HODGE THEORY

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ABSTRACT. Notes from a course on Hodge theory taught by Pierrick Bousseau in Spring 2025 at the University of Georgia.

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Remark 1.1. Textbook: *Hodge Theory and Complex Algebraic Geometry I: Volume 1 (Cambridge Studies in Advanced Mathematics)* by Voisin.

Remark 1.2. Recall that for a topological space X, we can assign invariants $H^i(X; \mathbf{Z})$ which are abelian groups. More generally, we can attach $H^i(X; F) := H^i(X; \mathbf{Z}) \otimes F$ for a field F, and with reasonable finiteness assumptions on X (e.g. compact manifolds), these will be finite-dimensional F-vector spaces.

Example 1.3. For X a genus g Riemann surface,

- $H^0(X; \mathbf{Z}) = \mathbf{Z}$,
- $H^1(X; \mathbf{Z}) = \mathbf{Z}^{2g}$,
- $H^2(X; \mathbf{Z}) = \mathbf{Z}$, and
- $H^i(X; \mathbf{Z}) = 0$ for $i \geq 3$.

This detects the fact that there are g holes in X.

Remark 1.4. Note that H^* is functorial: for $f : X \to Y$ a continuous map, there is a pullback in cohomology $f^* : H^i(Y; \mathbf{Z}) \to H^i(X; \mathbf{Z})$ which is a morphism of abelian groups. Over fields, this instead yields a linear map, which is generally easier to study.

A general theme: extra structure on X, which can include

- a complex structure,
- a symplectic structure,
- a Kähler structure,
- a $\mathbf{C}$-algebraic structure,

will induce extra structure on $H^*(X; \mathbf{C})$, i.e. a Hodge structure. We will look in particular at complex manifolds X, i.e. those locally identifiable with $\mathbf{C}^n$ for some n, which allows for a notion of holomorphic functions on X.

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Remark 1.5. Recall that a torus X can be written as $\mathbf{R}^2/\mathbf{Z}^2$, i.e. a quotient of the plane by the standard square lattice. We have $g = 1$ and $H^*(X;\mathbf{C}) = \mathbf{C} \oplus \mathbf{C}^2[1] \oplus \mathbf{C}[2]$. Putting a complex structure on X amounts to replacing $\mathbf{R}^2$ with $\mathbf{C}$ and yields a complex manifold of complex dimension 1. One can replace X with $X_\tau = \mathbf{C}/(\mathbf{Z} \oplus \tau\mathbf{Z})$ for any $\tau \in \mathbf{C}$. It is a theorem that $X_\tau \cong X_{\tau'}$ as complex manifolds iff $\exists a, b, c, d \in \mathbf{Z}$ with $ad - bc = 1$ such that $\tau' = \frac{a\tau+b}{c\tau+d}$. Thus these complex manifolds vary in continuous families, despite being identified as real manifolds.

Remark 1.6. We will examine the Hodge decomposition for compact Kähler manifold, which is a complex manifold with additional technical assumptions. This decomposition is of the form $H^i(X;\mathbf{C}) = \oplus_{p+q=i} H^{p,q}(X)$, which implies $\beta_i(X) = \sum_{p+q=i} h^{p,q}(X)$. The Hodge numbers $h^{p,q}$ thus refine the Betti numbers, and may contain more information.

Example 1.7. For X a genus g Riemann surface, one has

$$H^1(X;\mathbf{C}) = H^{1,0}(X) \oplus H^{0,1}(X)$$

where $h^{1,0}(X) = h^{0,1}(X) = g$. Thus the Hodge numbers alone don't see the complex structure, since they are always g. However, what will keep track of differences will be the interplay between the Hodge decomposition (as decompositions of vector spaces) as the integral structure of $H^i(X;\mathbf{Z}) \subseteq H^i(X;\mathbf{C})$.

Remark 1.8. Hodge structures will be related to period integrals $\int_\gamma \alpha$, which is where calculus enters the picture. The proof of the Hodge decomposition uses real analysis, in particular elliptic PDEs, in a crucial way.

Remark 1.9. Next time: a word about complex analysis, complex/Kähler manifolds, the Hodge decomposition.

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