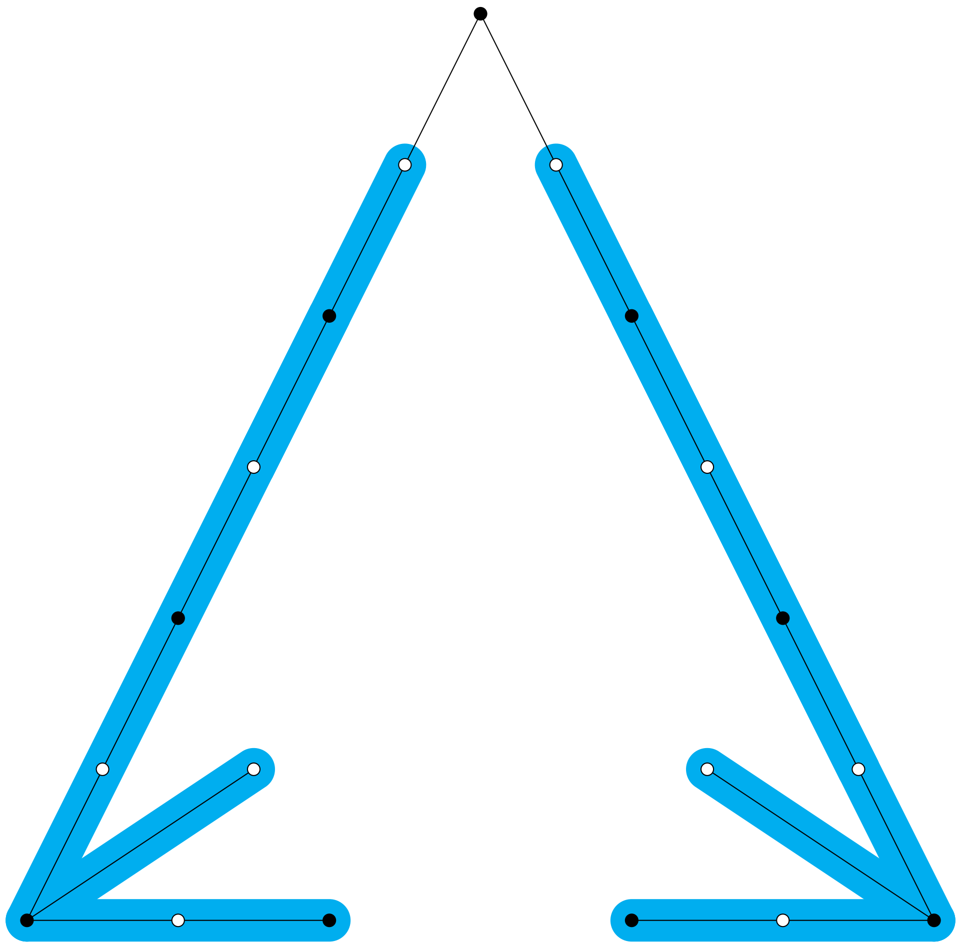
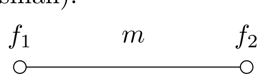


FIGURE 25. Caption

- Vertices corresponding to different elements $f_1, f_2 \in P\left(\mathcal{M}^{(2,4)}\right)$ are not connected by any edge if $f_1 \cdot f_2 = 0$.
- Simple edges of weight m (equivalently, by $m-2$ simple edges if $m > 2$ is small):

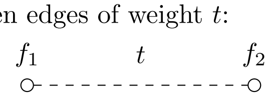


$$\Longrightarrow \qquad \frac{2f_1 \cdot f_2}{\sqrt{f_1^2 f_2^2}} = 2\cos\frac{\pi}{m}, \quad m \in \mathbb{N}$$

- Thick edges:

$$\Longrightarrow \qquad \frac{2f_1 \cdot f_2}{\sqrt{f_1^2 f_2^2}} = 2$$

- Broken edges of weight t:



$$\Longrightarrow \qquad \frac{2f_1 \cdot f_2}{\sqrt{f_1^2 f_2^2}} = t > 2$$

- A vertex corresponding to $f \in P^{(4)}\left(\mathcal{M}^{(2,4)}\right)$ is black:

$$\overset{f}{\bullet} \qquad \Longrightarrow \qquad f^2 = -4?$$

- It is white if $f \in P^{(2)}\left(\mathcal{M}^{(2,4)}\right)$:

$$\overset{f}{\circ} \qquad \Longrightarrow \qquad f^2 = -2?$$

- It is double-circled white if $f \in P(X)_{+I}$ (i.e. it corresponds to the class of a rational component of X^θ).

$$\overset{f}{\circledcirc} \qquad \Longrightarrow \qquad ??$$

Interpreting this geometrically: consider the cycle of $2\overline{k}$ white vertices cycling between plain and double-circled:

Second to last paragraph here: https://arxiv.org/pdf/2208.10383.pdf#page=41&zoom=140,99,262

- See [AE23]
- Each edge on the outer cycle corresponds to $\mathbb{P}^2$
- Single circle vertices (with odd i) corresponds to a line in $\mathbb{P}^2$
- Double-circled vertices (with even i) correspond to conics on the $\mathbb{P}^2$
- Explicit example worked out in [AET23, §5].

% https://arxiv.org/pdf/1903.09742.pdf#page=28&zoom=170,-70,442

- It seems like that from the Coxeter diagram, you draw the fan of a toric surface, you compute the charge, and then, if this is not 24, you fix it by blowing up some non-torus fixed points along the toric boundary.

13. Hermitian Symmetric Domains

13A. Cusp correspondence for Hermitian symmetric domains.

Definition 13.1 (Symmetric spaces)**.** A **locally symmetric space** is a connected Riemannian manifold such that every $x \in M$ is the fixed point of some involution $\gamma_x \in \mathrm{Isom}(U_x)$, the real algebraic Lie group of holomorphic automorphisms of an open subset $U_x \subseteq M$, which acts by -1 on T_xM. Equivalently, the covariant derivative of the curvature tensor vanishes, which is analogous to a constant curvature condition. It is a **symmetric space** if γ_x extends from $\mathrm{Isom}(U)$ to $\mathrm{Isom}(M)$.

If M is a symmetric space, then $M \cong G/K$ where $G = \mathrm{Isom}(M)$ is its group of isometries and $K := \mathrm{Stab}_G(x)$ is the stabilizer of any point $x \in M$. We say a manifold M is **homogeneous** if $M \cong G/K$ for some G and K.

Remark 13.2. Idea: Hermitian symmetric manifolds are manifolds that are homogeneous spaces such that every point has an involution preserving the Hermitian structure. These were first studied by Cartan in the context of Riemannian symmetric manifolds. They show up often as orbifold covers of moduli spaces, e.g. polarized abelian varieties (with or without level structure), polarized K3 surfaces, polarized irreducible holomorphic symplectic manifolds, etc. There is a structure theorem: any Hermitian symmetric manifold M decomposes as a product $M \cong \mathbf{C}^n/\Lambda \times M_c \times M_{nc}$ where Λ is some lattice, M_c is an HSM of compact type, and M_{nc} is an HSM of non-compact type. Every HSM of compact type is a flag manifold G/P for G a semisimple complex Lie group and P is a parabolic subgroup. Every HSM of non-compact type admits a canonical so-called Harish-Chandra embedding whose image is a bounded symmetric domain $D \subseteq \mathbf{C}^N$ for some N. Moreover, every HSM of non-compact type admits an associated Borel embedding into an associated HSM of compact type called its compact dual. Moreover, there is a Lie-theoretic classification of HSMs of compact and non-compact type – they are all of the form G/K for G a simple compact (resp. non-compact) Lie group and $K \leq G$ is a maximal compact subgroup with center isomorphic to $S^1 \cong \mathrm{U}_1(\mathbf{C})$.

By the Harish-Chandra embedding, non-compact HSMs can be realized as bounded domains $D \subseteq \mathbf{C}^N$ and admit a compactification by taking the closure $\overline{D} \supseteq D$ in $\mathbf{C}^N$. There is a partition of $\overline{D}$ by an equivalence relation related to being connected through chains of holomorphic discs, and each equivalence class is called a boundary component of D. Boundary components are in bijection with their normalizer subgroups, which are precisely maximal parabolic subgroups of $G \coloneqq \mathrm{Aut}(D)$.

A **Hermitian symmetric domain** is a Hermitian symmetric space of non-compact type.

Example 13.3. Some very basic examples of Hermitian symmetric manifolds:

- Tori $\mathbf{C}/\Lambda$ with Hermitian structure $g = dxdx + dydy$ induced from $\mathbf{R}^2$ (constant zero curvature).
- The upper half space $\mathcal{H}^1$ with Hermitian structure the hyperbolic metric $g = y^{-2}dxdy$ (constant negative curvature)
- $\mathbf{P}^1(\mathbf{C})$ with the Fubini-Study metric (constant positive curvature).

More advanced examples of symmetric spaces:

- $\mathbb{E}^n$, Euclidean space $\mathbf{R}^n$.
- $\mathbb{S}^n$, the spherical geometry,
- $\mathbb{H}^n$, hyperbolic space,
- $\mathrm{Sym}^{>0}_{n\times n}(\mathbf{R}) \leq \mathrm{SL}_n(\mathbf{R})$ the Riemannian manifold of positive-definite symmetric matrices with real entries
- X defined in the following way: let V be a Hermitian $\mathbf{C}$-module with Hermitian form h of signature (p, q) and let $X \subseteq \mathrm{Gr}_p(V)$ be the Grassmannian of p-dimensional subspaces W such that $h|_W$ is positive definite.

We first record their isometry groups:

- $\mathrm{Isom}(\mathbb{E}^n) = \mathbf{R}^n \rtimes \mathrm{O}_n(\mathbf{R})$.
- $\mathrm{Isom}(\mathbb{S}^n) = \mathrm{O}_{n+1}(\mathbf{R})$
- $\mathrm{Isom}(\mathbb{H}^n) = \mathrm{O}^+_{n+1}(\mathbf{R})$ the index 2 subgroup of $\mathrm{O}_{n+1}(\mathbf{R})$ which preserves the upper sheet $(\mathbb{H}^n)^+$.[8]
- $\mathrm{Isom}(\mathrm{Sym}^{>0}_{n\times n}(\mathbf{R})) = \mathrm{SL}_n(\mathbf{R})$.
- $\mathrm{Isom}(X) = \mathrm{SU}_{p,q}(\mathbf{C})$?

[8]Note that for $n = 1$, we can take the upper half-plane model which has isometry group $\mathrm{PSL}_2(\mathbf{R})$ or the disc model which has isometry group $\mathrm{PSU}_{1,1}(\mathbf{C})$. These are actually isomorphic as Lie groups.

Computing stabilizers of points, one can show

$$\mathbf{R}^n \cong \frac{\mathbf{R}^n \rtimes \mathrm{O}_n(\mathbf{R})}{\mathrm{O}_n(\mathbf{R})}$$

$$\mathbb{S}^n \cong \frac{\mathrm{O}_{n+1}(\mathbf{R})}{\mathrm{O}_n(\mathbf{R})}$$

$$\mathbb{H}^n \cong \frac{\mathrm{O}^+_{n+1}(\mathbf{R})}{\mathrm{O}_n(\mathbf{R})}$$

$$\mathrm{Sym}^{>0}_{n\times n}(\mathbf{R}) \cong \frac{\mathrm{SL}_n(\mathbf{R})}{\mathrm{SO}_n(\mathbf{R})}$$

$$X \cong \frac{\mathrm{SU}_{p,q}(\mathbf{C})}{\mathrm{SU}_p(\mathbf{C}) \times \mathrm{SU}_q(\mathbf{C})}$$

Note that taking $(p,q) = (1,1)$ yields $\mathbb{H}^2$.

Definition 13.4. % Probably need to cite:https://dept.math.lsa.umich.edu/~idolga/EnriquesOne.pdf# 136,697

A **Hermitian symmetric space** is a locally symmetric space M which is additionally equipped with an integrable almost-complex structure whose Riemannian metric is Hermitian.

We say M is **irreducible** if it is not the cartesian product of two symmetric Hermitian spaces; every irreducible such space is either $\mathbf{R}^n$ for some n or a homogeneous space G/K for G a real Lie group and K a maximal subgroup. We say M is of **compact type** if G is compact and K is a maximal proper subgroup, and of **non-compact type** if G is non-compact. If M is an irreducible Hermitian symmetric domain of non-compact type, there is an open embedding $M \hookrightarrow \mathcal{D}_L \subseteq \mathbf{C}^n$ onto a bounded subset $\mathcal{D}$ of complex n-space, in which case we call $\mathcal{D}$ a **bounded Hermitian symmetric domain**.

The simplest example is the upper half plane $\mathbb{H} := G/K$ for $G = \mathrm{SL}_2(\mathbf{R})$ and $K = \mathrm{SO}_2(\mathbf{R})$, which is biholomorphic to the bounded domain Δ via the Cayley transformation, which is a homogeneous space for $(G,K) = (\mathrm{SU}_{1,1}, B)$ where B is the subgroup of diagonal matrices. Note that $\mathbb{H} \cong \mathcal{H}_1$ is the Siegel upper half space of genus 1.

If (V,q) is a real quadratic space where $V := L_{\mathbf{R}}$ for L a lattice, we can define a corresponding domain

$$\mathcal{D}^\pm_L := \left\{ \mathbf{C}z \in \mathbf{P}(V_{\mathbf{C}}) \;\middle|\; z^2 = 0,\ |z| > 0 \right\},$$

the set of lines spanned by isotropic vectors of positive Hermitian norm $|z| := z\overline{z}$ in $V_{\mathbf{C}}$. If $\mathrm{sig}(L) = (2,n)$, so L is hyperbolic, this has an irreducible component decomposition into two parts $\mathcal{D}^\pm_L = \mathcal{D}^+_L \coprod \mathcal{D}^-_L$ interchanged by conjugation $z \mapsto \overline{z}$. Each component is an irreducible Hermitian symmetric domain of type

$$(G,K) = (\mathrm{SO}(V) := \mathrm{SO}_{2,n}(\mathbf{R}), \mathrm{SO}_2(\mathbf{R}) \times \mathrm{SO}_n(\mathbf{R})).,$$

i.e. a Type IV domain for $\mathrm{SO}_{2,n}$. We let $\mathcal{D}_L := \mathcal{D}^+_L$ be a choice of one component and write $\mathrm{O}^+(L) \le \mathrm{O}(L)$ for the subgroup which preserves $\mathcal{D}_L$ setwise. There is a distinguished divisor attached to $\mathcal{D}_L$, the **discriminant divisor**:

$$\mathcal{H}_L := \bigcup_{v \in R_2(L)} H_v \cap \mathcal{D}_L,$$

the hyperplane configuration defined by mirrors of roots. When Global Torelli is satisfied, there is a period map ϕ whose image is typically the complement of some hyperplane arrangement $\mathcal{H}$. In good cases, the relevant arrangement is precisely $\mathcal{H}_L$.

Note that $\mathcal{D}_L$ is isomorphic to a flag variety $G_{\mathbf{C}}/P$ for P some parabolic subgroup, and thus the compact form $\widetilde{\mathcal{D}}_L$ is a projective algebraic variety containing $\mathcal{D}_L$.

We say $\mathcal{D}_L$ as above is a **Hermitian symmetric domain of orthogonal type** or a **type IV Hermitian symmetric domain** in Cartan's classification. The period domains of K3 and Enriques surfaces are examples of such Type IV domains for $1 \leq n \leq 19$.

Definition 13.5. Let G be a simple linear algebraic group defined over $\mathbf{Q}$. We define

$$G(\mathbf{Z}) \coloneqq \mathrm{GL}_n(\mathbf{Z}) \cap G(\mathbf{Q})$$

where we use the natural embedding of algebraic groups $G \hookrightarrow \mathrm{GL}_n$ over $\mathbf{Q}$. A subgroup $\Gamma \leq G(\mathbf{Q})$ is **arithmetic** if $\Gamma \cap G(\mathbf{Z})$ has finite index in both Γ and $G(\mathbf{Z})$.

Definition 13.6 (Parabolic subgroups)**.** Let G be a linear algebraic group over $\mathbf{Q}$. We say $P \leq G$ is a parabolic subgroup if G/P is a projective variety.

Remark 13.7. As the notation suggests, there are other types of irreducible Hermitian symmetric domains. The following are some typical examples of the form $\Gamma \backslash \Omega$ for various definitions of Ω:

- Type III: Siegel modular varieties corresponding to $\Gamma \leq \mathsf{Sp}(\Lambda)$, the isometry group of a symplectic lattice, of rank $n \geq 3$.
- Type IV: Orthogonal modular varieties corresponding to $\Gamma \leq \mathrm{O}^+(\Lambda)$, a connected component of the isometry group of a lattice of signature $(2, n)$ for $n \geq 3$,
- Type $\mathrm{I}_{n,n}$: Hermitian modular varieties/Hermitian upper half spaces. These are attached to $\Gamma \leq \mathrm{U}(\Lambda)$ for Λ a Hermitian form q of signature (n, n) with $n \geq 2$. The compact dual is the Grassmannian $\mathrm{Gr}_{n,2n}$.
- Type II_{2n}: Quaternionic modular varieties/quaternionic upper half spaces. These are attached to $\Gamma \leq \mathsf{Sp}_{2n}(H)$ for H Hamilton's quaternions, attached to a skew-Hermitian space of dimension $2n$ with $n \geq 2$. The compact dual is the orthogonal Grassmannian $\mathrm{OGr}_{2n,4n}$

Where do $\mathcal{H}_g$ and $\Gamma \backslash \mathbb{H}^n$ fit in?

Remark 13.8. For Λ a lattice of signature $(2, n)$, the Hermitian symmetric domain attached to Λ is the following: define $Q \subseteq \mathbf{P}(\Lambda_{\mathbf{C}})$ be the quadric cut out by $(\omega, \omega) = 0$, then Ω_Λ is a choice of one of the two connected components of the open set Q defined by $(\omega, \overline{\omega}) > 0$. Letting $\mathrm{O}^+(\Lambda) \leq \mathrm{O}(\Lambda)$ be the subgroup preserving the component Ω_Λ and $\Gamma \leq \mathrm{O}^+(\Lambda)$ be any finite index subgroup, we obtain

$$X_\Lambda(\Gamma) \coloneqq \Gamma \backslash \Omega_\Lambda.$$

Embedding Ω_Λ in its compact dual, it has 0 and 1-dimensional boundary strata, corresponding to 1 and 2-dimensional isotropic subspaces of $\Lambda_{\mathbf{Q}}$. The BB compactification $\overline{X_\Lambda(\Gamma)}^{\mathrm{BB}}$ is the union of Ω_Λ and these rational boundary components, quotiented by the action of Γ, equipped with the Satake topology.

A toroidal compactification $\overline{X_\Lambda(\Gamma)}^{\mathrm{Tor}}$ is specificed by a finite collection of suitable fans $\{F_I\}$, one for each 0-cusp (i.e. each Γ-orbit of isotropic lines I in $\Lambda_{\mathbf{Q}}$).

For each I there is a tube domain realization given by taking the linear projection from the boundary point, which defines an isomorphism

$$\Omega_\Lambda/U(I)_{\mathbf{Z}} \cong U \subseteq T_I := U(I)_{\mathbf{C}}/U(I)_{\mathbf{Z}}$$

The partial compactifications for the 1-cusps are completely canonical, so the overall compactification is defined by gluing onto the boundary of $X_\Lambda(\Gamma)$ certain natural quotients of all of these partial compactifications to obtain $\overline{X_\Lambda(\Gamma)}^{\text{Tor}}$. This yields a compact algebraic space which is proper over Spec $\mathbf{C}$, and there is a natural morphism $\overline{X_\Lambda(\Gamma)}^{\text{Tor}} \to \overline{X_\Lambda(\Gamma)}^{\text{BB}}$. :::

Example 13.9.
```
Let $G \da \SL_2$ defined over $\QQ$ and let $\Gamma \leq \SL_2(\QQ)$ be an ari
\[
Y(\Gamma) \da \Gamma\backslash \HH^1
.\]
In this case, rational boundary components are given by $\PP^1(\QQ) = \QQ\union \ts{\infty}\s
\[
X(\Gamma) \da \overline{Y(\Gamma)} \da Y(\Gamma) \union \ts{\text{cusps}}
\]
topologized appropriately, where e.g. $\ts{\infty}$ is one such cusp.

Note that one typically takes the following groups for moduli of elliptic curves with level s
```

- $Y(N) := Y(\Gamma(N))$ where
$$\Gamma(N) := \ker\left(\phi_N : \mathrm{SL}_2(\mathbf{Z}) \to \mathrm{SL}_2(\mathbf{Z}/N\mathbf{Z})\right).$$
The level structure is a basis for $E[n]$.
- $Y_0(N) := Y(\Gamma_0(N))$ where $\Gamma_0(N) \supseteq \Gamma(N)$ is the pullback $\phi_N^{-1}\left(\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}\right)$. The level structure is an identification $\mu_N \hookrightarrow E_{\text{tors}}$.
- $Y_1(N) := Y(\Gamma_1(N))$ where $\Gamma_1(N)$ is the pullback $\phi_N^{-1}(1b01)$. The level structure is a point $p \in E$ of order N in the group structure.

  How parabolic subgroups appear here: for $G := \mathrm{SL}_2$, parabolic subgroups are all conjugate to the subgroup P of upper-triangular matrices, and $G(\mathbf{Q})/P(\mathbf{Q}) \cong \mathbf{P}^1(\mathbf{Q})$ parameterizes all such parabolic subgroups.

  Why automorphic forms matter: consider $\Gamma := \mathrm{SL}_2(\mathbf{Z})$. The graded ring of modular forms $\bigoplus_k M_k$ is graded-isomorphic to $\mathbf{C}[x,y]$ where $|x| = 4, |y| = 6$, and $\operatorname{Proj}\mathbf{C}[x,y] \cong \mathbf{P}^1(\mathbf{C})$. Letting $\{f_0, \cdots, f_N\}$ be a basis of M_k, we can write down a map

$$\begin{aligned}\phi_k : Y(\mathrm{SL}_2(\mathbf{Z})) &\to \mathbf{P}^n(\mathbf{C}) \\ z &\mapsto [f_0(z) : \cdots : f_N(z)]\end{aligned}$$

  For $k = 12$ this separates points and tangent directions, giving a projective embedding. Explicitly, the morphism is

$$\phi_{12}(z) = [E_4(z) : E_4(z)^3 - E_6(z)^2] \approx j(z)$$

  modulo some missing constants. In general, finding enough automorphic forms yields a projective embedding.

Would like to spell this out in terms of line bundles and linear systems too, in this easy case.

13B. **Misc.**

Remark 13.10. Let L be a lattice of signature $(2,n)$ and the associated period domain $\Omega_L^{\pm} = \Omega_L^{+} \coprod \Omega_L^{-}$. Let $\mathrm{O}(L)^+ \leq \mathrm{O}(L)$ be the finite index subgroup fixing Ω_L^+, equivalently the subgroup of elements of spinor norm one. A modular variety of orthogonal type is a homogeneous space of the form $F_L(\Gamma) := \Gamma \backslash \Omega_L^+$ for an arithmetic subgroup $\Gamma \leq \mathrm{O}(L_{\mathbf{Q}})^+$.

By general theory, such spaces admit BB compactifications $\overline{F_L(\Gamma)}^{\mathrm{BB}}$ where rational maximal parabolic subgroups correspond to stabilizers of isotropic subspaces of $L_{\mathbf{Q}}$; since $\mathrm{sig}(L) = (2,n)$ these are always isotropic lines or planes.

For period spaces of K3 surfaces, one takes $\Gamma := \mathrm{O}(L_{2d}) \cap \ker\left(\mathrm{O}(L) \to \mathrm{O}(A_L)\right)$. Boundary strata correspond to central fibers of KPP models of Type II and Type III.

Remark 13.11. Let L be a symplectic lattice of rank $2g$, ie.e. a free $\mathbf{Z}$-module with a nondegenerate alternating form $(-,-)$. Define the associated period space

$$D_L := \left\{ V \in \mathrm{Gr}_g(L_{\mathbf{C}}) \;\middle|\; (V,V) = 0,\, i(V, \overline{V}) > 0 \right\} \cong \mathsf{Sp}_{2g}(\mathbf{R})/\mathrm{U}_g(\mathbf{C}) \cong \mathcal{H}^g$$

which is a Hermitian symmetric domain of type III that can be identified with the Siegel upper half-space of dimension g. We can form the moduli space of PPAV as

$$\mathcal{A}_g := \mathsf{Sp}_{2g}(\mathbf{Z}) \backslash \mathcal{H}^g \cong \mathsf{Sp}_{2g}(\mathbf{Z}) \backslash \mathsf{Sp}_{2g}(\mathbf{R})/\mathrm{U}_g(\mathbf{C})$$

Rational boundary components of $\overline{\mathcal{A}_g}^{\mathrm{BB}}$ correspond to $\Gamma := \mathsf{Sp}_{2g}(\mathbf{Z})$ orbits of totally isotropic subspaces in $L_{\mathbf{Q}}$. Since Γ acts transitively, such spaces are indexed by their dimension $i = 0, 1, \cdots, g$ and there is a stratification

$$\overline{\mathcal{A}_g}^{\mathrm{BB}} = \coprod_{k=0}^{g} \mathcal{A}_k \implies \partial \overline{\mathcal{A}_g}^{\mathrm{BB}} = \coprod_{k=0}^{g-1} \mathcal{A}_k$$

Remark 13.12. The BB compactification of a locally symmetric domain D: write $D = H/K$ as a homogeneous space where $H := \mathrm{Hol}(D)^+$ and $K \leq H$ is a maximal compact subgroup. Then cusps in $\partial \overline{\Gamma \backslash D}^{\mathrm{BB}}$ correspond to rational maximal parabolic subgroups of H. To get boundary components: apply the Harish-Chandra embedding to D to embed $HC : D \hookrightarrow D^{cd}$ and let $F_P \in \overline{HC(D)}$ be a boundary component. Its normalizer $N(F_P) := \left\{ g \in H \;\middle|\; g(F_P) = F_P \right\} \leq H$ is a maximal parabolic in H. We say F_P is **rational** if $N(F_P)$ can be defined over $\mathbf{Q}$. Since Γ preserves such rational F_P, we can set $\partial D :=$ the disjoint union of all rational F_P and set $\overline{\Gamma \backslash D}^{\mathrm{BB}} = \frac{D \coprod \partial D}{\Gamma}$.

13B.1. *Explicit realizations of symmetric spaces.*

Remark 13.13. The symmetric space associated with a Lie group G is in some sense the most natural space G acts on. For $G = \mathrm{O}_{p,q}(\mathbf{R})$, the symmetric space is $\mathrm{Gr}^+(\mathbf{R}^{p,q})$, the Grassmannian of maximal positive-definite subspaces of $\mathbf{R}^{p,q}$. The right choice of maximal compact subgroup here is $K := \mathrm{O}_p(\mathbf{R}) \times \mathrm{O}_q(\mathbf{R})$,
Typo maybe.
the subgroup fixing $\mathbf{R}^{m,0}$. When $(p,q) = (2,n)$, these symmetric spaces admit special descriptions. Note that $\mathrm{O}_{n+1}(\mathbf{R})$ is the group of isometries of S^n, so its projectivization $\mathbf{P}\mathrm{O}_{n+1}(\mathbf{R})$ is the isometry group of an elliptic geometry. One can similarly obtain isometries of hyperbolic geometry:

- Start with $\mathbf{R}^{1,n}$

- Take the norm 1 vectors $H^{\pm} := \left\{ v \in \mathbf{R}^{1,n} \;\middle|\; v^2 = 1 \right\} = H^{+} \coprod H^{-}$ to get a 2-sheeted hyperboloid; the pseudo-Riemannian metric on $\mathbf{R}^{1,n}$ restricts to a Riemannian metric on H.
- Take one sheet H^{+}; this is a model of $\mathbb{H}^n$

Indexing might be off here

The group of isometries of H^{+} is now $\mathbf{P}\mathrm{O}_{1,n}(\mathbf{R})$. Note that in $\mathrm{O}_{1,n}(\mathbf{R})$ there is an index 2 subgroup[9]

$$\mathrm{O}_{1,n}(\mathbf{R})^{\pm} = \left\{ \gamma \in \mathrm{O}_{1,n}(\mathbf{R}) \;\middle|\; \gamma(H^{+}) = H^{+}, \gamma(H^{-}) = H^{-} \right\}.$$

Remark 13.14. Forming the symmetric spaces for $\mathrm{O}_{2,n}(\mathbf{R})$: the maximal compact is $K = \mathrm{O}_2(\mathbf{R}) \times \mathrm{O}_n(\mathbf{R})$ and $\mathrm{O}_2(\mathbf{R})$ is similar enough to $\mathrm{U}_1(\mathbf{C})$ that we should expect the associated symmetric space to be Hermitian. It will be an open subset of a certain quadric:

- Start with $\mathbf{P}(\mathbf{C}^{2,n})$.
- Take the quadric of isotropic vectors $Y = \left\{ z \in \mathbf{P}(\mathbf{C}^{2,n}) \;\middle|\; z^2 = 0 \right\}$.
- Take the open subset $U := \left\{ z \in Y \;\middle|\; (z, \overline{z}) > 0 \right\}$.

Why this matches the previous description: write $z = x + iy$, then $x^2 = y^2 > 0$ and $(x, y) = 0$, so $V := \mathbf{R}x \oplus \mathbf{R}y$ are an orthogonal basis for a positive definite subspace of $\mathbf{R}^{2,n}$. Multiplying by a scalar only changes basis, so we essentially get a map $\mathbf{P}(U) \to \mathrm{Gr}^{+}(\mathbf{R}^{2,n})$ naturally. This symmetric space can also be identified with points $z \in \mathbf{C}^{1,n-1}$ with $\Im(z) \in C^{+}$, one of two cones of $\mathbf{R}^{1,n-1}$, realizing this as a tube domain generalizing $\mathbb{H}$.

14. Hyperbolic Geometry

14A. Hyperbolic lattices.

Note: some of this mixes conventions, need to fix later.

Warning 14.1. There is a significant gap in the AG literature vs the physics literature for the terminology for hyperbolic spaces, and the traditional AG terminology can be "wrong" in some senses. For example, the AG literature will typically call $\left\{ v \in L_{\mathbf{R}} \;\middle|\; v^2 > 0 \right\}$ a "light cone", but this is not quite correct: the actual *light cone* in general relativity is $\left\{ v \in L_{\mathbf{R}} \;\middle|\; v^2 = 0 \right\}$. The following picture is the usual mnemonic:

[9] Apparently, these are elements whose spinor norm equals their determinant.

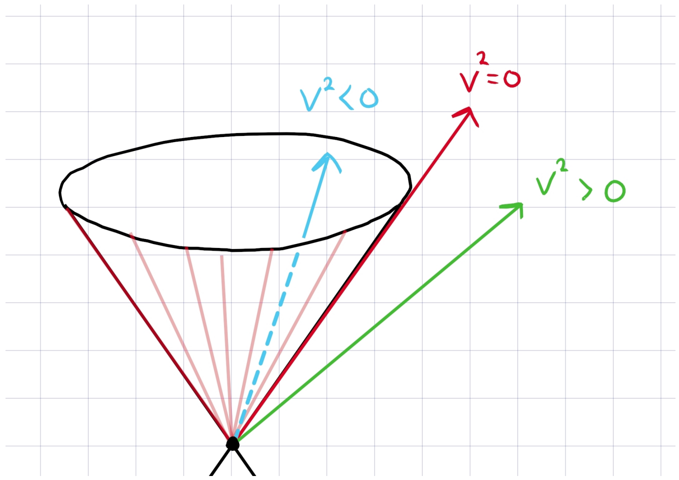


FIGURE 26. $\{v^2 = 0\}$ is the light cone, its interior is timelike and exterior spacelike.

Definition 14.2 (Hyperbolic lattices)**.** An indefinite lattice L is a **hyperbolic lattice** [10] if $\mathrm{sig}(L) = (1, n_-)$ or $(n_+, 1)$ for some $n_-, n_+ \geq 1$. By convention, by twisting L to $L(-1)$ if necessary, we assume hyperbolic lattices have signature $(1, n)$. In this convention, the single positive-definite direction is referred to as **timelike**, and the remaining directions are **spacelike**.

Definition 14.3 (Time/light/spacelike vectors)**.** Let L be a hyperbolic lattice of signature $(1, n)$. We say a vector $v \in L_{\mathbf{R}}$ is

- **timelike** if $v^2 < 0$,
- **lightlike** or **isotropic** if $v^2 = 0$.
- **spacelike** if $v^2 > 0$, More generally, a subspace $W \subseteq \mathbb{E}^{1,n}$ with the restricted form $(-,-)_W$ is
- **timelike** if $(-,-)_W$ is negative-definite, or is indefinite and non-degenerate,
- **lightlike** or **isotropic** if $(-,-)_W$ is degenerate, or
- **spacelike** if $(-,-)_W$ is positive-definite.

Define

$$L^{<0} := \left\{ v \in L_{\mathbf{R}} \;\middle|\; v^2 < 0 \right\} \quad \text{The timelike regime}$$
$$L^{=0} := \left\{ v \in L_{\mathbf{R}} \;\middle|\; v^2 = 0 \right\} \quad \text{The lightlike regime}$$
$$L^{>0} := \left\{ v \in L_{\mathbf{R}} \;\middle|\; v^2 > 0 \right\} \quad \text{The spacelike regime}$$

Remark 14.4. [AE22] refers to the non-spacelike regime $L^{\geq 0} := \left\{ v \in L_{\mathbf{R}} \;\middle|\; v^2 \geq 0 \right\}$ as the **round cone**; this is used for a model over $\overline{\mathbb{H}^n}$ with ideal points included, and is often used as the support of a semifan for a semitoroidal compactification.

Definition 14.5 (Past and future light cones)**.** Let L be a hyperbolic lattice of signature $(1, n)$. The spacelike regime $L^{>0}$ of L has an irreducible component

[10] Also called a **Lorentzian lattice**.

decomposition

$$L^{>0} := \left\{ v \in L_{\mathbf{R}} \;\middle|\; v^2 > 0 \right\} = C_L^+ \amalg C_L^-,$$

whose components we refer to as the **future light cone** and **past light cone** of L respectively, and can be distinguished by the sign of the coordinate in the negative-definite direction:

$$C_L^+ := \left\{ v \in L^{>0} \;\middle|\; v_0 > 0 \right\}, \qquad C_L^- := \left\{ v \in L^{>0} \;\middle|\; v_0 < 0 \right\}.$$

We write their closures in $L_{\mathbf{R}}$ as $\overline{C_L^+}$ and $\overline{C_L^-}$ respectively, and write $C_L := C_L^+$ for a fixed choice of a **future** light cone and $\overline{C_L}$ for its closure.

14B. **Models of hyperbolic space.**

Definition 14.6 (Euclidean upper-half space)**.** The upper-half space in $\mathbb{E}^n$ is

$$\mathbb{E}^n_+ := \left\{ (x_1, \cdots, x_n) \in \mathbb{E}^n \;\middle|\; x_1 > 0 \right\}.$$

Definition 14.7 (Minkowski space)**.** The **n-dimensional Minkowski space** $\mathbb{E}^{1,n}$ is the real vector space $\mathbf{R}^{n+1}$ equipped with a bilinear form of signature $(1, n)$ which can be explicitly written as

$$vw := -v_0 w_0 + \sum_{i=1}^{n} v_i w_i$$

with the associated quadratic form

$$v^2 := Q(v) := -v_0^2 + \sum_{i=1}^{n} v_i^2.$$

This induces a metric

$$\rho(v, w) := \operatorname{arccosh}(-vw).$$

Remark 14.8. If L is hyperbolic of signature $(1, n)$ then $L_{\mathbf{R}} \cong \mathbb{E}^{1,n}$ is a Minkowski space of dimension $n + 1 = \operatorname{rank}_{\mathbf{Z}} L$.

14B.1. *Half-plane models.*

https://arxiv.org/pdf/1908.01710.pdf#page=40&zoom=auto,-95,626

Definition 14.9 (de Sitter space and light cone of a lattice)**.** Let L be a hyperbolic lattice and consider the squaring functional

$$\begin{aligned} f_L : L_{\mathbf{R}} &\to \mathbf{R} \\ v &\mapsto v^2. \end{aligned}$$

One can show that ± 1 are regular values of f_L and thus define two canonical "hyperbolic unit spheres" which are regular surfaces. We define the **de Sitter space of** L as

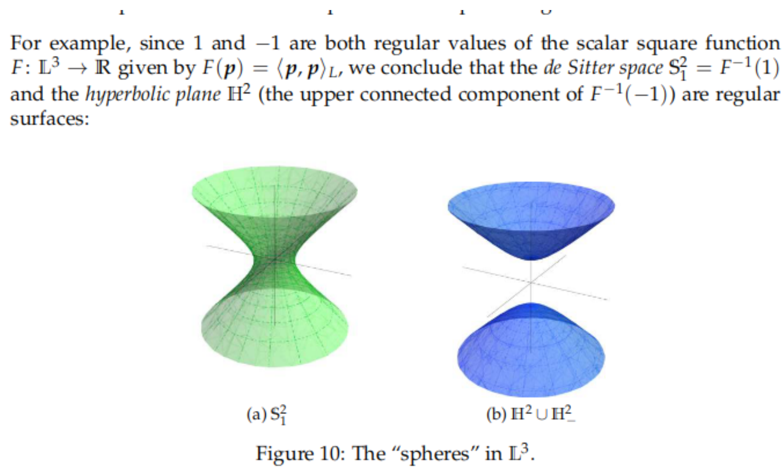
$$\mathrm{dS}_L := f_L^{-1}(1) = \left\{ v \in L_{\mathbf{R}} \;\middle|\; v^2 = 1 \right\} \subseteq L^{>0}$$

in the spacelike regime and the **unit hyperboloid of** L as the two-sheeted hyperboloid

$$H_L := f_L^{-1}(-1) = \left\{ v \in L_{\mathbf{R}} \;\middle|\; v^2 = -1 \right\} \subseteq L^{<0}$$

in the timelike regime.

Example 14.10. Figure 27 shows the de Sitter space and unit hyperboloid for a lattice L of signature $(2,1)$ in $\mathbb{E}^{2,1}$, visualized in $\mathbf{R}^3$.


For example, since 1 and -1 are both regular values of the scalar square function $F\colon \mathbb{L}^3 \to \mathbb{R}$ given by $F(\boldsymbol{p}) = \langle \boldsymbol{p}, \boldsymbol{p}\rangle_L$, we conclude that the *de Sitter space* $\mathbb{S}^2_1 = F^{-1}(1)$ and the *hyperbolic plane* $\mathbb{H}^2$ (the upper connected component of $F^{-1}(-1)$) are regular surfaces:

(a) $\mathbb{S}^2_1$ (b) $\mathbb{H}^2 \cup \mathbb{H}^2_-$

Figure 10: The "spheres" in $\mathbb{L}^3$.

FIGURE 27. The hyperbolic unit spheres: the de Sitter space and light cone for $\mathbb{E}^{2,1}$.

Definition 14.11 (Half-plane model/Lobachevsky space of a lattice)**.** Let L be a hyperbolic lattice. The **half-plane model of $\mathbb{H}^n$ associated to L** or **Lobachevsky space of L** is the unit hyperboloid of L intersected with its future light cone,

$$\mathbf{L}^n_L := H_L \cap C_L := \left\{ v \in L_{\mathbf{R}} \;\middle|\; v^2 = -1, v_0 > 0 \right\},$$

given the metric restricted from $L_{\mathbf{R}} \cong \mathbb{E}^{n,1}$. This more simply be described as the future sheet of the unit hyperboloid H_L, using the irreducible component decomposition

$$H_L = H_L^+ \amalg H_L^- = \left\{ v \in H_L \;\middle|\; v_0 > 0 \right\} \amalg \left\{ v \in H_L \;\middle|\; v_0 < 0 \right\}$$

and setting $\mathbf{L}^n_L := H_L^+$.

Remark 14.12. Note that H_L^+ is in the timelike regime. This gives a model of the hyperbolic space $\mathbb{H}^n$ which we often denote $\mathbb{H}_L$ when we do not fix a specific choice of model, or simply by $\mathbb{H}^n$ when the dependence on L is not important.

Remark 14.13 (The isometry group of hyperbolic spaces)**.** It can be shown that the isometries of the timelike regime $L^{<0}$ are restrictions of isometries of the ambient Minkowski space $\mathbb{E}^{1,n}$, and thus

$$\mathrm{Isom}(L^{<0}) \cong \mathrm{Isom}(\mathbb{E}^{1,n}) \cong \mathrm{O}_{1,n}(\mathbf{R}).$$

Using the half-plane model, we can thus naturally identify

$$\mathrm{Isom}(\mathbf{L}^n) \cong \mathrm{O}^+_{1,n}(\mathbf{R}) := \mathrm{Stab}_{\mathrm{O}_{1,n}(\mathbf{R})}(C_L),$$

the index 2 subgroup which stabilizes the future light cone C_L^+ of L. These are precisely the isometries of $\mathbb{E}^{1,n}$ of positive spinor norm.

14B.2. *Ball models.*

Definition 14.14 (The Poincar'e ball model)**.** Let L be a hyperbolic lattice. The **Poincar'e ball model of $\mathbb{H}^n$ associated to L** is defined as

$$\mathbf{B}_L^n := \mathbf{P}(L^{<0}),$$

the projectivization of the timelike regime of L, where

$$\begin{aligned}\mathbf{P}(-) : \mathbb{E}^{1,n} \setminus \{x_n \neq 0\} &\to \mathbb{E}^n \\ (x_0, \cdots, x_{n-1}, x_n) &\mapsto \left(\frac{x_0}{x_n}, \cdots, \frac{x_{n-1}}{x_n}\right).\end{aligned}$$

In this model, there is a natural compactification $\overline{\mathbb{H}^n}$ in $\mathbf{P}(S^n)$ such that the interior is given by $\mathbf{B}_L^n$ as above and the boundary by $\partial\overline{\mathbb{H}^n} = \mathbf{P}(L^{=0})$, i.e. ideal points correspond to (the projectivization of) the lightlike regime.

Remark 14.15 (An alternative construction)**.** It can be explicitly constructed by considering the future light cone C_L described in Theorem 14.5. Letting $\mathbf{R}_{>0}$ act on $L_{\mathbf{R}} \cong \mathbb{E}^{1,n}$ by scaling along the timelike direction (i.e. in the coordinate v_0), the ball model can be formed as the quotient

$$\mathbb{B}_L^n \cong C_L/\mathbf{R}_{>0} \subset \mathbf{P}(\mathbb{S}^n).$$

Remark 14.16. The advantage of $\mathbf{B}_L^n$ over $\mathbf{L}_L^n$ is that the former provides a natural compactification in $\mathbf{P}(\mathbb{S}^n)$. Moreover, it can be easier to work with hyperplanes in the ball model: let $\pi : \mathbb{E}^{1,n} \to \mathbf{P}(\mathbb{S}^n)$ be the natural projection, then every hyperplane $H_v := v^\perp$ for $v \in \mathbf{B}_L^n$ is of the form

$$H_v = \left\{ \pi(x) \;\middle|\; x \in C_L,\, xv = 0 \right\}$$

One can also concretely interpret the bilinear form geometrically in the ball model in the following way:

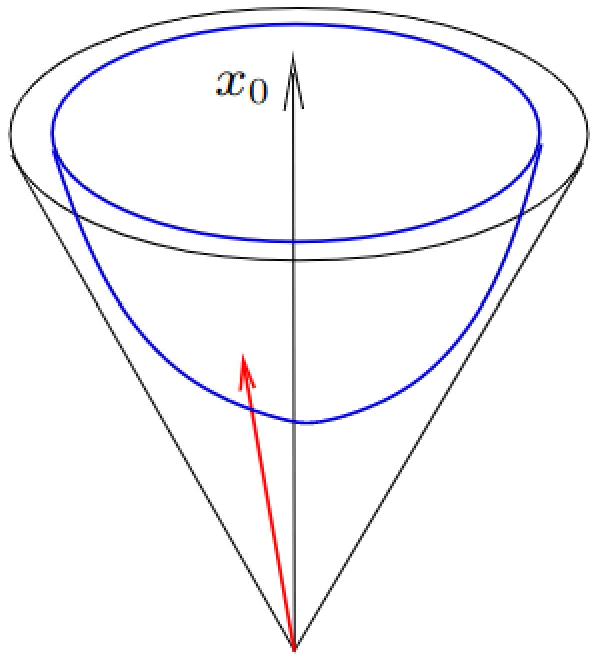
$$\begin{aligned} H_v \pitchfork H_w &\implies |vw| < 1 \implies -vw = \cos(\angle(H_v, H_w)) \\ H_v \parallel H_w &\implies |vw| = 1 \implies -vw = \cos(\angle(H_v, H_w)) \\ H_v \,)(\, H_w &\implies |vw| > 1 \implies -vw = \cosh(\rho(H_v, H_w)), \end{aligned}$$

where ρ is the hyperbolic metric described in Definition 14.7.

Remark 14.17. $\mathrm{Isom}(\mathbf{B}^n) = \mathbf{P}\mathrm{O}_{1,n}(\mathbf{R})$.

14B.3. *Ideal points.*

Remark 14.18. Let $H_L \cong \mathbb{H}^n$ be a model of hyperbolic space associated to a hyperbolic lattice L of signature $(1, n)$. Boundary points $\partial\overline{H_L}$ correspond to ideal points in $\mathbb{H}^n$, i.e. points "at infinity", which in turn correspond to 1-dimensional isotropic subspaces of L.



In this model, points in $\mathbb{H}^n$ are points in the interior of the cone and on the hyperboloid. Moreover points on $\partial\overline{\mathbb{H}^n}$ correspond to points on the surface of the cone:

$$\partial\overline{\mathbb{H}^n} \cong \left\{ v = (v_0, \cdots, v_{n+1}) \in L_{\mathbf{R}} \;\middle|\; v_0 > 0 \right\} \cap \left\{ v \in L_{\mathbf{R}} \;\middle|\; v^2 = 0 \right\}.$$

We interpret $uv = -\cos(\angle(H_u H_v))$, so $uv = -1$ means $H_u \cap H_v \in \partial\overline{\mathbb{H}}^n$, i.e. they are "parallel" planes. Hyperplanes in $\mathbb{H}^n$ correspond to branches of hyperbolas obtained by slicing the hyperboloid by a plane in $L_{\mathbf{R}}$.

Remark 14.19. Define Minkowski space as $\mathbb{E}^{1,n}$, which is $\mathbf{R}^n$ with the form $vw = v_0 w_0 - \sum v_i w_i$. Define Lobachevsky space $\mathbf{L}^n$ as the hyperboloid model of hyperbolic space, a certain "hyperbolic unit sphere":

$$\mathbf{L}^n := \left\{ v \in \mathbb{E}^{1,n} \;\middle|\; v^2 = 1, v_0 > 1 \right\}.$$

The geodesic curves are precisely intersections of the form $H_2 \cap \mathbf{L}^n$ where $H_2 \in \mathrm{Gr}_2(\mathbf{R}^{n+1})$ is a standard 2-plane passing through the origin in the ambient space. The hyperbolic metric on $\mathbf{L}^n$ is gotten by computing the length in the standard metric in $\mathbf{R}^{n+1}$ of any geodesic curve between two points. The associated Poincare ball model is contained in the standard Euclidean ball $\mathbf{B}^n \subset \mathbf{R}^{n+1}$ and is the projection of $\mathbf{L}^n$ onto the hyperplane $\{x_0 = 0\} \subset \mathbf{R}^{n+1}$ using rays passing through $(-1, 0, 0, \cdots, 0)$. Explicitly, the projection is

$$\begin{aligned} \phi : \mathbf{L}^n &\to \mathbf{B}^n \\ (v_0, \cdots, v_n) &\mapsto \frac{1}{1 + v_0}(v_1, \cdots, v_n) \end{aligned}$$

Geodesics are now straight lines through the origin or arcs of Euclidean circles intersecting $\partial \mathbf{B}^n$ orthogonally. Define the hyperbolic upper-half-space as

$$\mathbb{H}^n := \left\{ x = (x_1, \cdots, x_n) \in \mathbf{R}^n \;\middle|\; x_1 > 0 \right\}$$

which is obtained by taking inversions through certain spheres centered on $\partial \mathbf{B}^n$. Geodesics are now straight lines orthogonal to $\partial \mathbb{H}^n$ or half-circles centered on $\partial \mathbb{H}^n$.

14C. Root Systems.

Definition 14.20 (Primitive vectors)**.** `Let $L$ be any lattice. A finite set $S \da \ts{s_1,\cdots,`

Definition 14.21 (Roots and k-roots in lattices)**.** Let L be any lattice. For $k \in \mathbf{Z}_{>0}$, define the set of **k-roots in** L as

$$\Phi_k L := \left\{ v \in L_{\text{prim}} \;\middle|\; v^2 = k,\, 2(v, L) \subseteq k\mathbf{Z} \right\}$$

A **root** is by definition a 2-root. We write the set of roots in L as $\Phi(L)$, and the complete set of roots as

$$\Phi_\infty L := \bigcup_{k \geq 1} \Phi_k L.$$

Remark 14.22. In the theory of 2-elementary lattices, the roots consist of all (-2)-vectors along with any (-4)-vector v with $\operatorname{div}(v) = 2$.

Definition 14.23 (Reflections)**.** Let L be any lattice and $L_{\mathbf{R}}$ its associated $\mathbf{R}$-module. An element $s \in \mathrm{GL}(L_{\mathbf{R}})$ is a **reflection** if there exists a vector $v \in L_{\mathbf{R}}$ and an $\mathbf{R}$-linear functional $f \in \mathrm{Hom}_{\mathbf{R}}(L_{\mathbf{R}}, \mathbf{R})$, both depending on s,

$$s(x) = x - f(x)v \quad \forall x \in L_{\mathbf{R}}, \qquad f(v) = 2.$$

Concretely, s is an isometry of $L_{\mathbf{R}}$ which pointwise fixes a hyperplane and is an involution satisfying $\det(s) = -1$. Every reflection can be written in the form

$$s(u) = s_v(u) = u - \frac{uv}{v^2/2} v$$

for some $v^2 \neq 0$ in $L_{\mathbf{R}}$ determined up to scaling. The reflection in v is only well-defined when $2 \operatorname{div}(v) \in v^2 \mathbf{Z}$ where $\operatorname{div}(v)$ is the divisibility of v defined in **??**. The **reflection hyperplane** associated to s is the fixed subspace

$$H_v := \ker(f) = \ker(\mathrm{id} - s) \cong v^\perp.$$

:::{.remark} Alternatively: $s \in \mathrm{GL}_n(\mathbf{C})$ is a quasi-reflection if it 1 eigenvalue $\lambda \neq 1$ with an eigenspace of dimension 1 and the remaining eigenvalues all 1. It is a reflection if $\lambda = -1$. \end{remark}

Definition 14.24 (Mirrors in hyperbolic lattices)**.** Any root $v \in \Phi(L)$ defines a reflection s_v through the mirror

$$H_v := v^\perp := \left\{ x \in C_L^+ \;\middle|\; xv = 0 \right\}.$$

If H_v is the reflection hyperplane of a root, we say it is a **mirror** in L. Note that H_v is nonempty if and only if $v^2 < 0$.

Definition 14.25 (Weyl group)**.** Let L be any lattice. The **Weyl group of** L is defined as the group generated by reflections in 2-roots,

$$W_L := \left\{ s_v \;\middle|\; v \in \Phi(L) \right\} \leq \mathrm{O}_L(\mathbf{R})$$

Definition 14.26 (The discriminant locus)**.** For L a hyperbolic lattice, define the **discriminant locus of** L as the union of all mirrors of 2-roots,

$$\Delta(L) \coloneqq \bigcup_{v\in\Phi(L)} v^\perp \coloneqq \bigcup_{v\in\Phi(L)} H_v.$$

Forgot to write down what is C_L.

Definition 14.27 (Weyl chambers)**.** The **chamber decomposition** of C_L is defined as

$$C_L^\circ \coloneqq C_L \setminus \Delta(L) = C_L \setminus \left(\bigcup_{\delta\in\Phi(L)} \delta^\perp \right),$$

the complement of all mirrors. This further decomposes into connected components called **Weyl chambers**: fixing a chamber P, there is a decomposition into orbits

$$C_L^\circ = \coprod_{s_v\in W_L} s_v(P).$$

Remark 14.28. Any Weyl chamber P is a simplicial cone, so the orbit decomposition yields a decomposition of C_L° into simplicial cones. Since W acts on the set of Weyl chambers $\pi_0 C_L^\circ$ simply transitively and the closure $\overline{P}$ of any chamber is a fundamental domain for this action, there is a homeomorphism $\overline{P} \cong C_L^\circ/W$.

https://www.ms.u-tokyo.ac.jp/preprint/pdf/2007-12.pdf#page=7&zoom=100,88,601

Definition 14.29 (Fundamental chamber)**.** Let P be a Weyl chamber of L, define

$$\Phi(L)^+ \coloneqq \left\{ v\in\Phi(L) \,\middle|\, (v,P) > 0 \right\}$$
$$\Phi(L)^- \coloneqq \left\{ v\in\Phi(L) \,\middle|\, (v,P) < 0 \right\} = -\Phi(L)^+$$

which induces a decomposition

$$\Phi(L) = \Phi(L)^+ \coprod \Phi(L)^-.$$

Remark 14.30. Thus P can be written as

$$P = \left\{ v\in C_L \,\middle|\, (v, \Phi(L)^+) > 0 \right\} = \{\}.$$

This realizes P as an intersection of positive half-spaces and thus as a polytope.

Definition 14.31 (Walls)**.** Let $\overline{P}$ be the closure in $L_{\mathbf{R}}$ of P. We say a mirror $H_v \subseteq L_{\mathbf{R}}$ for $v\in\Phi(L)^+$ is a **wall of** P if $\operatorname{codim}_{L_{\mathbf{R}}}(H_v \cap \overline{P}) = 1$.

Definition 14.32 (Simple systems)**.** Let P be a Weyl chamber of P and let

$$\Pi(L,P) \coloneqq \left\{ v\in\Phi(L) \,\middle|\, H_v \text{ is a wall of } P \right\}$$

be the set of walls of P. We can more economically define P by

$$P = \left\{ v\in C_L \,\middle|\, (v, \Pi(L,P)) > 0 \right\},$$

Todo, messed up notation here a bit.

where no inequality is redundant. Moreover, $(P, \Pi(L,P))$ forms a Coxeter system, and $\overline{P}$ is a fundamental domain for $W(L) \curvearrowright L_{\mathbf{R}}$.

Definition 14.33 (Chambers and $\mathrm{O}^+(L)$)**.** The connected components of

$$V_L^+ \coloneqq \left\{ x\in L^\pm \,\middle|\, (\Phi(L), x) \neq 0 \right\}$$

are called **chambers** of L. Any positive isometry preserves L^+ and L^- set-wise, motivating the definition of the **group of positive isometries of** L

$$\mathrm{O}^+(L) := \left\{ \gamma \in \mathrm{O}(L) \;\middle|\; \gamma(L^+) = L^+, \gamma(L^-) = L^- \right\}$$

Definition 14.34 (Positive isometries)**.** We say an isometry $\gamma \in \Orth(L)$ is **positive** i

Definition 14.35 (Roots, root systems, root lattices)**.** A vector $v \in L$ is a **root** if $v^2 = -2$[11], and we write $\Phi(L)$ for the set of roots in L. If L is negative definite and $L = \mathbf{Z}\Phi(L)$[12], we say L is a **root lattice**. Any root lattice decomposes as a direct sum of root lattices of ADE type.

Definition 14.36 (Weyl group)**.** The **Weyl group** of L is the maximal subgroup of the orthogonal group of L generated by hyperplane reflections in roots,

$$W_L^2 := \left\langle s_v \;\middle|\; v \in \Phi(L) \right\rangle_{\mathbf{Z}} \leq \mathrm{O}(L).$$

One can similarly define the group of reflections in *all* vectors,

$$W_L := \left\langle s_v \;\middle|\; v \in L \right\rangle_{\mathbf{Z}} \trianglelefteq \mathrm{O}(L).$$

Since conjugating a reflection by any automorphism is again a reflection, this is a normal subgroup. If L is a hyperbolic lattice, we replace $\mathrm{O}(L)$ in the above definition by $\mathrm{O}^+(L)$, the isometries that preserve the future light cone.

Definition 14.37 (Mirrors/walls and chambers)**.** The **mirror** or **wall** associated with a root $v \in \Phi(L)$ is the hyperplane $H_v := v^\perp$. As v ranges over $\Phi(L)$, these partition $L_{\mathbf{R}}$ into subsets called **chambers**. The Weyl group acts on $L_{\mathbf{R}}$ by isometries and acts simply transitively on chambers, and we often distinguish a fundamental domain for this action called the **fundamental chamber**. We write D_L for the closure in $L_{\mathbf{R}}$ of a fundamental chamber. A **cusp** of L is a primitive isotropic lattice vector $e \in D_L \cap L$.

14D. **General Period Domains.**

Remark 14.38. Define $G^L := G_{L^\perp}$ for G any algebraic group determined by L.

Very useful convention: Ω^S involves $S^\perp$, while Ω_S involves just S.

Let $L \leq \mathrm{II}_{3,19}$ be a sublattice of signature $(1, r-1)$ so $\mathrm{sig}(L^\perp) = (2, 19-r+1)$. One can always form the period domain corresponding to L-polarized K3 surfaces as

$$\Omega^L := \Omega_{L^\perp} := \left\{ x \in (L^\perp)_{\mathbf{C}} \;\middle|\; x^2 = 0, x\overline{x} > 0 \right\},$$

The period domain can be described as a Hermitian symmetric space:

$$\Omega^L \cong \frac{\mathrm{SO}^L(\mathbf{R})}{\mathrm{SO}_2(\mathbf{R}) \times \mathrm{SO}_{20-r}(\mathbf{R})}$$

> http://content.algebraicgeometry.nl/2020-5/2020-5-021.pdf#page=10&zoom=auto,-85,607

For any arithmetic subgroup $\Gamma \leq \mathrm{O}^L(\mathbf{R})$ there is a complex-analytic isomorphism

$$\Gamma \backslash \Omega^L \cong \left(\Gamma \cap \mathrm{SO}^L(\mathbf{R})\right) \backslash \Omega^L.$$

[11] One occasionally calls any time-like vector $v^2 < 0$ a "root", in which case distinguishes between e.g. (-2)-roots $\Phi_2(L)$ and (-4)-roots $\Phi_4(L)$.

[12] i.e. if the roots form a $\mathbf{Z}$-generating set for L

In particular, for L a primitive sublattice of $\mathrm{II}_{3,19}$, letting F_L be the stack of L-polarized K3 surfaces, the period map τ_L yields an open immersion

$$\tau_L : F_L(\mathbf{C}) \hookrightarrow \widetilde{\mathrm{SO}^L}(\mathbf{Z})\backslash\Omega^L$$

where $\widetilde{\mathrm{SO}^L}$ are isometries of L which extend to an isometry of $\mathrm{II}_{3,19}$ which fixes L.

For $\operatorname{sig} L = (2, n)$ let $G_L \coloneqq \mathrm{SO}_{L_{\mathbf{Q}}}$ be its associated rational isometry group and let $\mathbf{X} \coloneqq \Omega_L$ the associated Hermitian symmetric space as above, forming a Shimura datum $(\mathbf{X}, G) \coloneqq (\Omega^L, \mathrm{SO}_{L_{\mathbf{Q}}})$. We can then realize

$$\mathrm{Sh}_L(\mathbf{C}) \coloneqq \mathrm{Sh}_{\mathbf{K}_L}[G_L, \mathbf{X}_L](\mathbf{C}) \cong \widetilde{\mathrm{SO}_L}(\mathbf{Z})\backslash\mathbf{X}_L$$

where $\mathbf{K}_L \coloneqq \ker\left(G_L \to \mathrm{Aut}(A_L)\right)(\widehat{\mathbf{Z}})$, the admissible morphism in $G_L(\mathbf{A}_f)$; the stack $\mathrm{Sh}_{\mathbf{K}}[G, \mathbf{X}]$ is a certain well-known quotient stack attached to a Shimura datum $(G, \mathbf{X})$ and a choice of a compact open subgroup $\mathbf{K} \leq G(\mathbf{A}_f)$ of the finite adeles.

Defining the compact dual:

$$\Omega^{L,\mathrm{cd}} \coloneqq \left\{ x \in (L^\perp)_{\mathbf{C}} \;\middle|\; x^2 = 0 \right\}.$$

15. Appendix

15A. Dynkin Diagram.

Remark 15.1. The following is a table of the classical and affine Dynkin diagrams:

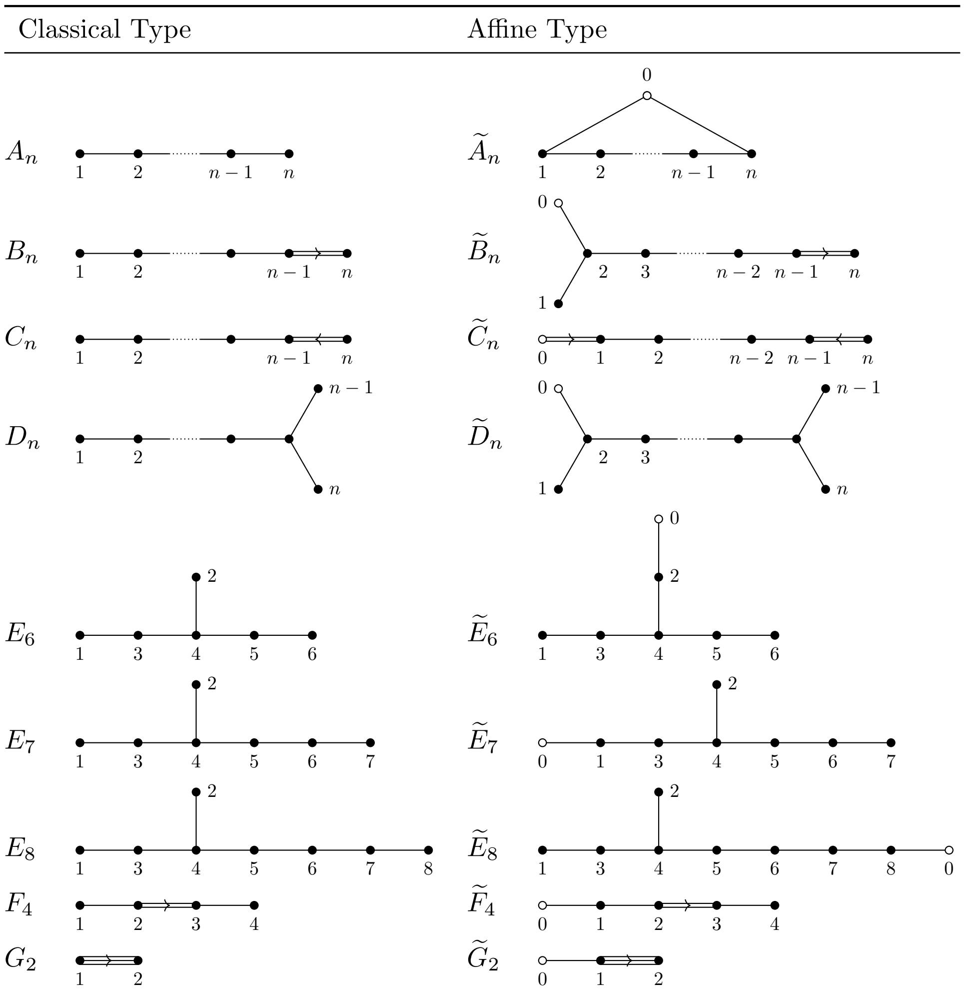
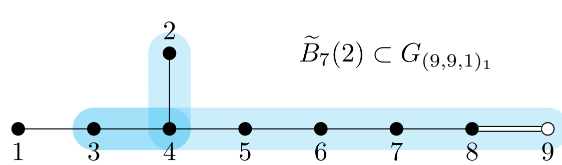
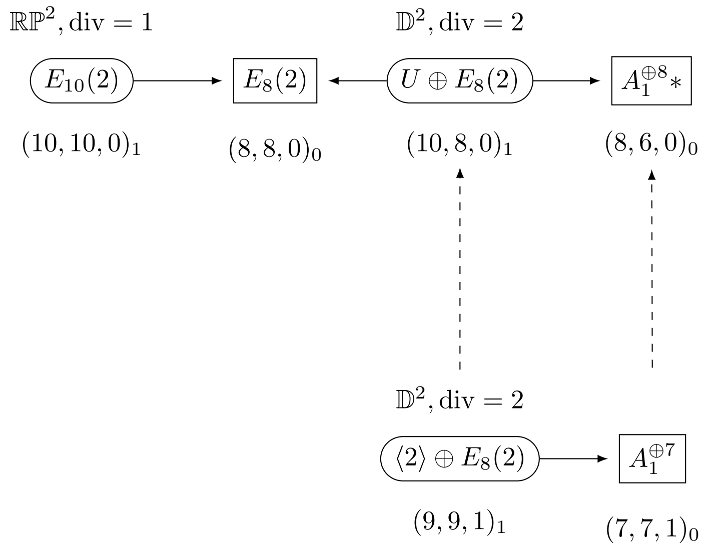


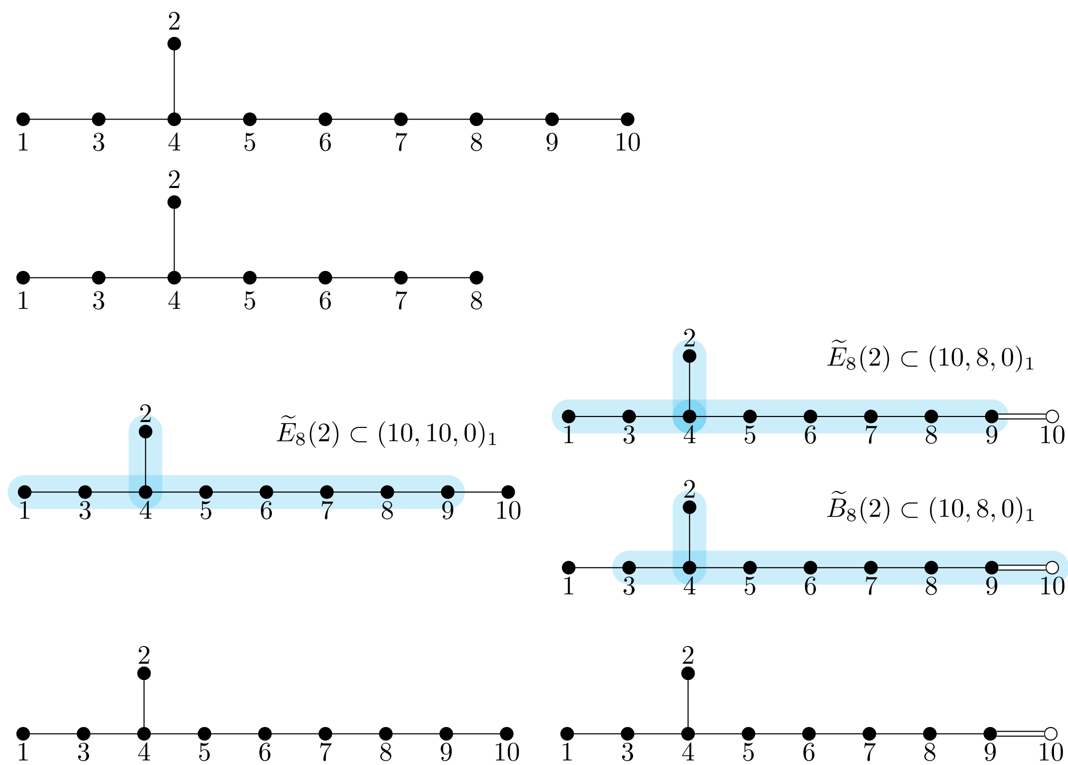
TABLE 7. Classical and Affine Dynkin Diagrams

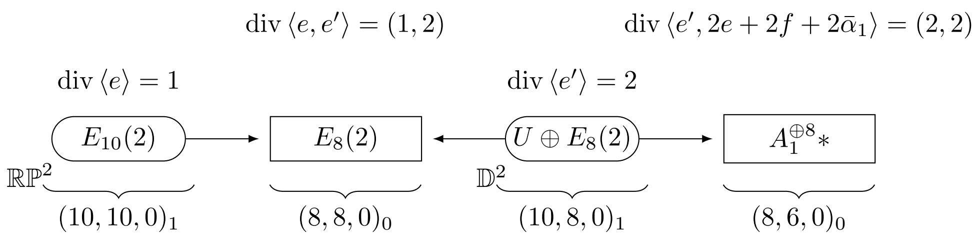
15B. **Images.**

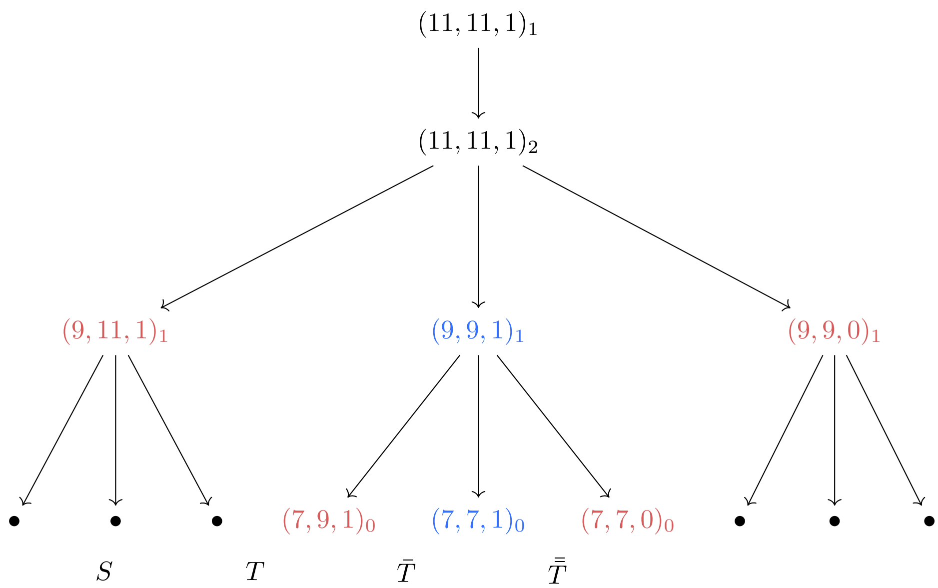
15B.1. *Tikz.*

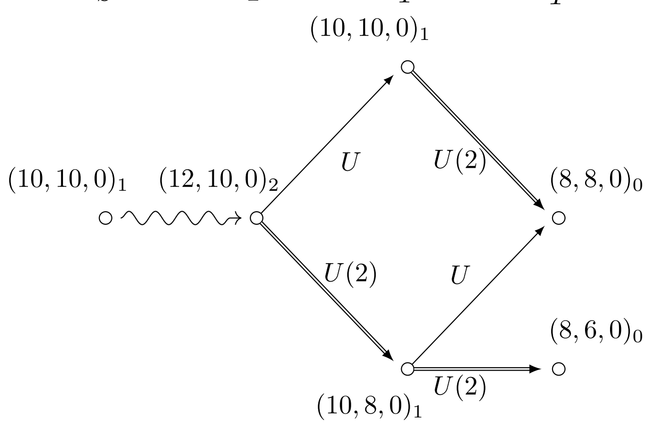



$\mathbb{RP}^2, \mathrm{div} = 1$
$\mathbb{D}^2, \mathrm{div} = 2$
$E_{10}(2)$
$E_8(2)$
$U \oplus E_8(2)$
$A_1^{\oplus 8} *$
$(10, 10, 0)_1$
$(8, 8, 0)_0$
$(10, 8, 0)_1$
$(8, 6, 0)_0$
$\mathbb{D}^2, \mathrm{div} = 2$
$\langle 2 \rangle \oplus E_8(2)$
$A_1^{\oplus 7}$
$(9, 9, 1)_1$
$(7, 7, 1)_0$


$\widetilde{E}_8(2) \subset (10, 8, 0)_1$
$\widetilde{E}_8(2) \subset (10, 10, 0)_1$
$\widetilde{B}_8(2) \subset (10, 8, 0)_1$


div ⟨e, e′⟩ = (1, 2)
div ⟨e′, 2e + 2f + 2ᾱ₁⟩ = (2, 2)
div ⟨e⟩ = 1
div ⟨e′⟩ = 2
E10(2)
E8(2)
U ⊕ E8(2)
A1⊕8 *
RP2
D2
(10, 10, 0)1
(8, 8, 0)0
(10, 8, 0)1
(8, 6, 0)0


(11, 11, 1)1
(11, 11, 1)2
(9, 11, 1)1
(9, 9, 1)1
(9, 9, 0)1
(7, 9, 1)0
(7, 7, 1)0
(7, 7, 0)0
S
T
T̄
T̿


(10, 10, 0)1
(10, 10, 0)1
(12, 10, 0)2
U
U(2)
(8, 8, 0)0
U(2)
U
(8, 6, 0)0
U(2)
(10, 8, 0)1

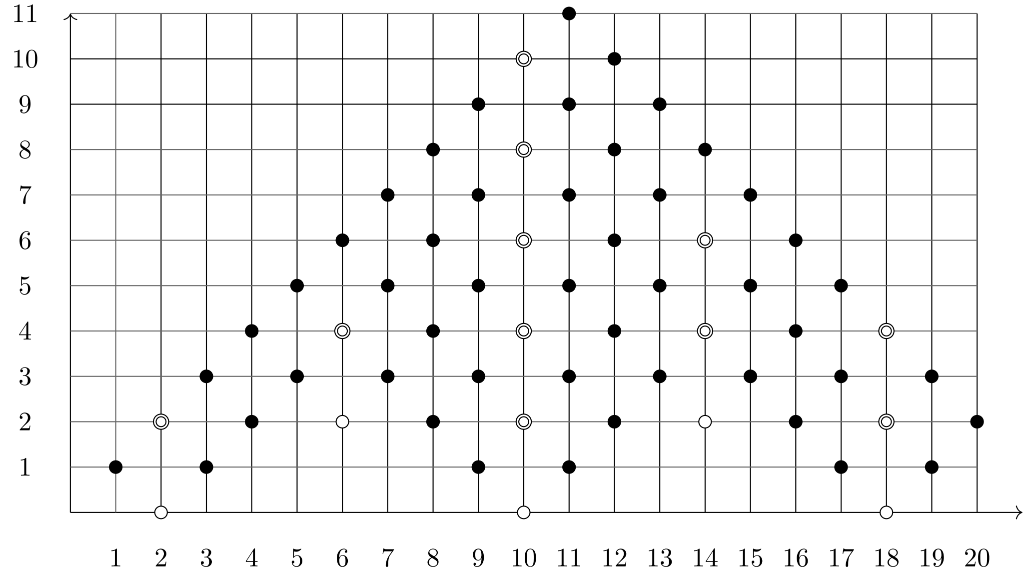
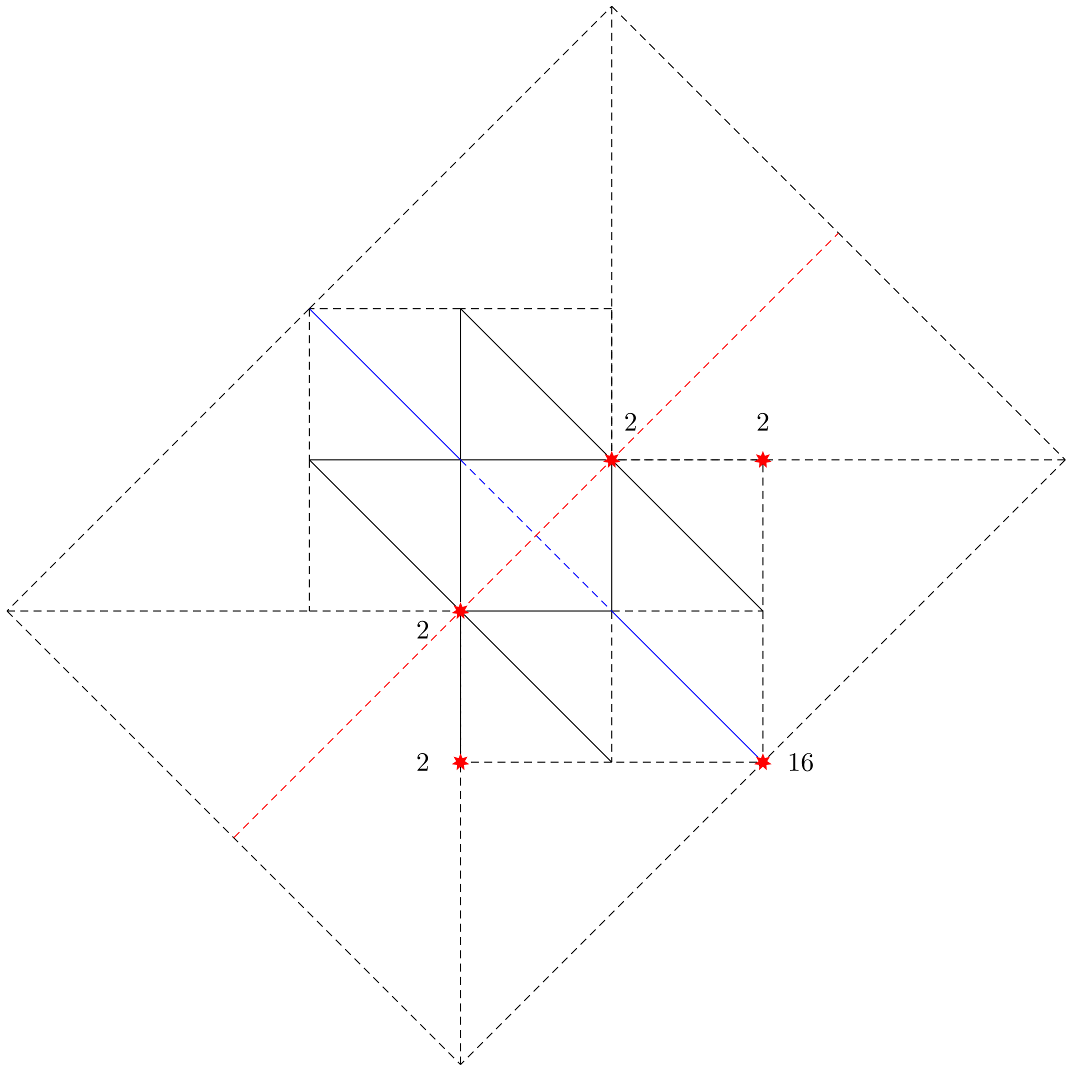
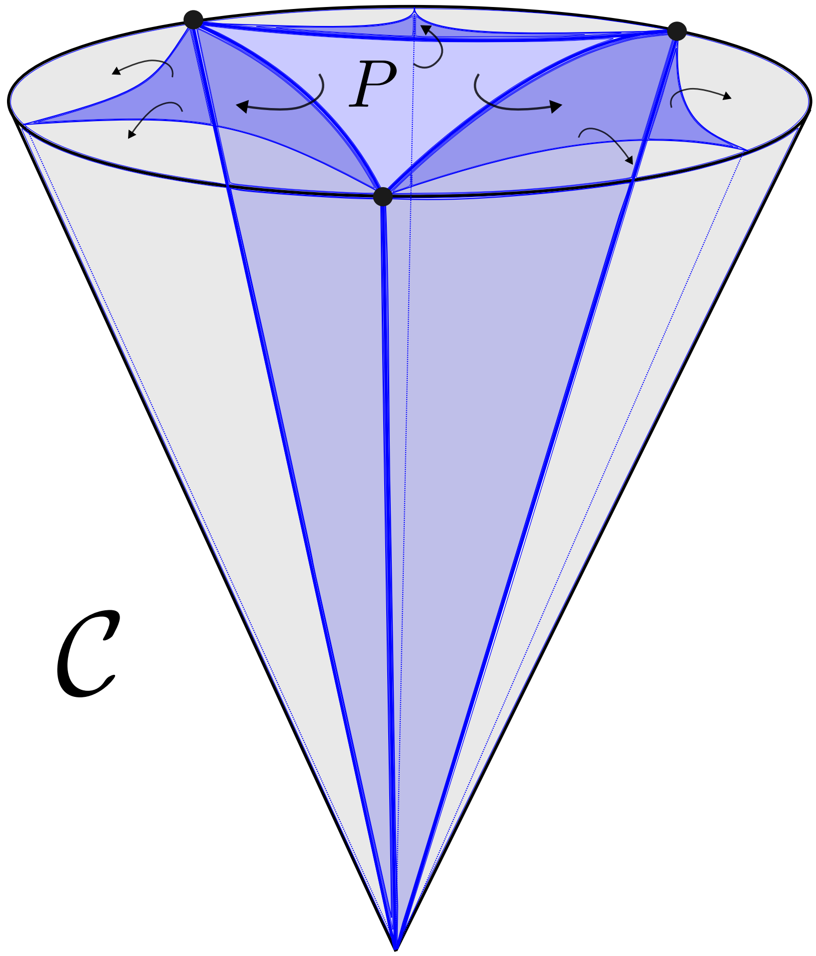
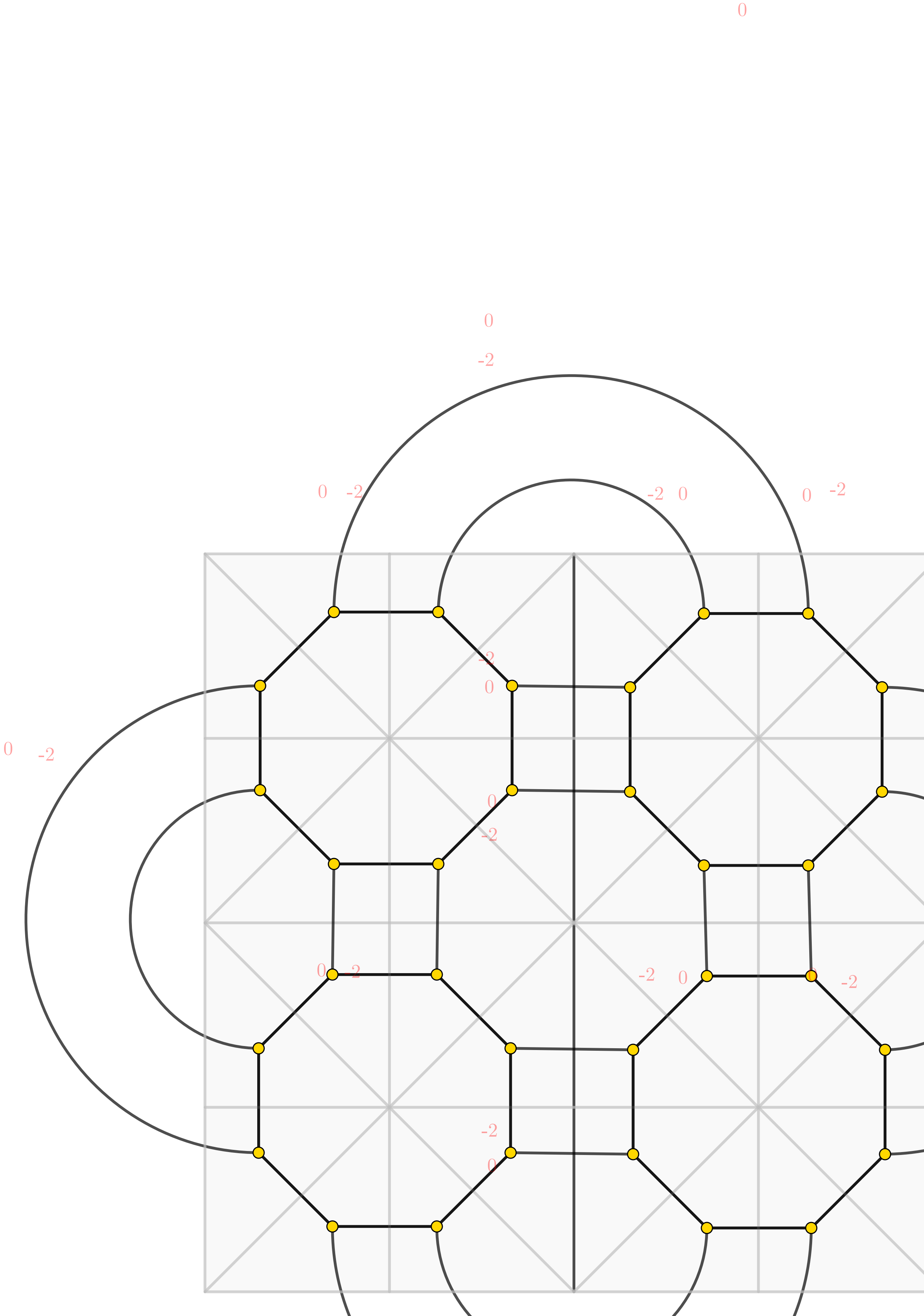


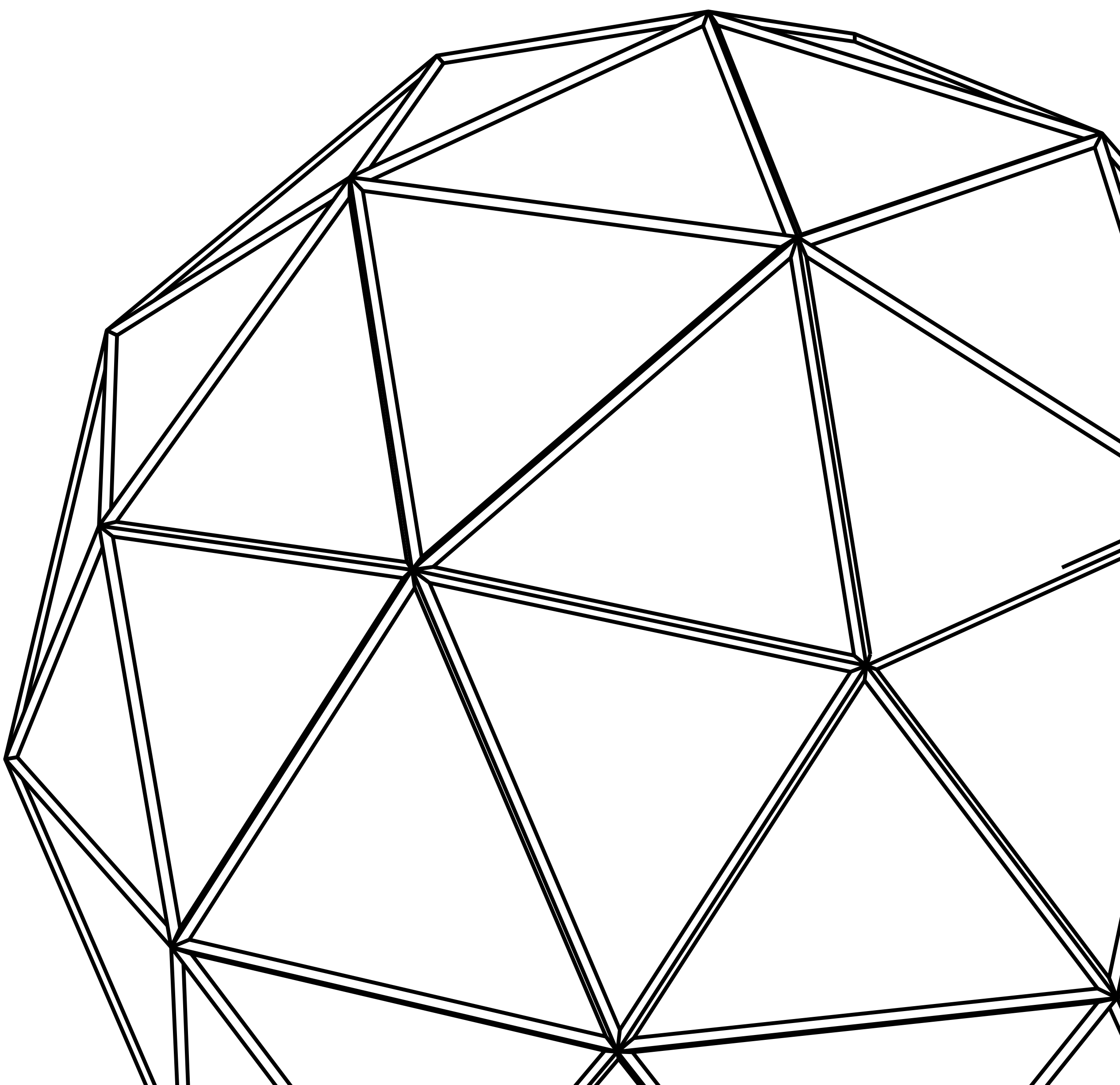
FIGURE 28. Nikulin's table of 2-elementary lattices.

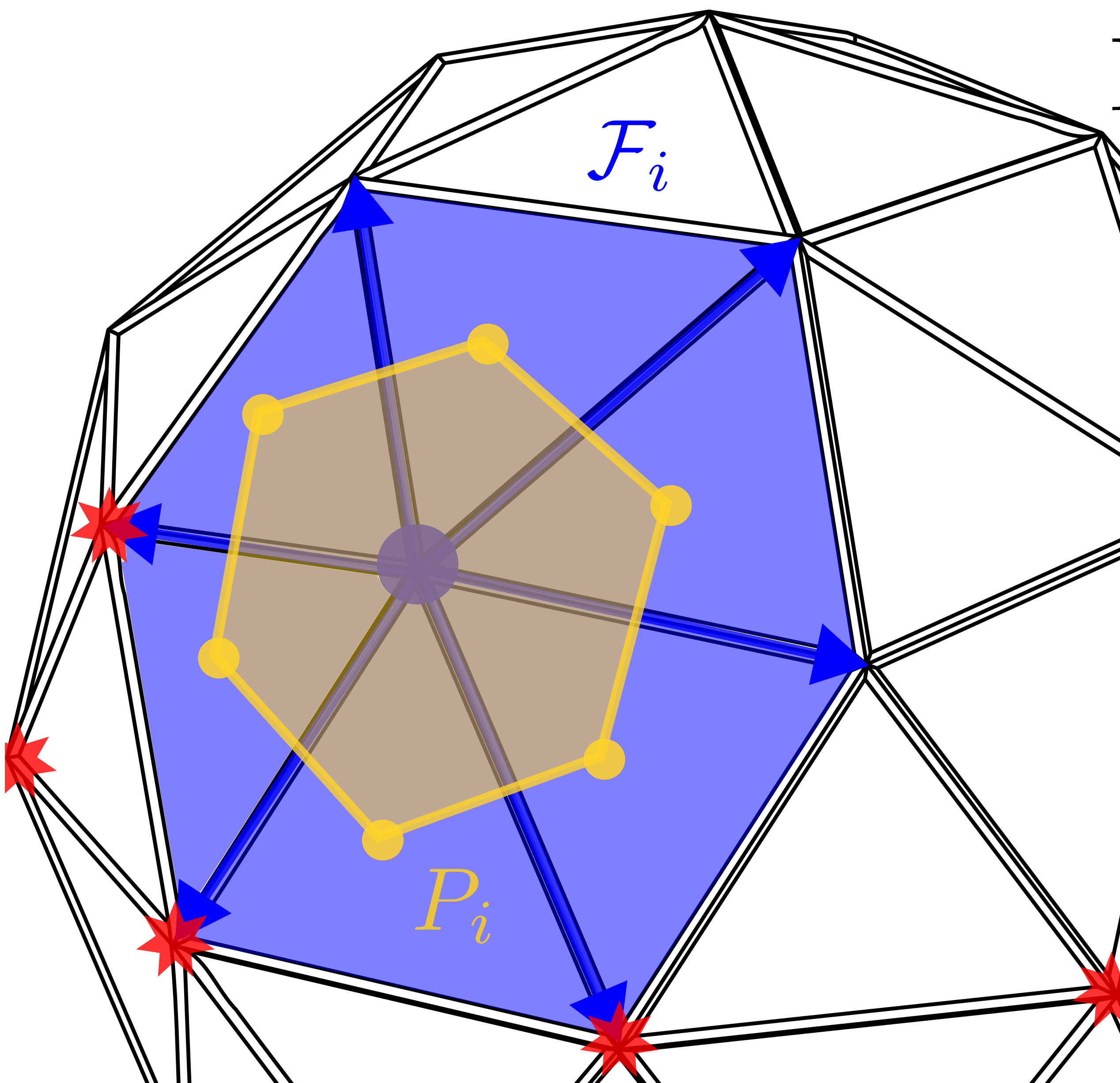

2
2
2
2
16

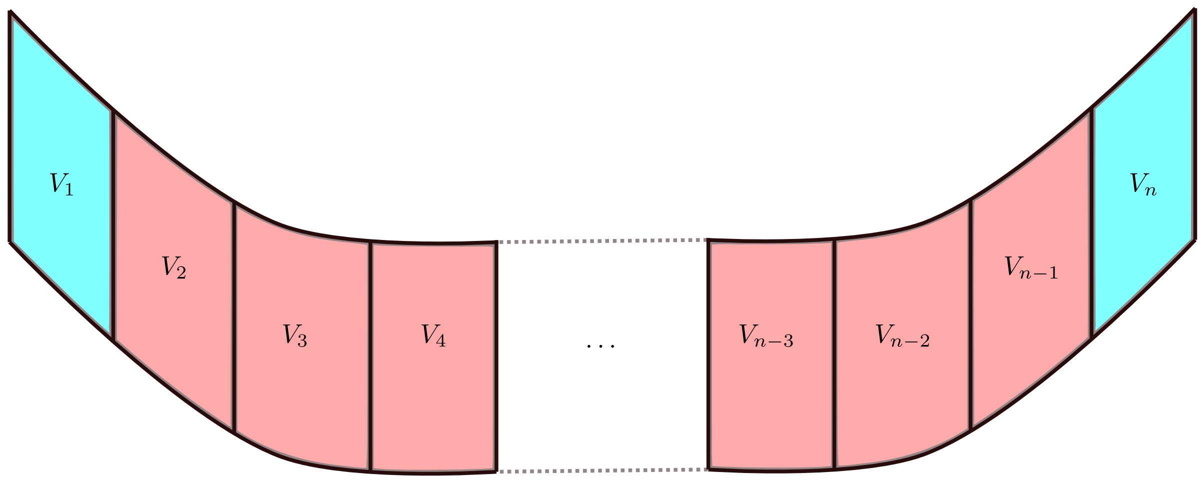


15B.2. *SVG.*


0
0
-2
0 -2
-2 0
0 -2
-2
0
0 -2
0
-2
0 -2
-2 0
0 -2
-2
0




$\mathcal{F}_i$
P_i



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