Floer Talk

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Contents

1 Background and Notation

From the text:

- $(W, \omega \in \Omega_2(W))$ is a (compact?) symplectic manifold
- $C^\infty(A, B)$ is the space of smooth maps with the C^∞ topology (idea: uniform convergence of a function and all derivatives on compact subsets)
- $C^\infty_{\text{loc}}(A, B)$ is the space with the C^∞ uniform convergence topology on compact subsets of A
- $H \in C^\infty(W; \mathbb{R})$ a Hamiltonian with X_H its vector field.
- $H \in C^\infty(W \times \mathbb{R}; \mathbb{R})$ given by $H_t \in C^\infty(W; \mathbb{R})$ is a time-dependent Hamiltonian.
- The action functional is given by

$$\begin{aligned} \mathcal{A}_H : \mathcal{L}W &\longrightarrow \mathbb{R} \\ x &\mapsto -\int_{\mathbb{D}} u^* \omega + \int_0^1 H_t(x(t))\ dt \end{aligned}$$

  where $\mathcal{L}W$ is the contractible loop space of W, $u : \mathbb{D} \longrightarrow W$ is an extension of $x : S^1 \longrightarrow W$ to the disc with $u(\exp(2\pi i t)) = x(t)$.

  - Example: $W = \mathbb{R}^{2n} \implies A_H(x) = \displaystyle\int_0^1 (H_t\ dt - p\ dq).$

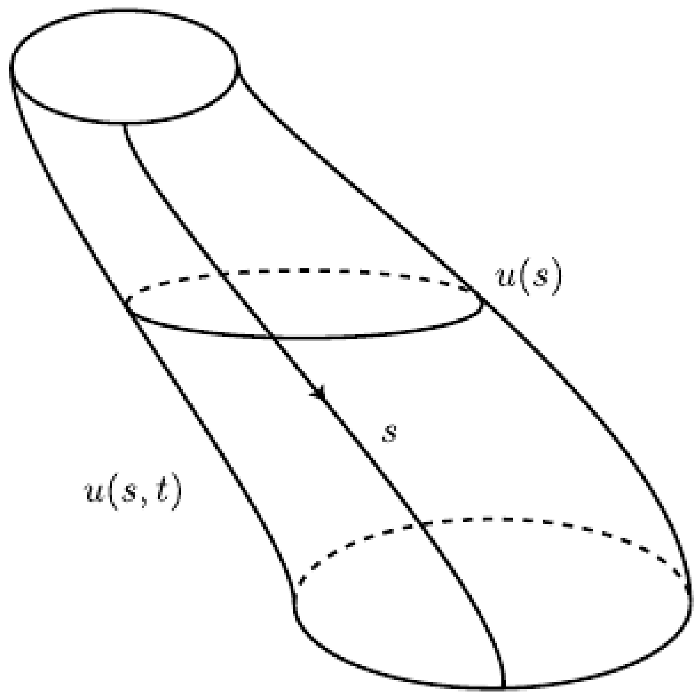
- Critical points of the action functional $\mathcal{A}_H$ are given by orbits, i.e. contractible loops $x, y \in \mathcal{L}W$
- In general, x, y are two periodic orbits of H of period 1.

- The Floer equation is given by

$$\frac{\partial u}{\partial s} + J(u)\frac{\partial u}{\partial t} + \operatorname{grad} H_t(u) = 0.$$

  This is a first-order perturbation of the Cauchy-Riemann equations, for which solutions would be J-holomorphic curves.

- Solutions are functions $u \in C^\infty(\mathbb{R} \times S^1; W) = C^\infty(\mathbb{R}; \mathcal{L}W)$
  - They correspond to "embedded cylinders" with sides u and contractible caps x, y regarded as loops in W.
  - They also correspond to paths in $\mathcal{L}W$ from $x \longrightarrow y$ (precisely: trajectories of the vector field $-\mathrm{grad}\mathcal{A}_H$)



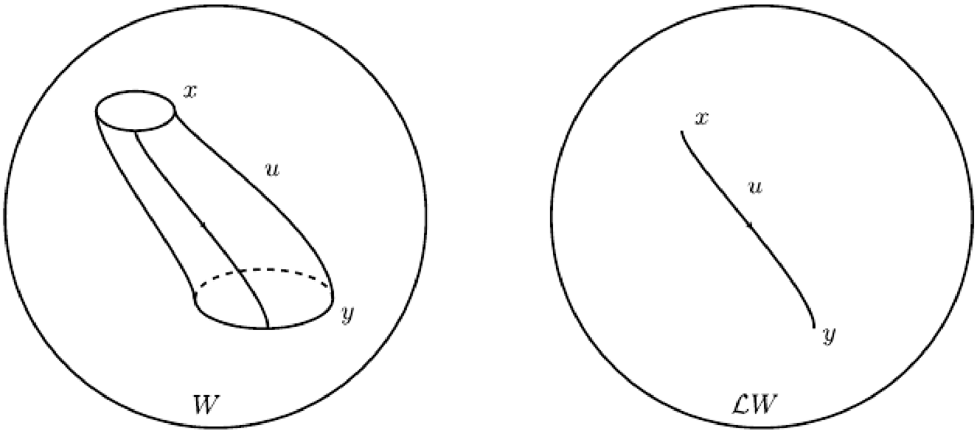


Fig. 6.5

Here $u(s) \in \mathcal{L}W$ is a loop with value at time t given by $u(s,t)$, and $\lim_{s \longrightarrow -\infty} u_s(t) = x$, $\lim_{s \longrightarrow \infty} u_s(t) = y$.

- The energy of a solution is $E(u) = \int_{\mathbb{R}\times S^1} |\partial_s u|^2 \, dsdt$.
- $\mathcal{M} = \left\{ u \in C^\infty(\mathbb{R}; \mathcal{L}W) \;\middle|\; E(u) < \infty \right\}$ (contractible solutions of finite energy), which is compact.
- $\mathcal{M}(x,y)$ is the space of solutions of the Floer equation connecting orbits x and y.
- $C_\searrow(x,y)$:

$$C_\searrow(x,y) := \{u \in C^\infty(\mathbb{R}\times S^1; W) \;\Big|\; \lim_{s\longrightarrow -\infty} u(s,t) = x(t), \quad \lim_{s\longrightarrow\infty} u(s,t) = y(t),$$
$$\left|\frac{\partial u}{\partial s}(s,t)\right| \le K e^{-\delta|s|}, \qquad \left|\frac{\partial u}{\partial t}(s,t) - X_H(u)\right| \le K e^{-\delta|s|}\}$$

where $K, \delta > 0$ are constants depending on u. So

$$|\partial_s u(s,t)|, |\partial_t u(s,t) - X_H(u)| \sim e^{|s|}.$$

From the Appendices

- Relatively compact: has compact closure.
- Compact operator: the image of bounded sets are relatively compact.
- Index of an operator: $\dim\ker - \dim\operatorname{coker}$.
- Fredholm operators: those for which the index makes sense, i.e. $\dim\ker < \infty, \dim\operatorname{coker} < \infty$.
- Elliptic operators: generalize the Laplacian Δ, coefficients of highest order derivatives are positive, principal symbol is invertible (???)
- Locally integrable: integrable on every compact subset

- Sobolev spaces: in dimension 1, define $\|u(t)\|_{s,p} = \sum_{i=0}^{s} \left\|\partial_t^i u(t)\right\|_{L^p}$ on $C^\infty(\overline{U})$, then take the completion and denote $W^{s,p}(\overline{U})$. Yields a distribution space, elements are functions with weak derivatives.
- Distribution: $C_c^\infty(U)^\vee$, the dual of the space of smooth compactly supported functions on an open set $U \subset \mathbb{R}^n$.

2 Talk

Overview: Analyze the space $\mathcal{M}(x, y)$ of solutions to the Floer equation connecting two orbits x, y of H. Show $\mathcal{M}(x, y)$ is in fact a manifold of dimension $\mu(x) - \mu(y)$.

Strategy:

1. Describe $\mathcal{M}(x, y)$ as the zero set of a section of a vector bundle over the Banach manifold $\mathcal{P}(x, y)$.
2. Apply the Sard-Smale theorem: perturb H to make $\mathcal{M}(x, y)$ the inverse image of a regular value of some map.
3. Show that the tangent maps (?) are Fredholm operators of index $\mu(x) - \mu(y) = \dim \mathcal{M}(x, y)$.

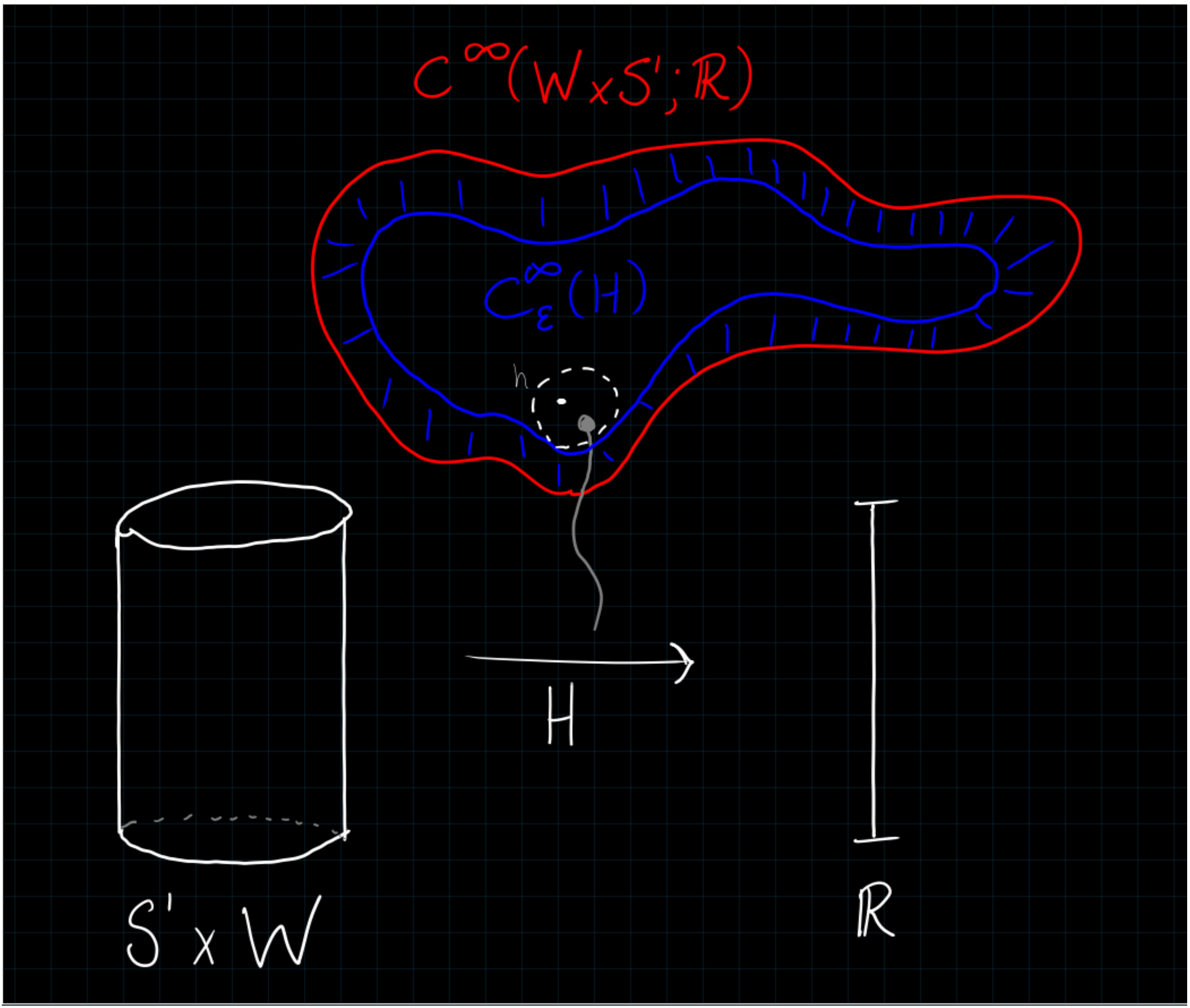
Goals:

- 8.3: Overview and big picture
- 8.4: Formula for linearization of $\mathcal{F}$.

2.1 8.3: The Space of Perturbations of H

Goal: given a fixed Hamiltonian $H \in C^\infty(W \times S^1; \mathbb{R})$, perturb it (without modifying the periodic orbits) so that $\mathcal{M}(x, y)$ are manifolds of the expected dimension.

Start by trying to construct a subspace $\mathcal{C}_\varepsilon^\infty(H) \subset \mathcal{C}^\infty(W \times S^1; \mathbb{R})$, the space of perturbations of H depending on a certain sequence $\varepsilon = \{\varepsilon_k\}$, and show it is a dense subspace.



Idea: similar to how you build $L^2(\mathbb{R})$, define a norm $\|\cdot\|_\varepsilon$ on $C^\infty_\varepsilon(H)$ and take the subspace of finite-norm elements.

- Let $h(\mathbf{x}, t) \in C^\infty_\varepsilon(H)$ denote a perturbation of H.
- Fix $\varepsilon = \left\{\varepsilon_k \;\middle|\; k \in \mathbb{Z}^{\geq 0}\right\} \subset \mathbb{R}^{>0}$ a sequence of real numbers, which we will choose carefully later.
- For a fixed $\mathbf{x} \in W, t \in \mathbb{R}$ and $k \in \mathbb{Z}^{\geq 0}$, define

$$\left|d^k h(\mathbf{x}, t)\right| = \max\left\{d^\alpha h(\mathbf{x}, t) \;\middle|\; |\alpha| = k\right\},$$

  the maximum over all sets of multi-indices α of length k.

  Note: I interpret this as

$$d^{\alpha_1, \alpha_2, \cdots, \alpha_k} h = \frac{\partial^k h}{\partial x_{\alpha_1}\, \partial x_{\alpha_2} \cdots \partial x_{\alpha_k}},$$

  the partial derivatives wrt the corresponding variables.

- Define a norm on $C^\infty(W \times S^1; \mathbb{R})$:

$$\|h\|_\varepsilon = \sum_{k\geq 0} \varepsilon_k \sup_{(x,t)\in W\times S^1} \left| d^k h(x,t) \right| .$$

- Since $W \times S^1$ is assumed compact (?), fix a finite covering $\{B_i\}$ of $W \times S^1$ such that

$$\bigcup_i B_i^\circ = W \times S^1.$$

  - Choose them in such a way we obtain charts

$$\Psi_i : B_i \longrightarrow \overline{B(0,1)} \subset \mathbb{R}^{2n+1} \text{ (?)}.$$

- Obtain the computable form

$$\|h\|_\varepsilon = \sum_{k\geq 0} \varepsilon_k \sup_{(x,t)\in W\times S^1} \sup_{i, z\in B(0,1)} \left| d^k (h \circ \Psi_i^{-1})(z) \right| .$$

- Define

$$C_\varepsilon^\infty = \left\{ h \in C^\infty(W \times S^1; \mathbb{R}) \;\middle|\; \|h\|_\varepsilon < \infty \right\} \subset C^\infty(W \times S^1; \mathbb{R}),$$

  which is a Banach space (normed and complete).

- Show that the sequence $\{\varepsilon_k\}$ can be chosen so that C_ε^∞ is a *dense* subspace for the C^∞ topology, and in particular for the C^1 topology.

Proposition 2.1.
Such a sequence $\{\varepsilon_k\}$ can be chosen.

Lemma 2.2.
$C^\infty(W \times S^1; \mathbb{R})$ with the C^1 topology is separable as a topological space (contains a countable dense subset).

Proof (of Lemma, Sketch).

First prove for C^0:

- **Idea:** reduce to polynomials in $\mathbb{R}^m$.
- Embed $W \times S^1 \hookrightarrow [-M, M]^m \cong I^m \subset \mathbb{R}^m$ for some large m, reduces to proving it for $C^\infty(I^m; \mathbb{R})$.
- Recall Stone-Weierstrass:

  > For $A \leq C^0(X; \mathbb{R})$ a subalgebra with X compact Hausdorff and A containing a nonzero constant function, A is dense iff it separates points (for all $a \neq b \in X$ there exists $f \in A$ such that $f(a) \neq f(b)$)

- Apply to $A = \mathbb{Q}[x_1, \cdots, x_m]$ the subalgebra of polynomial functions, the nonzero constant function $c(x) = 1$, and show it separates points via $f(x) = x - a$, then $f(a) = 0$ and $f(b) = a - b \neq 0$ by assumption.

- Thus A is a countable dense subset.

Then prove for C^1:

- **Idea**: Take polynomials convolved with a countable sequence of bump functions, which is still a countable dense subset.
- Choose a smooth bump function χ supported on $B(0,1)$
- Define the sequence $\chi_k(x) := k^m \chi(kx)$.
- Prove that $(f * \chi_k) \overset{k \longrightarrow \infty}{\longrightarrow} f$ in the C^0_{loc} sense (?)
- Show that for a fixed k, any other sequence $g_\ell \longrightarrow f$ in C^∞_{loc}, we have $g_\ell * \chi_k \longrightarrow f * \chi_k$ in the C^0_{loc} sense using

$$|g_\ell - f| \longrightarrow 0 \implies \sup_K \left| \frac{\partial}{\partial x_i}(g_\ell - f) * \chi_k \right| \leq \sup_k |g_\ell - f| \cdot (\cdots) \longrightarrow 0 \quad \forall i$$

- Conclude $\lim_\ell \lim_k g_\ell * \chi_k = f$.
- Taking g_ℓ to be polynomial approximations, the following subset is countable and dense:

$$\bigcup_{k \in \mathbb{Z}^{\geq 0}} \left\{ P * \chi_k \;\middle|\; P \in \mathbb{Q}[x_1, \cdots, x_m] \right\}$$

  which are pushed through the charts Ψ_i to actually compute.

■

The second part of this proof generalizes to C^∞.

Proof (of Proposition, Sketch).

- By the lemma, produce a sequence $\{f_n\} \subset C^\infty(W \times S^1; \mathbb{R})$ dense for the C^1 topology.
- Using the norm on $C^n(W \times S^1; \mathbb{R})$ for the f_n, define

$$\frac{1}{\varepsilon_n} = 2^n \max \left\{ \|f_k\| \;\middle|\; k \leq n \right\} \implies \varepsilon_n \sup |d^n f_k(x,t)| \leq 2^{-n}$$

  which is summable.

■

Why does this imply density? I don't know.

The next proposition establishes a version of this theorem with compact support:

Proposition 2.3.
For any $(\mathbf{x}, t) \subset U \in W \times S^1)$ there exists a $V \subset U$ such that every $h \in C^\infty(W \times S^1; \mathbb{R})$ can be approximated in the C^1 topology by functions in C^∞_ε supported in U.

Then fix a time-dependent Hamiltonian H_0 with nondegenerate periodic orbits and consider

$$\left\{ h \in C^\infty_\varepsilon(H_0) \;\middle|\; h(x,t) = 0 \text{ in some } U \supseteq \text{the 1-periodic orbits of } H_0 \right\}$$

Then supp(h) is “far” from Per(H_0), so

$$\|h\|_\varepsilon \ll 1 \implies \mathrm{Per}(H_0 + h) = \mathrm{Per}(H_0)$$

and are both nondegenerate.

2.2 Review 8.2

What is $\mathcal{F}$?

We started with the unadorned Floer map:

$$\begin{aligned}\mathcal{F} : \mathcal{C}^\infty\left(\mathbb{R} \times S^1; W\right) &\longrightarrow \mathcal{C}^\infty\left(\mathbb{R} \times S^1; TW\right) \\ u &\mapsto \frac{\partial u}{\partial s} + J\frac{\partial u}{\partial t} + \mathrm{grad}_u\,(H_t)\end{aligned}$$

and promoted this to a map of Banach spaces

$$\begin{aligned}\mathcal{F} : \mathcal{P}^{1,p}(x, y) &\longrightarrow \mathcal{L}^p(x, y) \\ \mathcal{F}(u) &= \frac{\partial u}{\partial s} + J(u)\frac{\partial u}{\partial t} + \mathrm{grad}\, H_t(u).\end{aligned}$$

What is the LHS? It is the space of maps

$$\begin{aligned}\mathcal{P}^{1,p}(x, y) :? &\longrightarrow ? \\ (s, t) &\mapsto \exp_{w(s,t)} Y(s, t).\end{aligned}$$

where $Y \in W^{1,p}(w^*TW)$ and $w \in C^\infty_\searrow(x, y)$.

2.3 8.4: Linearizing the Floer equation: The Differential of $\mathcal{F}$

Choose $m > n = \dim(W)$ and embed $TW \hookrightarrow \mathbb{R}^m$ to identify tangent vectors (such as Z_i, tangents to W along u or in a neighborhood B of u) with actual vectors in $\mathbb{R}^m$.

Why? Bypasses differentiating vector fields and the Levi-Cevita connection.

We can then identify

$$\mathrm{im}\ \mathcal{F} = C^\infty(\mathbb{R} \times S^1; \mathbb{R}^m) \quad \text{or} \quad L^p(\mathbb{R} \times S^1; W),$$

and we seek to compute its differential $d\mathcal{F}$.

We’ve just replaced the codomain here.

Recall that

- x, y are contractible loops in W that are nondegenerate critical points of the action functional $\mathcal{A}_H$,
- $u \in \mathcal{M}(x, y) \subset C^\infty_{\mathrm{loc}}$ denotes a fixed solution to the Floer equation,
- $C_\searrow(x, y)$ was the set of solutions $u : \mathbb{R} \times S^1 \longrightarrow W$ satisfying some conditions.

Recall:

$$C_{\searrow}(x,y) := \{u \in C^\infty(\mathbb{R} \times S^1; W) \mid \lim_{s \longrightarrow -\infty} u(s,t) = x(t), \quad \lim_{s \longrightarrow \infty} u(s,t) = y(t)\}$$
$$\left|\frac{\partial u}{\partial t}(s,t)\right| \quad \text{and} \quad \left|\frac{\partial u}{\partial t}(s,t) - X_H(u)\right| \sim \exp(|s|)$$

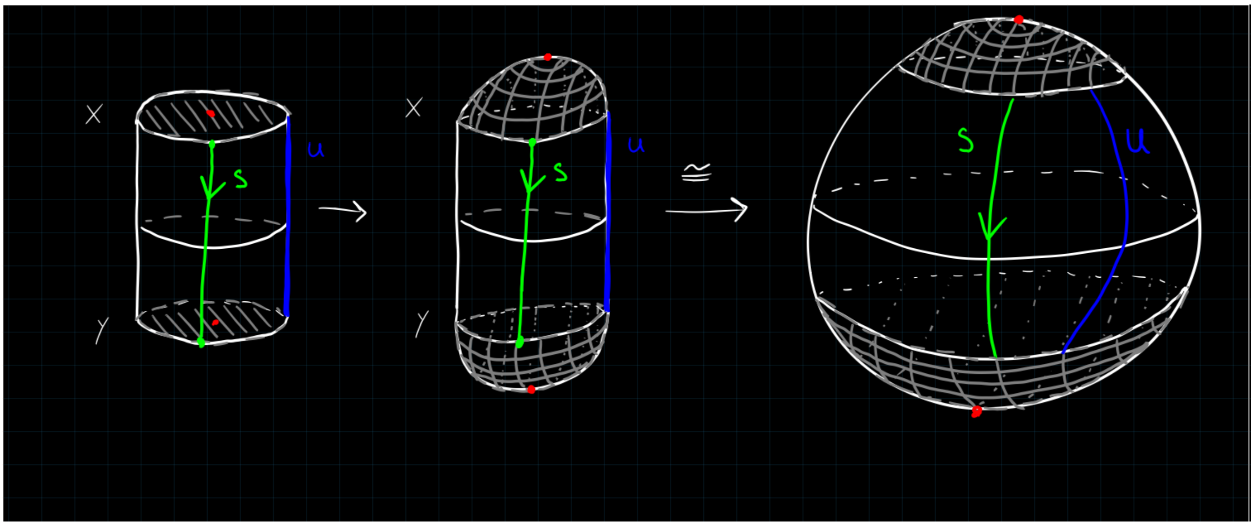
Fix a solution

$$u \in \mathcal{M}(x,y) \subset C^\infty_{\text{loc}}(\mathbb{R} \times S^1; W).$$

We lift each solution to a map

$$\tilde{u} : S^2 \longrightarrow W$$

in the following way: the loops x, y are contractible, so they bound discs. So we extend by pushing these discs out slightly::



From earlier in the book, we have

Assumption (6.22):

For every $w \in C^\infty(S^2, W)$ there exists a symplectic trivialization of the fiber bundle w^*TW, i.e. $\langle c_1(TW),\ \pi_2(W)\rangle = 0$ where c_1 denotes the first Chern class of the bundle TW.

Note: I don't know what this pairing is. The top Chern class is the Euler class (obstructs nowhere zero sections) and are defined inductively:

$$c_1(TW) = e(\Lambda^n(TW)) \in H^2(W; \mathbb{Z})$$

Assumption is satisfied when all maps $S^2 \longrightarrow W$ lift to $B^3 \iff \pi_2(W) = 0$.

We have a pullback that is a symplectic fiber bundle:

$$\begin{tikzcd} \tilde{u}^*TW \arrow[r, "d\tilde{u}"] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & TW \arrow[d] \\ S^2 \arrow[r, "\tilde{u}"] & W \end{tikzcd}$$

- Using the assumption, trivialize the pullback $\tilde{u}^*TW$ to obtain an orthonormal unitary frame

$$\{Z_i\}_{i=1}^{2n} \subset T_{u(s,t)}W$$

  where
  - The frame depends smoothly on $(s,t) \in S^2$,
  - $\lim_{s \longrightarrow \infty} Z_i$ exists for each i.
  -

$$\frac{\partial}{\partial s}, \quad \frac{\partial^2}{\partial s^2}, \quad \frac{\partial^2}{\partial s\, \partial t} \quad \curvearrowright Z_i \overset{s \longrightarrow \pm\infty}{\longrightarrow} 0 \quad \text{for each } i$$

  Claim: such trivializations exist, "using cylinders near the spherical caps in the figure".

Recall what $\mathcal{P}^{1,p}(x,y), J, X_t$ are here.

- Use this frame to define a chart centered at u of $\mathcal{P}^{1,p}(x,y)$ given by

$$\begin{aligned} \iota : W^{1,p}\left(\mathbb{R} \times S^1; \mathbb{R}^{2n}\right) &\longrightarrow \mathcal{P}^{1,p}(x,y) \\ \mathbf{y} = (y_1, \ldots, y_{2n}) &\longmapsto \exp_u\left(\sum y_i Z_i\right). \end{aligned}$$

  - Note that the derivative at zero is $\sum_{i=1}^{2n} y_i Z_i$.

- Define and compute the differential of the composite map $\tilde{\mathcal{F}}$ defined as follows:

$$\begin{tikzcd} \mathcal{P}^{1,p}(x,y) \arrow[r, "\mathcal{F}"] \arrow[rr, bend left, dotted, "\tilde{\mathcal{F}}"] & L^p\left(\mathbb{R} \times S^1; TW\right) \arrow[r] & L^p\left(\mathbb{R} \times S^1; \mathbb{R}^m\right) \end{tikzcd}$$

$$u \xrightarrow{\quad\quad\quad \tilde{\mathcal{F}} \quad\quad\quad} \frac{\partial u}{\partial s} + J(u)\left(\frac{\partial u}{\partial t} - X_t(u)\right)$$

  - From now on, let $\mathcal{F}$ denote $\tilde{\mathcal{F}}$.

- Take the vector

$$Y(s,t) := (y_1(s,t), \cdots) \in \mathbb{R}^{2n} \subset \mathbb{R}^m$$

  - View Y as a vector in $\mathbb{R}^m$ tangent to W, given by $Y = \sum_{i=1}^{2n} y_i Z_i$.

- Plug $u + Y$ into the equation for $\mathcal{F}$, directly yielding

$$\mathcal{F}(u) = \quad \frac{\partial u}{\partial s} \quad + J(u)\frac{\partial u}{\partial t} \quad - J(u)X_t(u)$$

$$\Longrightarrow \mathcal{F}(u+Y) = \frac{\partial(u+Y)}{\partial s} \quad + J(u+Y)\frac{\partial(u+Y)}{\partial t} \quad - J(u+Y)X_t(u+Y)$$

- Extract the part that is linear in Y and collect terms:

$$(d\mathcal{F})_u(Y) = \frac{\partial Y}{\partial s} + (dJ)_u(Y)\frac{\partial u}{\partial t} + J(u)\frac{\partial Y}{\partial t} - (dJ)_u(Y)X_t - J(u)\left(dX_t\right)_u(Y)$$

$$= \left(\frac{\partial Y}{\partial s} + J(u)\frac{\partial Y}{\partial t}\right) + \left((dJ)_u(Y)\frac{\partial u}{\partial t} - (dJ)_u(Y)X_t - J(u)\left(dX_t\right)_u(Y)\right)$$

.

  - This is a sum of two differential operators:
    * One of order 1, one of order 2 (Perspective 1)
    * The Cauchy-Riemann operator, and one of order zero (Perspective 2, not immediate from this form)

- Now compute in charts. Need a lemma:

Lemma 2.4*(Leibniz Rule).*
For any source space X and any maps

$$\begin{aligned} J : X &\longrightarrow \mathrm{End}(\mathbb{R}^m) \\ Y, v : X &\longrightarrow \mathbb{R}^m \end{aligned}$$

we have

$$(dJ)(Y) \cdot v = d(Jv)(Y) - Jdv(Y).$$

Proof .
Differentiate the map

$$\begin{aligned} J \cdot v : X &\longrightarrow \mathbb{R}^m \\ x &\mapsto J(x) \cdot v(x) \end{aligned}$$

to obtain

$$\begin{aligned} J(x+Y)v(x+y) &= (J(x) + (dJ)_x(Y)) \ \cdot \ (v(x) + (dv)_x(Y)) + \cdots \\ &= J(x) \cdot v(x) + J(x) \cdot (dv)_x(Y) + (dJ)_x(Y) \cdot v(x) + (dJ)_x(Y) \cdot (dv)_x(Y) + \cdots \end{aligned}$$

$$\implies d(J \cdot v)_x(Y) = (dJ)_x(Y) \cdot v(x) + J(x) \cdot (dv)_x(Y).$$

∎

- Using the chart ι defined by $\{Z_i\}$ to write $Y = \sum_{i=1}^{2n} y_i Z_i$ and thus

$$(d\mathcal{F})_u(Y) = O_0 + O_1$$

where O_0 are order 0 terms ("they do not differentiate the y_i") and the O_1 are order 1 terms:

$$O_1 = \sum_{i=1}^{2n} \left(\frac{\partial y_i}{\partial s} Z_i + \frac{\partial y_i}{\partial t} J(u) Z_i \right)$$

$$O_0 = \sum_{i=1}^{2n} y_i \left(\frac{\partial Z_i}{\partial s} + J(u) \frac{\partial Z_i}{\partial t} + (dJ)_u(Z_i) \frac{\partial u}{\partial t} - J(u)(dX_t)_u Z_i - (dJ)_u(Z_i) X_t \right).$$

Note: the book seems to be incorrect here, or at least ambiguously worded:

$$(d\mathcal{F})_u(Y) = \sum \left(\frac{\partial y_i}{\partial s} Z_i + \frac{\partial y_i}{\partial t} J(u) Z_i \right) + \sum y_i \left(\frac{\partial Z_i}{\partial s} + J(u) \frac{\partial Z_i}{\partial t} + (dJ)_u(Z_i) \frac{\partial u}{\partial t} - J(u)(dX_t)_u Z_i - (dJ)_u(Z_i) X_t \right).$$

The terms on the first line are "of order 0", that is, they do not differentiate the y_i. We begin by studying the "order 1" terms, the remaining ones. It is

- Study O_1 first, which (claim) reduces to

$$O_1 = \sum_{i=1}^{2n} \left(\frac{\partial y_i}{\partial s} + J_0 \frac{\partial y_i}{\partial t} \right) Z_i = \bar{\partial}(y_1, \cdots, y_{2n}).$$

  where J_0 is the standard complex structure on $\mathbb{R}^{2n} = \mathbb{C}^n$

  - The second equality follows from the assumption that the Z_i are symplectic and orthonormal.
  - Note that this writes $(d\mathcal{F})_u(Y) = O_0 + O_C R$, a sum of an order zero and a Cauchy-Riemann operator.

- Note that since we've computed in charts, we have actually computed the differential of $\mathcal{F}_u$ in the following diagram

$$\begin{tikzcd}
W^{1,p}(\mathbb{R}\times S^1;\mathbb{R}^{2n}) \arrow[r, "\iota"] \arrow[rrr, dashed, bend left, "\mathcal{F}_u"] & \mathcal{P}^{1,p}(x,y) \arrow[r, "\mathcal{F}"] \arrow[rr, dotted, bend left, "\tilde{\mathcal{F}}"] & L^p\left(\mathbb{R}\times S^1;TW\right) \arrow[r] & L^p\left(\mathbb{R}\times S^1;\mathbb{R}^m\right)
\end{tikzcd}$$

$$u \xrightarrow{\quad\tilde{\mathcal{F}}\quad} \frac{\partial u}{\partial s}+J(u)\left(\frac{\partial u}{\partial t}-X_t(u)\right)$$

$$(y_1,\ldots,y_{2n}) \longrightarrow \exp_u\left(\sum y_i Z_i\right)$$

So we've technically computed $(dF_\mu)_0$.

- Remark on the decomposition

$$(d\mathcal{F})_u=\left(\frac{\partial Y}{\partial s}+J(u)\frac{\partial Y}{\partial t}\right)+\left((dJ)_u(Y)\frac{\partial u}{\partial t}-(dJ)_u(Y)X_t-J(u)\left(dX_t\right)_u(Y)\right)$$

$$:=\bar{\partial}Y+SY$$

where $S\in C^\infty(\mathbb{R}\times S^1;\mathrm{End}(\mathbb{R}^n))$ is a linear operator of order 0.

Proposition 2.5 ***(8.4.4, CR + Symmetric in the Limit).***
If u solves Floer's equation, then

$$(d\mathcal{F})_u=\bar{\partial}+S(s,t)$$

where

- S is linear
- S tends to a symmetric operator as $s\longrightarrow\pm\infty$, and
- $$\frac{\partial S}{\partial s}(s,t)\overset{s\longrightarrow\pm\infty}{\longrightarrow}0\quad\text{uniformly in } t$$

Proof .
Omitted – S is exactly O_0 from before:

$$\begin{aligned}
O_0&=\sum_{i=1}^{2n}y_i\left(\frac{\partial Z_i}{\partial s}+J(u)\frac{\partial Z_i}{\partial t}+(dJ)_u(Z_i)\frac{\partial u}{\partial t}-J(u)(dX_t)_uZ_i-(dJ)_u(Z_i)X_t\right)\\
&=\sum_{i=1}^{2n}y_i\left(\frac{\partial Z_i}{\partial s}+(dJ)_u(Z_i)\left(\frac{\partial u}{\partial t}-(Z_i)X_t\right)+J(u)\frac{\partial Z_i}{\partial t}-J(u)(dX_t)_uZ_i\right).
\end{aligned}$$

- The term in blue vanishes as $s\longrightarrow\pm\infty$
  - Using the fact that u is a solution
  - Uses $\dfrac{\partial u}{\partial s}\longrightarrow 0$ uniformly (as do its derivatives?)

- Suffices to show the remaining part is symmetric in the limit, i.e. write as

$$A(y_1, \cdots, y_{2n}) = \cdots \implies A_{ij} = A_{ji}$$

  using inner product calculations
- Uses the fact the Z_i needed to be chosen to be unitary and symplectic. ■

- Write O_1 as a map $Y \mapsto S \cdot Y$, so $S \in C^\infty(\mathbb{R} \times S^1; \mathrm{End}(\mathbb{R}^{2n}))$ and define the symmetric operators

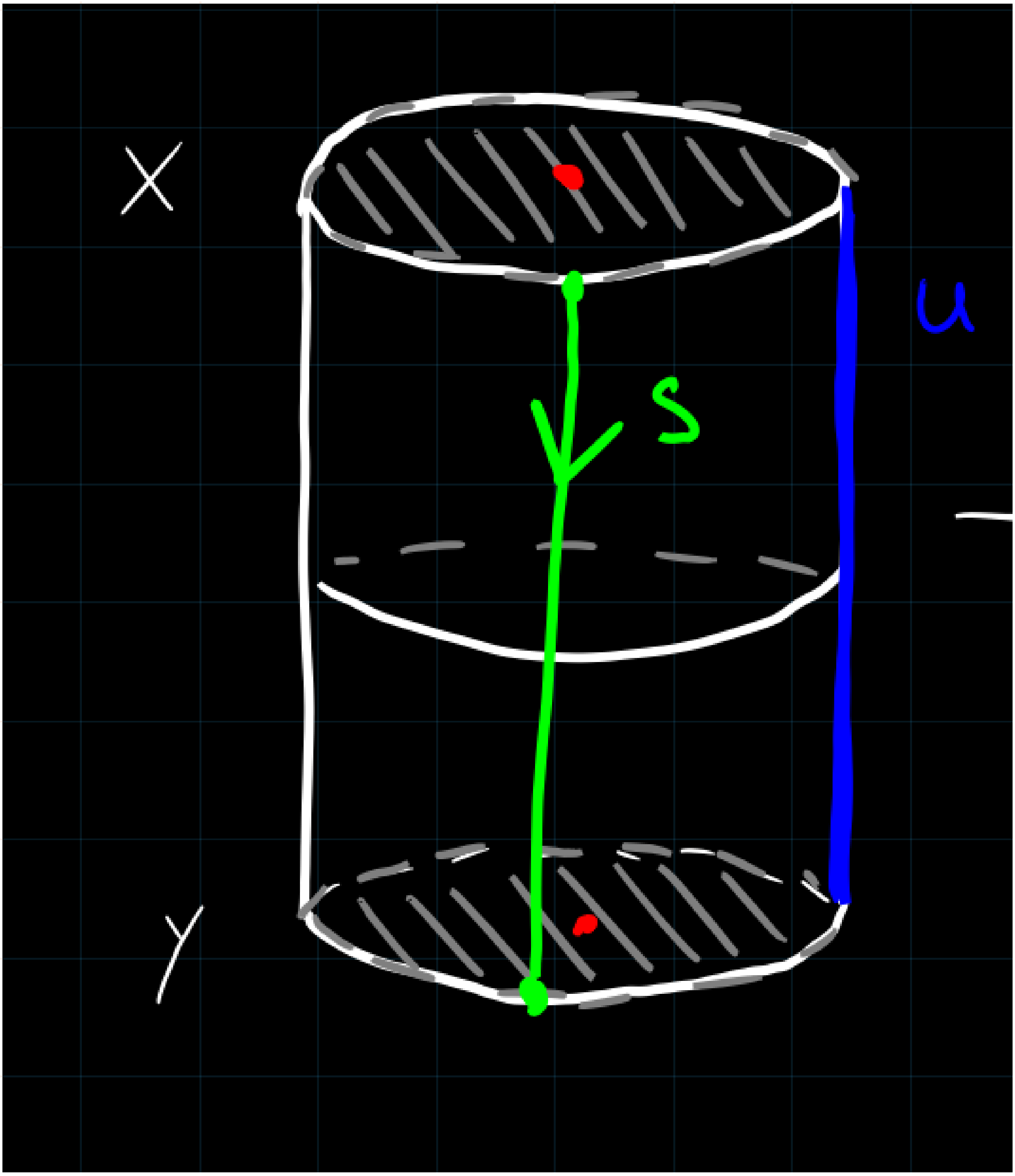
$$S^{\pm} := \lim_{s \longrightarrow \pm\infty} S(s, \cdot) \quad \text{respectively}$$

Proposition 2.6.
The equation

$$\partial_t Y = J_0 S^{\pm} Y$$

linearizes Hamilton's equation

$$\frac{\partial z}{\partial t} = X_t(z) \quad \text{at} \quad \begin{cases} x = \lim\limits_{s \longrightarrow -\infty} u & \text{for } S^- \\ y = \lim\limits_{s \longrightarrow \infty} u & \text{for } S^+ \end{cases} \quad \text{respectively.}$$



Proof .
Omitted. Sketch:

- Use the fact that $\dfrac{\partial Y}{\partial t} = (dX_t)_x Y$
- Expand $\sum \dfrac{\partial y_i}{\partial t} Z_i$ in the Z_i basis (roughly) to write $\dfrac{\partial y_i}{\partial t} = \sum b_{ij} y_j$ for some coefficients

b_{ij}.

- Collect terms into a matrix/operator $B^{\mp}$ for x, y respectively to write

$$\frac{\partial Y}{\partial t} = B^{-} \cdot Y$$

- Write $(d\mathcal{F})_u = \bar{\partial} + S$ where S is zero order and symmetric in the limit
- Get the corresponding operator A in coordinates
- Expand in a basis (roughly) as $A(\sum y_i Z_i) = \sum s_{ij} y_j Z_i$
- Check that $s_{ij} = \pm b_{i \pm n, j}$
- This implies

$$S^{-} = -J_0 B^{-} \quad S^{+} = -J_0 B^{+} \implies \frac{\partial Y}{\partial t} = J_0 S^{\pm} Y$$

■

- Given a solution u, we have a right $\mathbb{R}$-action, so for $s \in \mathbb{R}$,

$$\begin{aligned} u \cdot s \in C^{\infty}(\mathbb{R} \times S^1; W) \\ (\sigma, t) \mapsto u(\sigma + s, t) \end{aligned}$$

  is also a solution, so $\mathcal{F}(u \cdot s) = 0$ for all s.

  - In other words: we can flow solutions?

- **Punchline**: $\frac{\partial u}{\partial s}$ **is a solution of the linearized equation**, since

$$0 = \frac{\partial}{\partial s} \mathcal{F}(u \cdot s) = (d\mathcal{F})_u \left(\frac{\partial u}{\partial s} \right).$$

  - Along any nonconstant solution connecting x and y, $\dim \ker (d\mathcal{F})_u \geq 1$.