Background in scissors congruence

Remark 0.1. How do we assign a number to area? In ancient Greece, numbers were things which were associated to lengths, so asking for area to correspond to a number didn't really make sense. Euclid defined area as "that which does not change under decomposition." In more modern language, we have the following definition:

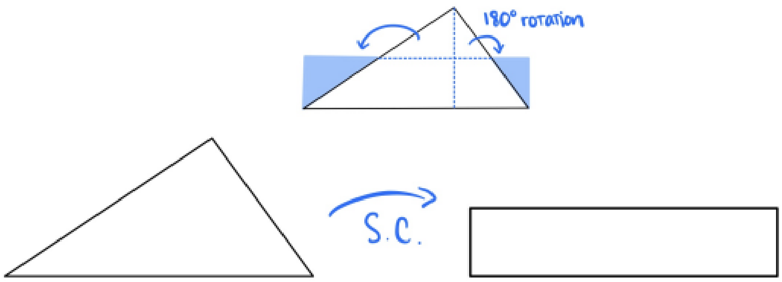
Definition 0.2 (Scissors congruence)**.** Two polygons P and Q are **scissors congruent** if $P = \uplus P_i$ and $Q = \uplus Q_i$ such that $P_i \cong Q_i$ for all i. That is, there is an isometry $g_i \in \text{Isom}(E^2)$ such that $g_i P_i = Q_i$.

Notation. By $P = \uplus P_i$, we mean that $P = \cup P_i$ and the intersections $P_i \cap P_j$ have measure zero.

Theorem 0.3. *P and Q are scissors congruent iff* $\text{area}(P) = \text{area}(Q)$.

Remark 0.4. This theorem tells us Euclid's notion of area was well-defined. The forward implication of the theorem is not too bad so we will focus on the converse.

any triangle into a base$\times\frac{\text{height}}{2}$ rectangle like this:



Question 0.7 (Hilbert's 3rd Problem)**.** Is scissors congruence a well-defined notion of volume? That is, if two polyhedra[1] have the same volume, are they scissors congruent? Can we find a counterexample?

Remark 0.8. The first question to ask is how many cuts we're allowed to make. If we allow *infinitely* many cuts, then the techniques of calculus tell us yes, that two polyhedra of the same volume are scissors congruent. But what if we only allow finitely many cuts? In 1901, shortly(!) after Hilbert proposed this problem, it was answered by his student(!!) Dehn.

Theorem 0.9 (Dehn, 1901)**.** *The cube and the regular tetrahedron are not scissors congruent.*

Remark 0.10. To prove this, Dehn constructs something called the **Dehn invariant** D and shows that $D(\text{cube}) = 0$ but $D(\text{tetrahedon}) \neq 0$.

Dehn invariant

Definition 0.11 (Dehn invariant)**.** The **Dehn invariant** of a polyhedron P is

$$D(P) = \sum_{\text{edges } e} \text{length}(e) \otimes \text{angle}(e)/\pi.$$

Remark 0.12. This invariant $D(P)$ lives in the tensor product $\mathbb{R} \otimes_{\mathbb{Z}} \mathbb{R}/\mathbb{Z}$, but tensor products were not defined until 1938! In 1965, [Syd65] showed that the Dehn invariant and volume completely characterize scissors congruence:

Theorem 0.13 (Sydler, 1965)**.** *If* $\text{vol}(P) = \text{vol}(Q)$ *and* $D(P) = D(Q)$ *then* P *is scissors congruent to* Q.[2]

Definition 0.14 (The polytope algebra)**.** Let X be a geometry, usually hyperbolic $\mathbb{H}^n$, spherical $\mathbb{S}^n$, or Euclidean $\mathbb{E}^n$, and let G be a group of isometries, for example $G = \text{Isom}(X)$). The **polytope algebra** is defined as

$$\mathcal{P}(X, G) := \mathbb{Z}[\text{Polytopes in } X]/\sim$$

where $[P \uplus Q] \sim [P] + [Q]$ and $[P] \sim [gP]$ for all $g \in G$.

Remark 0.15. The first relation lets us decompose elements into smaller pieces and the second relation lets us consider isomorphism classes under the group action.

Theorem 0.16. *If* X *is Euclidean, spherical, or hyperbolic, then* P *and* Q *are scissors congruent iff* $[P] = [Q]$ *in* $\mathcal{P}(X, G)$.

Question 0.18 (Generalized Hilbert's 3rd Problem)**.** Can we understand $\mathcal{P}(X, G)$ for $X = \mathbb{E}^n, \mathbb{S}^n, \mathbb{H}^n$ and $G \leq \text{Isom}(X)$ a subgroup of isometries?

Goncharov's conjecture

Conjecture 0.20 (Goncharov's Conjecture)**.** In dimension $2n + 1$, we have $2n$ Dehn invariants D_i. We can intersect their kernels and map into $\mathbb{R}$ via volume,

$$\bigcap_{i=1}^{n} \ker D_i \xrightarrow{\text{vol}} \mathbb{R}.$$

If the Dehn invariants and volume tell us everything about scissors congruence, the volume map should be injective. Goncharov conjectured that the volume factored through a somewhat mysterious group:

$$\begin{array}{ccc} \bigcap_{i=1}^{n} \ker D_i & \xhookrightarrow{\;f\;} & (\text{gr}^{\gamma}_{n+1} \mathsf{K}_{2n+1}(\mathbb{C}) \otimes \varepsilon(n+1))^{-} \\ & \searrow^{\text{vol}} \quad \mathbb{R} \quad \swarrow^{B_r} & \end{array}$$

where B_r is the **Borel regulator**. We have have the following conjectures concerning this diagram:

- The map f exists[3],
- f is injective, and
- The Borel regulator B_r is injective for $\mathbb{C}$.

The Borel regulator is injective in many cases, e.g. if $\mathbb{C}$ is replaced with a number field, although injectivity over $\mathbb{C}$ is currently unknown.

Remark 0.21. The formula for scissors congruence naturally leads us into the world of algebraic K-theory, where we have the definition for any (commutative) ring R with unit

$$\mathsf{K}_0(R) := \mathbb{Z}[\text{finitely generated projective } R\text{-modules}]/\sim$$

where $[B] = [A] + [C]$ for every exact sequence $A \hookrightarrow B \twoheadrightarrow C$. In particular, notice that this implies $[A] = [A']$ if $A \cong A'$, so this should be reminiscent of Definition 0.14.

different for different fields.

A breakthrough came when Quillen constructed a space (really, a spectrum) $\mathsf{K}(R)$ and defines $\mathsf{K}_n(R) := \pi_n \mathsf{K}(R)$. To construct this space, we make two observations:

K theory of varieties

Definition 7.1 (Grothendieck ring of varieties)**.** The **Grothendieck ring of varieties** $\mathsf{K}_0(\mathsf{Var}_{/S}) = \langle \mathrm{Iso}_S \rangle / \sim$ is the free abelian group generated by isomorphism classes of S-varieties, modulo the relation $[X] = [Y] + [X \setminus Y]$, where Y is a closed subscheme of X, a representative of a class in Iso_S. The ring structure on $\mathsf{K}_0(\mathsf{Var}_{/S})$ is given by $[X] \cdot [Y] = [X \underset{S}{\times} Y]$.

Notation. If $S = \operatorname{Spec} k$ for a field k, we will denote $\mathsf{K}_0(\mathsf{Var}_k) := \mathsf{K}_0(\mathsf{Var}_{/S})$. The **Lefschetz motive** is $\mathbb{L} := [\mathbb{A}^1_{/S}]$.

Constructing from assemblers:

Definition 8.5 (The Grothendieck ring of varieties)**.** Let Sp be a category of spectra – concretely, one can take the category of symmetric spectra of simplicial sets along with its stable model structure with levelwise cofibrations. Let $\mathcal{V}_k$ to be the assembler whose objects are the objects of $\mathsf{Var}_{/k}$ and whose morphisms are closed inclusions of varieties, or equivalently locally closed embeddings of schemes. Since the field k will be fixed in the statements of most theorems, we will suppress the base field and write $\mathcal{V}$.

Let $\mathsf{K}(\mathcal{V})$ be its associated K-theory spectrum. The group $\mathsf{K}_0(\mathcal{V}) := \pi_0 \mathsf{K}(\mathcal{V})$ has a ring structure and can be shown to coincide with the **Grothendieck ring of varieties** as in Michael's talk. We will write elements in this ring using square brackets, so if X is a variety, $[X]$ denotes its equivalence class in $\mathsf{K}_0(\mathcal{V})$.

Examples in this ring

Remark 7.2. $\mathsf{K}_0(\mathsf{Var}_{/S})$ has the following properties.

1. $[\varnothing] = 0$ and $[S] = 1$.
2. $[\mathbb{P}^n_{/S}] = 1 + \mathbb{L} + \mathbb{L}^2 + \cdots + \mathbb{L}^n$, which can be shown inductively.

- $[\mathbb{G}_m] := [\mathbb{A}^n \setminus \{0\}] = \mathbb{L} - [\text{pt}]$,

- $[\mathbb{P}^1] = \mathbb{L} + [\text{pt}]$,
- For $\mathcal{E} \to X$ a rank n vector bundle[10], $[\mathcal{E}] = [X] \cdot [\mathbb{A}^n] = [X] \cdot \mathbb{L}^n$.

The last example shows that $\mathsf{K}_0(\mathcal{V})$ does not distinguish between trivial and nontrivial bundles. [Bor15] profitably uses this fact and similar computations to prove that a cut-and-paste conjecture of Larsen-Lunts fails, which conjecturally has applications to rationality of motivic zeta functions.

Ring structure

7.3 The ring structure of $\mathsf{K}_0(\mathsf{Var}_k)$

Remark 7.6. $\mathsf{K}_0(\mathsf{Var}_k)$ is a badly behaved ring:

- $\mathsf{K}_0(\mathsf{Var}_k)$ is infinite,
- $\mathsf{K}_0(\mathsf{Var}_k)$ is not Noetherian (Liu–Sebag '10),
- $\mathsf{K}_0(\mathsf{Var}_k)$ is not an integral domain (Poonen '02),
- $\mathsf{K}_0(\mathsf{Var}_k)/\mathbb{L} \cong \mathbb{Z}[\text{SB}]$, where SB denotes stably birational equivalence classes of k-varieties (birational after multiplication by a large projective space),
- $\mathbb{L}$ is a zero divisor over $\mathbb{C}$ (Borisov '18). Borisov constructs varieties X, Y over $\mathbb{C}$ such that

$$[X](\mathbb{L}^2 - 1)(\mathbb{L} - 1)\mathbb{L}^7 = [Y](\mathbb{L}^2 - 1)(\mathbb{L} - 1)\mathbb{L}^7$$

  but $[X] \neq [Y]$.

Birationality and PW iso

Definition 7.3 (Piecewise isomorphic varieties)**.** We say that two S-varieties X and Y are **piecewise isomorphic** if there exist locally closed subvarieties $\{X_i\}_{i\in I}$ of X and $\{Y_j\}_{j\in J}$of Y such that $X = \bigcup_i X_i$, $Y = \bigcup_j Y_j$, and there is a bijection $\sigma\colon I \to J$ such that $X_i \cong Y_{\sigma(i)}$.

Notation. Denote
$$\mathsf{K}_0(\mathsf{Var}_{/S})[\mathbb{L}^{-1}] = \mathcal{M}_S.$$

Definition 8.9 (Birational varieties)**.** Two varieties X, Y are **birational** if and only there is an isomorphism of $\varphi : U \xrightarrow{\sim} V$ of nonempty dense[11] open subschemes. Note that φ need not extend to a well-defined function on all of X and Y, and does not generally imply $X \cong Y$.

Definition 8.11 (Stable birationality)**.** Two varieties X, Y are **stably birational** if and only if there is a birational isomorphism
$$X \times \mathbb{P}^N \overset{\sim}{\dashrightarrow} Y \times \mathbb{P}^M$$
for some N, M large enough.

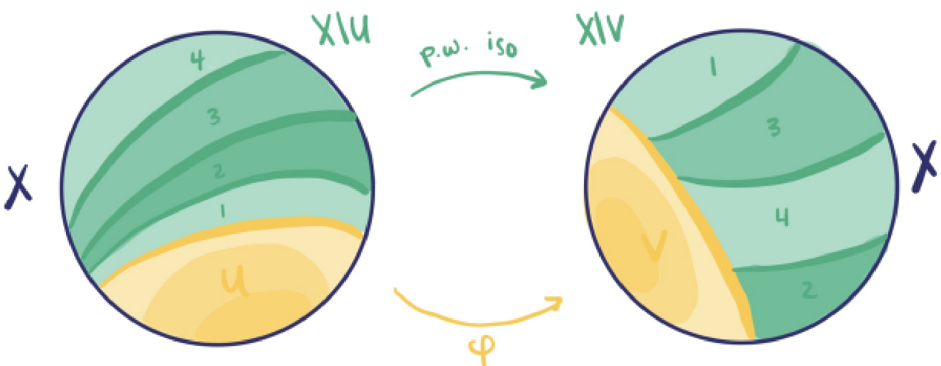
Remark 8.12. Many interesting invariants of birational geometry are in fact *stable* birational invariants. Some examples include:

- The Hodge number
$$h^{0,1}(X) = \dim_{\mathbb{C}} H^{0,1}(X^{\mathrm{an}})$$
where X^{an} as the analytic space associated to X and $H^{p,q}(X^{\mathrm{an}}) \coloneqq H^0(X^{\mathrm{an}}; \Omega^1_{X^{\mathrm{an}}})$,

- the (analytic) fundamental group $\pi_1(X^{\mathrm{an}})$, and
- the zeroth Chow group $\mathrm{CH}_0(X)$.

A recent exposition of other applications of stable birationality is given in [Voi16].

Remark 8.14. This definition of a piecewise isomorphism is meant to capture the notion of cut-and-paste equivalence of varieties. To see how this relates to K-theory, note that if X and Y are piecewise isomorphism, then their classes are equal in $\mathsf{K}_0(\mathcal{V})$. On the other hand, if X and Y are birational, it is not generally the case that their classes are equal in $\mathsf{K}_0(\mathcal{V})$. However, if there is a birational morphism $X \dashrightarrow Y$ defined on $U \subseteq X$ and $V \subseteq Y$ and one *additionally* requires that $X \setminus U \cong Y \setminus V$, then X and Y are in fact piecewise isomorphic and thus have equal classes in $\mathsf{K}_0(\mathcal{V})$.



Motivating questions:

Question 8.15 (Motivating question 1)**.** When is the canonical ring localization morphism $\mathsf{K}_0(\mathcal{V}) \to \mathsf{K}_0(\mathcal{V})[1/\mathbb{L}]$ injective? In particular, when can equations in the localization be pulled back to valid equations in the original ring?

More philosophically, what does equality in $\mathsf{K}_0(\mathcal{V})$ actually *mean* geometrically? What geometric information is the Grothendieck ring capturing, and what conclusions can be drawn from equations in this ring?

Question 8.16 (Motivating question 2)**.** When is $\operatorname{Ann}(\mathbb{L})$ nonzero?

Conjecture 8.20 (A cut-and-paste conjecture of Larsen-Lunts)**.** If $[X] = [Y]$ is an equality the Grothendieck ring K_0, then there is a piecewise isomorphism $X \underset{\text{pw}}{\cong} Y$.

Remark 8.21. This conjecture is now known to be false – Borisov and Karzhemanov construct counterexamples for fields k that embed in $\mathbb{C}$, and [Zak17] shows that this additionally fails for a wider class of *convenient*[13] fields.

Conjecture 8.22. This is almost true, and the only obstructions come from $\operatorname{Ann}(\mathbb{L})$.

Conjecture 8.23. For certain varieties, equality $[X] = [Y]$ in the Grothendieck ring implies that X, Y are **stably birational**.

Remark 8.24. For the second motivating question, why might one care about this *particular* ring-theoretic property? Recall that this condition is equivalent to the injectivity of the map $\cdot\mathbb{L}$, so one answer is that having a nonzero annihilator allows cancellation of $\mathbb{L}$ in equations. Thus computations like the following can be carried out:

$$[X]\cdot\mathbb{L} = [Y]\cdot\mathbb{L} \implies ([X]-[Y])\cdot\mathbb{L} = 0 \overset{\operatorname{Ann}(\mathbb{L})=0}{\implies} [X]-[Y] = 0 \implies [X] = [Y],$$

and so equality "up to a power of $\mathbb{L}$" implies honest equality. A separate mo-

The latter ring appears in conjectures concerning rationality of motivic zeta functions $\zeta_X(t)$. The recent paper [LL20] exhibits a K3 surface X in such that $\zeta_X(t)$ is *not* rational over $\mathsf{K}_0(\mathcal{V})$, and discuss the possibility of its rationality as a formal power series in $\mathsf{K}_0(\mathcal{V})[1/\mathbb{L}]$ instead.

Result using mirror symmetry (Grassmannian-Pfaffian correspondence)

a coincidence at all.

Proposition 8.1 (Borisov)**.** *The cut-and-paste conjecture of Larsen and Lunts is false.*

[13] This is a technical condition to be described later.

Inna's theorem

8.3 Theorems and proof sketches

Theorem 8.28 ([Zak17] Theorem A)**.** *There is a homotopical enrichment of* $\mathsf{K}_0(\mathcal{V})$ *with a simple associated graded. Let*

- $\mathcal{V}^{(n)}$ *be the nth filtered assembler of* $\mathcal{V}$ *generated by varieties of dimension* $d \leq n$,
- $\mathrm{Aut}_k\, k(X)$ *be the group of birational automorphisms of the variety* X,
- B_n *be the set of birational isomorphism classes of varieties of dimension* $d = n$.

There is a spectrum $\mathsf{K}(\mathcal{V})$ *such that* $\mathsf{K}_0(\mathcal{V}) := \pi_0 \mathsf{K}(\mathcal{V})$ *coincides with the previously defined Grothendieck group of varieties, and* $\mathcal{V}^{(n)}$ *induces a filtration on* $\mathsf{K}(\mathcal{V})$ *such that*

$$\mathsf{gr}_n \mathsf{K}(\mathcal{V}) = \bigvee_{[X] \in B_n} \Sigma^\infty_+ \mathbf{B}\mathrm{Aut}_k\, k(X),$$

with an associated spectral sequence

$$E^1_{p,q} = \bigvee_{[X] \in B_n} (\pi_p \Sigma^\infty \mathbf{B}\mathrm{Aut}_k\, k(X) \oplus \pi_p \mathbb{S}) \Rightarrow \mathsf{K}_p(\mathcal{V})$$

Remark 8.29. Note that the $p = 0$ column converges to $\mathsf{K}_0(\mathcal{V})$.