Linearization Continued

Section 8.4 Follow-Up

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Review

Definitions

- The Floer equation is given by

$$\frac{\partial u}{\partial s} + J(u)\frac{\partial u}{\partial t} + \operatorname{grad} H_t(u) = 0.$$

- We fixed a solution and lifted it to a sphere:

$$u \in C^\infty(S^1 \times \mathbb{R}; W) \quad \mapsto \quad \tilde{u} \in C^\infty(S^2; W)$$

- We use the assumption:
  For every $w \in C^\infty(S^2, W)$ there exists a symplectic trivialization of the fiber bundle $w^* TW$, i.e. $\langle c_1(TW),\ \pi_2(W)\rangle = 0$ where c_1 denotes the first Chern class of the bundle TW.

- We use this trivialize the pullback $\tilde{u}^* TW$ to obtain an orthonormal unitary frame

$$\{Z_i\}_{i=1}^{2n} \subset T_{u(s,t)} W$$

The Frame

- We used the chosen frame $\{Z_i\}$ to define a chart centered at u of $\mathcal{P}^{1,p}(x,y)$ given by

$$\begin{aligned}\iota : W^{1,p}\left(\mathbb{R}\times S^1; \mathbb{R}^{2n}\right) &\longrightarrow \mathcal{P}^{1,p}(x,y)\\ Y=(y_1,\ldots,y_{2n}) &\longmapsto \exp_u\left(\sum y_i Z_i\right).\end{aligned}$$

- We regard $Y(s,t)$ as a tangent vector to W in some Euclidean embedding.

Charts

- We seek to compute the composite map in charts:

$$W^{1,p}(\mathbb{R}\times S^1;\mathbb{R}^{2n}) \xrightarrow{\iota} \mathcal{P}^{1,p}(x,y) \xrightarrow{\mathcal{F}} L^p\left(\mathbb{R}\times S^1;TW\right) \longrightarrow L^p\left(\mathbb{R}\times S^1;\mathbb{R}^m\right)$$

$$\mathcal{F}_u : W^{1,p}(\mathbb{R}\times S^1;\mathbb{R}^{2n}) \dashrightarrow L^p\left(\mathbb{R}\times S^1;\mathbb{R}^m\right), \qquad \bar{\mathcal{F}} : \mathcal{P}^{1,p}(x,y) \dashrightarrow L^p\left(\mathbb{R}\times S^1;\mathbb{R}^m\right)$$

$$u \xrightarrow{\bar{\mathcal{F}}} \frac{\partial u}{\partial s} + J(u)\left(\frac{\partial u}{\partial t} - X_t(u)\right)$$

$$(y_1,\ldots,y_{2n}) \longrightarrow \exp_u\left(\sum y_i Z_i\right)$$

Add a Tangent

$$\mathcal{F}(u) = \frac{\partial u}{\partial s} + J(u)\frac{\partial u}{\partial t} - J(u)X_t(u)$$

$$\mathcal{F}(u+Y) = \frac{\partial (u+Y)}{\partial s} + J(u+Y)\frac{\partial (u+Y)}{\partial t} - J(u+Y)X_t(u+Y)$$

Extract the part that is linear in Y and collect terms:

$$\begin{aligned}
(d\mathcal{F})_u(Y) \\
&= \frac{\partial Y}{\partial s} + (dJ)_u(Y)\frac{\partial u}{\partial t} + J(u)\frac{\partial Y}{\partial t} - (dJ)_u(Y)X_t - J(u)\left(dX_t\right)_u(Y) \\
&= \left(\frac{\partial Y}{\partial s} + J(u)\frac{\partial Y}{\partial t}\right) \\
&\quad + \left((dJ)_u(Y)\frac{\partial u}{\partial t} - (dJ)_u(Y)X_t - J(u)\left(dX_t\right)_u(Y)\right)
\end{aligned}$$

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8.4.1: Leibniz Rule

Leibniz Rule

Recall the Leibniz rule

$$(dJ)(Y)\cdot v = d(Jv)(Y) - Jdv(Y)$$

$$\begin{aligned}
(d\mathcal{F})_u(Y) &= \left(\frac{\partial Y}{\partial s} + J(u)\frac{\partial Y}{\partial t}\right) \\
&\quad + \left((dJ)_u(Y)\frac{\partial u}{\partial t} - (dJ)_u(Y)X_t - J(u)\left(dX_t\right)_u(Y)\right) \\
&= \sum_{i=1}^{2n}\left(\frac{\partial y_i}{\partial s}Z_i + \frac{\partial y_i}{\partial t}J(u)Z_i\right) \\
&\quad + \sum_{i=1}^{2n} y_i\left(\frac{\partial Z_i}{\partial s} + J(u)\frac{\partial Z_i}{\partial t} + (dJ)_u(Z_i)\frac{\partial u}{\partial t}\right. \\
&\qquad\qquad \left. - J(u)(dX_t)_u Z_i - (dJ)_u(Z_i)X_t\right).
\end{aligned}$$

Use the fact that this is $O_1 + O_0$ in Y.

Order 1

Study O_1 first, which (claim) reduces to

$$O_1 = \sum_{i=1}^{2n} \left(\frac{\partial y_i}{\partial s} + J_0 \frac{\partial y_i}{\partial t} \right) Z_i = \bar{\partial}(y_1, \cdots, y_{2n}).$$

where J_0 is the standard complex structure on $\mathbb{R}^{2n} = \mathbb{C}^n$

Use this to write

$$(d\mathcal{F})_u = \bar{\partial} Y + SY$$

where $S \in C^\infty(\mathbb{R} \times S^1; \mathrm{End}(\mathbb{R}^n))$ is a linear operator of order 0.

Order 0 Part is Symmetric in the Limit

8.4.4: Order 0 Part is Symmetric in the Limit

Theorem (8.4.4, CR + Symmetric in the Limit)

If u solves Floer's equation, then

$$(d\mathcal{F})_u = \bar{\partial} + S(s,t)$$

where

1. *S is linear*
2. *S tends to a symmetric operator as $s \longrightarrow \pm\infty$, and*
3. *We have the limiting behavior*

$$\frac{\partial S}{\partial s}(s,t) \overset{s \longrightarrow \pm\infty}{\longrightarrow} 0 \quad \textit{uniformly in } t$$

Proof

Collect terms in the order zero part:

$$\begin{aligned} O_0 = S(y_1, \cdots, y_{2n}) &= \sum_{i=1}^{2n} y_i \Bigg[\frac{\partial Z_i}{\partial s} + J(u)\frac{\partial Z_i}{\partial t} + (dJ)_u(Z_i)\frac{\partial u}{\partial t} \\ &\qquad - J(u)(dX_t)_u Z_i - (dJ)_u(Z_i)X_t \Bigg] \\ &= \sum_{i=1}^{2n} y_i \Bigg[\frac{\partial Z_i}{\partial s} + (dJ)_u(Z_i)\left(\frac{\partial u}{\partial t} - (Z_i)X_t\right) \\ &\qquad + J(u)\frac{\partial Z_i}{\partial t} - J(u)(dX_t)_u Z_i \Bigg]. \end{aligned}$$

- Claim: the terms in blue and orange vanish in the limit $s \longrightarrow \pm\infty$, so it suffices to prove that the red term limits to a symmetric operator.

Proof: Blue Term Vanishes

$$(dJ)_u(Z_i)\left(\frac{\partial u}{\partial t} - (Z_i)X_t\right) \longrightarrow 0$$

The term in blue vanishes: since u is a solution and

$$\frac{\partial u}{\partial s} \overset{s \longrightarrow \pm\infty}{\longrightarrow} 0 \quad \text{uniformly}$$

as do its derivatives, we have

$$\left(\frac{\partial u}{\partial t} - X_t(u)\right) \overset{s \longrightarrow \pm\infty}{\longrightarrow} 0$$

This seems to be the full argument for the blue term.

Proof: Orange Term Vanishes (1 and 3)

$$\frac{\partial Z_i}{\partial s} \xrightarrow{s \longrightarrow \pm\infty} 0$$

Follows since the frame Z_i was chosen such that

$$\frac{\partial}{\partial s}, \quad \frac{\partial^2}{\partial s^2}, \quad \frac{\partial^2}{\partial s\, \partial t} \quad \curvearrowright Z_i \xrightarrow{s \longrightarrow \pm\infty} 0 \quad \text{for each } i$$

This also implies

$$\frac{\partial S}{\partial s} \xrightarrow{s \longrightarrow \pm\infty} 0.$$

This shows parts (1) and (3) of the theorem: linearity and limits to zero uniformly in t?

Proof: Symmetry

Write the remaining red term as

$$A := A(y_1, \ldots, y_{2n}) = \sum y_i \left(J(u) \frac{\partial Z_i}{\partial t} - J(u) \left(dX_t\right)_u (Z_i) \right).$$

Extract the jth component:

$$A_j = \sum y_i \left\langle J(u) \frac{\partial Z_i}{\partial t} - J(u) \left(dX_t\right)_u (Z_i), \quad Z_j \right\rangle.$$

We'll show that

$$\lim_{s \to \pm\infty} \left\langle J(u) \frac{\partial Z_i}{\partial t} - J(u) \left(dX_t\right)_u (Z_i), Z_j \right\rangle - \left\langle J(u) \frac{\partial Z_j}{\partial t} - J(u) \left(dX_t\right)_u Z_j, \quad Z_i \right\rangle = 0.$$

Proof: Symmetry

Use the fact that the frame $\{Z_i\}$ is unitary:

$$\begin{aligned}
0 &= \frac{\partial}{\partial t} \langle J(u) Z_i, Z_j \rangle \\
&= \left\langle (dJ)_u \left(\frac{\partial u}{\partial t}\right) Z_i, Z_j \right\rangle + \left\langle J(u) \frac{\partial Z_i}{\partial t}, Z_j \right\rangle + \left\langle J(u) Z_i, \frac{\partial Z_j}{\partial t} \right\rangle \\
&= \left\langle (dJ)_u \left(\frac{\partial u}{\partial t}\right) Z_i, Z_j \right\rangle + \left\langle J(u) \frac{\partial Z_i}{\partial t}, Z_j \right\rangle - \left\langle Z_i, J(u) \frac{\partial Z_j}{\partial t} \right\rangle .
\end{aligned}$$

Proof: Symmetry

Therefore it suffices to show

$$
\begin{aligned}
& -\left\langle J(u)\left(dX_t\right)_u\left(Z_i\right), \quad Z_j\right\rangle \\
& +\left\langle J(u)\left(dX_t\right)_u\left(Z_j\right), \quad Z_i\right\rangle \\
& -\left\langle (dJ)_u\left(\frac{\partial u}{\partial t}\right) Z_i, \quad Z_j\right\rangle
\end{aligned}
$$

$$
\overset{s\longrightarrow\pm\infty}{\longrightarrow} 0.
$$

Proof: Symmetry

Using the fact that

$$\left(\frac{\partial u}{\partial t} - X_t(u) \right) \overset{s \longrightarrow \pm\infty}{\longrightarrow} 0$$

we can equivalently show

$$\begin{aligned} &- \left\langle J(u) \left(dX_t \right)_u (Z_i), \quad Z_j \right\rangle \\ &+ \left\langle J(u) \left(dX_t \right)_u (Z_j), \quad Z_i \right\rangle \\ &- \langle (dJ)_u (X_t) Z_i, \quad Z_j \rangle \end{aligned}$$

$$\overset{s \longrightarrow \pm\infty}{\longrightarrow} 0$$

For a fixed (s, t), this expression only depends on Z_i at the point $u(s, t)$.

Lemma That Concludes the Proof

Lemma

Lemma: For $p \in W$, $\{Z_i\}$ a unitary basis of $T_p W$,

$$\begin{aligned} & -\left\langle J(p)\left(dX_t\right)_p\left(Z_i\right), \quad Z_j\right\rangle \\ & +\left\langle J(p)\left(dX_t\right)_p\left(Z_j\right), \quad Z_i\right\rangle \\ & -\left\langle (dJ)_p\left(X_t\right) Z_i, \qquad Z_j\right\rangle \\ & = 0. \end{aligned}$$

Claim: This lemma immediately concludes the previous proof?

Proof of Lemma

Extend $\{Z_i\}$ to a chart containing p and use the Leibniz rule to rewrite

$$-\left\langle J(p)\left(dX_t\right)_p\left(Z_i\right), Z_j\right\rangle + \left\langle J(p)\left(dX_t\right)_p\left(Z_j\right), Z_i\right\rangle - \langle (dJ)_p\left(X_t\right) Z_i, Z_j\rangle = 0$$

as

$$-\left\langle J\left(dX_t\right)\left(Z_i\right), Z_j\right\rangle + \left\langle J\left(dX_t\right)\left(Z_j\right), Z_i\right\rangle + \left\langle J\left(dZ_i\right)\left(X_t\right), Z_j\right\rangle - \left\langle d\left(JZ_i\right)\left(X_t\right), Z_j\right\rangle$$

$$= \left\langle J\left[X_t, Z_i\right], Z_j\right\rangle + \left\langle J\left(dX_t\right)\left(Z_j\right), Z_i\right\rangle - \left\langle d\left(JZ_i\right)\left(X_t\right), Z_j\right\rangle.$$

where we'll rewrite the red terms.

Proof of Lemma

Now use

$$X_t\langle JZ_i,\ Z_j\rangle = 0 \implies \langle d\left(JZ_i\right)\left(X_t\right), Z_j\rangle + \langle JZ_i, \left(dZ_j\right)\left(X_t\right)\rangle = 0.$$

We now rewrite the RHS from before:

$$\langle J\left[X_t, Z_i\right], Z_j\rangle + \langle J\left(dX_t\right)\left(Z_j\right), Z_i\rangle + \langle JZ_i, \left(dZ_j\right)\left(X_t\right)\rangle$$

$$\begin{aligned} &= \langle J\left[X_t, Z_i\right], Z_j\rangle + \langle J\left(dX_t\right)\left(Z_j\right) - J\left(dZ_j\right)\left(X_t\right), Z_i\rangle \\ &= \langle J\left[X_t, Z_i\right], Z_j\rangle - \langle J\left[X_t, Z_j\right], Z_i\rangle \\ &= \omega\left(\left[X_t, Z_i\right], Z_j\right) - \omega\left(\left[X_t, Z_j\right], Z_i\right). \end{aligned}$$

The symmetry follows from ω being closed and

$$\begin{aligned} 0 &= d\omega\left(X_t, Z_i, Z_j\right) \\ &= X_t \cdot \omega\left(Z_i, Z_j\right) - Z_i \cdot \omega\left(X_t, Z_j\right) + Z_j \cdot \omega\left(X_t, Z_i\right) \\ &\quad - \omega\left(\left[X_t, Z_i\right], Z_j\right) + \omega\left(\left[X_t, Z_j\right], Z_i\right) - \omega\left(\left[Z_i, Z_j\right], X_t\right) \\ &= -X_t \cdot \langle Z_i, JZ_j\rangle + Z_i \cdot \left(dH_t\right)\left(Z_j\right) - Z_j \cdot \left(dH_t\right)\left(Z_i\right) \\ &\quad - \left(dH_t\right)\left(\left[Z_i, Z_j\right]\right) - \omega\left(\left[X_t, Z_i\right], Z_j\right) + \omega\left(\left[X_t, Z_j\right], Z_i\right) \\ &= d\left(dH_t\right)\left(Z_i, Z_j\right) - \omega\left(\left[X_t, Z_i\right], Z_j\right) + \omega\left(\left[X_t, Z_j\right], Z_i\right) \\ &= -\omega\left(\left[X_t, Z_i\right], Z_j\right) + \omega\left(\left[X_t, Z_j\right], Z_i\right). \end{aligned}$$

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8.4.6: Linearization of Hamilton's Equation

Linearization of Hamilton's Equation

Recall

$$(d\mathcal{F})_u = \bar{\partial} Y + SY = \left(\bar{\partial} + S\right) Y$$

Now think of S as a map $Y \mapsto S \cdot Y$, so $S \in C^\infty(\mathbb{R} \times S^1; \mathrm{End}(\mathbb{R}^{2n}))$ and define the symmetric operators

$$S^\pm := \lim_{s \longrightarrow \pm\infty} S(s, \cdot) \quad \text{respectively}$$

Theorem

The equation

$$\partial_t Y = J_0 S^\pm Y$$

is a linearization of Hamilton's equation

$$\frac{\partial z}{\partial t} = X_t(z) \quad \textit{at} \quad \begin{cases} x = \lim_{s \longrightarrow -\infty} u & \textit{for } S^- \\ y = \lim_{s \longrightarrow \infty} u & \textit{for } S^+ \end{cases} \quad \textit{respectively.}$$

Proof

We first linearize Hamilton's equation at x:

$$\frac{\partial z}{\partial t} = X_t(z) \quad \overset{\text{linearized}}{\Longrightarrow} \quad \frac{\partial Y}{\partial t} = (dX_t)_x Y.$$

So write $Y = \sum y_i Z_i$ to obtain

$$\begin{aligned}
\sum_i \frac{\partial y_i}{\partial t} Z_i &= \sum_i y_i \left(-\frac{\partial Z_i}{\partial t} + (dX_t)(Z_i) \right) \\
&= \sum_i \sum_j y_i \left\langle -\frac{\partial Z_i}{\partial t} + (dX_t)(Z_i), Z_j \right\rangle Z_j \\
&= \sum_i \sum_j y_j \left\langle -\frac{\partial Z_i}{\partial t} + (dX_t)(Z_j), Z_i \right\rangle Z_i
\end{aligned}$$

$$\implies \frac{\partial y_i}{\partial t} = \sum_j \left\langle -\frac{\partial Z_j}{\partial t} + (dX_t)(Z_j), Z_i \right\rangle y_j.$$

Proof

Thus we can rewrite the linearized equation as

$$\frac{\partial Y}{\partial t} = (dX_t)_x Y = B^- \cdot Y, \quad b_{ij} = \left\langle -\frac{\partial Z_j}{\partial t} + (dX_t)_x (Z_j), Z_i \right\rangle.$$

Recall

$$A := A(y_1, \dots, y_{2n}) = \sum y_i \left(J(u) \frac{\partial Z_i}{\partial t} - J(u) (dX_t)_u (Z_i) \right).$$

Now take $s \longrightarrow -\infty$ and look at the order zero part of $(d\mathcal{F})_u$. By the proof of 8.4.4, we have

$$\begin{aligned}
A\left(\sum y_i Z_i\right) &= \sum_i \left(J(x) \frac{\partial Z_i}{\partial t} - J(x) (dX_t)_x (Z_i) \right) \\
&= \sum_i \sum_j y_i \left\langle J \frac{\partial Z_i}{\partial t} - J(dX_t)(Z_i), Z_j \right\rangle Z_j \\
&= \sum_i \sum_j y_j \left\langle J \frac{\partial Z_j}{\partial t} - J(dX_t)(Z_j), Z_i \right\rangle Z_i \\
&= \sum_i \sum_j \left\langle -\frac{\partial Z_j}{\partial t} + (dX_t)(Z_j), JZ_i \right\rangle y_j Z_i.
\end{aligned}$$

Proof

Deduce that

$$S = (s_{ij}), \quad s_{ij} = \left\langle -\frac{\partial Z_j}{\partial t} + (dX_t)_x (Z_j), \quad J(x) Z_i \right\rangle$$

$$JZ_i = \begin{cases} Z_{i+n} & i \leq n \\ -Z_{i-n} & i \geq n+1 \end{cases} \implies s_{ij} = \begin{cases} b_{i+n,j} & i \leq n \\ -b_{i-n,j} & i \geq n+1 \end{cases}$$

$$\iff S^- = -J_0 B^-.$$

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