Algebraic Topology 2

Monday, 8 January, 2018 02:01 PM

Review Singular homology, Mayer-Vietoris

Also reduced homology: Augment

$\tilde{C}^* X.$ $\quad C_1 X \to C_0 X \longrightarrow \underset{\text{"}C_{-1}X\text{"}}{\mathbb{Z}} \longrightarrow 0$

$\to \tilde{H}_*(pt) = 0$

$\left(H_*(pt) = \mathbb{Z} \cdot \mathbb{1}[\deg = 0] \right)$

Relative homology

For $A \subseteq X$ $\quad$ "relative chain complex"

$$0 \to C_* A \to C_* X \longrightarrow \overbrace{C_*(X,A)}$$

$$:= C_* X / C_* A$$

$x \in C_n(X,A) \to \{x\} = \underbrace{x + C_n A}_{\text{coset}}$ where $x \in C_n X$

Elts of $H_n(X,A)$: cycles where $\partial x \in C_{n-1} A$

Immediately get les:

$$\cdots \longrightarrow H_n X \to H_n(X,A) \xrightarrow{\delta} H_{n-1}(A) \xrightarrow{i_*} H_{n-1}(X) \to \cdots$$

$$A \overset{i}{\hookrightarrow} X$$

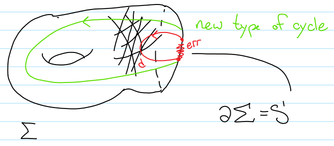
Note: i injective $\not\Rightarrow$ i_* injective!!
think $S^1 \hookrightarrow$ Disc

So $H_n(X,A)$ are the obstructions to i_* all being isomorphisms

but $H_n(X,A) = 0 \not\Rightarrow H_n(A) \cong H_n(X)$!! $\quad \left(H_n(X,A) \neq H_n X / H_n A \right)$

only if all zero.

Ex. 1



z a cycle $\to \exists y$? s.t. $\partial(y + C_{n+1} A) = z + C_n A$

$\to z - \partial y \in \underbrace{C_n A}_{\text{up to an "error"}}$

Consider simplicially

$$\begin{tikzcd}
0 \arrow[r] & H_2(\Sigma) \arrow[r] \arrow[d, equal] & H_2(\Sigma,\partial\Sigma) \arrow[r] \arrow[d, equal] & H_1(\partial\Sigma) \arrow[r, "0 \text{ map}"] \arrow[d, equal] & H_1(\Sigma) \arrow[r] \arrow[d, equal] & H_1(\Sigma,\partial\Sigma) \arrow[r] & H_0(\partial\Sigma) \arrow[r] \arrow[d, equal] & H_0(\Sigma) \arrow[r, "0"] \arrow[d, equal] & H_0(\Sigma,\partial\Sigma) \arrow[r] & 0 \\
& 0 & \mathbb{Z} & \mathbb{Z} & \mathbb{Z}\oplus\mathbb{Z} & & \mathbb{Z} & \mathbb{Z} & =0 &
\end{tikzcd}$$

Sum all clockwise triangles, $\neq 0$.

$\hookrightarrow 0 \to ? \to \mathbb{Z} \to 0$

since $0 \to ? \to 0$

not always $\mathbb{Z}$!

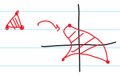
Ex. 2 $\quad H_2(\mathbb{R}, \mathbb{R}^2 - 0)$

LES

small $\to$ big $\to$ rel

$$\begin{tikzcd}
& & {[\sigma]} \arrow[d, "\in"] & \cong \arrow[d] & & & \\
\underbrace{H_2(\mathbb{R}^2-0)}_{\overset{\cong}{S^1}} \arrow[r] & H_2(\mathbb{R}) \arrow[r] & H_2(\mathbb{R}^2, \mathbb{R}^2-0) \arrow[r, "\delta"'] & H_1(\mathbb{R}^2-0) \arrow[r] & H_1(\mathbb{R}) \arrow[r] & H_1(\mathbb{R}^2) \arrow[r] & H_1(\mathbb{R}^2, \mathbb{R}^2-0)
\end{tikzcd}$$

$\underbrace{H_2(\mathbb{R}^2-0)}_{=0}$, $H_2(\mathbb{R}) = 0$, $H_1(\mathbb{R}^2-0) = \mathbb{Z}$



Use snake lemma iso

$\longrightarrow \cong \mathbb{Z}$!

Depends only on space & pt, gives local info

Reduced Relative Homology

Makes $H_0\,pt = 0$, more uniform behavior

$$\tilde{C}_*(X,A) = C_*(X,A),\ C_{-1}(X,A) = \frac{\mathbb{Z}}{\mathbb{Z}} = 0$$

Given $B \subseteq A \subseteq X$, ses

$$0 \to \frac{C_* A}{C_* B} \to \frac{C_* X}{C_* B} \twoheadrightarrow \frac{C_* X}{C_* A} \to 0$$

/small /bigger

"les of a triple"

$$\cdots H_n(A,B) \to H_n(X,B) \to H_n(X,A) \xrightarrow{\partial} H_{n-1}(A,B) \cdots$$

Next up: Excision!

Math 187A Login
alice
mirror

$X =$ CW complex

Define $C_n^{cell} X = H_n(X_n, X_{n-1})$ ← Look @ LES

$$\cong H_n(\vee_{\alpha \in I^n} S^n_\alpha)$$

$$\cong \bigoplus_\alpha \mathbb{Z} \quad \text{(number of n-cells)}$$

∂. From LES of triple

$$H_n(X^n, X^{n-1}) \xrightarrow[\delta]{} H_{n-1}(X^{n-1}, X^{n-2})$$

$$[z] \longrightarrow [\partial z]$$

Thm: This homology is iso to singular homology

(X^n, X^{n-1})

$$H_{i+1}(X^n, X^{n-1}) \to H_i(X^{n-1}) \hookrightarrow H_i(X^n) \longrightarrow H_i(X^n, X^{n-1}) \longrightarrow$$

$= 0 \longleftarrow$ when $i \neq n, n-1 \longrightarrow = 0$; $\nearrow \cong$

- If $i > n$, $H_i(X^n) \cong H_i(X^{n-1}) \cong \cdots H_i(X^0) = 0$ ↖ set of pts
  No homology! (In singular homology, there could be many maps from $\Delta^k \to X$!)
- If $i < n$, $H_i X^n \cong H_i X^{n+1} \cong \cdots\cdots \cong^? H_i(X)$ not a priori, but for CW complexes it works
  (stops if fin dim)
  n^{th} hom only depends on $n+1$ skeleton!

Use direct limit $\varinjlim_n H_i X^n = L =^? H_i(X)$

Always have

$$\begin{tikzcd}
H_i X^n \arrow[r, hook] \arrow[drr] \arrow[ddrr, bend right] & H_i X^{n+1} \arrow[r, hook] \arrow[dr] \arrow[ddr] & H_i X^{n+2} \arrow[r, hook] \arrow[d] \arrow[dd, bend right] & \cdots \\
 & & L \arrow[d, "\exists!\Theta"] & \\
 & & H_i X &
\end{tikzcd}$$

Use universal property

$L \cong$ the stable group

Claim: Θ is an iso

$\twoheadrightarrow$: Take $[z] \in H_i X$, $z \in C^i X$ with $\partial z = 0$

$L \cong \bigoplus_n H_i X^n /$ id elt with image

CW complex: compact set $\cap$ only fin many interiors

Any singular i-chain is in a compact subset of X,

Union of only finitely many images → lives in a fin skeleton X^N

So $[z] \in H_i X^N$ and is just included in X

$\hookrightarrow$: Let $[y] \in L$, $[y] \mapsto 0^{\in H_i X}$, some i chain so y is a boundary ; zero in homology

Since y is a fin lin combo, eventually becomes zero farther down the chain

Standard/common trick!

Doesn't work for cohomology due to direct products (compactness fails) cohomology due to direct products (compactness fails)

So $[z] \in H_i X$ and is just induced in X

$\hookrightarrow$: Let $[y] \in L$, $[y] \mapsto 0 \in H_i X$, some i chain so y is a boundary (zero in homology)
Since y is a fin lin combo, eventually becomes zero farther down the chain
So $y = \partial v$ for some $v \in C_{i+1} X$, compactness $\longrightarrow v \in C_{i+1} X^M \longrightarrow \operatorname{im}([y]) = 0$ in L!

$$\boxed{\begin{array}{l} H_i X \cong H_i X^{i+1} \\ H_{n+k} X^n = 0 \quad \forall k \geq 1 \end{array}}$$

0:

$$\begin{tikzcd}
 & & & H_n(X^{n+1}, X^n) & & \\
H_n X^{n-1} \arrow[dr] & & \boxed{H_n X^{n+1}} \arrow[ur] & & & \\
 & H_n X^n \arrow[ur, "i_n"] \arrow[drr, "j_n"] & & & & \cdots \\
H_{n+1}(X^{n+1}, X^n) \arrow[ur, "\delta_{n+1}"] \arrow[rr, "\partial_{n+1}"] & & H_n(X^n, X^{n-1}) \arrow[rr, "\partial_n"] \arrow[dr, "\delta_n"] & & H_{n-1}(X^{n-1}, X^{n-2}) \arrow[ur] & \\
 & & & H_{n-1} X^{n-1} \arrow[ur, "j_{n-1}"] \arrow[dr] & & \\
 & & H_{n-1}(X^{n-2}) \arrow[ur] & & \cdots &
\end{tikzcd}$$

LES

Want

LES

LES

$$H_n X \cong H_n X^{n+1} \cong H_n X^n / \ker i_n \cong H_n X^n / \operatorname{im} \delta_{n+1} \cong \operatorname{im} j_n / \operatorname{im} \partial_n \cong \operatorname{Ker} \delta_n / \operatorname{im} \partial_{n+1} \cong \operatorname{Ker} \partial_n / \operatorname{im} \partial_{n+1} = H_n(C_*^{cell} X)$$

Examples

- $\mathbb{CP}^n = e^0 + e^2 + \ldots e^{2n}$

$$\frac{\mathbb{C}^{n+1} - \{\bar{0}\}}{\mathbb{C}^*} - \frac{(\mathbb{C}^n \times \{0\}) - \{\bar{0}\}}{\mathbb{C}^*} = \mathbb{C}^n = [(z_0\, z_1 \cdots z_n\, 1)]$$

$\longrightarrow \mathbb{CP}^n \supseteq \mathbb{CP}^{n-1} \supseteq \mathbb{CP}^{n-2} \supseteq \ldots$

$\longrightarrow C_*^{cell}\, \mathbb{CP}^n = (0 \to \mathbb{Z} \to 0 \to \mathbb{Z} \to \ldots)$

$\longrightarrow H_i\, \mathbb{CP}^n = \begin{cases} \mathbb{Z}, & i \text{ even}, i \leq n \\ 0, & \text{else} \end{cases}$

- S^n: e^0, e^n, $\mathbb{Z}$ in 0 & n

- $\mathbb{RP}^n$: $0 \to \underset{n}{\mathbb{Z}} \xrightarrow{d_n} \underset{n-1}{\mathbb{Z}} \xrightarrow{d_{n-1}} \ldots \xrightarrow{d_1} \underset{0}{\mathbb{Z}} \xrightarrow{d_0} 0$

$$\partial_i = \begin{cases} 0, & i \text{ odd} \\ \times 2, & i \text{ even} \end{cases}$$

Eg n even

$$\mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0$$

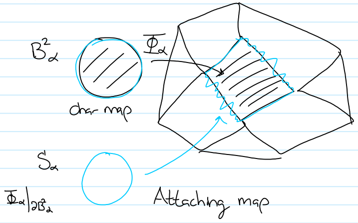
$$H_* X = \begin{cases} 0, & \text{even dims } 2 \ldots n \\ \mathbb{Z}, & \dim 0 \\ \mathbb{Z}_2, & \text{odd dims } 1 \ldots n-1 \end{cases}$$

n odd

$$\mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \cdots \mathbb{Z} \to \mathbb{Z} \to 0$$

$$H_* X = \begin{cases} \mathbb{Z} & \dim 0, n \\ \mathbb{Z}_2 & \text{in odd dims} \\ 0 & \text{else} \end{cases}$$

Using H^{cell} to compute H^{sing}



$$H^n\Big(\coprod_\alpha (B^n_\alpha, \partial B^n_\alpha)\Big) \xrightarrow{\Phi_n} \boxed{H^n(X^n, X^{n-1})}$$ want

$\cong \bigoplus H^n(B^n_\alpha, \partial B^n_\alpha)$ $\Big|$ $\cong$ collapse $\Big|$

$$\begin{tikzcd}
H^n(\coprod_\alpha (B^n_\alpha, \partial B^n_\alpha)) \arrow[r, "\Phi_n"] \arrow[d, "\cong"] \arrow[rd, phantom, "\text{collapse}"] & \boxed{H^n(X^n, X^{n-1})} \arrow[d] & \text{want} \\
H^n(\vee_\alpha (B^n_\alpha / \partial B^n_\alpha)) \arrow[r, "\Phi_n"] \arrow[d, phantom, "\shortparallel"] & H^n(X^n / X^{n-1}) & \\
\vee_\alpha S^n_\alpha & &
\end{tikzcd}$$

$\cong \bigoplus H^n(B^n_\alpha, \partial B^n_\alpha)$

$$\begin{tikzcd}
H_n(B^n, \partial B^n) \arrow[r, phantom, "\overset{\delta}{\cong}"] \arrow[d, phantom, "\shortparallel"] & H_{n-1}(\partial B^n) \arrow[d, phantom, "\shortparallel"] \\
H_n(B^n / \partial B^n) \arrow[d, phantom, "\shortparallel"] & H_{n-1}(S^{n-1}) \arrow[d, bend right] \\
H_n(S^n) \arrow[r, bend right] & \cong \mathbb{Z}
\end{tikzcd}$$

Orientation is a choice of a generator

Allows writing $C_*^{cell}(X) = \bigoplus_{\alpha \in I^n} \mathbb{Z} \longrightarrow d^n_{\alpha,\beta} = \begin{bmatrix} \mathbb{Z}\text{-matrix} \\ \wr \end{bmatrix} \bigoplus_{\alpha \in I^n} \mathbb{Z} \longrightarrow \bigoplus_{\beta \in I^{n-1}} \mathbb{Z}$

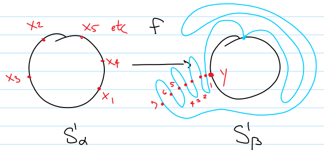
$$\begin{tikzcd}
\bigoplus H_n(B, \partial B) \arrow[r] & H_n(X^n, X^{n-1}) \arrow[r, "\delta_1"] \arrow[rd, phantom, "\circlearrowright"] & H_{n-1}(X^n, X^{n-1}) \arrow[r] & H_{n-1}(X^n / X^{n-1}) \arrow[r] & H_{n-1}(\vee S^{n-1}_\alpha) \arrow[r] & \bigoplus \mathbb{Z} \\
 & H_n(B, \partial B) \arrow[lu, dashed, "\text{include one}"] \arrow[u, "\Phi_n^*"'] \arrow[r, "\delta_2 \text{ from LES}"', "\text{of } (B^n, \partial B^n, \emptyset)"] & H_{n-1}(B^n, \partial B^n) \arrow[u, "\Phi^*_{n-1}"'] \arrow[rr, "\text{collapse all non-}\beta\ n\text{-1 cells}"'] & & H_{n-1}\left(\frac{\vee S_\beta}{\vee_{\alpha \neq \beta} S_\alpha} \right) \arrow[d, phantom, "\shortparallel"] & \\
 & & & & S^n_\beta &
\end{tikzcd}$$

So $d^n_{\alpha,\beta}$ = degree of map $S^{n-1}_\alpha \to S^{n-1}_\beta$ (multiplication by some integer)

choice in top homology

But how to calculate degree?

Look @ circles



1) Choose orientation

2) Count preimages

- Look at local orientations on source, push to target
  +1 if match, else -1

In C^∞ setting: Sard's thm.

Geometric Calculation of degree

Sp. $\exists f : S^n_\alpha \to S^n_\beta$

where $\exists y \in S^n_\beta \mid f^{-1}(y) = \{x_i\}_{i=1}^m$, $x_i \in \underset{\text{open}}{U_i \subset} S^n_\alpha$

with $y \in V_i \in S^n_\alpha$, $f(V_i) \cong U_i$

$$\underset{\in \mathbb{Z}}{\deg f} = \sum_{i=1}^m \underset{\in \{\pm 1\}}{\deg_{x_i} f}$$

Local degree: gen of $H_n(V, V - \{y\})$

$$\begin{tikzcd}
H_n(V, V - \{y\}) \arrow[d, phantom, "\cong \text{ exc}"] \arrow[dddd, bend right] & & \\
H_n(S^n, S^n - \{y\}) \arrow[r, phantom, "\cong"] & H_n(S^n, \mathbb{R}^n) \arrow[d, phantom, "\cong"] \\
 & H_n(S^n) \\
 & \\
H_n(S^n_\alpha) &
\end{tikzcd}$$

induces choic of gen in $H_n(S^n_\alpha, S_\alpha - \{y\})$

$\cong H_n(U_i, U_i - \{x_i\})$

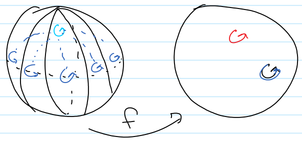
Same for U_i.

induces choic of gen in $H_n(S^n_\alpha, S_\alpha - \{y\})$
$\qquad\qquad\qquad\qquad\qquad \overset{\cong}{H_n}(U_i, U_i - \{x_i\})$

Same for U_i,

$$H_n(U_i, U_i - x_i) \xrightarrow[\cong]{(F|_{U_i})_*} H_n(V_i, V_i - y_i)$$

$$g_{\alpha,i} \to \pm 1 \cdot g_{\beta,i}$$



$\sum \begin{pmatrix} z \to z^p \\ S^1 \to S^1 \end{pmatrix}$

Wrap p segment
$\to \deg f = z^p$

Thm $C_n^{\text{cell}} \xrightarrow{d} C_{n-1}^{\text{cell}}$

$$\bigoplus_{\alpha \in I^n} \mathbb{Z} \xrightarrow[\boxed{d_{\alpha\beta}}]{} \bigoplus_{\beta \in I^{n-1}} \mathbb{Z}$$

$\hookrightarrow$ = degree of map $S^{n-1} \to S^{n-1}$
$S^{n-1}_\alpha = \partial B^n_\alpha \longrightarrow X^{n-1} \longrightarrow \dfrac{X^{n-1}}{X^{n-2}} = S^{n-1}_\beta$

$\deg F = \sum \underbrace{\deg_{x_i}(F)}_{=\pm 1}$] local degrees

Pf

Look at

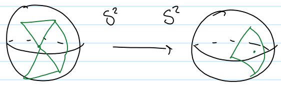
$$\begin{tikzcd}
H_n(S^n_\alpha) \arrow[r, "f_*"] \arrow[d, "\text{from LES}"'] \arrow[dr, phantom, "g_\alpha \mapsto ?"] & H_n(S^n_\beta) \arrow[d, "\text{LES of rel pair}"] \\
H_n(S^n_\alpha, S^n_\alpha - \bigcup_i \{x_i\}) \arrow[r, "f_*"] \arrow[d, "\text{Exc}"] & H_n(S^n_\beta, S^n_\beta - \{y\}) \\
H_n(\coprod (U_i, U_i - x_i)) \arrow[r, "(f|_{\sqcup U_i})_*"] \arrow[d, phantom, "\shortparallel"] & H_n(V_i, V_i - y\} \\
\bigoplus H_n(U_i, U_i - x_i) \arrow[ur, "\bigoplus (f|_{U_i})_*"'] &
\end{tikzcd}$$

$\sim \cong$ since LES $\to H_n(\mathbb{R}^n) = 0$

Claim: $g_\alpha \longmapsto \bigoplus g_{\alpha,i} = \sum (\deg_{x_i} f)\, g_{\alpha,i}$

Idea:

$$\begin{tikzcd}
\text{Global} \arrow[r, "f"] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & \text{Global} \arrow[d] \\
\text{Local} \arrow[r, "f_*"'] & \text{Local}
\end{tikzcd}$$

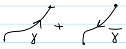


Delete point, only
need to keep simplex
w/ point in it

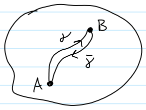
g_α = oriented triangle decomp

(Remember: $\uparrow = - \downarrow$ works in simplicial hom,
not in singular)

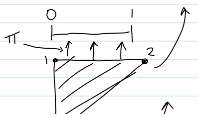
but "up to boundaries" works



$= \partial(2 \text{ dim})$



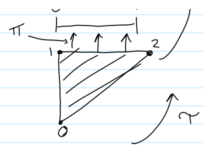
Find a 2-chain z
with $\gamma + \bar{\gamma} = \partial z$



$\sigma := \gamma \circ \pi$

$\partial\sigma = \sum \sim = +\gamma - \gamma + \boxed{\varepsilon}$ — degenerate 1-simplex mapping to A

so $\varepsilon = \partial\sigma$, not quite what we want

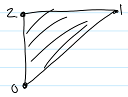


degenerate 1-simplex mapping to A

$\partial\sigma = \sum \sim = +\gamma - \gamma + \boxed{\varepsilon}$

so $\varepsilon = \partial\sigma$, not quite what we want

Fix:



Want -thing to degenerate

$$\rightarrow \partial\tau = \gamma - \varepsilon + \bar{\gamma} \rightarrow \underline{\gamma + \bar{\gamma}} = \partial\tau + \overset{\overset{\partial\sigma)}{\shortparallel}}{\varepsilon}$$

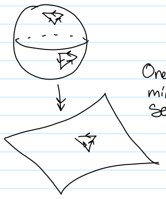
$$\boxed{= \partial(\tau + \sigma)}$$

So modulo bd, $\gamma \sim -\bar{\gamma}$

Triangulation on $S^n \rightarrow$ triangulation on S^{n-1}

$\rightarrow H_n S^n \cong H_{n-1} S^{n-1}$

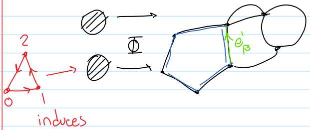
Never really need a model in mind, but convenient to sometimes (actually take bds)



One projects with minus signs so some cancellation

Ex

2-complex

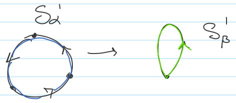


induces orientation

$0 \rightarrow C_2^{\text{cell}} \rightarrow C_1^{\text{cell}} \rightarrow C_0^{\text{cell}} \rightarrow 0$

$\partial(\rightarrow) = +\text{end} - \text{start}$

Collapse attaching map



compute winding number

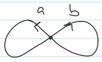
$\rightarrow H_1(X) = Ab(\pi_1(X))$

No longer care about order of edge traversal

Works for 2-complex ⇒ works for all spaces

H_n only sees $k \leq n$ chains

Eg Van Kampen might have



attach via $\underbrace{ababa^{-1}b^{-1}a^{-1}b^{-1}}$

Free group w/this relation

Homology only count exponents!

$0 \rightarrow \mathbb{Z}_e \rightarrow \mathbb{Z}_a \oplus \mathbb{Z}_b \rightarrow$

$e \mapsto 0$

Analogously, can consider C_*^{cell} for a simplicial complex

$\Rightarrow C_*^{\text{cell}} =$ n map of affine space

$\cong$

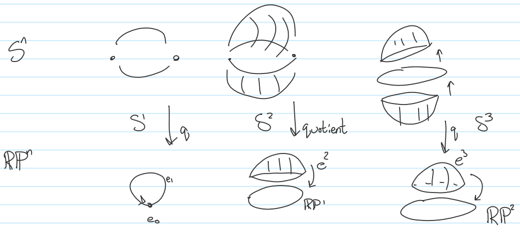
C_*^{singular}

$H^*_{\text{simplicial}} \cong H^*_{\text{singular}}$

Ex $H_*(\mathbb{RP}^n)$: $C_*^{\text{cell}}(\mathbb{RP}^n)$

$0 \rightarrow \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\times 2} \cdots \rightarrow \mathbb{Z} \rightarrow \mathbb{Z} \rightarrow 0$

x2/0 depends on parity



Want deg of attaching map

$$\begin{tikzcd} S^2_\alpha \arrow[r] \arrow[rrrrr, bend right, "\text{want deg of this map}"'] & \partial B^3 \arrow[r] & S^2 \arrow[r] & \mathbb{RP}^2 \arrow[r] & \mathbb{RP}^2/\mathbb{RP}^1 \arrow[r] & S^2_B \end{tikzcd}$$

Compare local degrees. Choose $y \in S^2_\beta$, $x_1, x_2 \in f^{-1}(\{y\}) \subset S^2_\alpha$

$$g \longmapsto g_1 \longmapsto g_1 \longmapsto h$$

$$\begin{tikzcd} H_2(S^2_\alpha) \arrow[r] \arrow[d, "\text{antipodal}\ \alpha^*"] & H_2(S^2_\alpha, S^2_\alpha - \{x_1\}) \arrow[r, "exc"] \arrow[d, "\alpha^*"] \arrow[ld, phantom, "\circlearrowleft"] & H_2(U_1, U_1 - \{x_1\}) \arrow[r, "\phi"] \arrow[d, "\alpha^*"] \arrow[ld, phantom, "\circlearrowleft"] & H_2(V_1, V_1 - \{y\}) \\ H_2(S^2_\alpha) \arrow[r] & H_2(S^2_\alpha, S^2_\alpha - \{x_2\}) \arrow[r] & H_2(U_2, U_2 - \{x_2\}) \arrow[ru, "\phi"] & \end{tikzcd}$$

$g \mapsto (-1)^\alpha g$, $\deg \alpha$

$(-1)^\alpha g_2$ $(-1)^\alpha g_2$

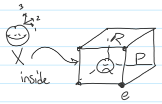
Because antipodal map commutes with q

$\rightarrow \phi((-1)^\alpha g_0) \overset{?}{=} (-1)^\alpha \deg_{x_2}(\phi)$

$\rightarrow \deg_{x_1}(\phi) = (-1)^\alpha \deg_{x_2}(\phi)$ for attaching odd-dim cell

gives x0

$X = S^1 \times S^1 \times S^1 \cong (e_0 \times e_1) \times (f_0 \times f_1) \times (g_0 \times g_1)$



Faces glued "torically"

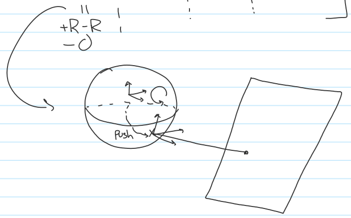
1 vertex

$$0 \to \mathbb{Z} \to \mathbb{Z}^3 \to \mathbb{Z}^3 \to \mathbb{Z} \to 0$$

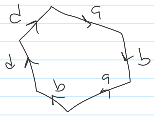
$\partial X = \begin{matrix} \pm P \pm P \\ \pm Q \pm Q \\ \pm R \pm R \end{matrix}$ $\begin{matrix} P \\ Q \to 0 \\ R \end{matrix}$ $\begin{matrix} a \quad 0 \\ b \mapsto 0 \\ c \quad 0 \end{matrix}$

$\pm R \pm R = +R - R = 0$

Zero maps → homology is just this chain



For example, pair polygon sides



Nonzero homology iff any 2 with same dir iff nonorientable

Homology w/ rational coefs → Chain complex of $\mathbb{Q}$-vector spaces

$V_n \to V_{n-1} \to \dots V_0 \to 0$, $H_i = \frac{Z_i}{B_i}$

All exact sequences split! (+ no torsion, just use dimension)

Given chain complex, $\sum_i (-1)^i \dim V_i = \boxed{\sum_i \dim H_i(V_*)}$ ← homotopy invariant

$:= \chi(V_*)$, euler char $\in \mathbb{Z}$

easy to compute!

Proof: Very easy!

$0 \to B_i \to Z_i \to H_i \to 0$ is exact

$0 \to Z_i \to V_i \to B_{i-1} \to 0$

$$\rightarrow \begin{cases} Z_i = B_i \oplus H_i \\ V_i = Z_i \oplus B_{i-1} \end{cases}$$

$$\rightarrow \begin{cases} \dim Z_i = \dim B_i + \dim H_i \\ \dim V_i = \dim B_{i-1} + \dim Z_i . \end{cases}$$

Homotopy invariant, since it involves H_i

But easy to compute, just count cells!

$\dim\left[H_i(X, \mathbb{Q})\right] = b_i$, betti number

$\rightarrow$ Euler characteristic $= \sum_i (-1)^i \dim H_i(X, \mathbb{Q}) = \chi(X) \in \mathbb{Z}$

Makes sense when finite dim $\forall i$, and only finitely many terms

CW complex (Finite): $\sum (-1)^i b_i = \sum (-1)^i \cdot |\{e_i\}|$

$\hookrightarrow$ number of i-cells

$\rightarrow$ Homotopy invariant!

(defined using homology)

$\chi(X \cup_Z Y) = \chi(U) + \chi(Y) - \chi(Z)$

Proof: Count cells or

use Mayer-Vietoris

Statements about $\mathbb{N}$ or $\mathbb{Z}$ might lift to

ones about chain maps and/or alt. sums of V-dims!

- $\chi(X \times Y) = \chi(X) \cdot \chi(Y)$
- Multiplicative for covers

$$\begin{tikzcd} E \arrow[d, "d\text{-fold}"] \\ B \end{tikzcd} \qquad \rightarrow \chi(E) = d \cdot \chi(B)$$

(count cells)

Ex $\mathbb{CP}_2$: $\chi = 3$ $\quad \rightarrow \nexists$ a free action of $\mathbb{Z}_2$ on X when $\chi(X)$ is odd!

$e_0 \cup e_2 \cup e_4$

Ex Gauss-Bonet

$\Sigma_g \in \mathbb{E}^3$

$\int K \, d(\text{Area}) = 2\pi \cdot \chi(X)$

$\uparrow$ Scalar curvature

Similar: Aatiyah-Singer index thm] Equivalence of "Euler char

Grothendieck Riemann Roch] style" calculations

$\boxed{\text{Lefshetz Fixed point thm}}$

Finite CW complex, look at $f: K \to K$ (geometric realization)

Define $L(f) = \sum (-1)^i \operatorname{Tr}(L)$

~ induces $f^*: H_*(X, \mathbb{Q}) \circlearrowleft$

$\operatorname{Tr}(f^*) \in \mathbb{Q}$, basis invariant

Then $L(f) \neq 0 \Rightarrow f$ has a fixed point

Proof : $0 \to B_i \to Z_i \to H_i \to 0$

$0 \to Z_i \to C_i \to B_{i-1} \to 0$

$\mathbb{Q}$-vector spaces, so these split

f induces self maps $C_i \circlearrowleft f_*$

Use simplicial appx/subdivide to homotop f to a simplicial map

f^* commutes with ∂, so
cycles $\circlearrowleft$
bds $\circlearrowleft$

Action of $\mathbb{N}$ on bottom seq $(f_*, f_*^2, \cdots)$

Can restrict to direct sum pieces

$C_i = Z_i \oplus B_{i-1}$

$f_*|_{C_i} = f_*|_{Z_i} + f_*|_{B_{i-1}}$ } Representation-theoretic statement (characters, eg χ is char(id))

If f has no fixed point, there is a minimum translation distance

$\exists \varepsilon$ s.t $d(x, f(x)) > \varepsilon$

So choose simplicial appx $< \varepsilon$ so f fixes no simplex

Then in matrix rep of map has zeros on diagonal $\longrightarrow$ $\mathrm{Tr}\, f_* = 0$

Can use this kind of idea to find fixed points of interesting ($\neq$ id) maps

Look at Torus

$H_0 \ \mathbb{Z}$
$H_1 \ \mathbb{Z}^2$
$H_2 \ \mathbb{Z}$

$L(f) = 1 - tr([\,]) + d$

Look at homology w/coefficients
($\longrightarrow$ Generic chain complexes in Hom Al_g)

Given any abelian group A, can define

$$C^{Sing}(X:A) = \{\sum_{fin} a_i \sigma_i^n \mid a_i \in A\}$$, finite A-linear combinations

Define ∂ using A-linear extensions

$$\partial(\sum a_i \sigma_i^n) = ? \quad = \sum a_i \sigma_i^{n-1}$$
Take $\partial(\sigma_i^n) = \sum \sigma_i^{n-1}$

Can define $H_*(X:A)$ right mod left mod
Can look at $(C_*)_{\mathbb{Z}}$, ${}_{\mathbb{Z}}(A_{\mathbb{Z}})$, $\mathbb{Z}$-modules
$\rightarrow {}_{\mathbb{Z}}C_* \otimes_{\mathbb{Z}} {}_{\mathbb{Z}}A$

Can convert right R-module to right $\mathbb{Z}$-modules
(Review: Bimodule - M with an R action + S action)

Taking homology doesn't commute with $\otimes$!

$$H_*(S^n : A) = (\overset{0}{A}, 0, \ldots, \overset{n}{A}, 0 \longrightarrow)$$
$$H_*(\{*\} : A) = (A, 0 \longrightarrow)$$
$$\tilde{H}_*(X:A) \Rightarrow C_1(X:A) \rightarrow C_0(X:A) \rightarrow \cancel{0}$$
$$\xrightarrow{\varepsilon} A \rightarrow 0$$
$$\varepsilon: \sum a_i \sigma_i \mapsto \sum a_i$$

Recall degree, $F: \mathbb{Z} \rightarrow \mathbb{Z}$, $1 \rightarrow d \cdot 1$, but Aut(A) can be way more

Lemma: $S^n \circlearrowleft f \rightsquigarrow H_n(S^n : A) \circlearrowleft f_* = A \circlearrowleft f_*$

$\rightarrow \deg f \in \mathbb{Z}$, <u>same as $A = \mathbb{Z}$ case.</u>

<u>PF</u>: Define $\alpha_a : \mathbb{Z} \rightarrow A$, $1 \rightarrow a$

$$\left[\text{General: } \underset{\text{abelian}}{B} \xrightarrow{\varphi} C \rightsquigarrow C_*(X:A) \xrightarrow{\varphi^*} C_*(X:B) \qquad \varphi^*\Sigma = \Sigma\varphi^* \right]$$

yields $C_*(X,\mathbb{Z}) \xrightarrow{\alpha_a} C_*(X,A)$

$$\begin{tikzcd}
\arrow[r] & H_*(S^n:\mathbb{Z}) \arrow[r, "\alpha_a"] \arrow[d, "F^*"'] \arrow[dr, phantom, "\circlearrowright"] & H_*(S^n:A) \arrow[d, "F^*"] \\
& H_*(S^n:\mathbb{Z}) \arrow[r, "\alpha_a"] & H_*(S^n:A)
\end{tikzcd}$$

$$\begin{tikzcd}
x \; (=1\in\mathbb{Z}) \arrow[r] \arrow[d] & a \arrow[d] \\
d\cdot x \arrow[r, bend right] & f_*(a)
\end{tikzcd}$$

$d\alpha_a(x) = d\cdot a = f_*(a)$ ▨

"Algebra thing" independent from "Geometric thing"!

If $C_*^{cell}(X:\mathbb{Z})$ is computed using degrees $d_{\alpha\beta}$ <u>then</u>

$$C_*^{cell}(X:A) = C_*^{cell}(X:\mathbb{Z}) \otimes_{\mathbb{Z}} A$$

(The naive hope works)

<u>Ex</u>

$$C_*^{cell}(\mathbb{RP}^n:\mathbb{Z}) = 0 \to \mathbb{Z} \xrightarrow[\text{even } 2]{\text{odd } 0} \mathbb{Z} \xrightarrow[0]{2} \cdots \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0$$

$$C_*^{cell}(\mathbb{RP}^n:\mathbb{Z}_2) \quad 0 \to \mathbb{Z}_2 \xrightarrow{0} \mathbb{Z}_2 \xrightarrow{0} \quad \xrightarrow{0} \mathbb{Z}_2 \xrightarrow{0} \mathbb{Z}_2 \xrightarrow{0} 0$$

$$\to H_*(\mathbb{RP}^n,\mathbb{Z}_2) = (\underset{0}{\mathbb{Z}_2}, \cdots, \underset{n}{\mathbb{Z}_2}, 0 \longrightarrow)$$

$\xrightarrow{} (-)\otimes_{\mathbb{Z}} A$

homology, $\otimes$: $0 \cdots \mathbb{Z}_1$

$\otimes$, homology: $\mathbb{Z}_2 \text{ -- } \mathbb{Z}_2$

$\neq$!!

Niceness: going from $\mathbb{Z}$-mod $\to$ $\mathbb{Z}$ mod by $\otimes_{\mathbb{Z}}$ with Abelian group (so bimodule)

To what extent does the commuting fail? General setting, rings & R mod $\to$ S mod

$$\begin{tikzcd}
& C_*(X:\overset{\text{free}}{\mathbb{Z}}) \arrow[r, "\substack{\text{take coefs}\\ \text{mod } 2}"'] & C_*(X,\mathbb{Z}_2) = C_*(X;\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}_2 \arrow[d, bend left] \\
C_*(X:\mathbb{Z}) \arrow[ur, "\boxed{\times 2}"] & & 0
\end{tikzcd}$$

Works nicely b/c no torsion,
singular chain gps $C_*(X,\mathbb{Z})$ are
free abelian
General: Kernel might not be $im(\times 2)$

$$\to 0 \to \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \xrightarrow{\text{mod } 2} \mathbb{Z}_2 \to 0$$

map on H_* induced by
mod 2 chain map

$\to$ LES (Bockstein)

$$\to H_n(X;\mathbb{Z}) \xrightarrow{\times 2} H_n(X;\mathbb{Z}) \xrightarrow{\text{"mod 2"}} H_n(X;\mathbb{Z}_2) \xrightarrow{\beta} H_{n-1}(X:\mathbb{Z}) \to H_{n-1}(X;\mathbb{Z}) \to H_{n-1}(X,\mathbb{Z}_2)$$

Can always break this up into SESs by looking @ kernels/cokernels.

<u>1-29-18</u>

Bockstein Sequence (Begins hom. alg)

$C_*(X,\mathbb{Z})$ ~ simp/sing/cell complex

$$X_1 \xrightarrow{\varphi} X_2 \Rightarrow C_*(X_1,A) \xrightarrow{\varphi} C_*(X_2,A)$$

Take A to be a field (linear alg instead of modules)

$\mathbb{Z}_p$ yields

$$0 \to \mathbb{Z} \xrightarrow{\times p} \mathbb{Z} \xrightarrow{\text{mod } p} \mathbb{Z}/\mathbb{Z}_p \to 0$$

Since A is free, yields (when $A=\mathbb{Z}$)

$$0 \to C_*(X,\mathbb{Z}) \xrightarrow{\times p} C_*(X,\mathbb{Z}) \xrightarrow{\text{mod } p} C_*(X,\mathbb{Z}_p) \to 0$$

yields LES

$$H_n(X,\mathbb{Z}) \xrightarrow{\times p} H_n(X,\mathbb{Z}) \xrightarrow{j} \underbrace{H_n(X,\mathbb{Z}_p)} \xrightarrow{\beta_p} H_{n-1}(X,\mathbb{Z}) \xrightarrow{j}$$

first iso

Note: $im\, j = \dfrac{H_n(X,\mathbb{Z})}{\ker j} = \dfrac{H_n(X,\mathbb{Z})}{im(\cdot p)} = coker(\cdot p)$

yields LES

$$H_n(X,\mathbb{Z}) \xrightarrow{\times p} H_n(X,\mathbb{Z}) \xrightarrow{j} \underbrace{H_n(X,\mathbb{Z}_p)}_{\downarrow} \xrightarrow{\beta_p} H_{n-1}(X,\mathbb{Z}) \xrightarrow{j}$$

Note: $\operatorname{im} j = \frac{H_n(X,\mathbb{Z})}{\ker j} \overset{\text{first iso}}{=} \frac{H_n(X,\mathbb{Z})}{\operatorname{im}(\cdot p)} = \operatorname{coker}(\cdot p)$

break into SES

$$0 \to \frac{H_n(X,\mathbb{Z})}{\operatorname{im}(\cdot p : H_n(X,\mathbb{Z})^{\circlearrowleft})} \longrightarrow H_n(X,\mathbb{Z}_p) \to \underbrace{\ker\{\cdot p : H_{n-1}(X,\mathbb{Z})^{\circlearrowleft}\}}_{\text{subgroup of p-torsion}} \to 0$$

$$\|$$

$$\begin{tikzcd} 0 \arrow[r] & H_n(X,\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}_p \arrow[r, "i"'] & H_n(X,\mathbb{Z}_p) \arrow[l, dashed, bend right, "\text{r, a retraction}"'] \arrow[r] & \operatorname{Tor}_1^{\mathbb{Z}}(\mathbb{Z}_p, H_{n-1}(X,\mathbb{Z})) \arrow[l, dashed, bend right, "\text{s, a section}"'] \arrow[r] & 0 \end{tikzcd}$$

$\circlearrowleft_{=id}$ $\circlearrowleft_{id}$

$A \otimes \mathbb{Z}_p \xrightarrow{\sim} A/pA$

$a \otimes \lambda \longrightarrow [\lambda a]$

$\mathbb{Z}$ has dim 1, so higher Ext/Tor vanish (like S^1, or H_0 is known a priori)

Naive hope: homology commutes w/ $\otimes$

Theorem: this splits, $H_n(X,\mathbb{Z}_p) = \overbrace{\operatorname{Tor}(\mathbb{Z}_p, \underline{\underline{H_{n-1}}}(X,\mathbb{Z}))}^{\text{p-torsion "correction"}} \oplus [H_n(X,\mathbb{Z}) \otimes \mathbb{Z}_p]$

So integer homology determines mod p homology - not interesting?

But allows working one prime at a time → assemble
(F_p vector spaces, seq. split)

Constructing a retraction $r : H_n(X,\mathbb{Z}_p) \to \frac{H_n(X,\mathbb{Z})}{p \cdot H_n(X,\mathbb{Z})}$ ← i, from SES

Find a chain map $r^* : C_*(X,\mathbb{Z}_p) \to C_*(X,\mathbb{Z})$ with im r^* well defined mod p

$$\underset{\in \mathbb{Z}_p}{\underset{\downarrow}{\sum \lambda_i \sigma_i}} \longmapsto \sum \hat{\lambda}_i \sigma_i \quad (\text{just lift residue to } \mathbb{Z})$$

Problem: Not quite a chain map - taking ∂ doesn't work with random choices

Better method: try mapping cycles directly $(z \in H_n(X,\mathbb{Z}_p))$

$z = \sum \lambda_i \sigma_i$ lift to $\hat{z} = \sum \hat{\lambda}_i \sigma_i \in C_n(X,\mathbb{Z})$

Consider

$$\begin{tikzcd} 0 \arrow[r] & Z_n(X,\mathbb{Z}) \arrow[r] & C_n(X,\mathbb{Z}) \arrow[l, dashed, bend right, "\exists \pi"'] \arrow[r, "\partial"] & B_{n-1}(X,\mathbb{Z}) \arrow[r] & 0 \end{tikzcd}$$

Splits. $\underbrace{B_{n-1}(X,\mathbb{Z})}_{\text{Free}} \leq C_n(X,\mathbb{Z})$

Choose a splitting, project $\hat{z}$ using π.

$C_n \cong Z_n \oplus B_{n-1}$

Then look at homology class

Check: $r \circ i = id \big|_{H_n(X,\mathbb{Z})/p \cdot H_n \cdots}$

① Start with $\omega = \sum \lambda_i \sigma_i \in Z_n(X,\mathbb{Z})$

$(r \circ i)(\omega) = id(\omega)$

② Well defined

Different choice of $\hat{\lambda}_i$ differ by cycles in $p \cdot Z(X,\mathbb{Z})$

which project to $p \cdot Z_n(X,\mathbb{Z}) \longrightarrow$ ends up in $p \cdot H_n(X,\mathbb{Z})$

Had to choose splittings, so not quite functorial

Choice of splitting / Choice of iso

$$H_n(X,\mathbb{Z}_p) \xrightarrow{\sim} \underbrace{(\sim) \oplus (\sim)}$$

no systematic choices

Changing $X \to Y$ doesn't induce maps on RHS

Cohomology

X a space, make a chain complex

$$\begin{tikzcd} \cdots \arrow[r] & C_{n+1} \arrow[r, "\partial_{n+1}"] & C_n \arrow[r, "\partial_n"] & C_{n-1} \arrow[r, "\partial_{n-1}"] & \cdots \end{tikzcd}$$

] Finite lin combos of {simplices} with $\mathbb{Z}$-coefs

$\downarrow$ Dualize/take adjoint

$\rightsquigarrow \mathrm{Hom}(\cdot, \mathbb{Z})$

$$\begin{tikzcd} \cdots & C^{n+1} \arrow[l] & C^n \arrow[l, "\delta^{n+1}"'] & C^{n-1} \arrow[l, "\delta^n"'] & \cdots \arrow[l, "\delta^{n-1}"'] \end{tikzcd}$$

} No finiteness condition
integer-valued Function on {simplices}

Assign an integer to every $\Delta^n \xrightarrow{\sigma} X$ (Large!!)

$\delta^n: C^{n-1} \to C^n$

$(\xi: C^{n-1} \to \mathbb{Z}) \longmapsto \xi \circ \partial_n$

Yields pairing $\langle \cdot , \cdot \rangle . \ \mathrm{Hom}(C_n, \mathbb{Z}) \times C_n \to \mathbb{Z}$

$(\xi, x) \longmapsto \langle \xi, x \rangle := \xi(x)$

Koszul rule of signs

$$(\overbrace{\underset{C^{n+1}}{\delta}}^{\deg +1} \circ \overbrace{\underset{C_{n+1}}{\xi}}^{\deg n})(x) := (-1)^{\overbrace{n}^{n = \deg \delta \cdot \deg \xi}} \xi(\partial x)$$

for $\delta \in C^n$

Note: No dif b/w chains & cochain

Just define "C_{-n}" $:= C^n$

$\delta: C_n \to C_{n+1}$ increasing, deg +1

$\downarrow$

$\delta: C_{-n} \to C_{-n-1}$ dec, deg -1

Given $f: X \to Y$

$\downarrow$

$f_*: C^*X \to C^*Y \Rightarrow f_*: H_*X \to H_*Y$

$\xi \circ f \leftarrow\!\shortmid \xi$

→ Contravariance

$$\begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr, phantom, "\rightsquigarrow"] & H^*X \\ Y \arrow[r] & H^*Y \arrow[u, "f^*"'] \end{tikzcd}$$

$$\begin{tikzcd} \mathbb{Z}\text{-mod} = Ab \arrow[r, "D"] & Ab \end{tikzcd}$$

$A \longmapsto \mathrm{Hom}_{\mathbb{Z}}(A, \mathbb{Z})$

But $D^2 \neq I$!

$\mathbb{Z}_2 \to \mathrm{Hom}(\mathbb{Z}_2, \mathbb{Z}) = 0$!

$$\overbrace{\underset{\text{countable}}{\bigoplus} \mathbb{Z}}^{\text{Countable}} \xrightarrow{D} \overbrace{\underset{\text{countable}}{\prod} \mathbb{Z}}^{\text{Uncountable}} \xrightarrow{D} \overbrace{\oplus \mathbb{Z}}^{\text{wat}}$$

n^{th} chain group of some CW complex

Usual properties

- Hty invariance
- Mayer-Vietoris
- Relative versions, ie restriction

$A \overset{i}{\hookrightarrow} X$

by precomposition

$C^*(X) \xrightarrow{i_*} C^*(A)$

$C^*(X, A) := \mathrm{Ker}\ i_*$

SES

$\downarrow$

$$\begin{tikzcd} 0 \arrow[r] & C^*(X,A) \arrow[r] & C^*(X) \arrow[r] & C^*(A) \arrow[loop right, "\delta"] \end{tikzcd}$$

SES $0 \to C^*(X,A) \to C^*(X) \to C^*(A) \curvearrowright \delta$

$\downarrow$

LES $\cdots \to H^*(X,A) \to H^*(A) \to H^*(X) \xrightarrow{\delta} H^{*+1}(X,A) \to \cdots$

- Excision

$H^* S^n = (\underset{0}{\mathbb{Z}}, 0, 0, \dots, \underset{n}{\mathbb{Z}}, 0, \dots)$

- Reduced

$$\begin{tikzcd} C^{-1} \arrow[d, phantom, "\parallel"] & C^0 \arrow[l, "\varepsilon"'] \arrow[r] & C^1 \\ \mathbb{Z} & & \end{tikzcd}$$

Natural choice $f: C^0 \to \mathbb{Z}$

$\sigma_0 \to 1$

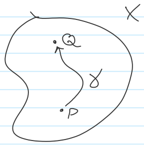
$\tilde{H}^*(\{*\}) = 0$ {gets rid of $\mathbb{Z}$ in deg 0}

Ex

$0 \to C^0 \to C^1 \to \cdots$

ξ fns on pts

η fns on paths $\gamma: \Delta^1 \to \mathbb{O}$



$\delta(\xi) = 0$ means

$\to \xi(\partial\gamma) = 0 \quad \forall \gamma$

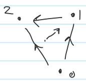
$\to \xi(Q-P) = 0 \quad \forall (Q,P)$ connected by a path

$\to \xi$ const on pts in a path component

$H_0 X$: spanned by path cpts

$H^0 X$: spanned by $\mathbb{Z}$-valued fns on paths cpts

$\delta(\xi)(\sigma: \Delta^2 \to X) = \xi$(alt sum of faces of σ)



01 - 02+12

$\to \xi(01) - \xi(12) = \xi(02)$

fns that vanish on bds of triangles

Then $Z^1 = \{$1-cocycles$\} = \{$additive fns on paths$\}$

≈ winding number

Rmk: $H_1(X) = \text{Hom}(\pi_1(X), \mathbb{Z})$, "winding # fns"

Ex

$H^*(\mathbb{RP}^n, \mathbb{Z})$

$C_*^{\text{cell}}(\mathbb{RP}^n) \quad 0 \to \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \cdots \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0$

$\to H_* = \begin{cases} \mathbb{Z}^2, & \text{odd dims} \\ \mathbb{Z}, & 0, n \\ 0, & \text{else} \end{cases}$

$[2]^* = [2]^T$

$C^*_{\text{cell}}(\mathbb{RP}^n) \quad 0 \leftarrow \mathbb{Z} \xleftarrow{0} \mathbb{Z} \xleftarrow{2} \cdots \xleftarrow{2} \mathbb{Z} \xleftarrow{0} \mathbb{Z} \leftarrow 0$

$\to H^* = \begin{cases} \mathbb{Z}^2 & \text{even dim} \\ \mathbb{Z}, & 0, n \\ 0, & \text{else} \end{cases}$ swaps even/odd

$H^i(X) \neq \text{Hom}(H_i(X), \mathbb{Z})$ in general!

{Notice that this doesn't work in the even case above

$\text{Hom}(0, \mathbb{Z}) \neq \mathbb{Z}_2$}

We found that $\cdot \otimes \mathbb{Z}$ doesn't commute with $\text{Hom}(\cdot, \mathbb{Z})$, needed error terms – similar result here?

C_* = singular chain complex over $\mathbb{Z}$

$C^n = \mathrm{Hom}(C_n, \mathbb{Z})$

$$\delta : C^n \to C^{n+1}$$
$$\xi \mapsto (-1)^n (\xi \circ \partial)$$

Note $H^n(X) \ncong \mathrm{Hom}(H_n(X), \mathbb{Z})$

Always <u>some</u> map

$$H^n(X) \longrightarrow \mathrm{Hom}(H_n(X), \mathbb{Z})$$

$$\langle \cdot , \cdot \rangle : H^n(X) \times H_n(X) \to \mathbb{Z}$$
$$([\eta], [z]) \mapsto \eta(x)$$

$C^n \times C_n \to \mathbb{Z}$ induces $\frac{Z^n}{B^n} \times \frac{Z_n}{B_n} \longrightarrow \mathbb{Z}$

$(\xi, x) \mapsto \xi(x)$

$$(\eta + \delta\phi, z + \partial y)$$
$$\mapsto \eta(z) + \overbrace{(\delta \circ \phi)(z)}^{\pm\phi(\partial z)} + \overbrace{\eta(\partial y)}^{=0} + \overbrace{(\delta\phi)(\partial y)}^{?}$$
$$= (\eta, z)$$

Not an iso <u>but</u>

over a field - Kronecker duality

$$\boxed{H^n(X, \mathbb{F}) \xrightarrow[\text{iso}]{\sim} \mathrm{Hom}(H_n(X, \mathbb{F}), \mathbb{F})}$$
$$\shortparallel$$
$$H_n(X, \mathbb{F})^\vee$$

Perfect pairing (M, N)

$M \cong N^\vee$

Surjective: Given ξ

$$\xi : \quad Z_n \to H_n \to \mathbb{F}$$
$$z \to [z] \to \xi(z)$$

Extend to $\hat{\xi}$ C_n

$$0 \to Z_n \to C_n \underset{\curvearrowleft s}{\to} B_{n-1} \to 0$$

$\to C_n = Z_n \oplus s(B_{n-1})$ $\qquad\qquad$ δ cobound.

Check $\delta \hat{\xi} = 0$

<u>Inj:</u> Given $\xi \in Z^n$, $\xi(z) = 0$,

want $\phi \in C^{n-1}$ st. $\xi = \delta\phi$ $(\to [\xi] = 0 \in H^n)$

$$\boxed{0 \to B_{n-1} \to Z_{n-1} \boxed{\underset{\curvearrowleft s}{\to} H_{n-1}} \to 0}$$

$\to Z_{n-1} = B_n \oplus s H_{n-1}$ (extend by 0)

Might not work, H_n may not be free (unless torsion-free)

eg Finite index $G \leq F_{ree}$

$$\boxed{0 \to Z_{n-1} \to C_{n-1} \underset{\curvearrowleft}{\to} B_{n-2} \to 0}$$

$\to C_{n-1} = Z_{n-1} \oplus B_{n-2}$

Free modules

$\to$ splitting: okay

giving $\tilde{\tilde{\phi}} : C_{n-1} \to \mathbb{F}$

<u>Cup product</u>

$H^* = \bigoplus_k H^k$.

Cup product $\underset{H^i}{\underset{\shortparallel}{\alpha}} \cup \underset{H^j}{\underset{\shortparallel}{\beta}} = (-1)^{i \cdot j} \beta \cup \alpha$

Note $(\alpha^1 + \alpha^2) \cup \beta \neq \beta \cup (\alpha^1 + \alpha^2)$

$$\shortparallel$$
$$(\alpha^1 \cup \beta) + (\alpha^2 \cup \beta) \qquad =$$
$$\nparallel \qquad\qquad \shortparallel$$
$$(-\beta \cup \alpha^1) + (\beta \cup \alpha^2)$$

Define front p-face map

$$\alpha_p : \Delta^p \hookrightarrow \Delta^{p+q}$$

$$(t_0 \cdots t_p) \to (t_0 \cdots t_p, 0, 0 \cdots 0)$$

back q-face $\beta_q : \Delta^q \to \Delta^{p+q}$

$$(t_0 \cdots t_q) \to (0 \cdots 0\, t_0 \cdots t_q)$$

Given cochains

$$\xi \in C^p$$

$$\phi \in C^q$$

def $(\xi \cup \phi)(\sigma : \Delta^{p+q} \to X)$

$$= \underset{\overset{\cap}{\mathbb{R}}}{\xi(\sigma \circ \alpha_p)} \cdot_{\mathbb{R}} \underset{\overset{\cap}{\mathbb{R}}}{\phi(\sigma \circ \beta_q)}$$

Very non commutative at cochain level! Only gains commutativity @ H^* level