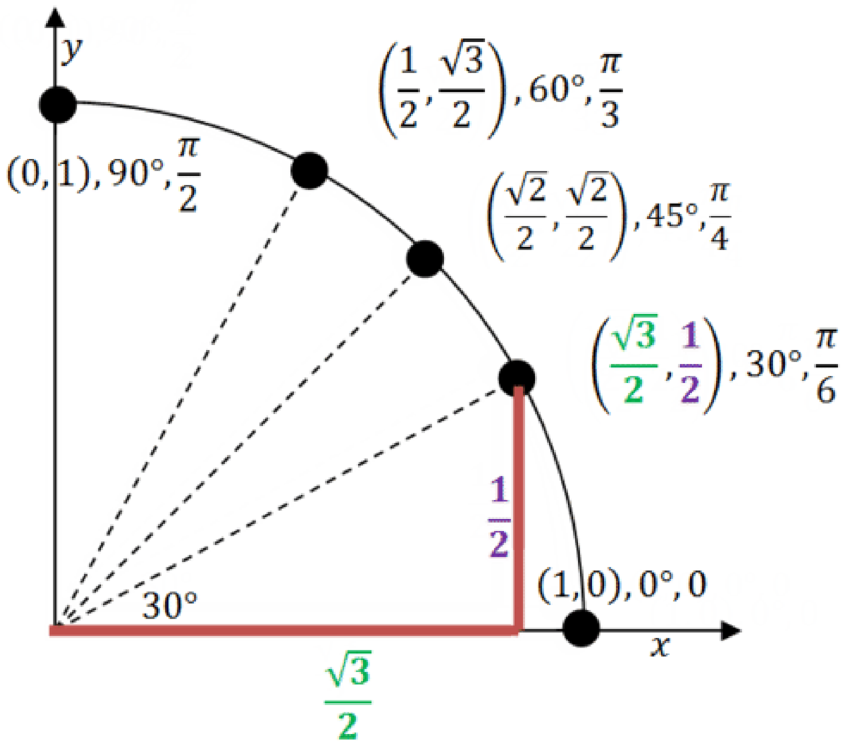

Notes: These are rough notes for the Math 1113 Precalculus course at the University of Georgia

Precalculus

University of Georgia, Spring 2021

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1 Preface

2 Unit 1: Functions

Theorem 2.0.1 ***(The Pythagorean Theorem).***
If a, b are the legs of a right triangle with hypotenuse c, there is a relation

$$a^2 + b^2 = c^2.$$

Theorem 2.0.2 ***(The Distance Formula).***
If $p = (x_1, y_1)$ and $q = (x_2, y_2)$ are points in the Cartesian plane, then there is a **distance function**

$$\begin{aligned} d : \{\text{Pairs of points } (p,q)\} &\to \mathbb{R} \\ (p,q) &\mapsto d(p,q) := \sqrt{(x_2 - x_1)^2 + (y_2 - y_q)^2}. \end{aligned}$$

Law of cosines

Definition 2.0.3 (Linear Functions)
A function $f : \mathbb{R} \to \mathbb{R}$ is **linear** if and only if f has a formula of the following form:

$$f(x) = \alpha x + \beta \qquad \alpha, \beta \in \mathbb{R}.$$

Definition 2.0.4 (Intercepts)
Given a function $f : \mathbb{R} \to \mathbb{R}$, an x-**intercept** of f is a point $(x_0, 0)$ on the graph of f, so $f(x_0) = 0$. Equivalently, it is a point on the intersection of the graph and the x-axis.

A y-**intercept** of f is a point $(0, y_0)$ on the graph of f, so $f(0) = y_0$. Equivalently, it is a point on the intersection of the graph and the y-axis.

Definition 2.0.5 (Relation)
A **relation** on two sets X and Y is a set of ordered pairs $(x, y) \in X \times Y$, so R can be described as a set:

$$R = \{(x_0, y_0), (x_1, y_2), \cdots\}.$$

The **domain** of the relation is the set of all $x \in X$ that occur in the first slot of these pairs, and the **range** is the set of all $y \in Y$ that occur in the second slot.

Definition 2.0.6 (Function)
A relation R is a **function** if it satisfies the following *deterministic property*: for every $x_0 \in$

> dom(R), there is exactly *one* pair of the form $(x_0, y_0) \in R$.

Remark 2.0.7: This says we can think of X as "inputs" and Y as "output", and a function is a way to unambiguously assign inputs to outputs. It can be useful to think of functions like programs: if I send in an x, what y should the program return to me? If I run this program today, tomorrow, and 100 years from now, sending in the same x every time, we might want it to give the same output every time, which is the *deterministic* property: I can *determine* a single unique output if I know what the input is. If my program tells me that $2 + 2 = 4$ today but $2 + 2 = 5$ tomorrow, who knows what it will return in 100 years! We can't "determine" it.

Slogan 2.0.8
For domains and ranges:

- Domains: the set of *meaningful* inputs that the function "knows" how to handle.
- Ranges: the set of *attainable* outputs that we can expect.

Remark 2.0.9: To determine a domain:

1. Naively hope it is *all* of $\mathbb{R}$.
2. Throw out "problematic" points.
3. Draw a number line and write out what you are left with in interval notation.

Example 2.0.10 ***(?):*** Define

$$\begin{aligned} f &: \mathbb{R} \to \mathbb{R} \\ x &\mapsto \frac{1}{x}. \end{aligned}$$

Then $\text{dom}(f) = \mathbb{R} \setminus \{0\} = (-\infty, 0) \cup (0, \infty)$ and $\text{range}(f) = \mathbb{R}$.

Example 2.0.11 ***(?):*** Define

$$\begin{aligned} f &: \mathbb{R} \to \mathbb{R} \\ x &\mapsto \sqrt{x}. \end{aligned}$$

Then $\text{dom}(f) = \mathbb{R} \setminus (-\infty, 0) = [0, \infty)$ and $\text{range}(f) = [0, \infty)$.

3 Unit 2: Exponential and Logarithmic Functions

4 Unit 3: Trigonometric Functions

4.1 General Notes

- In this section, always draw a picture! Virtually 100% of the time.
  - In particular, a unit circle should almost always show up.
- Use exact ratios wherever possible.
- There are too many details and formulas to just memorize in this unit: focus on the **processes**.

4.2 Common Mistakes

Some facts to remember:

- $\sin^{-1}(\theta) \neq 1/\sin(\theta)$. Mnemonic: reciprocals of trigonometric functions already have a better name, here $\csc(\theta)$.

4.3 Basic Trigonometric Functions

Sin/cos/etc as ratios

4.4 Proportionality Relationships

Definition 4.4.1 (Radian)

What is a 1 radian?

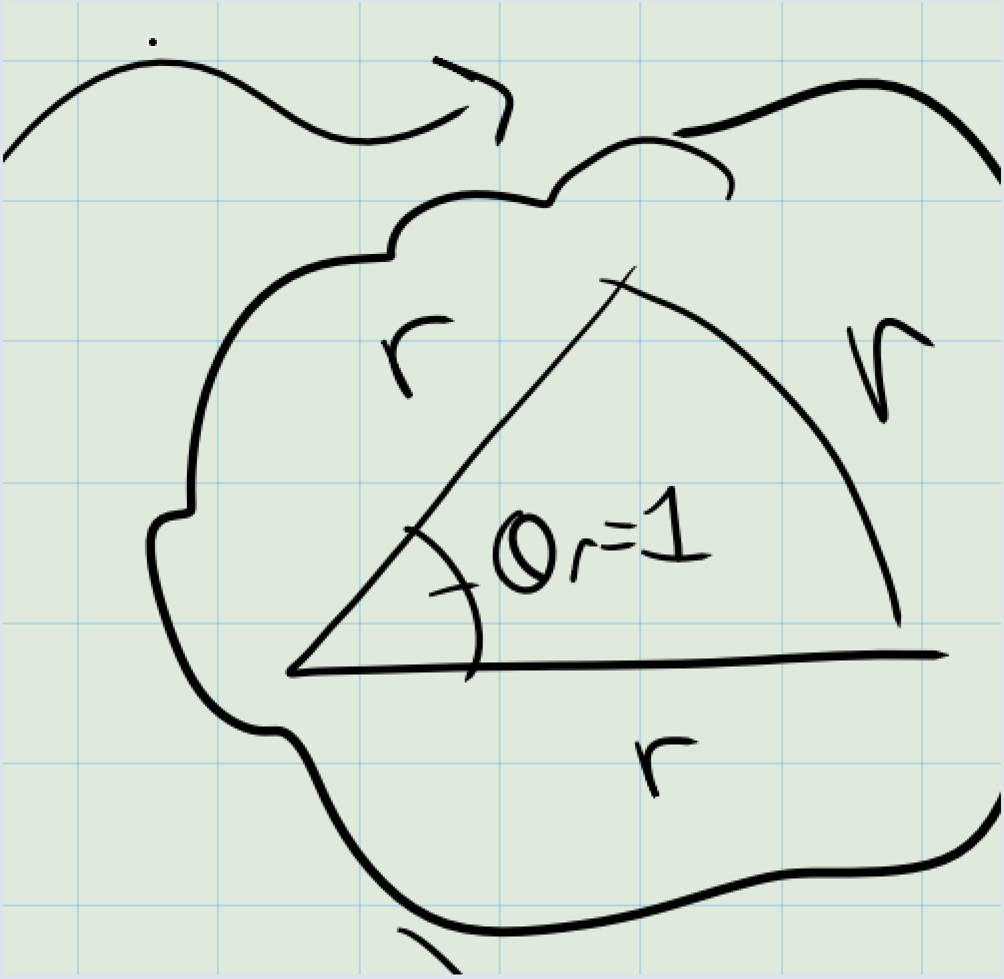


Figure 1: image_2021-04-18-21-51-59

Remark 4.4.2: In geometric terms, an angle in radians in the ratio of the arc length $s(\theta, R)$ to the radius R, so

$$\theta_R = \frac{s(\theta, R)}{R}.$$

Definition 4.4.3 (Coterminal Angles)
If θ is an abstract angle, we will say $\theta + k\,\mathrm{rev} \simeq \theta$ for any integer $k \in \mathbb{Z}$. Any such angle is said to be **coterminal** to θ.

Remark 4.4.4: In radians:

$$\theta_R \simeq \theta_R + k \cdot 2\pi \qquad k \in \mathbb{Z}.$$

In degrees:

$$\theta_D \simeq \theta_D + k \cdot 360^\circ \qquad k \in \mathbb{Z}.$$

Proposition 4.4.5 ***(Degrees are related to radians).***

todo

$$\frac{\theta}{1\,\text{rev}} = \frac{\theta_R}{2\pi\,\text{rad}} = \frac{\theta_D}{360^\circ}.$$

Proposition 4.4.6 ***(Arc length and sector area are related to radians).***

todo

$$\frac{\theta}{1\,\text{rev}} = \frac{s(R,\theta)}{2\pi R} = \frac{A(R,\theta)}{\pi R^2}.$$

This implies that

$$A(R,\theta) = \frac{R^2\theta}{2}$$
$$s(R,\theta) = R\theta.$$

4.5 Trigonometric Functions as Ratios

Definition 4.5.1 (?)
There are 6 trigonometric functions defined by the following ratios:

soh-cah-toa, cho-sha-cao

Function	Domain	Range
sin	$\mathbb{R}$	$[-1, 1]$
cos	$\mathbb{R}$	$[-1, 1]$
tan	$\mathbb{R} \setminus \left\{\pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \cdots\right\}$	?
csc	$\mathbb{R} \setminus \{0, \pm\pi, \pm 2\pi, \cdots\}$	?

sec	$\mathbb{R} \setminus \left\{\pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \cdots\right\}$	?
cot	$\mathbb{R} \setminus \{0, \pm\pi, \pm 2\pi, \cdots\}$	?

Proposition 4.5.2 ***(Domains of trigonometric functions).***

4.6 Polar Coordinates

Definition 4.6.1 (Unit Circle)
The **unit circle** is defined as

$$S^1 := \left\{\mathbf{p} = (x, y) \in \mathbb{R}^2 \;\middle|\; d(\mathbf{p}, \mathbf{0}) = 1\right\} = \left\{(x, y) \in \mathbb{R}^2 \;\middle|\; x^2 + y^2 = 1\right\},$$

the set of all points in the plane that are distance exactly 1 from the origin.

Theorem 4.6.2 ***(Polar Coordinates).***
If a vector $\mathbf{v}$ has at an angle of θ in radians and has length R, the corresponding point $\mathbf{p}$ at the end of $\mathbf{v}$ is given by

$$\mathbf{p} = [x, y] = [R\cos(\theta), R\sin(\theta)].$$

Conversely, if (x, y) are known, then the corresponding R and θ are given by

$$[R, \theta] = \left[\sqrt{x^2 + y^2}, \arctan\left(\frac{y}{x}\right)\right].$$

Corollary 4.6.3 ***(Polar Coordinates on S^1).***
If $R = 1$, so $\mathbf{v}$ is on the unit circle S^1, then

$$[x, y] = [\cos(\theta), \sin(\theta)].$$

Remark 4.6.4: This is a very important fact! The x, y coordinates on the unit circle *literally* corresponding to cosines and sines of subtended angles will be used frequently.

Slogan 4.6.5
Cosines are like x coordinates, sines are like y coordinates.

Example 4.6.6 ***(?):*** Given $\theta_R = 4\pi/3$, what is the corresponding point on the unit circle S^1?

⚠ **Warning 4.6.7**
Note that $\sin(\theta), \cos(\theta)$ work for any θ at all. However, $\cos(\theta) = 0$ sometimes, so $\tan(\theta) := \sin(\theta)/\cos(\theta)$ will on occasion be problematic. Similar story for the other functions.

4.7 Special Angles

For reference: the unit circle.

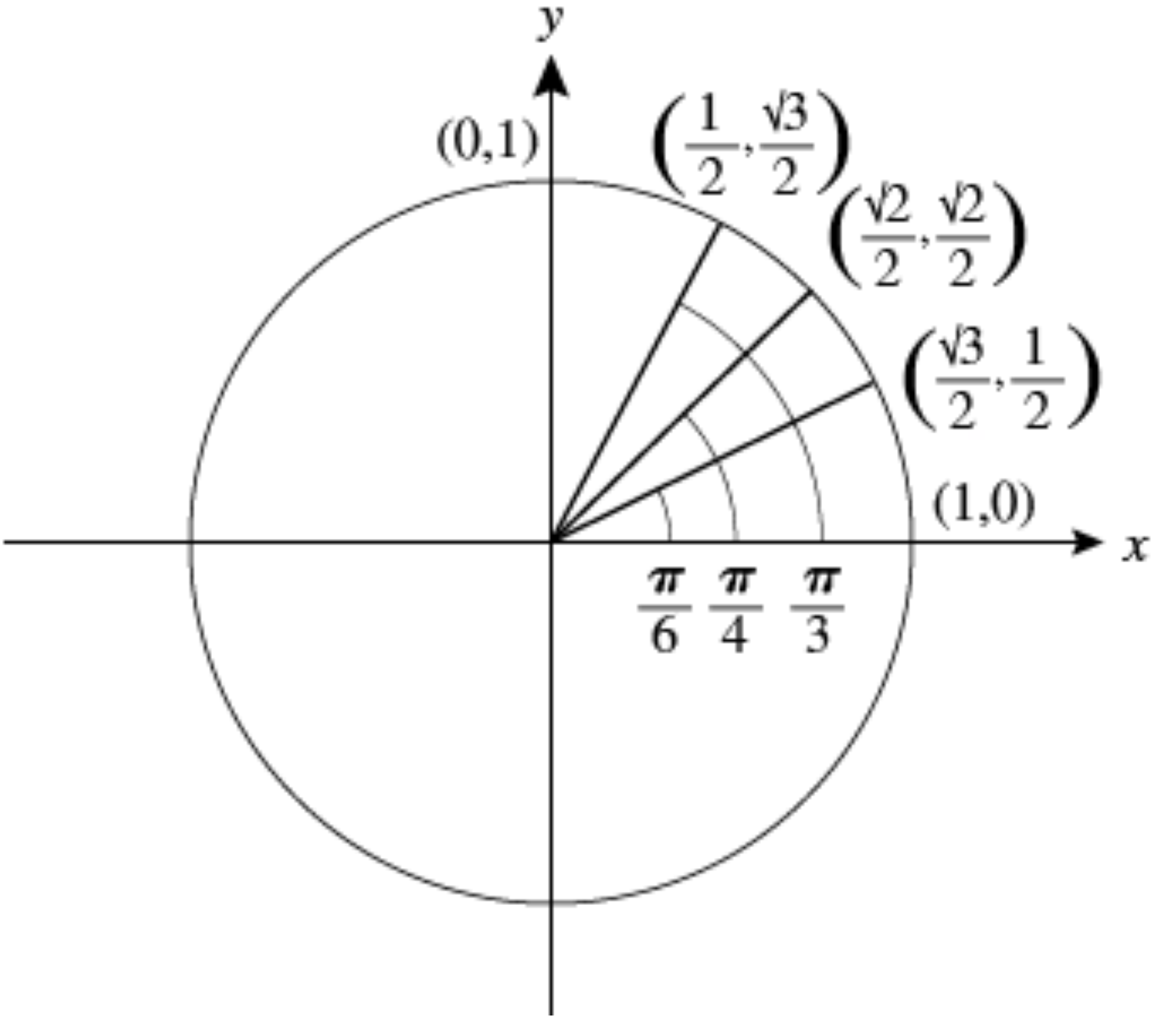


Figure 2: image_2021-04-18-21-06-45

Remark 4.7.1: Idea: we want to partition the circle simultaneously

- Into 8 pieces, so we increment by $2\pi/8 = \pi/4$
- Into 12 pieces, so we increment by $2\pi/12 = \pi/6$.

Proposition 4.7.2 ***(Trick to memorize special angles).***

Table of special angles, increasing/decreasing

4.8 Reference Angles and the Flipping Method

Definition 4.8.1 (Reference Angle)
Given a vector at of length R and angle θ, the **reference angle** θ_{Ref} is the acute angle in the triangle formed by dropping a perpendicular to the nearest horizontal axis.

Proposition 4.8.2 ***(?).***
Reference angles for each quadrant:

$$\begin{aligned}\text{Quadrant II}: &\quad \theta + \theta_{\text{Ref}} = \pi \\ \text{Quadrant III}: &\quad \pi + \theta_{\text{Ref}} = \theta \\ \text{Quadrant IV}: &\quad \theta + \theta_{\text{Ref}} = 2\pi.\end{aligned}$$

Example 4.8.3 ***(?):*** Given $\sin(\theta) = 7/25$, what are the five remaining trigonometric functions of θ?

Method:

1. Draw a picture! Embed θ into a right triangle.
2. Find the missing side using the Pythagorean theorem.
3. Use definition of trigonometric functions are ratios.

Remark 4.8.4: Note that you can not necessarily find the angle θ here, but we didn't need it. If we *did* want θ, we would need an inverse function to free the argument:

$$\begin{aligned}\sin(\theta) &= 7/25 \\ \implies \arcsin(\sin(\theta)) &= \arcsin(7/25) \\ \implies \theta &= \arcsin(7/25) \\ &.\end{aligned}$$

4.9 Identities Using Pythagoras

Proposition 4.9.1 ***(?).***

$$\begin{aligned}(\sin(\theta))^2 + (\cos(\theta))^2 &= 1 \\ 1 + (\cot(\theta))^2 &= (\csc(\theta))^2 \\ (\tan(\theta))^2 + 1 &= (\sec(\theta))^2.\end{aligned}$$

Proof (?).
Derive first from Pythagorean theorem in S^1. Obtain the second by dividing through by $(\sin(\theta))^2$. Obtain the third by dividing through by $(\cos(\theta))^2$. ■

4.10 Even/Odd Properties

Question 4.10.1
Thinking of $\cos(\theta)$ as a function of θ, is it

- Even?
- Odd?
- Neither?

Remark 4.10.2: Why do we care? The Fundamental Theorem of Calculus.

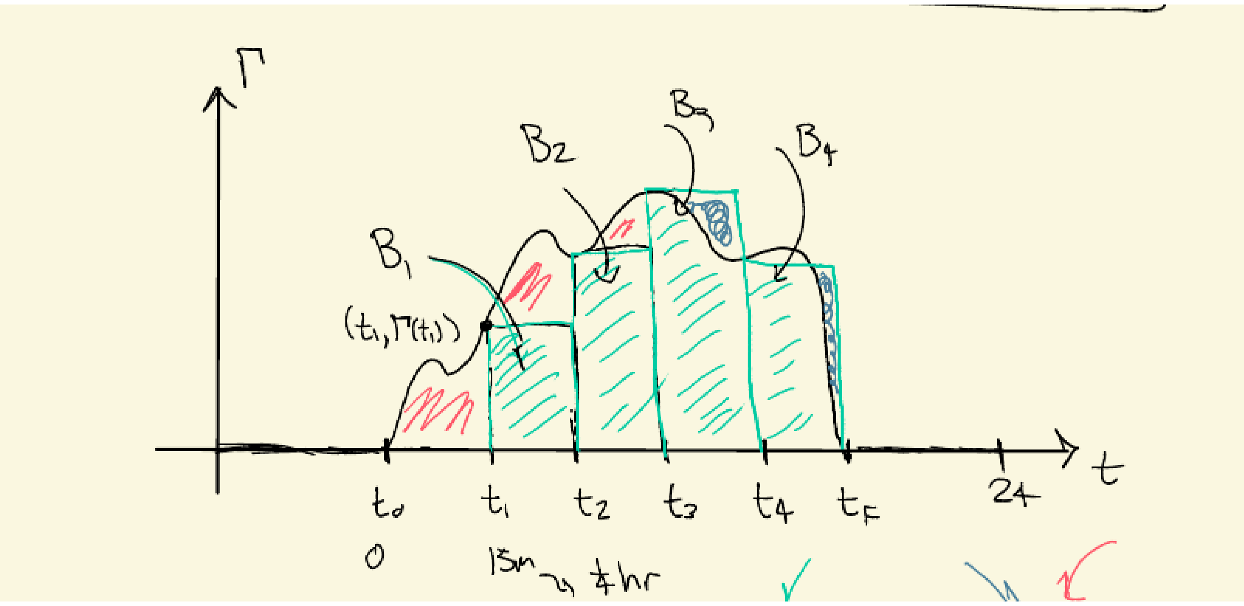


Figure 3: image_2021-04-18-22-39-08

Proposition 4.10.3 *(?).*

- $f(\theta) := \cos(\theta)$ is an even function.
- $g(\theta) := \sin(\theta)$ is an odd function.

Proof (?).
Plot vectors for $\theta, -\theta$ on S^1 and flip over the x-axis. ■

Corollary 4.10.4 *(?).*

- $\cos(t), \sec(t)$ are even.
- $\sin(t), \csc(t), \tan(t), \cot(t)$ are odd.

4.11 Wave Function

Remark 4.11.1: Motivation: let a vector run around the unit circle, where we think of θ as a time parameter. What are its x and y coordinates? What happens if we plot $x(t)$ in a new θ plane?

Definition 4.11.2 (Standard Form of a Wave Function)
The **standard form** of a wave function is given by

$$f(t) := A\cos(\omega(t-\varphi)) + \delta,$$

where

- A is the **amplitude**,
- ω is the **frequency**,
- φ is the **phase shift**, and
- δ is the **vertical shift**.
- $P := 2\pi/\omega$ is the **period**, so $f(t+kP) = f(t)$ for all $k \in \mathbb{Z}$.

Insert plot

Remark 4.11.3: Note that this is nothing more than a usual cosine wave, just translated/dilated in the x direction and the y direction.

⚠ **Warning 4.11.4**
Don't memorize equations like $y = \sin(Bt + C)$ and e.g. the phase shift if $\varphi = -C/B$. Instead, use a process: always put your equation in standard form, then you can just read off the parameters. For example:

$$\begin{aligned} f(t) &= \cos(Bt + C) \\ &= \cos(B(t + \frac{C}{B})) \\ &= \cos(\omega(t - \varphi)) \end{aligned}$$

$$\implies B = \omega, \varphi = -\frac{C}{B}.$$

Example 4.11.5 ***(?):*** Put the following wave in standard form:

$$f(t) := 4\cos(3t + 2).$$

Example 4.11.6 ***(?):*** Put the following wave in standard form:

$$f(t) := \alpha\cos(\beta t + \gamma).$$

Proposition 4.11.7 ***(?).***
How to plot the graph of a wave equation:

1. Put in standard form.
2. Read off the parameters to build a rectangular box of width P and height $2|A|$ about the line $y = \delta$.
3. Break the box into 4 pieces using the key points $t = \varphi + \frac{k}{4}P$ for $k = 0, 1, 2, 3, 4$.

Example 4.11.8 ***(Plotting):*** Plot the following function in the t plane:

$$f(t) = 2\cos\left(5t - \frac{\pi}{2}\right) + 7.$$

Example 4.11.9 ***(?):*** Plot the following:

$$f(t) = -2\sin(3t - 7).$$

Proposition 4.11.10 ***(Determining the equation of a sine wave).***
Given a picture of a graph of a sine wave,

1. Draw a horizontal line cutting the wave in half. This will be δ.
2. Measure the distance from this midline to a peak. This will be $|A|$.
3. Restrict to one full period, starting either at a peak (if you want to match $\cos(t)$) or a zero (if you want to match $\sin(t)$). Pick the period starting as close as possible to the y-axis.
4. Measure the period P and reverse-engineer it to get ω: $P = 2\pi/\omega \implies \omega = 2\pi/P$.
5. Measure the distance from the starting point to the y-axis: this is φ.

Example 4.11.11 ***(?):*** Determine the equation of the following wave function:

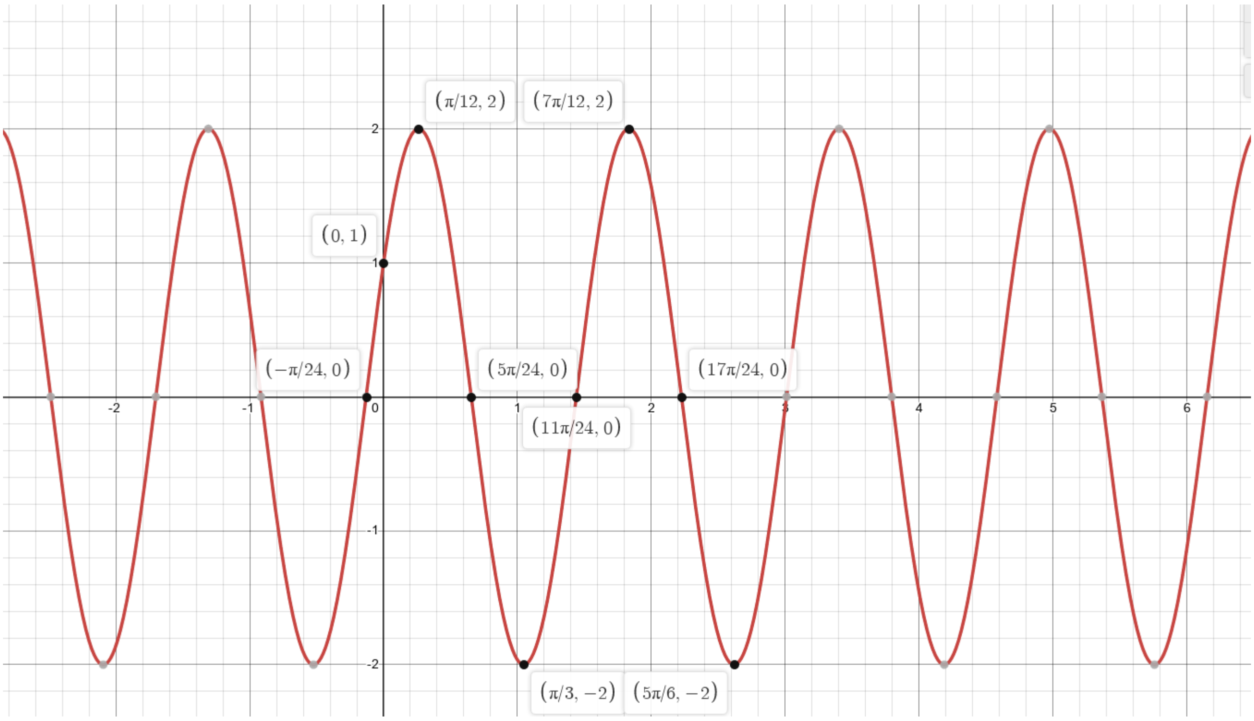


Figure 4: image_2021-04-18-20-51-34

Solution:

$$f(t) = 2\sin\left(4t + \frac{\pi}{6}\right).$$

Remark 4.11.12: Note that we can graph other trigonometric functions: they get pretty wild though.

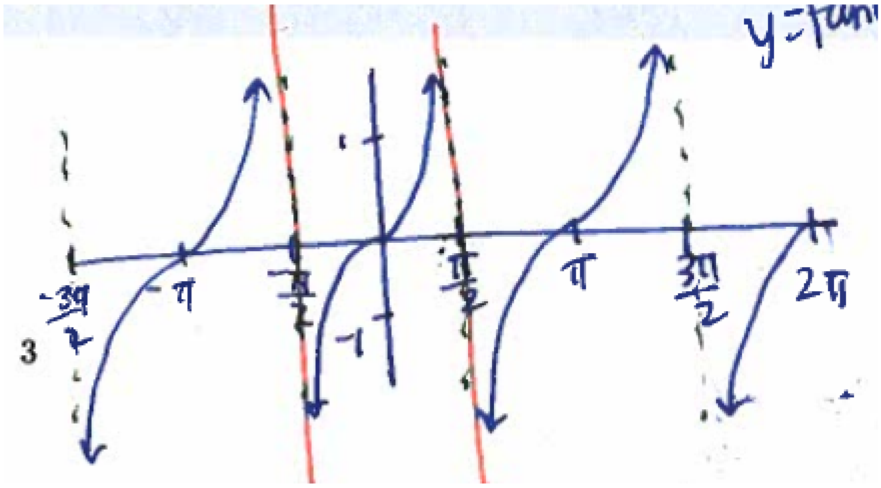
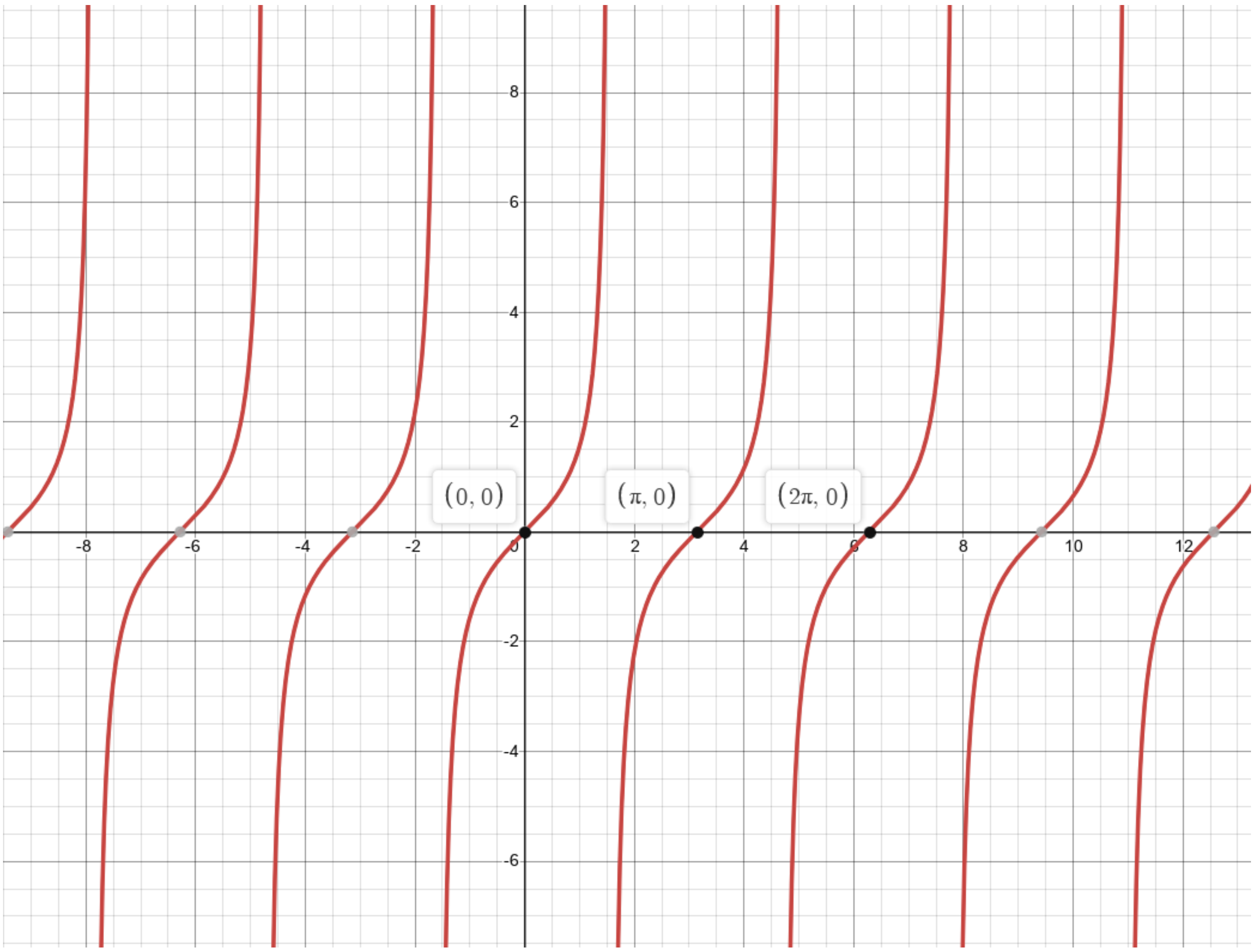


Figure 5: Tangent



4.12 Simplifying Identities

Remark 4.12.1: The goal: reduce a complicated mess of trigonometric functions to something as simple as possible. We'll use a **boxing-up method**.

Remark 4.12.2: On verifying identities: if you want to show $f(\theta) = g(\theta)$, start at one and arrive at the other:

$$\begin{aligned} f(\theta) &= \text{simplify } f \\ &= \cdots \\ &= \cdots \\ &= \cdots \\ &= g(\theta) \end{aligned}.$$

⚠ **Warning 4.12.3**
If you end up with something like $1 = 1$ or $0 = 0$, this is hinting at a problem with your logic.

Exercise 4.12.4 (?)
Simplify the following:

$$F(\theta) := \left(\frac{\sin(\theta)\cos(\theta)}{\cot(\theta)}\right)\cos(\theta)\csc(\theta).$$

Solution:

$$F = s\left(\frac{s}{c}\right).$$

Remark 4.12.5: As an alternative, you can use the **transitivity of equality**: show that $f(\theta) = h(\theta)$ for some totally different function h, and then show $g(\theta) = h(\theta)$ as well.

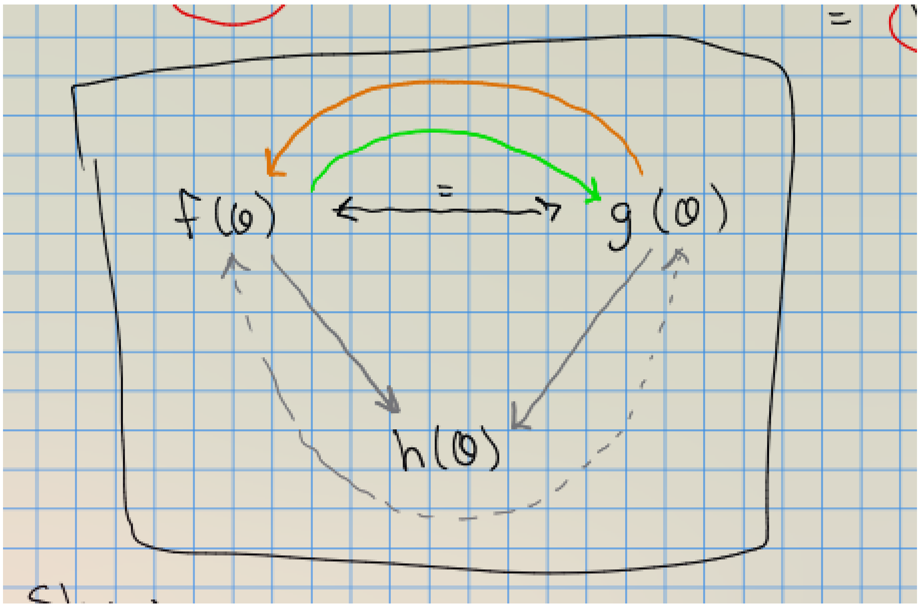


Figure 6: image_2021-04-18-21-58-52

Exercise 4.12.6 (Reducing both sides to a common expression)
Show the following identity:

$$\sin(-\theta) + \csc(\theta) = \cot(\theta)\cos(\theta)$$

by showing both sides are separately equal to $h(\theta) := \csc(\theta) - \sin(\theta)$.

4.13 Inverse Functions

4.13.1 Motivation

Remark 4.13.1: Motivation: we want a way to solve equations where the unknown θ is stuck in the argument of a trigonometric function. For example, for $\sin : \mathbb{R}_A \to \mathbb{R}_B$, this would be some function $f : \mathbb{R}_B \to \mathbb{R}_A$ such that

$$f(\sin(\theta)) = \mathrm{id}(\theta) = \theta$$
$$\sin(f(y)) = \mathrm{id}(y) = y.$$

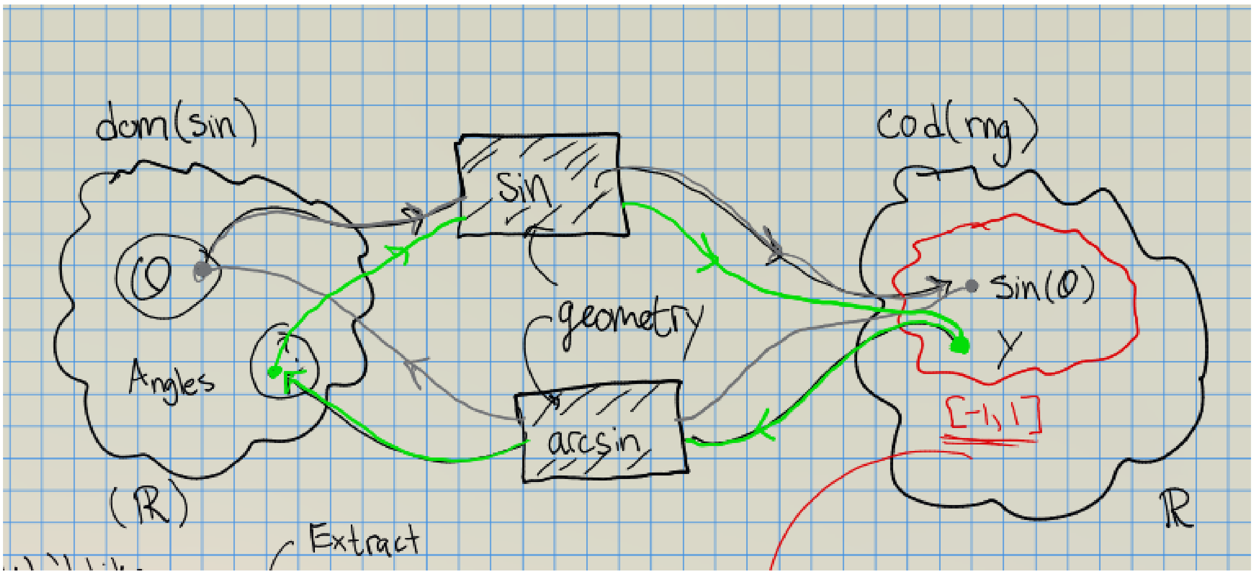
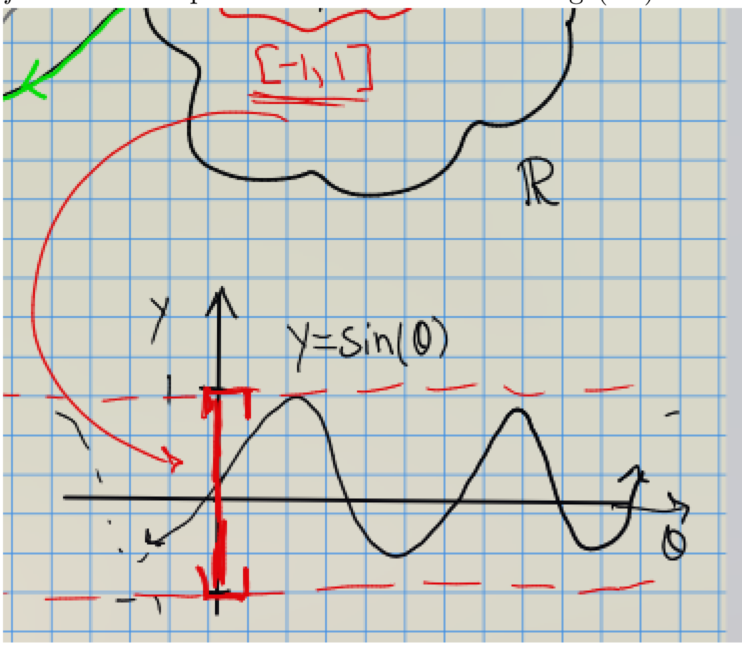


Figure 7: Input-Output perspective: important!

Note that we only ever have to define f on range(sin), since we're only ever sending outputs of f in as the inputs of sin. So we need range(sin) $\subset$ dom(f), noting that range(sin) $= [-1, 1]$:



Similarly, we need range(f) $\subset$ dom(sin).

4.13.2 Using Triangles

Remark 4.13.2: Optimistically imagine that we had some such inverse function. Then we could evaluate some expressions without even knowing anything else about it. The trick:

$$\begin{aligned} &\theta = \arccos(p/q) \\ \implies &\cos(\theta) = \cos(\arccos(p/q)) \\ \implies &\cos(\theta) = p/q. \end{aligned}$$

Now embed this in a triangle. We can't solve for θ, but we can solve for other trigonometric functions.

Exercise 4.13.3 (Using functional inverse property)

$$\cos\left(\arccos\left(\frac{\sqrt{5}}{5}\right)\right) = \frac{\sqrt{5}}{5}$$
$$\arccos\left(\cos\left(\frac{\sqrt{5}}{5}\right)\right) = \frac{\sqrt{5}}{5}.$$

Exercise 4.13.4 (Using a triangle)

$$\tan\left(\arcsin\left(\frac{p}{q}\right)\right) = \frac{p}{\sqrt{q^2 - p^2}}.$$

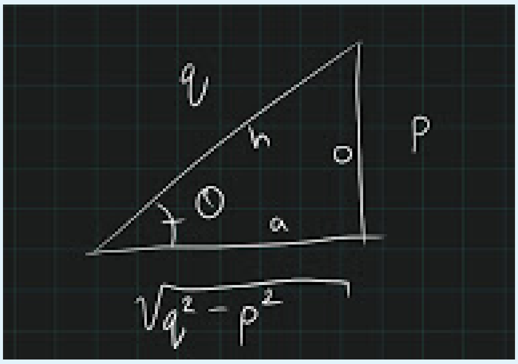


Figure 8: image_2021-04-22-22-14-13

Exercise 4.13.5 (Can't extract angles)
Compute $\arcsin(3/5)$.
⚠ **Warning 4.13.6**
This is equal to $\sin^{-1}(3/5)$, which is *not* equal to $\dfrac{1}{\sin(3/5)}$! One way to remember this is that we have another name for reciprocals, here $\csc(3/5)$.

Solution:

$$\begin{aligned} \theta &= \arcsin(3/5) \\ \implies \sin(\theta) &= (3/5) && \text{roughly by injectivity} \\ \implies &= \cdots ?. \end{aligned}$$

We are out of luck, since this isn't a special angle. So we can't find a numerical value of θ. We can find other trig functions of θ though:

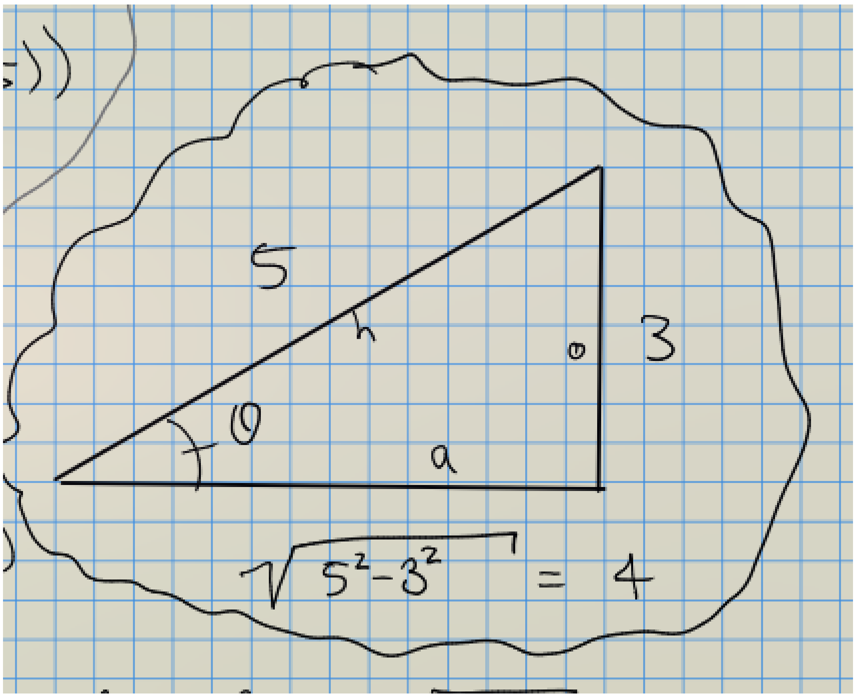


Figure 9: image_2021-04-18-22-30-09

So for example, $\cos(\arcsin(3/5)) = 4/5$.

Remark 4.13.7: Most inverse trigonometric functions can *not* be exactly solved! We'll have to approximate by calculator if we want the actual angle. If we just want *other* trigonometric functions though, we can always embed in a triangle.

Example 4.13.8 ***(Using triangles):*** Show the following:

- $\cos(\arcsin(24/26)) = 10/26$
  - Write $\theta = \arcsin(24/26)$, note θ is in $[-\pi/2, \pi/2] = \text{range}(\arcsin)$.
- $\tan(\arccos(-10/26)) = 10/26$
  - Write $\theta = \arccos(-10/26)$, note θ is in $[0, \pi] = \text{range}(\arccos)$

4.13.3 Defining Inverses

Remark 4.13.9: The setup: try swapping y and θ in the graph of $y = \sin(\theta)$:

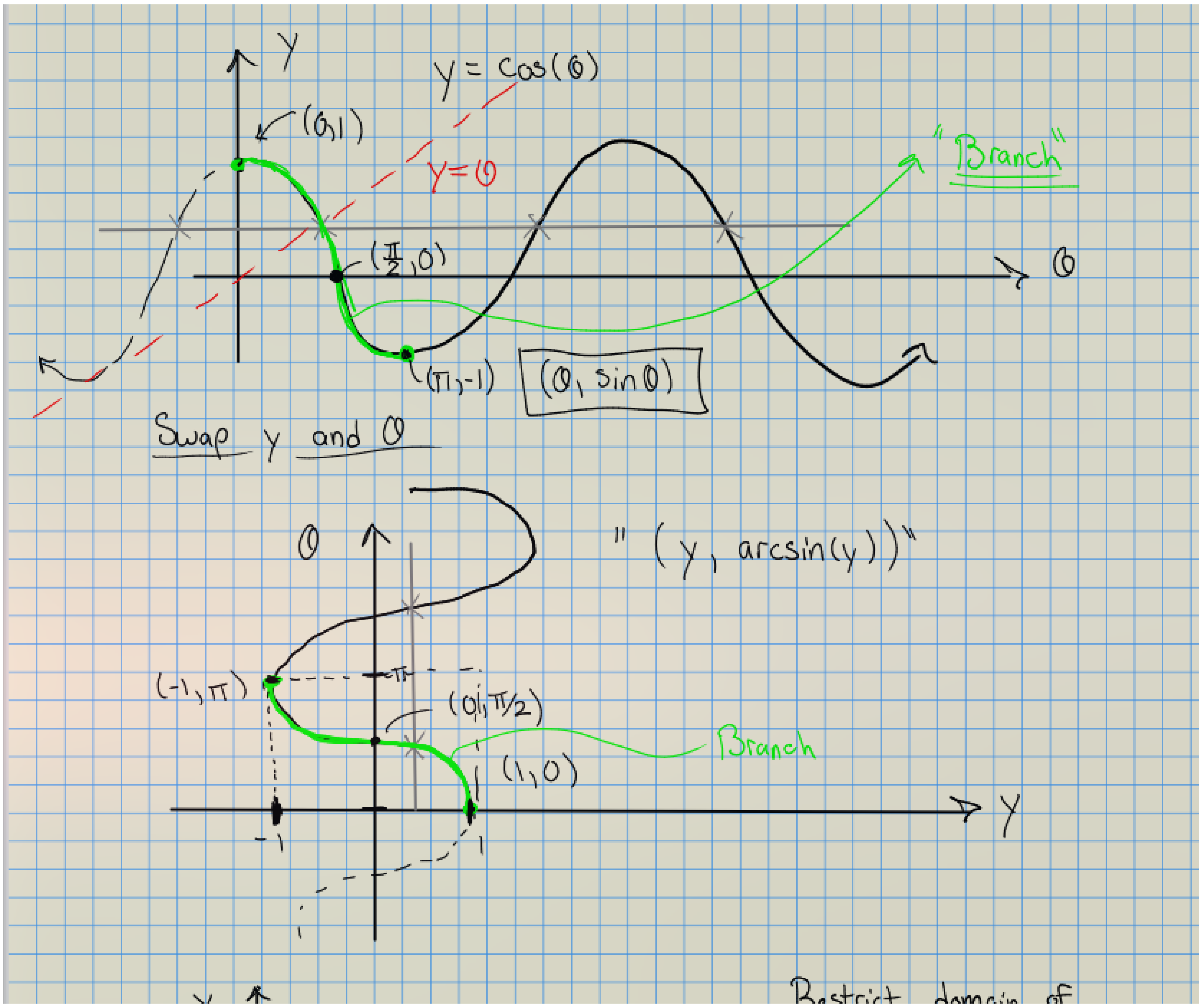


Figure 10: image_2021-04-18-22-32-36

Note that the latter is a function (vertical line test) iff the former is injective (horizontal line test). So we take the largest branch where the inverse is a function:

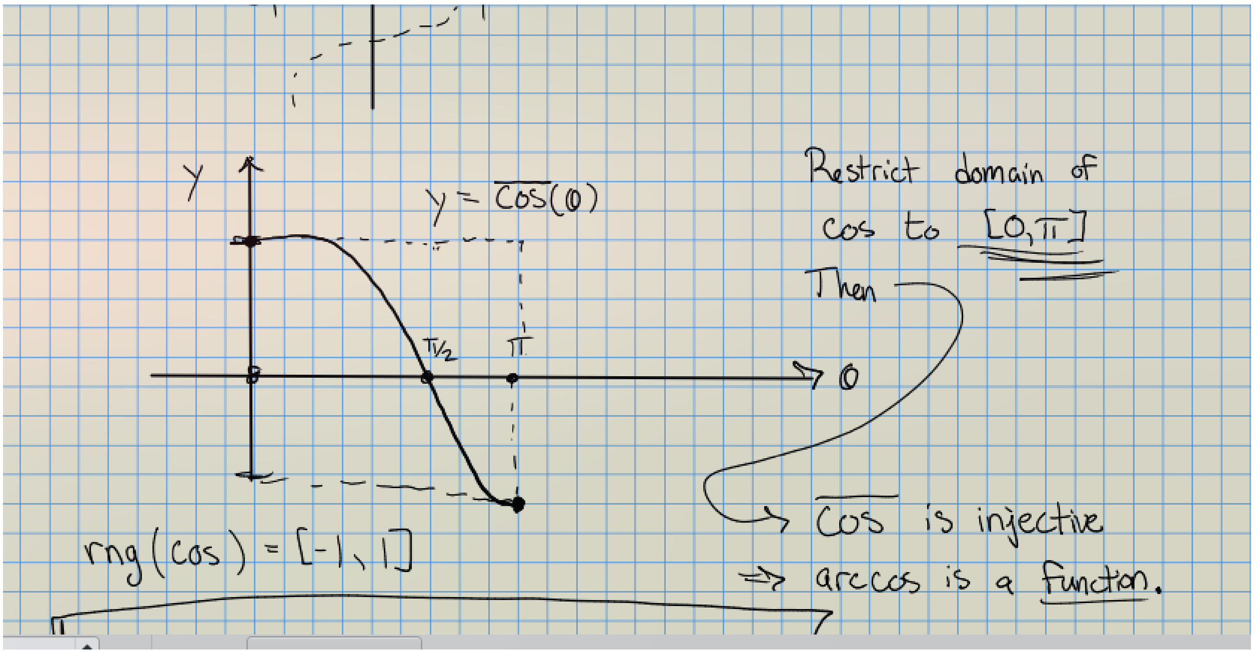


Figure 11: image_2021-04-18-22-33-27

Back on our original graph, this looks like the following:

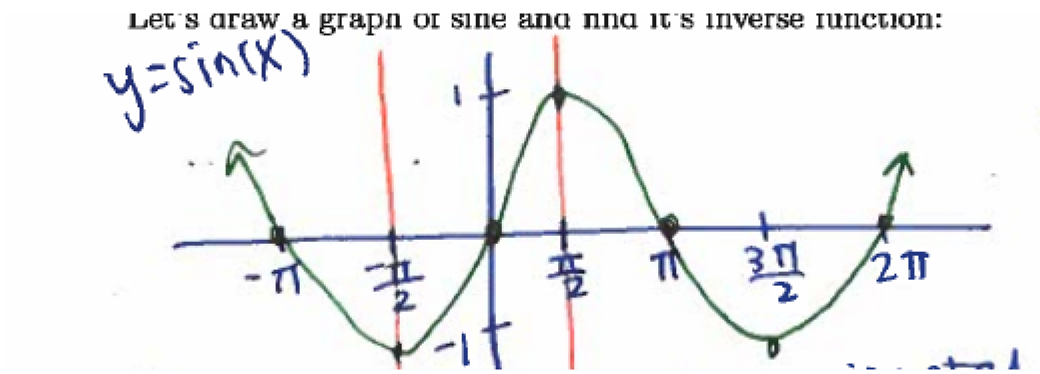


Figure 12: image_2021-04-18-20-53-25

Restricting, we get

- dom(arccos) := range(cos) = $[-1, 1]$.
- range(arccos) := dom(cos) = $[0, \pi]$.

Remark 4.13.10: A similar analysis works for $\sin(\theta)$:

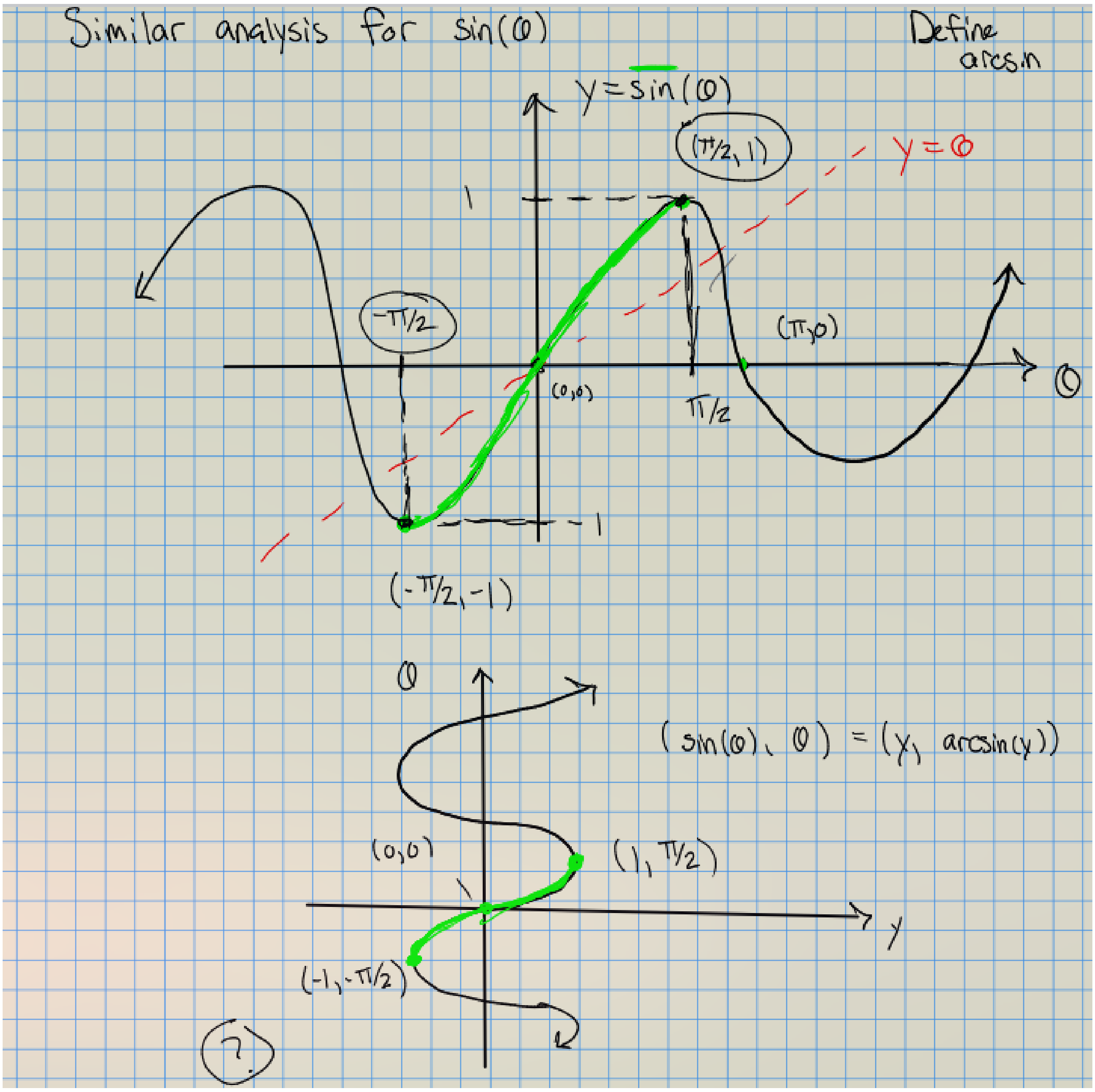


Figure 13: image_2021-04-18-22-34-21

Restricting, we get

- dom(arcsin) := range(sin) = $[-1, 1]$.
- range(arcsin) := dom(sin) = $[-\pi/2, \pi/2]$.

Remark 4.13.11: This gives us a new tool to solve equations:

$$\begin{aligned} \vdots &= \vdots \\ \implies \cos(x) &= b \\ \implies \arccos(\cos(x)) &= \arccos(b) \\ \implies x &= \arccos(b), \end{aligned}$$

but only if we know this makes sense based on domain/range issues.

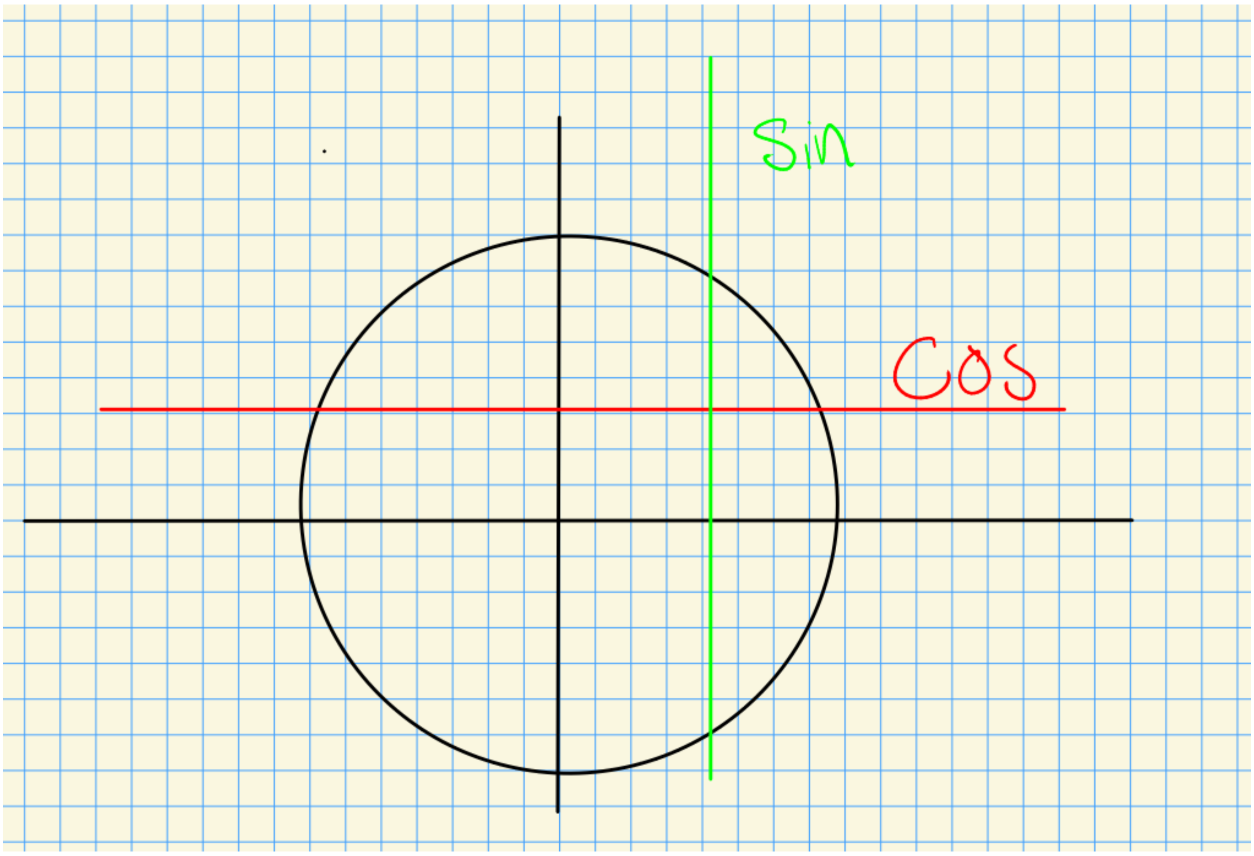
Proposition 4.13.12 ***(Domains of inverse trigonometric functions).***
Restrict domains in the following ways:

- sin: $[-\pi/2, \pi/2]$
- cos : $[0, \pi]$
- tan : $[-\pi/2, \pi/2]$

Function	Domain	Range
arcsin	$[-1, 1]$	$[-\pi/2, \pi/2]$
arccos	$[-1, 1]$	$[0, \pi]$
arctan	$\mathbb{R}$	$(-\pi/2, \pi/2)$
arccsc	$\mathbb{R} \setminus \{0, \pm\pi, \pm 2\pi, \cdots\}$	$[-\pi/2, \pi/2] \setminus \{0\}$
arcsec	$\mathbb{R} \setminus \left\{\pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \cdots\right\}$	$[0, \pi] \setminus \{\pi/2\}$
arccot	$\mathbb{R}$	$(0, \pi)$

Slogan 4.13.13
There is an easy way to remember this:

- Cosines are x-values, pick the upper (or lower) half of the circle to make them unique.
- Sines are y-values, pick the right (or left) half of the circle to make them unique.



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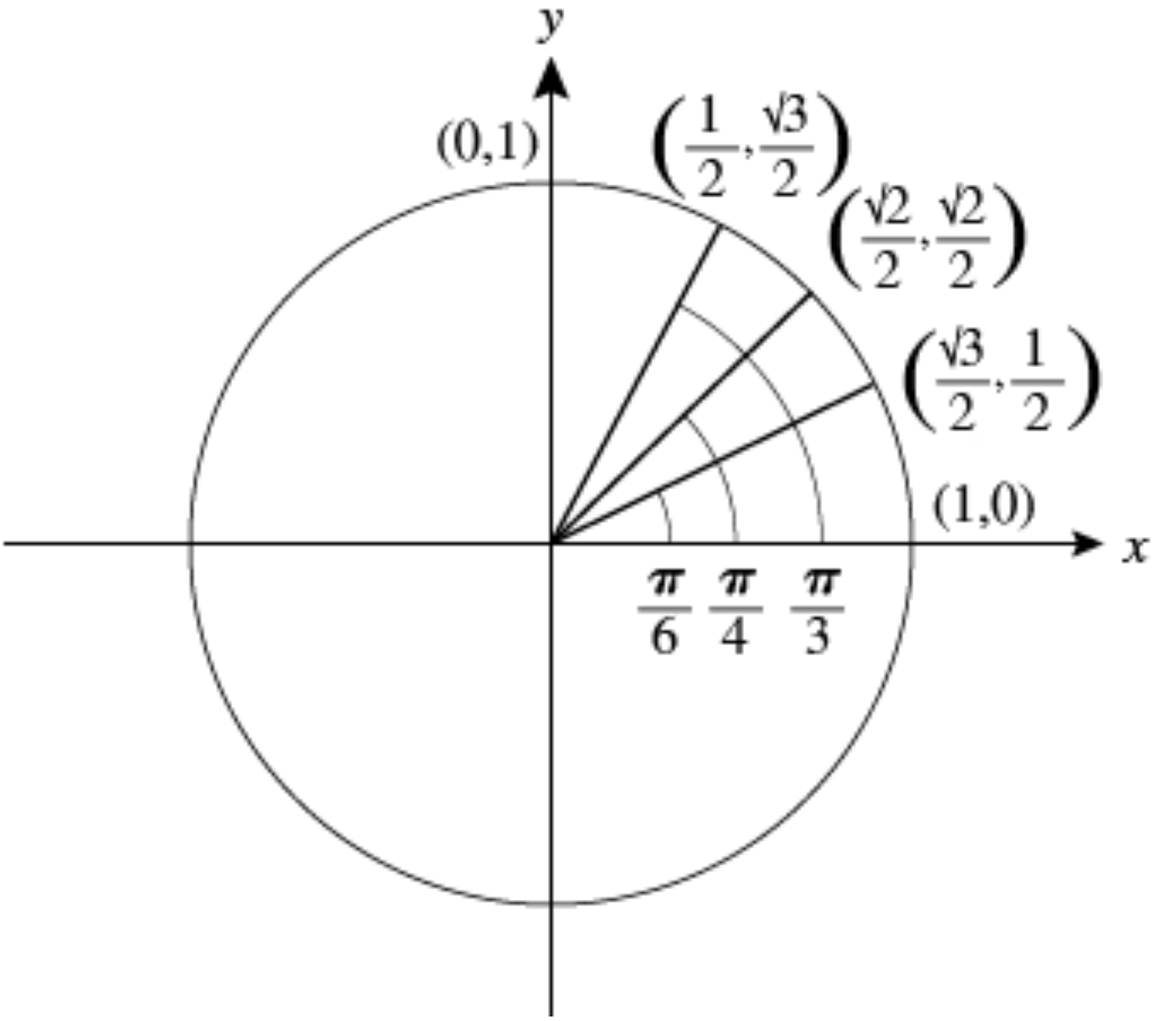

Figure 15: Unit Circle

Example 4.13.14 ***(Using special angles):*** We have some exact values.

Sines should be in QI or QIV:

- $\arcsin(1/2) = \pi/6$
- $\arcsin(\sqrt{3}/2) = \pi/3$
- $\arcsin(-1/2) = -\pi/6$

Cosines should be in QI or QII:

- $\arccos(\sqrt{3}/2) = \pi/6$
- $\arccos(-\sqrt{2}/2) = 3\pi/4$
- $\arccos(1/2) = \pi/3$

Tangents should be in QI or QIV:

- $\arctan(\sqrt{3}/3) = \pi/6$
- $\arctan(0) = 0$
- $\arctan(1) = \pi/4$

Warning 4.13.15
Note that if f, g are an inverse pair, we have

$$f \circ g = \text{id} \iff f(g(x)) = x, \quad g(f(x)) = x.$$

However, we have to be careful with domains for trigonometric functions:

- $\arcsin(\sin(x)) = x \iff x \in [-\pi/2, \pi/2]$ (restricted domain of sin)
- $\sin(\arcsin(x)) = x \iff x \in [-1, 1]$ (domain of arcsin)
- $\arccos(\cos(x)) = x \iff x \in [0, \pi]$ (restricted domain of cos)
- $\cos(\arccos(x)) = x \iff x \in [-1, 1]$ (domain of arccos)
- $\arctan(\tan(x)) = x \iff x \in [0]$ (restricted domain of tan)
- $\tan(\arctan(x)) = x \iff x \in \mathbb{R}$
  - Domain of arctan, then range is $[-\pi/2, \pi/2]$, which is in the domain of tan.

4.14 Double/Half-Angle Identities

Remark 4.14.1: Sometimes we are interested in **superposition** of waves, see Desmos for an example. Mathematically this is modeled by adding wave functions together. Similarly, we are sometimes interested in **modulating** or **enveloping** waves, which is modeled by multiplying a wave with another function: see Desmos.

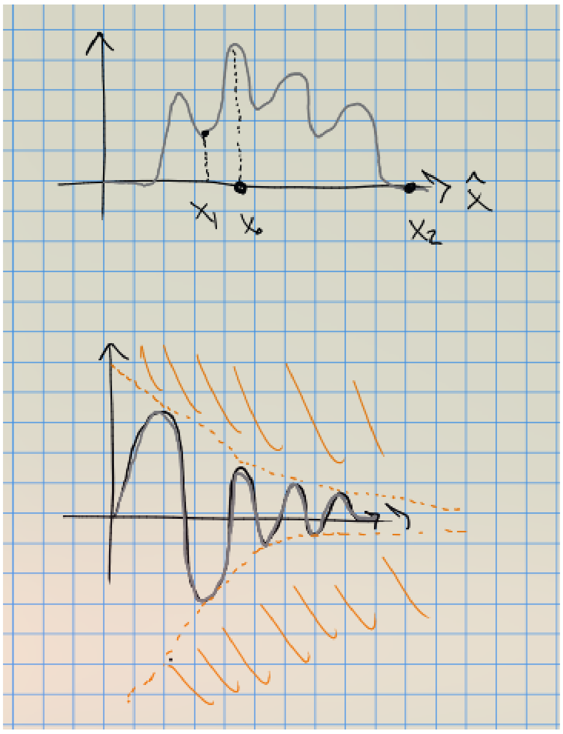

Figure 16: image_2021-04-18-22-06-08

We can sometimes rewrite these as a *single* wave with a phase shift.

Proposition 4.14.2 ***(Angle Sum Identities).***
Identities:

$$\begin{aligned}\sin(\theta+\psi) &= \sin(\theta)\cos(\psi)+\cos(\theta)\sin(\psi)\\ \cos(\theta+\psi) &= \cos(\theta)\cos(\psi)+\sin(\theta)\sin(\psi).\end{aligned}$$

Note that you can divide these to get

$$\tan(\theta+\psi)=\frac{\tan(\theta)+\tan(\psi)}{1-\tan(\theta)\tan(\psi)},$$

and replace ψ with $-\psi$ and use even/odd properties to get formulas for $\sin(\theta-\psi), \cos(\theta-\psi)$

Slogan 4.14.3
Sines are friendly and cosines are clique-y!

Corollary 4.14.4 ***(Double angle identities).***

Taking $\theta = \psi$ is the above identities yields

$$\begin{aligned}\sin(2\theta) &= \sin(\theta)\cos(\theta) + \cos(\theta)\sin(\theta) \\ &= 2\sin(\theta)\cos(\theta)\end{aligned}$$

$$\begin{aligned}\cos(2\theta) &= \cos(\theta)\cos(\theta) + \sin(\theta)\sin(\theta) \\ &= \cos^2(\theta) - \sin^2(\theta).\end{aligned}$$

Warning 4.14.5
The latter is not equal to 1! That would be $\cos^2(\theta) + \sin^2(\theta)$.

Remark 4.14.6: Why do we care? We had 16 special angles, this gives a lot more. For example,

$$\cos(\pi/12) = \cos(\pi/3 - \pi/4) = \cdots \text{ plug in.}$$

By allowing increments of $\pi/12$, we have 24 total angles.

Corollary 4.14.7*(?)*.
Starting from the following:

$$\begin{aligned}\cos(2\theta) &= \cos^2(\theta) - \sin^2(\theta) \\ &= \cos^2(\theta) - \left(1 - \cos^2(\theta)\right) \\ &= 2\cos^2(\theta) - 1 && \text{using } s^2 + c^2 = 1,\end{aligned}$$

one can solve for

$$\cos^2(\theta) = \frac{1}{2}\left(1 + \cos(2\theta)\right).$$

Similarly

$$\begin{aligned}\cos(2\theta) &= \cos^2(\theta) - \sin^2(\theta) \\ &= \left(1 - \sin^2(\theta)\right) - \sin^2(\theta) \\ &= 1 - 2\sin^2(\theta) && \text{using } s^2 + c^2 = 1,\end{aligned}$$

solving yields

$$\sin^2(\theta) = \frac{1}{2}(1 - \cos(2\theta)).$$

Remark 4.14.8: These are very important in Calculus! This gives us a way to reduce the exponents on expressions like $\sin^n(\theta)$.

4.15 Bonus: Complex Exponentials

Question 4.15.1
We spent one entire unit studying the function $f(x) = e^x$, and another studying the functions $g(x) = \cos(x), h(x) = \sin(x)$. They seem completely unrelated, but miraculously they are both just shadows of of unifying concept.

Remark 4.15.2: Components of vectors: every $\mathbf{v} \in \mathbb{R}^2$ breaks up as the sum of two vectors, i.e. $\mathbf{v} = \mathbf{v}_x + \mathbf{v}_y$. In coordinates, if $\mathbf{v} = (a, b)$, we have $\mathbf{v}_x = (a, 0)$ and $\mathbf{v}_y = (0, b)$. Alternatively, we can drop the ordered pair notation and write $\mathbf{v} = a\widehat{\mathbf{x}} + b\widehat{\mathbf{y}}$.

Remark 4.15.3: We've worked with the *Cartesian plane* all semester. One powerful tool is replacing this with the *complex* plane. We formally define a new symbol i and replace the $\widehat{\mathbf{y}}$ direction with the i direction – this amounts to replacing ordered pairs $(a, b) := a\widehat{\mathbf{x}} + b\widehat{\mathbf{y}}$ by a single number $a + ib$.

Example 4.15.4*(How to work with complex numbers):* Complex numbers can be added:

$$(a + bi) + (c + di) = (a + c) + (b + d)i.$$

This is perhaps easier to understand in the ordered pair notation: you just add the components in each component:

$$[a, b] + [c, d] = [a + c, b + d].$$

Complex numbers can be multiplied:

$$\begin{aligned}(a + bi)(c + di) &= a(c + di) + bi(c + di)\\ &= ac + adi + bci + bdi^2\\ &= (ac - bd) + (ad + bc)i.\end{aligned}$$

This is harder to see in the ordered pair notation.

We can compare complex numbers: they are equal iff their components are equal:

$$a + bi = c + di \iff a = c \text{ and } b = d,$$

or in ordered pair notation,

$$[a, b] = [c, d] \iff a = c \text{ and } b = d.$$

Remark 4.15.5: The symbol i happens to have another algebraic property. Consider the family of equations $f(x, t) = x^2 + t$, and think about finding the roots. Finding a root is solving $f(x, t) = 0$, which is the exact same thing as finding the intersection points with the graph of $g(x) = 0$. Taking $t = 0$ yields $f(x) = x^2$, which has a root at zero. Taking $t < 0$ yields two roots. However, taking $t > 0$ yields no roots – at least not in $\mathbb{R}$. As it turns out, the function $f_1(x) = x^2 + 1$ and $g(x) = 0$ *do* intersect in some other, bigger space, and we're only seeing a shadow of this! In other words, $x^2 + 1 = 0$ didn't have solutions in $\mathbb{R}$, but *will* have a solution in $\mathbb{C}$.

Remark 4.15.6: The following is the main link between exponentials and waves:

Proposition 4.15.7 ***(Euler's Formula).***

$$e^{i\theta} = \cos(\theta) + i\sin(\theta).$$

Remark 4.15.8: Really, this is just polar coordinates on the unit circle: if we go back to ordered pair notation, this is just giving a point $(\cos(\theta), \sin(\theta)) \in S^1$. So the *complex number* $e^{i\theta}$ is also a *vector* pointing at an angle θ from the origin and landing on the unit circle.

Proposition 4.15.9 ***(Euler's Identity).***

$$e^{i\pi} = -1.$$

Remark 4.15.10: This is remarkable! It relates some of the most fundamental constant numbers in mathematics:

- $e = 2.718\ldots$
- $\pi = 3.14159\ldots$
- -1

Proof: just plug π into Euler's equation. Geometric interpretation: π radians is directly to the left.

Example 4.15.11 ***(?):*** An application: proving the angle sum formulas algebraically. We start by considering the angle $\alpha + \beta$. On one hand, Euler's formula says

$$e^{i(\alpha+\beta)} = \cos(\alpha+\beta) + i\sin(\alpha+\beta) = [\cos(\alpha+\beta), \sin(\alpha+\beta)].$$

On the other hand, we can use properties of exponentials first and expand:

$$\begin{aligned}
e^{i(\alpha+\beta)} &= e^{i\alpha}e^{i\beta} \\
&= (\cos(\alpha) + i\sin(\alpha)) \cdot (\cos(\beta) + i\sin(\beta)) \\
&= \cos(\alpha)\left(\cos(\beta) + i\sin(\beta)\right) + i\sin(\alpha)\left(\cos(\beta) + i\sin(\beta)\right) \\
&= \cos(\alpha)\cos(\beta) + i\cos(\alpha)\sin(\beta) + i\sin(\alpha)\cos(\beta) + i^2\sin(\alpha)\sin(\beta) \\
&= (\cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)) + i\left(\cos(\alpha)\sin(\beta) + \sin(\alpha)\cos(\beta)\right) \\
&= \left[\cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta), \quad \cos(\alpha)\sin(\beta) + \sin(\alpha)\cos(\beta)\right].
\end{aligned}$$

Now we just equate components:

$$[\cos(\alpha+\beta), \sin(\alpha+\beta)] = [\cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta), \quad \cos(\alpha)\sin(\beta) + \sin(\alpha)\cos(\beta)]$$

$$\implies \cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$$
$$\implies \sin(\alpha+\beta) = \cos(\alpha)\sin(\beta) + \sin(\alpha)\cos(\beta).$$

Remark 4.15.12: The analogy goes farther: polar coordinates are essentially just a shadow of complex numbers. Since $e^{i\theta} \in S^1$, we can scale by a radius r to write $z = re^{i\theta}$ and get any point in the plane. If we just draw a vector $\mathbf{v}[r\cos(\theta), r\sin(\theta)]$, note that Euler's formula gives us a way to get a complex number z that corresponds to it:

$$z := re^{i\theta} = r(\cos(\theta) + i\sin(\theta)) = r\cos(\theta) + i \cdot r\sin(\theta) = [r\cos(\theta), r\sin(\theta)] = \mathbf{v}.$$

Remark 4.15.13: Results like these are at the heart of mathematics: having a bunch of equations, seeing patterns, and trying to find some common, unifying, and hopefully simpler structure that underlies all of it. An example you'll see in Calculus: all of the graphs we've been looking at in this class are "shadows" of intersecting shapes in some higher dimensional space!

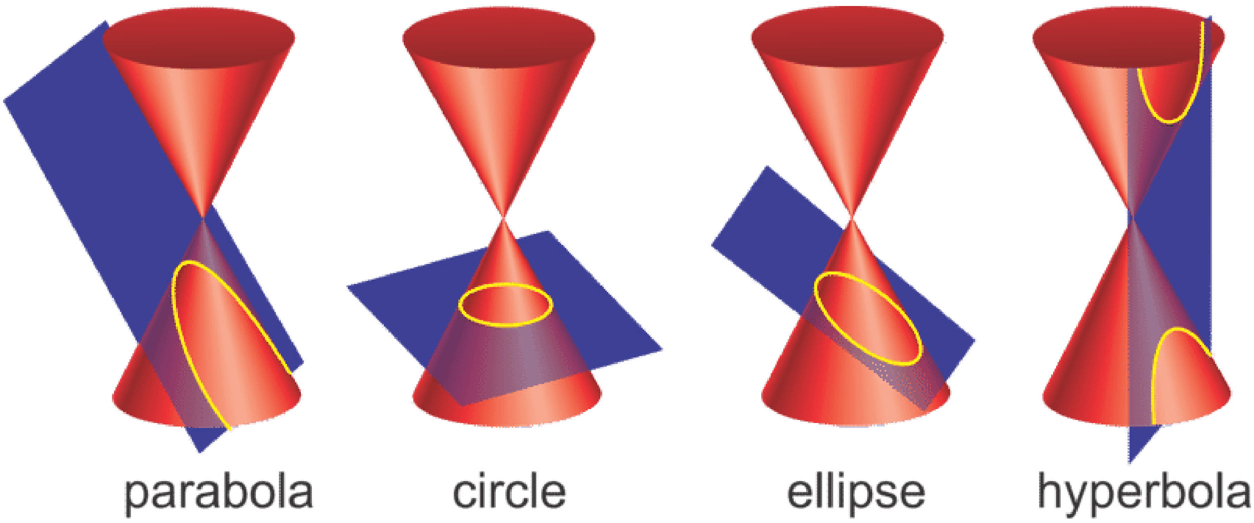

Figure 17: Conic Sections

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