1.

$$\underbrace{\cdots F^{-*} \to \cdots F^{-3} \to F^{-2} \to F^{-1} \to F^{0}}_{} \to M$$

The free resolution, all F^i free

exists $\forall M \in R\text{-mod}$

2. Any map $f: M \to N$ lifts to a map $F^*_M \to F^*_N$

3. unique up to htpy

4. Any two F^*M are canonically hty-equiv

$$\begin{array}{ll} & F^{-2} \to F^{-1} \to F^{0} \to M \\ \mathrm{id}_F \quad \theta_{-2} & \quad\;\;\downarrow \qquad\; \downarrow \\ & G^{-2} \to G^{-1} \to G^{0} \to M \\ \phi_{-2} & \quad\;\;\downarrow \qquad\; \downarrow \\ & F^{-2} \to F^{-1} \to F^{0} \to M \\ \mathrm{id}_G \quad \theta_{-1} & \\ \text{Repeat} & G^{-2} \to G^{-1} \;\cdots\cdots\; - \end{array}$$

by uniqueness

$\underbrace{\phi_{-2} \circ \theta_{-2}} \simeq \mathrm{id}_{F^*}$

chain hty

$\rightsquigarrow$ Cat of hty classes of chain complexes

R: $R\text{-mod} \overset{\text{Replace by } F^*}{\longmapsto} \underbrace{H_0}(Ch(R\text{-mod}))$

"hty cat of"

Objects: Chain complexes

Morphisms: Chain hty

<u>Note on replacement</u>

$F^* \to M \to 0$ is acyclic iff

M is a complex ($R\text{-mod} \hookrightarrow$ Chain complexes)

$$\begin{array}{ccccccc} & & & (\deg 0) & & & \\ 0 & \to & 0 & \to & M & \to & 0 \\ \uparrow 0 & & \uparrow 0 & & \uparrow \varepsilon & & \uparrow 0 \\ F^{-2} & \to & F^{-1} & \xrightarrow{\partial^{-1}} & F^{0} & \xrightarrow{\partial^0} & 0 \end{array}$$

a quasiisomorphism

homology here

$F^0 / \mathrm{im}\, \partial^{-1} = M$ by defn

$\to$ $\varepsilon \circ \partial^{-1} = 0$, replace $d^2 = 0$ by $\circlearrowleft$

$\underbrace{\text{Quasi iso}} \to$ Chain hty equivalence

weaker, just have same homology - not an equiv relation

$\exists ! f: \mathbb{Z}_2 \to \mathbb{Z}$

$a \to 0$

$$\begin{array}{ccccccc} 0 \to & \mathbb{Z} & \xrightarrow{\times 2} & \mathbb{Z} & \to & 0 \\ & 0 \downarrow & & f \uparrow \downarrow \mathrm{mod}\, 2 & & \\ 0 \to & 0 & \to & \mathbb{Z}_2 & \to & 0 \end{array}$$

is a quasi-iso with no inverse

Don't care much about actual complex (CW, Singular, w/e) when computing homology

Replace random R-modules with free ones!

What can we do with F^*?

$$0 \to A \to B \to C \to 0$$

SES $\in$ Ab

$M \in Ab \longrightarrow F: (\cdot) \otimes M$ is right exact

$$\boxed{0 \dashrightarrow ? \dashrightarrow} A \otimes M \to \overbrace{B \otimes M \twoheadrightarrow C \otimes M \to 0}^{\text{Preserved}}$$

What goes here?

ker = im

may not be injective!

$$0 \to \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \xrightarrow{\text{mod } 2} \mathbb{Z}_2 \to 0$$

$$\downarrow (-) \otimes \mathbb{Z}_2$$

$$\cancel{0 \dashrightarrow} \underbrace{\mathbb{Z}_2 \xrightarrow{0} \mathbb{Z}_2 \xrightarrow{id} \mathbb{Z}_2 \to 0}$$

$K(A, B, C, M)$ to restore exactness

$\hookrightarrow Tor(C, M)$

Def: R a ring, take $F: (-) \otimes_R (-)$ a bifunctor

$$\text{mod-}R \times R\text{-mod} \longrightarrow Ab$$

$$(M_R, {}_R N) \longrightarrow M \otimes_R N$$

Take $F_R^* \to M_R \to 0$, apply $F \implies F_* \otimes_R N$] possibly no longer acyclic so take homology

$\uparrow$ almost acyclic

so def $Tor_R^{-n}(M, N) = h^{-n}(F^* \otimes_R M)$

$\hookrightarrow$ take ker/im, forget co/contravariance

$$\underbrace{0 \to \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z}}_{\otimes N} \xrightarrow{\text{mod } 2} \mathbb{Z}_2 \to 0$$

$$0 \to N \xrightarrow{\times 2} N \to 0$$

$\to h^0 = N/2N$

$h^{-1} = \ker(\times 2) = 2\text{-torsion} \in N$

$= N \otimes \mathbb{Z}_2$

$$F^{-1} \to F^0 \to M \to 0$$

$$F^{-1} \otimes N \to F^0 \otimes N \to M \otimes N \to 0$$

$\to Tor_R^0(M, N) = M \otimes_R N$

$\longrightarrow Tor^0_R(M,N) = M \otimes_R N$

For $R=\mathbb{Z}$, the new thing is

$$Tor^{-1}_{\mathbb{Z}}(M,N) := \text{"}Tor(M,N)\text{"}$$

Can always choose F^* with $* \leq -2 \to F^* = 0$

$$\cdots F(G) \cong F(G) \longrightarrow G \longrightarrow 0$$

$$0 \to F^{-1} \to F^0 \to M \longrightarrow 0$$

when $R = \mathbb{Z}$ (or a PID)

Show that this is qhty invariant

Measure failure of exactness of $\cdot \otimes M$ by how many Tors extend off to left

---

Last time: Free Resolution

$M \in R$-mod, we can replace M with

$$F^* \xrightarrow{\epsilon} M \to 0$$

acyclic, exact, no homology

$$\begin{array}{ccccccc} \to F^{-2} & \to & F^{-1} & \to & F^0 & \to & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ \to 0 & \to & 0 & \to & M & \to & 0 \end{array}$$

quasi-iso of chain complexes

Any two resolutions are hty equiv

$$F^* \underset{\varphi}{\overset{\Theta}{\rightleftarrows}} G^*$$

$$\Theta \circ \varphi \simeq id_{G^*}$$
$$\varphi \circ \Theta \simeq id_{F^*}$$

Given M_R, ${}_RN$

$$\boxed{Tor^{-n}_R(M_R, {}_RN) := h^{-n}(F^*_R \otimes_R {}_RN)}$$

choose any free resolution

take homology of result

- Well defined, indep of choice of F^*

  $(\cdot) \otimes_R N$ preserves chain hty equiv

  $h \to h \otimes id_N$, $\partial h + h\partial = (\Theta \circ \varphi) - id$

- Bifunctor mod-$R \times R$-mod $\longrightarrow Ab$

$$Ab = \mathbb{Z}\text{-mod}$$

• Bifunctor mod-R $\times$ R-mod $\longrightarrow$ Ab $= \mathbb{Z}$-mod

With $M \in \mathbb{Z}$-mod, only Tor^0 & Tor^{-1} are needed — Only interesting one for $M \in \mathrm{Ab}$, $\mathrm{Tor}^{-1}_{\mathbb{Z}}(\cdot,\cdot) := $ "$\mathrm{Tor}(\cdot,\cdot)$"

just choose $F^{-1} = \ker \varepsilon : F_0 \longrightarrow M$

Given $0 \to A \to B \to C \to 0$, ${}_R N \in R$-mod, apply $(\cdot) \otimes_R N$ to yield les

$$\mathrm{Tor}^{-1}(A,N) \to \mathrm{Tor}^{-1}(B,N) \to \mathrm{Tor}^{-1}(C,N) \to A \otimes_R N \to B \otimes_R N \to C \otimes_R N \to 0$$

not inj, surjects onto $\ker = k(A,B,C,N) = \mathrm{Tor}^{-1}(C,N) / \mathrm{im}\,\mathrm{Tor}^{-1}(B,N)$

$$\begin{tikzcd} & A \arrow[dr] & \\ C \arrow[ur] & & B \arrow[ll] \end{tikzcd} \implies \begin{tikzcd} & \mathrm{Tor}(A,N) \arrow[dr] & \\ \mathrm{Tor}(C,N) \arrow[ur] & & \mathrm{Tor}(B,N) \arrow[ll] \end{tikzcd}$$

PF Want SES

Pick free res of $A, B, C \rightsquigarrow A^*, B^*, C^*$

↳ Problem: Choice!

Can lift maps $A \to B \uparrow A^* \to B^*$

but is $0 \to A^* \otimes N \to B^* \otimes N \to C^* \otimes N \to 0$ exact?

not in general, can pick adversarially to mess up kernels

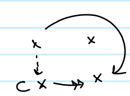
Fix: First pick A^*, C^*, then pick B^* suitably

$$\begin{tikzcd} 0 \arrow[r] & A^{-1} \arrow[r] \arrow[d, "\partial"'] & A^{-1} \oplus C^{-1} \arrow[r] \arrow[d, dotted, "\text{choice}"] & C^{-1} \arrow[r] \arrow[d, "\partial"] & 0 \\ 0 \arrow[r] & A^0 \arrow[r] \arrow[d, two heads] & A^0 \oplus C^0 \arrow[r] \arrow[d, dotted, "\text{by freeness of } C^0"] & C^0 \arrow[r] \arrow[d, two heads] & 0 \\ & A \arrow[r] & B \arrow[r] & C & \end{tikzcd}$$

df df

No choice, $B^0 \overset{?}{=} A^0 \oplus C^0$

Inductive diagram chase



Take basis of C^{-1}, Find $c \in A^0 \oplus C^0$ that maps to $(\partial \circ \pi)(c) \in C^0$

Moral: Need to pick resolutions with nice properties in general

UCT for spaces

X a space, use $R = \mathbb{Z}$, $\exists$ a ses

$$0 \longrightarrow \underbrace{H_n(X;\mathbb{Z}) \otimes G \xrightarrow{(\cong ?)^{\text{no}}} \underbrace{H_n(X;G)}_{\text{want}}}_{\text{guess}} \longrightarrow \underbrace{\mathrm{Tor}(H_{n-1}(X,\mathbb{Z}); G)}_{\text{error term}} \to 0$$

guess error term

Splits, but non-canonically

$$H_n(X;G) \cong (H_n(X;\mathbb{Z}) \otimes G) \oplus Tor(H_{n-1}(X,\mathbb{Z});G)$$

PF:

Recall $(C_*^{sing}(X,G), \partial) := (\underbrace{C_*^{sing}(X,\mathbb{Z})}_{\text{complex of free abelian gps}} \otimes G, \partial \otimes id_G)$

Want: $h(C_*) \otimes_R M$ related to $h(C^* \otimes_R M)$ - complicated!

$\sim$ Need UCT spectral sequence

① Form...

$0 \to (Z_*, 0) \to (C_*, \partial) \to (B_{*-1}, 0) \to 0$

use 0 as differentials

$$\begin{array}{ccccccccc} 0 \to & Z_n & \xhookrightarrow{i} & C_n & \xrightarrow{\pi} & B_{n-1} & \to 0 \\ & {\scriptstyle 0}\downarrow \ G & & \downarrow{\scriptstyle \partial} \ G & & \downarrow{\scriptstyle 0} & \\ 0 \to & Z_{n-1} & \xhookrightarrow[i]{} & C_{n-1} & \xrightarrow[\pi]{} & B_{n-2} & \to 0 \end{array}$$

② Tensor with G

$0 \to (Z_* \otimes G, 0) \to (C_* \otimes G, \partial \otimes id) \to (B_* \otimes G, 0) \to 0$

Not obviously exact, but it is

Each level splits, $C_n \cong Z_n \oplus B_{n-1}$

Tensoring preserves direct sum

- Note $C_* \neq Z_* \oplus B_*$!

③ So take les

$$\xrightarrow{\delta_n} Z_n \otimes G \to H_n(C_* \otimes G) \to B_{n-1} \otimes G \xrightarrow{\delta_{n-1}} Z_{n-1} \otimes G \to 0$$

So $\delta = (\cdot \otimes G) \circ i$, check elts using snake lemma

SES, $\rightsquigarrow$

$$0 \to \underset{\substack{\| \\ \frac{Z_n \otimes G}{B_n \otimes G}}}{\mathrm{coker}\,\delta_n} \to H_n(C_* \otimes G) \to \underbrace{\ker \delta_{n-1}}_{} \to 0$$

This is the Tor group

but why? Defined using free resolution..

Cycles & boundaries are free!

UCT

$$0 \to H_n(X;\mathbb{Z})\otimes G \to H_n(X;G) \to \mathrm{Tor}(H_{n-1}(X;\mathbb{Z}),G) \to 0$$

?

$$B_n \otimes G \to Z_n \otimes G \to H_n(X;G) \to B_{n-1}\otimes G \xrightarrow{i\otimes id_G} Z_{n-1}\otimes G$$

Note $0 \to B_n \to Z_n \to H_n(X;\mathbb{Z})$

is a free resolution of H_n

$\to$ $\ker(i\otimes id_G) = \underline{\mathrm{Tor}(H_n(X;\mathbb{Z}),G)}$

$\mathrm{coker}(i\otimes id_G) = \underline{H_n(X;\mathbb{Z})}\otimes G$

So $0 \to \mathrm{coker} \to H_n \to \ker \to 0$

Note $0 \to Z_n \to C_n \to \underbrace{B_n}_{\text{free}} \to 0$] splits, so choose s, yields r

$\to$ $C_n = \underbrace{Z_n}_{\text{project onto this factor}} \oplus s(B_n)$

Projection: $C_n \xrightarrow{r} Z_n \xrightarrow{\text{quotient}} H_n = Z_n/B_n$

$\to$ $\cdot\otimes G$ yields $C_n\otimes G \to H_n(X;\mathbb{Z})\otimes G$

$z\otimes g \to [z\otimes g]$

Computations

$\rightsquigarrow$ $\mathrm{Tor}(A,B) = A^*\otimes B$

(For Ab) $\mathrm{Tor}(\oplus C_i, G) = \oplus \mathrm{Tor}(C_i;G)$

$\mathrm{Tor}(\mathbb{Z}_n, G) = \ker\{\cdot n : G\to G\}$

$\mathrm{Tor}(\mathbb{Z}, G) = 0$

$\mathrm{Tor}_R(M,N) = \mathrm{Tor}(N,M)$ for bimodules (not obvious)

Problem: $\mathrm{Tor}(\mathbb{Q},N) = ?$

- Choose a resolution with a pattern of generators

	$\mathbb{Z}_n$	$\mathbb{Z}$	$\mathbb{Q}$
$\mathbb{Z}_n$	·	·	·
$\mathbb{Z}$	·	·	·
$\mathbb{Q}$	·	·	·

($\mathbb{Q}$ useful eg for Euler char)

Fill in

Another UCT: $H_* \leftrightarrows H^*$

$\mathrm{Hom}_{R\text{-mod}}({}_RM, {}_RN) \in \mathrm{Ab}$

$\mathrm{Hom}_{R\text{-mod}}(-,-): R\text{-mod} \times R\text{-mod} \longrightarrow Ab$

Analogous to $(-)\otimes_R M : \text{Mod-}R \to Ab$
is <u>right</u> (not fully) exact, then

- $\mathrm{Hom}({}_RM, -)$ covariant $R\text{-mod} \to Ab$
- $\mathrm{Hom}(-, {}_RM)$ contravariant, or $R\text{-mod}^{op} \to Ab$
  $\rightsquigarrow$ covariant

are left exact

$0 \to A \to B \to C \to 0 \quad \in Ch\, R\text{-mod}$

$0 \to \mathrm{Hom}(M,A) \to \mathrm{Hom}(M,B) \to \mathrm{Hom}(M,C) \dashrightarrow$ might fail exactness here

may not surject

Left exact = exact here

Used more $\longrightarrow$ $0 \to \mathrm{Hom}(C,M) \to \mathrm{Hom}(B,M) \to \mathrm{Hom}(A,M) \dashrightarrow$

<u>Df</u> $\mathrm{Ext}^n_R({}_RM, {}_RN) = h^n(\mathcal{H}om(F^*, N))$
where $F^* \to M \to 0$

1) Take free res
2) Take $\mathrm{Hom}(-,N)$

So $\mathcal{H}om^i(F^*, N) = \mathrm{Hom}(F^{-i}, N)$
where dif. $\delta : \mathrm{Hom}(F^{-i}, N) \to \mathrm{Hom}(F^{-i+1}, N)$ (increasing degree)
$= (-1)^i [(-)\circ \partial]$

<u>Thm</u>: Ext is error term, $\exists$ LES

$\mathrm{Hom}\,C \to \mathrm{Hom}\,B \to \mathrm{Hom}\,A \to \mathrm{Ext}^1 C \to \mathrm{Ext}^1 B \to \dots$

"$\mathrm{Ext}(G,H)$" $:= \mathrm{Ext}^1_{\mathbb{Z}}(G,H)$

Like Tor, build a table

⚠ Not symmetric! Would usually take $(\cdot\,,\ \cdot)$
proj res, inj res

Free res

In general, derived functors, injective/projective resolution

Special Cx: X = space, $C_*(X;\mathbb{Z}) \to C^*(X;\mathbb{Z}) = \mathrm{Hom}(C_n(X;\mathbb{Z}), G)$

Recall kronecker pairing, evaluate cohom on hom

$$0 \to \mathrm{Ext}(H_{n-1}(X;\mathbb{Z}), G) \longrightarrow H^n(X;G) \longrightarrow \mathrm{Hom}(H_n(X;\mathbb{Z}), G) \longrightarrow 0$$

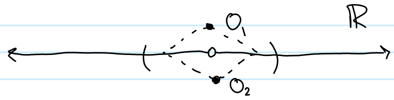
So torsion in neighboring degrees contribute to failure of expected isomorphism.

<u>Next</u>:
- Interesting resolutions, not over $\mathbb{Z}$
- Homology on groups
- Back to spaces
- See problem sheets, look at resolutions
  (Pick good indexing/grading scheme!)

See md notes

Today: Manifolds

- Locally homeomorphic to $\mathbb{R}^n$
- Hausdorff $\longrightarrow$ <u>Counter</u>



Not hausdorff - $\nexists\ O_1 \in N(O_2)$

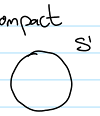
- 2nd Countable $\longrightarrow$ $S_1 \times S_1$
  (Has countable base of open sets)

$\mathbb{R} \times [0,1)$
long line
lex order

Hausdorff w/ lower section topology but no countable basis (not obvious) despite locally $\cong (a,b)$

IFF subspace of $\mathbb{R}^N$ locally $\cong \mathbb{R}^n$ for some $n \leq N$

- Compact vs non compact



$(0,1) \subseteq \mathbb{R}^1$

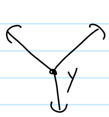
- Connected vs. not
- Generalization: Mfd w/bd (not a special case of mfds!)
  Also allow $x \in U \cong \mathbb{R}^n_{\geq 0}$
- <u>Defn</u>: $\partial M = \{x \in M \mid \nexists\ (N_x, x) \xrightarrow{\phi} (\mathbb{R}^n, 0)\}$
  $M^\circ = (\partial M)^c \sim \partial M = (M^\circ)^c$

<u>Note</u>: Another way to distinguish pts: Local homology

$$H_n(M, M-x) \cong H_n(U, U-x) \cong H_n(\mathbb{R}^n, \mathbb{R}^n - 0) \overset{LES}{\cong} \underline{H_{n-1}(\mathbb{R}^n - 0) = \mathbb{Z}}$$

$$H_n(\mathbb{R}^n_{\geq 0}, \mathbb{R}^n_{\geq 0} - 0) \cong H_{n-1}(\mathbb{R}^n_{\geq 0} - 0) = H_{n-1}(B^n) = \underline{0}$$

<u>Ex</u>: $Y =$



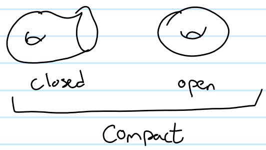
not a mfd

$$\longrightarrow H_1(Y, Y-x) = H_0(Y-x) = \mathbb{Z}^3$$

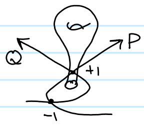
but $H_0(\longleftrightarrow) = \mathbb{Z}^2$ ∎

- Closed: Compact, $\partial M = \emptyset$

• Closed: Compact, $\partial M = \emptyset$



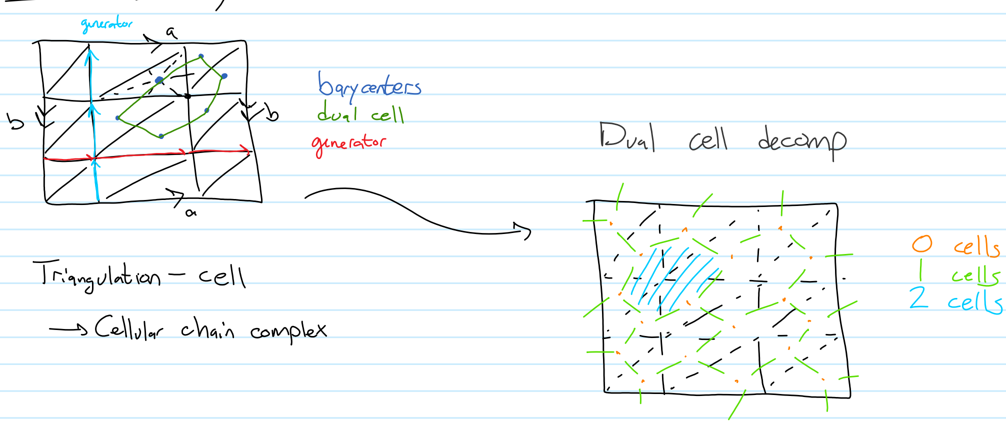
• Fact: $\partial^2 M = 0$ → Suggests nice homology



$P \cap Q = 0$
algebraically - does this mean P, Q can be made disjoint?

Works in high dims, 2/3/4 tough
known handle decomp

Poincaré Duality



Triangulation - cell
→ Cellular chain complex

yields cellular chain complex D_*
with $H_* C_* = H_* D_*$
but reverses chain complex
and $C^* = D_*$

→ Moral: $H^i(M; \mathbb{F}) = H_{n-i}(M, \mathbb{F}) = H^{n-i}(M; \mathbb{F})$
need vector space for dim iso

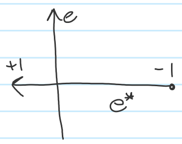
Take $\mathbb{F} = \mathbb{Q}$ → $b_i = b_{n-i}$

C_* (cell complex) ⟶ D_* (dual complex)

Need a canonical way to assign
Issue: Orientation

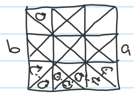
Have unique transverse edge in above construction

Choice of



requires global orientation, works for $\mathbb{Z}_2$ coefs

Note for fundamental square



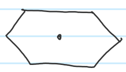
works if $a=\uparrow$, $b=\uparrow$
not if $a=\uparrow$, $b=\downarrow$

Push orientation through to cancel edges and produce cycle.

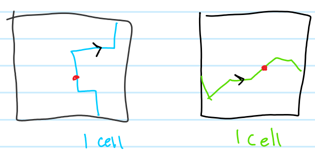
Expect complementary-dimensional intersections

2-cells:

3-cells



Bilinear cell pairing



Add up all intersections with sign

Well defined:

$\longrightarrow \exists\ H_i(T,\mathbb{Z}) \otimes H_{n-i}(T,\mathbb{Z}) \to \mathbb{Z}$

Algebraic intersection number can be computed using homology (perturb paths)

$\longrightarrow H_i \otimes H^i \to \mathbb{Z}$
$\langle \cdot , \cdot \rangle$

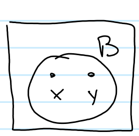
View: Homology looks more like simplices ($H_2 \sim$ surfaces)

Orientation

Given M_n, orientation @ p is generator of $H_n(M, M-p) \cong \mathbb{Z}$ by excision

Define orientation cover
$$\begin{array}{ccc} \tilde{M} & & (x, \mu_x) \\ \downarrow \pi & & \downarrow \\ M & & x \end{array}$$
orientation at x

Topologize using base of open sets in $\mathbb{R}^n$



$$\begin{tikzcd} & H(M-B) \arrow[dl, "\cong"', "f\,l_y"] \arrow[dr, "f\,l_y"] & \\ H(M-x) \arrow[rr, bend right, "\sim"] & & H(M-y) \end{tikzcd}$$

Can use sets $(B, \pm 1)$ as base for $\tilde{M}$
$B_1 \cap B_2 = \bigcup_{i \in I} B_i$

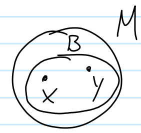
Moral: $\begin{array}{c}\tilde{M} \\ | \\ M\end{array}$ determined by $W_1 : \pi_1(M, p) \longrightarrow \mathbb{Z}$

Orientable iff W_1 trivial.

Orientations

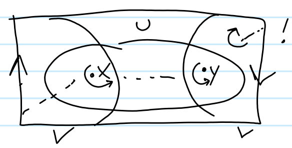
A generator of $\mu_x \in H_n(M, M-x)$

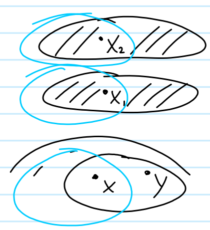
$$\begin{tikzcd} & H_n(M, M-B) \arrow[dl, "e_x"'] \arrow[dr, "e_y"] & \\ H_n(M, M-x) \arrow[rr, "\sim", "e_y \circ e_x^{-1}"'] & & H_n(M, M-y) \end{tikzcd}$$



An <u>orientation</u> is a family of $\{\mu_x\}_{x\in M}$ of local orientations with local consistency

$x, y \in U \Rightarrow \mu_x, \mu_y$ are related by propagation





Recall orientation double cover $\tilde{M} = \{(x, \mu_x)\}$

$$\begin{array}{c} \tilde{M} \\ \downarrow \\ M \end{array}$$

$\tilde{M}$ is classified by the orientation hom. $w_1 : \Pi_1(M) \longrightarrow \mathbb{Z}_2$ (groupoid)

$\pi_1(M, p) \longrightarrow \{\pm 1\}$

<u>Ex</u>: Mobius strip $w_1(M, x) \longrightarrow \mathbb{Z}_2$, $\alpha \longmapsto -1$

M orientable iff w_1 is trivial

iff $\tilde{M} = M \amalg \mathbb{Z}_2$

iff $\tilde{M}$ is disconnected

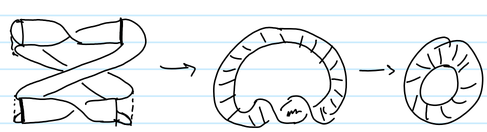
iff $\tilde{M}$ admits a section $s: M \to \tilde{M}$, $x \to (x, \mu_x)$

$$\begin{tikzcd} \tilde{M} \arrow[d, shift right, "\pi"'] \\ M \arrow[u, shift right, "s"'] \end{tikzcd}$$

so $(\pi \circ s)(x) = x$ $\quad$ {cts!}

· $\tilde{M}$ is orientable

$$\begin{array}{c} S^1 \times I \\ \uparrow \\ M \end{array}$$



<u>Pf</u> Take $(x, \mu_x) \longrightarrow ((x, \mu_x), \mu_x)$

it's cts.

$$\begin{array}{c} H_n(\tilde{M}, \tilde{M} - (x, \mu_x)) \\ \cong \ \pi^* \\ H_n(M, M-x) \end{array}$$

<u>Ex</u>: Related to problem sheet: 3x cover of torus

An orientable mfd can not cover a non-orientable mfd with odd degree

$$\begin{tikzcd} N \arrow[d, "\pi", "2"'] \\ M \end{tikzcd} \rightsquigarrow w_1^N = w_1^M \circ \pi_*$$

$\pi_1 N \hookrightarrow \pi_1 M$ kernel is index 2

$\boxed{\text{Think through! good test}}$ multiplicativity of indices

---

Towards Poincaré Duality

<u>Thm</u> Existence of fundamental class

- MFd compact, closed, orientable, $\underbrace{\text{w/o boundary}}_{\text{closed}}$ + connected

$$H_n(M) \cong \begin{cases} \mathbb{Z}, & M \text{ compact + orientable} \\ 0, & \text{else} \end{cases}$$

$H_{n>0}(M) = 0$

generator of $\mathbb{Z}$ = fundamental class

$[M]$ or μ_M

Local $\longrightarrow$ Global

- Need inductive process on patches
- But $U \subseteq M$ not compact, so $H_n \neq \mathbb{Z}$
  - $\hookrightarrow$ Need to say something nontrivial for non-compact mfds
  - Use "relative" version of thm first

Let M^n be connected, $\partial M = 0$ {no compactness or orientability}

$K \subseteq M^n$ compact

- $H_{i>n}(M, M-K) = 0$
- $H_n(M, M-K) = \mathbb{Z} \cdot \mathbb{1}[M \text{ orientable}]$

<u>Idea</u>: Show true for

$$\left.\begin{array}{l} \cdot K_1 \\ \cdot K_2 \\ \cdot K_1 \cap K_2 =: K_{12} \end{array}\right\} \Longrightarrow \text{true for } K_1 \cup K_2$$

Use this M-V sequence (relative)

$(M, M-K_1) \cap (M, M-K_2) = (M, M-(K_1 \cup K_2))$

$= (M, M-K)$

$\sim \quad \cup \quad \sim \longrightarrow (M, M-(K_1 \cap K_2))$

$$\downarrow \qquad \downarrow \qquad = (M, M-K)$$

$$\sim \quad \cup \quad \sim \longrightarrow (M, M-(K_1 \cap K_2))$$

$$= (M, M-K_{12})$$

$$\to H_{i+1} K_{12} \to H_i K_1 \oplus H_i K_2 \to H_i K \to H_i K_{12} \to$$

Know inductive step, but what is base case?

$\sim$ $\mathbb{R}_n$, cpct subset: Simplex

$\sim$ Pull back to M

① $H_{i>n}(M, M-K) = 0$

② $b \in H_n(M, M-K)$, $b = 0$

$\longrightarrow$ All restrictions to pts are zero

$$\text{MV} \quad H_{n+1}(M, M-K_{12}) \to H_n(M, M-K) \xhookrightarrow{i \oplus -i} H_n(M, M-K_1) \oplus H_n(M, M-K_2)$$

$$0$$

Induction Arg: ② is true for K_1, K_2, K_{12}

① is true

$\Longrightarrow$

$b \in H_n(M, M-K) \mapsto \rho_1 b \oplus -\rho_2 b$ restrictions

$\rho_x(\rho_1 b) = \rho_x b = 0$ by assum.

$\rho_1 b = \rho_2 b$ by assum.

$b = 0$ by exactness. ∎

$\Longleftarrow$ $\mu_{K_1} \oplus -\mu_{K_2} \longmapsto \underbrace{\rho_{12}\mu_{K_1} - \rho_{12}\mu_{K_2}} = b$

$\rho_x(\quad) = \rho_x \mu_{K_1} - \rho_x \mu_{K_2} \in H(M, M-x)$

$= \mu_x - \mu_x$ by assum.

$= 0$

$\therefore$ $b = 0$ by ②

$\therefore$ $\mu_{K_1} \oplus -\mu_{K_2}$ is img of some elt by exactness

③ Given an orientation $\{\mu_x\}_{x \in K}$, $\exists!\, \mu$ s.t. $\mu|_x = \mu_x \ \forall x$

$\hookrightarrow \in H_n(M, M-K)$

$$H_n(M; \mathbb{Z}) = \begin{cases} \mathbb{Z}, & M \text{ cpt/orientable} \\ 0, & \text{else} \end{cases}$$

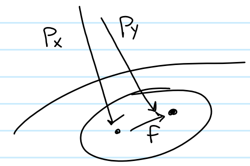
<u>Pf</u> Show $\rho_x \; H_n M \to H_n(M, M-x) \cong \mathbb{Z}$

is injective

$\rho_x \; \backslash \; \backslash \rho_y$

Pf Show $\rho_x \; H_n M \to H_n(M, M-x) \cong \mathbb{Z}$

is injective



Sp. $b \in H_n M$ has restriction $\rho_x b = 0$

and know $\rho_y b = 0 \;\; \forall y \in N(b)$

And if $\rho_x b \neq 0$ then $\rho_y \neq 0$

So $M = \{x \mid \rho_x b = 0\} \amalg \{x \mid \rho_x b \neq 0\}$

$F =$ "$\rho_x \circ \rho_y^{-1}$" by comm

connectedness $\to$ one is empty $\to$ injectivity

$\to H_n M \in \{0, \mathbb{Z}\}$. Which?,

M cpt + orientable $\to$ take $K = M$, use ③

$\to \exists! \; \mu_M \neq 0 \in H_n(M, \varnothing) = H_n(M)$

Must be a generator, and restricts to a

generator μ_x at each x

$\to \mu_M$ is the fundamental class (orientable)

choice of generator $\to$ oriented

M non cpt

Suppose $[z] \in H_n M$

let $Z =$ image of z ($\bigcup_{i \in I}$ simplexes, cpt)

pick $x \notin Z$, then

Restricting $[z]$ to $H_n(M, M-x) = 0$

$\to [z] = 0$ since ρ_x is injective

Sp. $\exists$ a nonzero class $\mu \in H_n M$

Restrict to get $\{\rho_x \mu\}_{x \in M} \rightsquigarrow$ Nonzero, locally consistent

Are they actually generators?

If $\rho_x \mu = k \cdot 1$, 1 a gen of $H_n(M, M-x)$

just redefine $\mu_x := \frac{1}{k} \rho_x \mu$

These give an orientation

$\boxed{\therefore \; M \text{ non orientable} \to H_n M = 0}$ Contrapositive

Also proves $\rho_x \; H_n M \xrightarrow{\sim} H_n(M, M-x)$

$\text{gen} \mapsto \text{gen}$

is an isomorphism

Important part - restricting locally to compute

(Induction is mostly fiddling)

Want a version of $H^i M = H_{n-i} M$ $\quad$ } closed, orientable }

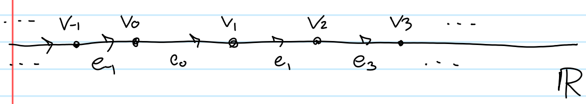
(Induction is mostly fiddling)

Want a version of $H^i M = H_{n-i} M$ {closed, orientable}

for non-compact

$\rightsquigarrow H_c$, compactly supported homology

Ex Cellular example



$$0 \to \oplus \mathbb{Z} e_i \to \oplus \mathbb{Z} v_i \to 0$$

$\partial e_i = v_{i+1} - v_i \to v_i \approx v_j \; \forall i, j$

- $H_0 = \mathbb{Z}\langle [v_i]_\sim \rangle \cong \mathbb{Z}$
- $H_1 = 0$ {no finite lin. comb. with $\partial = 0$}

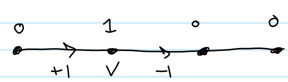
Dual

$$0 \to \mathbb{Z}^{\text{verts}} \to \mathbb{Z}^{\text{edges}} \to 0$$

$$\mathbb{Z}^{\text{verts}} = \coprod_{\text{verts}} \mathbb{Z} \qquad \mathbb{Z}^{\text{edges}} = \coprod_{\text{edge}} \mathbb{Z}$$

$(\partial f)(e_i) = f(v_{i+1}) - f(v_i)$ — discrete derivative of f

$e_v = \mathbb{1}[\{v\}]$



- $H^0 = \mathbb{Z}\langle \text{const fns} \rangle \cong \mathbb{Z}$
  $v_i \mapsto +1$
- $H^1 = ?$

Any fn ϕ : edges $\to \mathbb{Z}$

is δ of some f

Define $f(n) = \int_a^n \phi(e)$

$\to \delta f = \phi$ (1)

$\boxed{\to H^1 = 0 \,!!}$

Change to compactly supported

$\to$ (1) may not hold $\forall f$

$\to H^1 \neq 0$

Previously : $\underbrace{H_c^i}_{\text{compact support}} \mathbb{R}^n \cong H_{n-i} \mathbb{R}^n$

What is the map? Cap product {Adjoint to cup product}

Have: $H^i X \times H_i X \longrightarrow \mathbb{Z}$

$$\Big(\underset{\in C^n X}{[\xi]}\ ,\ \underset{\in C_n X}{[z]}\Big) \longrightarrow \langle \xi, z\rangle$$

Use partial application

$$\cap : C^i X \times C_i X \longrightarrow C_{n-i} X$$

$$\left(\xi\ ,\ \sigma : \Delta^n \to X\right) \longmapsto \langle \xi, \sigma \circ \beta_i\rangle \cdot (\sigma \circ \alpha_{n-i})$$

$\alpha_j : \Delta^j \hookrightarrow \Delta^n$ front face

$\beta_j : \Delta^{n-j} \hookrightarrow \Delta^n$ back face

Note $\langle \xi \cup \phi, \sigma\rangle = \langle \xi, \sigma \circ \alpha_i\rangle \cdot \langle \phi, \beta_i\rangle$

$$= \langle \xi, \phi \cap \sigma\rangle$$

So "adjoints": $\phi \cap (-) \leftrightsquigarrow (-) \cup \phi$

① $\partial(\xi \cap x) = (\delta\xi) \cap x + (-1)^{q} \xi \cap (\partial x)$

So $\cap$ is a chain map

$$\begin{array}{ccc} (C^*, \delta) \otimes (C_*, \partial) & \xrightarrow{\ \cap\ } & (C_*, \partial) \\ {\scriptstyle (\delta \otimes 1) \pm (1 \otimes \partial)} \downarrow & \circlearrowright & \downarrow \partial \\ (C^*, \partial) \otimes (C_*, \partial) & \xrightarrow{\ \cap\ } & (C_*, \partial) \end{array}$$

② Get a map $\underbrace{H^i \times H_n}_{} \longrightarrow H_{n-i}$

Factors through $H(C^* \otimes C_*)$

$\cap$: Reduces degree

③ Still get adjunction

$$\langle [\xi]^p \cup [\phi]^q, [z]_{p+q}\rangle = \langle [\xi], [\phi] \cap [z]\rangle$$

So $[\theta] \cup ([\phi] \cap [z]) = ([\theta] \cup [\phi]) \cap [z]$

$[1] \cap [x] = [x]$

$\in H^0 X$

$\Longrightarrow$ $\underline{H_* X \text{ is an } \underbrace{H^* X}_{} \text{-module}}$

$\Longrightarrow$ $\underline{H_* X \text{ is an } \underbrace{H^* X}_{\text{ring}} \text{-module}}$

④ All functorial

$$X \xrightarrow{f} Y \quad \begin{array}{l} \nearrow H_* X \xrightarrow{f_*} H_* Y \quad \text{push forward} \\ \searrow H^* X \xleftarrow{f^*} H^* Y \quad \text{pull back} \end{array}$$

$$\begin{array}{ccccc} \overset{[z]}{} & & & & \\ H^i X & \otimes & H_n X & \longrightarrow & H_{n-i} X \\ \uparrow f^* & & \downarrow f_* & & \downarrow f_* \\ H^i Y & \otimes & H_n X & \longrightarrow & H_{n-i} Y \\ [\xi] & & & & \end{array}$$

$$\Longrightarrow f_*\left(f^*[\xi] \cap [Z]\right) = [\xi] \cap f_*[Z]$$

Given $\begin{array}{c} X \\ \downarrow f \\ Y \end{array}$, exists $\begin{array}{c} RX \\ \uparrow f^* \\ RY \end{array}$ so view any RX module as an RY module

then $F_* : {}_{RY}RX \longrightarrow {}_{RY}RY$

is an RY-module map.

⑤ Relative versions for $A \subseteq X$

$$\begin{array}{ccccc} C^i X & & C_n X & \longrightarrow & C_{n-i} X \\ \vdots & & \vdots & & \\ \downarrow & & \downarrow & & \\ C^i(X,A) & & C_n(X,A) & \longrightarrow & \boxed{C_{n-i} X} \end{array}$$

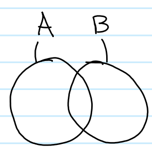
$C^i(X,A) = \ker\left(C^i X \xrightarrow{\iota^*} C^i A\right)$, $\quad C_n(X,A) = \dfrac{C_n X}{C_n A}$

Yields

$$C^i X \times C_n(X,A) \longrightarrow C_{n-i}(X,A)$$

$$C^i(X,A) \times C_n(X,A) \longrightarrow C_{n-i} X$$

or generalized,



$$C^p(X,A) \times C^q(X,B) \xrightarrow{\cup} C^{p+q}(X, \underbrace{A+B}_{})$$

Vanishes when in A or B,
not quite $A \cup B$ (quasi-iso though)

Think: like functions that vanish on A, or on B

- $\underline{\text{Thm}}$. Let M be $\underbrace{\text{connected}}_{\text{not essential}}$, $\underbrace{\text{oriented}}_{\text{fund class}}$ n-mfd, $\partial M = \emptyset$,

can define

$D: H^i M \xrightarrow{\sim} H M$

fund class

not essential

Can define

$$D_M : H^i_c M \xrightarrow{\sim} H_{n-i} M$$

cap with fundamental class

$$H^i_c M = \lim_{K \subseteq M} H^i(M, M-K)$$

$$K \subseteq L \to L^c \subseteq K^c$$

$$[\xi] \longmapsto i^*[\xi]$$

$$\begin{tikzcd} H^i(M,M-K) \arrow[r] \arrow[dr, bend right, "\cdot \cap \mu_K"'] & H^i(M,M-L) \arrow[d, bend left, "\cdot \cap \mu_L"] \\ & \mathcal{L} \end{tikzcd}$$

$$i_*(i^*[\xi_K] \cap \mu_L) = [\xi_K] \cap i_* \mu_L$$

i_*: id; $i_* \mu_L = \mu_K$

acting on $H_{n-i}(-) \xrightarrow{id} H_{n-i} M$

Make map out of compatible family

Take $\mu_K \in H_n(M, M-K)$

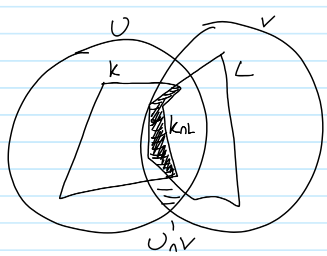
Use $C^i(X,A) \times C_n(X,A) \xrightarrow{\cap} C_{n-i} X$

M^n connected, oriented mfd

We defined

$$\begin{tikzcd} H^i_c M \arrow[r, "D_M"] \arrow[d, equal] & H_{n-i} M \\ \lim H^i(M, M-K) \arrow[ur, dashed, "- \cap \mu_K"'] & \end{tikzcd}$$

Prove it's an iso!



MV, exact

excision

exact

MV, exact

$$\begin{tikzcd}[column sep=small] H^i(M, M-K\cap L) \arrow[r] \arrow[d] & H^i(M,M-K) \oplus H^i(M,M-L) \arrow[r] \arrow[d] & H^i(M, M-K\cup L) \arrow[d] \\ H(U\cap V, U\cap V - K\cap L) \arrow[r] \arrow[d, "\cap \mu_{K\cap L}"] & H^i(U, U-K) \oplus H^i(V, V-L) \arrow[r] \arrow[d, "\cap\mu_K \,\oplus\, \cap\mu_L"] & H^i(U\cap V, U\cup V - K\cup L) \arrow[d, "\cap \mu_{K\cup L}"] \\ H_{n-i}(U\cap V) \arrow[r] & H_{n-i}(U) \oplus H_{n-i}(V) \arrow[r] & H_{n-i}(U\cap V) \end{tikzcd}$$

Commutes: See Hatcher, 5-lemma, choices in excision maps (pull back to chains)

Take direct limit ($\varinjlim K \subseteq U$, $\varinjlim L \subseteq V$)

Every compact $C \subseteq U \cup V$ is eventually in some $K \cup L$

$$\to H^i_c(U\cap V) \to H^i_c(U) \oplus H^i_c(V) \to H^i_c(U \cup V)$$

But this is the $\varinjlim$ of an exact sequence - is it still exact?

Yes: $\varinjlim_{\alpha} \{ \cdots \to A_\alpha \to B_\alpha \to C_\alpha \to \cdots \mid \text{exact} \}_{\alpha \in I}$ is exact

$\varinjlim$ commutes w/homology (Think as chain complex w/zero cohom)

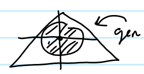
$$\cdots \to (\varinjlim A_\alpha) \to (\varinjlim B_\alpha) \to (\varinjlim C_\alpha) \to \cdots \quad \text{is exact.} \blacksquare$$

$$\begin{tikzcd}
\to H^i_c(U \cap V) \arrow[r] \arrow[d, "\cong\, D_{U\cap V}"] & H^i_c(U) \oplus H^i_c(V) \arrow[r] \arrow[d, dashed, "\cong \text{ by } 5\text{-lemma}", "D_U \oplus D_V"'] & H^i_c(U \cup V) \arrow[d, "\cong\ D_{U\cup V}"] \\
H_{n-i}(U \cap V) \arrow[r] & H_{n-i}(U) \oplus H_{n-i}(V) \arrow[r] & H_{n-i}(U \cup V)
\end{tikzcd}$$

Steps now are

① Show $D_U \cong$ in $\mathbb{R}^n$ $\qquad H^i_c(\mathbb{R}^n) \to H_{n-i}(\mathbb{R}^n)$

$$H^n(\mathbb{R}^n, \mathbb{R}^n - B) \times H_n(\mathbb{R}^n, \mathbb{R}^n - B) \to H_0 \mathbb{R}^n$$



use

$$H^n(X) \times \mathrm{Hom}(H_n X, \mathbb{Z}) \to \mathbb{Z}$$

(known, non-degenerate)

② True for any convex open $\subseteq \mathbb{R}^n$ (homeomorphic)

③ True for finite union of ② (induction, eg write n-dim simplicial complex as $\Delta^n \cup \{\Delta^i : i < n\}$)

~ Aside: Need infinite gluing lemma (more than pairwise → finite)

Let $U_1 \subseteq U_2 \subseteq \cdots$ be a chain of opens in M^n for which P.D. holds, show it holds for $\bigcup U_i$

$$\begin{tikzcd}
H^i_c U_1 \arrow[r, dashed, "\text{should exist}"] \arrow[d, "\cong\ D_{U_1}"] & H^i_c U_2 \arrow[d, "\cong\ D_{U_2}"] \\
H_{n-i} U_1 \arrow[r] & H_{n-i} U_2
\end{tikzcd}$$

Claim: $\varinjlim_n H^i_c U_n = H^i_c U$

$\varinjlim_n H_{n-i} U_n = H_{n-i} U$

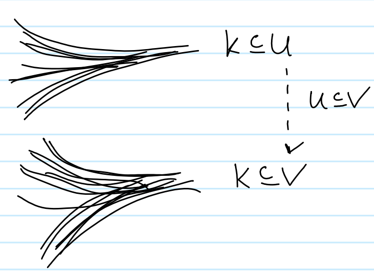
Pf: If $H^i_c(U) = \varinjlim_{K \subseteq U} H^i(U, U-K)$

$\cong$ exc

$= \varinjlim_{K \subseteq U} H^i(V, V-K)$

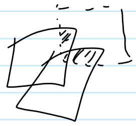
$\downarrow$ by univ. prop

$H^i_c V = \varinjlim_{K \subseteq V} H^i(V, V-K)$



Convexity necessary! eg.

is this actually a union of a smaller # of open balls?



This is just union of intersections; at most n-1

④ True for <u>any</u> open $\subseteq \mathbb{R}^n$, can express as countable union of convex opens

⑤ Glue mfd as countable union of such opens.

$H^i M^n \overset{PD}{\cong} H_{n-i} M^n$ ⟵ Doesn't depend on ring, but beware: orientation! Mostly use $\mathbb{Z}, \mathbb{Q}, \mathbb{Z}_2$

① n odd $\to \chi(M) = 0$

$\chi(X) = \Sigma (-1)^i \dim H_i(X, \mathbb{Q})$

$= b_0 - b_1 + b_2 - \dots (-1)^n b_n$

$\overset{PD}{\rightsquigarrow} \dim H^i(X,\mathbb{Q}) = \dim (H^i(X,\mathbb{Q}))^*$

$= \dim (H_{n-i}(X,\mathbb{Q}))$

n odd $\to$ 0 ... n even # of terms

$\to b_k = -b_{n-k}$

$\to$ All cancel

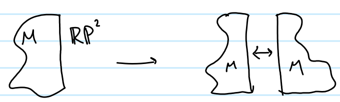
- $\chi(M^{odd}) = 0$. Why?

Take $\tilde{M}$ orientation cover

$\tilde{M} \xrightarrow{2:1} M$ $\quad \chi$ is multiplicative

- $\mathbb{RP}^2 \neq \partial M^3$ for any compact 3-dim mfd

Sp



$M_D = M \sqcup_{\partial M} M$

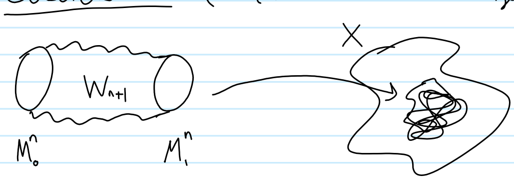
M_D closed $\to \chi(M_D) = 0$

But $\chi(M_D) = \chi(M) + \chi(M) - \chi(\mathbb{RP}^2)$

$= 2\chi(M) - 1 \longrightarrow 0$ is odd ※

<u>Generalization</u>: M^n, n even, $\chi(M^n)$ odd $\to$ not a boundary

$\rightsquigarrow$ <u>Cobordism!</u> $(\sim \{f: \text{Closed n-mfds} \to \text{Target space}\})$



$M^n \xrightarrow{f} X \sim M^n \xrightarrow{g} X$ ...

M_0^n $\quad$ M_1^n

$M_0^n \xrightarrow{f} X \cong M_1^n \xrightarrow{g} X$ if $\exists\, W^{n+1}$ s.t. $\partial W^{n+1} = M_0 \cup M_1$

and $\exists\, W^{n+1} \xrightarrow{F} X$

extending f, g

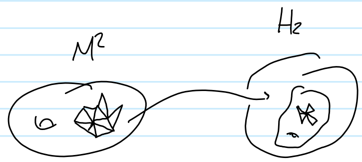
Yields $\Omega_n(X)$, n^{th} bordism gp. of X

Map: $\Omega_n \to H_n$

$[M^n \xrightarrow{f} X] \to f_* \mu_M$ push forward fundamental class

Not all classes in H_n can be represented this way

Not generally injective or surjective



$\to \Omega_n^{\text{Unorient}}(X) \to H_n(X, \mathbb{Z})$

not injective.

<u>PD & cup product</u>

$H^i(M, \mathbb{Z}) \times H^{n-i}(M, \mathbb{Z}) \to \mathbb{Z}$

$(\alpha, \beta) \to \langle \alpha \cup \beta, [M] \rangle$ ← Fund class, ← Kronecker pairing

$\qquad\qquad H^n \qquad H_n$

$:= " \int_M \alpha \cup \beta "$

$= \langle \alpha, \beta \cap [M] \rangle$

$= \langle \alpha, D([M]) \rangle \quad D :=$ pd map

$\rightsquigarrow$
$$\begin{array}{ccccc} H^i & \times & H^{n-i} & \to & \mathbb{Z} \\ \| & & \downarrow D & & \| \\ H^i & \times & H_i & \to & \mathbb{Z} \end{array}$$
] this nondegenerate when $H^i = (H_i)^*$

(by UCT, when $H_{i-1} = 0$)

Ex: $H^*(\mathbb{CP}^n, \mathbb{Z}) \overset{\text{as rings}}{\cong} \mathbb{Z}[\alpha] / (\alpha^{n+1})$ ← $\alpha \cup \alpha \cup \dots$

CP^n: $e_0 \cup e_2 \cdots \cup e_{2n-2} \cup e_{2n}$

CP^{n-1}: $e_0 \cup e_2 \cdots \cup e_{2n-2}$

$i: H_i CP^{n-1} \hookrightarrow H_i\, CP^n$

For $i, k \leq 2n-2$

$\downarrow$ dual

$i_*\; H^k CP^n \hookrightarrow H^k CP^{n-1}$

$\alpha^{n-1} \cup \alpha \to \cdot$ generator

$$\underset{H^{2n-2}}{\alpha^{n-1}} \cup \underset{H^{2}}{\alpha} \longrightarrow \underset{\in H^{2n}}{\cdot} \text{ generator}$$

From non-deg. of intersection pairing

$\rightsquigarrow \exists \beta \in H^2$ st. $\alpha^{n-1} \cup \beta = \text{gen } H^{2n}$

$\beta = \lambda\alpha$, $\lambda = \pm 1$ else $\alpha^{n-1} \cup \lambda\alpha \neq \text{gen}$