Linearization and Transversality

Sections 8.3 and 8.4

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Review

Recalling Notation

- The Floer equation is given by

$$\frac{\partial u}{\partial s} + J(u)\frac{\partial u}{\partial t} + \operatorname{grad} H_t(u) = 0.$$

- Critical points of the action functional $\mathcal{A}_H$ are given by orbits, i.e. contractible loops $x, y \in \mathcal{L}W$
- In general, x, y are two periodic orbits of H of period 1.
- Solutions are functions $u \in C^\infty(\mathbb{R} \times S^1; W) = C^\infty(\mathbb{R}; \mathcal{L}W)$
- $\mathcal{M}(x, y)$ is the moduli space of solutions of the Floer equation connecting orbits x and y.
- $W^{1,p}(x, y)$ and $\mathcal{P}^{1,p}(x, y)$ were completions of $C^\infty(?)$ with respect to certain norms.

The "Program" for Chapter 8

- Show that $\mathcal{M}(x, y)$ is a manifold of dimension $\mu(x) - \mu(y)$
- Define $\mathcal{M}(x, y)$ as the inverse image of a regular value of some map
- Perturb H to apply the Sard-Smale theorem
- Show the tangent maps are Fredholm operators and compute their index.

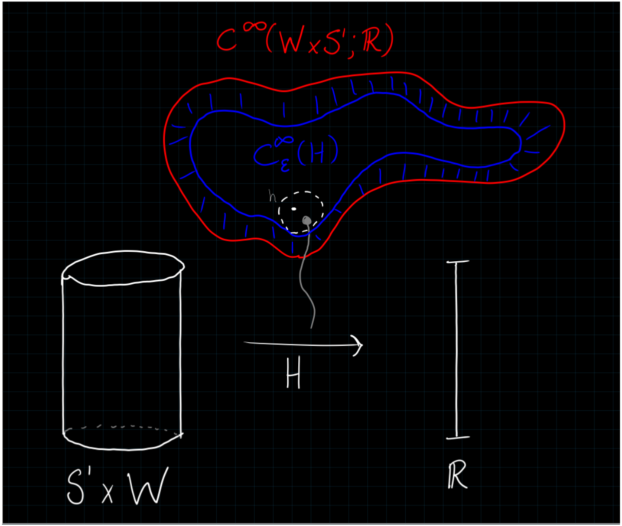
Section 8.3: The Space of Perturbations of H

Goal

Goal: Given a fixed Hamiltonian $H \in C^\infty(W \times S^1; \mathbb{R})$, perturb it (without modifying the periodic orbits) so that $\mathcal{M}(x, y)$ are manifolds of the expected dimension.

Goal

Start by trying to construct a subspace $\mathcal{C}^{\infty}_{\varepsilon}(H) \subset \mathcal{C}^{\infty}(W \times S^1; \mathbb{R})$, the space of perturbations of H depending on a certain sequence $\varepsilon = \{\varepsilon_k\}$, and show it is a dense subspace.



Define an Absolute Value

Idea: similar to how you build $L^2(\mathbb{R})$, define a norm $\|\cdot\|_\varepsilon$ on $C^\infty_\varepsilon(H)$ and take the subspace of finite-norm elements.

- Let $h(\mathbf{x}, t) \in C^\infty_\varepsilon(H)$ denote a perturbation of H.
- Fix $\varepsilon = \left\{ \varepsilon_k \,\middle|\, k \in \mathbb{Z}^{\geq 0} \right\} \subset \mathbb{R}^{>0}$ a sequence of real numbers, which we will choose carefully later.
- For a fixed $\mathbf{x} \in W$, $t \in \mathbb{R}$ and $k \in \mathbb{Z}^{\geq 0}$, define

$$\left| d^k h(\mathbf{x}, t) \right| = \max \left\{ d^\alpha h(\mathbf{x}, t) \,\middle|\, |\alpha| = k \right\},$$

  the maximum over all sets of multi-indices α of length k.
  Note: I interpret this as

$$d^{\alpha_1, \alpha_2, \cdots, \alpha_k} h = \frac{\partial^k h}{\partial x_{\alpha_1} \, \partial x_{\alpha_2} \cdots \partial x_{\alpha_k}},$$

  the partial derivatives wrt the corresponding variables.

Define a Norm

- Define a norm on $C^\infty(W \times S^1; \mathbb{R})$:

$$\|h\|_{\text{"}} = \sum_{k \geq 0} \varepsilon_k \sup_{(x,t) \in W \times S^1} \left| d^k h(x,t) \right| .$$

- Since $W \times S^1$ is assumed compact (?), fix a finite covering $\{B_i\}$ of $W \times S^1$ such that

$$\bigcup_i B_i^\circ = W \times S^1.$$

  - Choose them in such a way we obtain charts

$$\Psi_i : B_i \longrightarrow \overline{B(0,1)} \subset \mathbb{R}^{2n+1} \text{ (?)}.$$

- Obtain the computable form

$$\|h\|_{\text{"}} = \sum_{k \geq 0} \varepsilon_k \sup_{(x,t) \in W \times S^1} \sup_{i, z \in B(0,1)} \left| d^k (h \circ \Psi_i^{-1})(z) \right| .$$

Define a Banach Space

- Define

$$C_\varepsilon^\infty = \left\{ h \in C^\infty(W \times S^1; \mathbb{R}) \;\middle|\; \|h\|_\varepsilon < \infty \right\} \subset C^\infty(W \times S^1; \mathbb{R}),$$

  which is a Banach space (normed and complete).
- Show that the sequence $\{\varepsilon_k\}$ can be chosen so that C_ε^∞ is a *dense* subspace for the C^∞ topology, and in particular for the C^1 topology.

Theorem

Such a sequence $\{\varepsilon_k\}$ can be chosen.

Lemma

$C^\infty(W \times S^1; \mathbb{R})$ with the C^1 topology is separable as a topological space (contains a countable dense subset).

Sketch Proof of Theorem

- By the lemma, produce a sequence $\{f_n\} \subset C^\infty(W \times S^1; \mathbb{R})$ dense for the C^1 topology.
- Using the norm on $C^n(W \times S^1; \mathbb{R})$ for the f_n, define

$$\frac{1}{\varepsilon_n} = 2^n \max \left\{ \|f_k\| \;\middle|\; k \leq n \right\} \implies \varepsilon_n \sup |d^n f_k(x,t)| \leq 2^{-n}$$

which is summable.

∎

Why does this imply density? I don't know.

Modified Theorem

The next proposition establishes a version of this theorem with compact support:

Theorem

For any $(\mathbf{x}, t) \subset U \in W \times S^1)$ *there exists a* $V \subset U$ *such that every* $h \in C^\infty(W \times S^1; \mathbb{R})$ *can be approximated in the* C^1 *topology by functions in* C_ε^∞ *supported in* U*.*

Then fix a time-dependent Hamiltonian H_0 with nondegenerate periodic orbits and consider

$$\left\{ h \in C_\varepsilon^\infty(H_0) \;\middle|\; h(x, t) = 0 \text{ in some } U \supseteq \text{the 1-periodic orbits of } H_0 \right\}$$

Then $\operatorname{supp}(h)$ is "far" from $\text{Per}(H_0)$, so

$$\|h\|_\varepsilon \ll 1 \implies \text{Per}(H_0 + h) = \text{Per}(H_0)$$

and are both nondegenerate.

Section 8.4: Linearizing the Floer Equation: The Differential of F

Goal

Choose $m > n = \mathbf{dim}(W)$ and embed $TW \hookrightarrow \mathbb{R}^m$ to identify tangent vectors (such as Z_i, tangents to W along u or in a neighborhood B of u) with actual vectors in $\mathbb{R}^m$.

Why? Bypasses differentiating vector fields and the Levi-Cevita connection.

We can then identify

$$\operatorname{im} \mathcal{F} = C^\infty(\mathbb{R} \times S^1; \mathbb{R}^m) \quad \text{or} \quad L^p(\mathbb{R} \times S^1; W),$$

and we seek to compute its differential $d\mathcal{F}$.

We've just replaced the codomain here.

Definitions

Recall that

- x, y are contractible loops in W that are nondegenerate critical points of the action functional $\mathcal{A}_H$,
- $u \in \mathcal{M}(x, y) \subset C^\infty_{\text{loc}}$ denotes a fixed solution to the Floer equation,
- $C_\searrow(x, y) \subset \left\{u \in C^\infty(R \times S^1; W)\right\}$ is the set of smooth solutions $u : \mathbb{R} \times S^1 \longrightarrow W$ satisfying some conditions:

$$\lim_{s \longrightarrow -\infty} u(s, t) = x(t), \quad \lim_{s \longrightarrow \infty} u(s, t) = y(t)$$

$$\text{and } \left|\frac{\partial u}{\partial t}(s, t)\right|, \quad \left|\frac{\partial u}{\partial t}(s, t) - X_H(u)\right| \sim \exp(|s|)$$

Compactify to Sphere

Fix a solution

$$u \in \mathcal{M}(x, y) \subset C^{\infty}_{\text{loc}}(\mathbb{R} \times S^1; W).$$

We lift each solution to a map

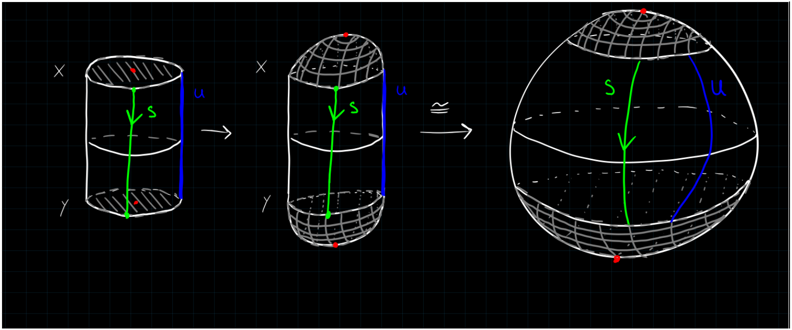
$$\tilde{u} : S^2 \longrightarrow W$$

in the following way:

The loops x, y are contractible, so they bound discs. So we extend by pushing these discs out slightly:

Lift to 2-Sphere

$$u \in C^\infty(S^1 \times \mathbb{R}; W) \quad \mapsto \quad \tilde{u} \in C^\infty(S^2; W)$$



Trivial the Pullback

From earlier in the book, we have

Assumption (6.22):

For every $w \in C^\infty(S^2, W)$ there exists a symplectic trivialization of the fiber bundle $w^* TW$, i.e. $\langle c_1(TW), \ \pi_2(W)\rangle = 0$ where c_1 denotes the first Chern class of the bundle TW.

Note: I don't know what this pairing is. The top Chern class is the Euler class (obstructs nowhere zero sections) and are defined inductively:

$$c_1(TW) = e(\Lambda^n(TW)) \in H^2(W; \mathbb{Z})$$

Assumption is satisfied when all maps $S^2 \longrightarrow W$ lift to $B^3 \iff \pi_2(W) = 0$.

We have a pullback that is a symplectic fiber bundle:

$$\begin{array}{ccc} \tilde{u}^* TW & \xrightarrow{d\tilde{u}} & TW \\ \downarrow & \lrcorner & \downarrow \\ S^2 & \xrightarrow{\tilde{u}} & W \end{array}$$

Choose a Frame

- Using the assumption, trivialize the pullback $\tilde{u}^* TW$ to obtain an orthonormal unitary frame

$$\{Z_i\}_{i=1}^{2n} \subset T_{u(s,t)} W$$

where

  - The frame depends smoothly on $(s,t) \in S^2$,
  - $\lim_{s \longrightarrow \infty} Z_i$ exists for each i.
  -

$$\frac{\partial}{\partial s}, \quad \frac{\partial^2}{\partial s^2}, \quad \frac{\partial^2}{\partial s \, \partial t} \quad \curvearrowright Z_i \overset{s \longrightarrow \pm\infty}{\longrightarrow} 0 \quad \text{for each } i$$

Claim: such trivializations exist, "using cylinders near the spherical caps in the figure".

Define "Banach Manifold Charts"

Recall we had $W^{1,p}(x, y)$ a completion of C^∞

$$\mathcal{M}(x, y) \subset C^\infty_\searrow(x, y) \subset \mathcal{P}^{1,p}(x, y) \underset{\text{defn}}{\subset} \left\{ (s, t) \xrightarrow{\varphi} \exp_{w(s,t)} Y(s, t) \right\}.$$

where we restrict to

- $Y \in W^{1,p}(w^* TW)$,
- $w \in C^\infty_\searrow(x, y)$

Use the chosen frame $\{Z_i\}$ to define a chart centered at u of $\mathcal{P}^{1,p}(x, y)$ given by

$$\begin{aligned} \iota : W^{1,p}\left(\mathbb{R} \times S^1; \mathbb{R}^{2n}\right) &\longrightarrow \mathcal{P}^{1,p}(x, y) \\ \mathbf{y} = (y_1, \ldots, y_{2n}) &\longmapsto \exp_u \left(\sum y_i Z_i \right). \end{aligned}$$

- Note that the derivative at zero is $\sum_{i=1}^{2n} y_i Z_i$.

Define the Floer Map in Charts

Define and compute the differential of the composite map $\tilde{\mathcal{F}}$ defined as follows:

$$\mathcal{P}^{1,p}(x,y) \xrightarrow{\mathcal{F}} L^p\left(\mathbb{R}\times S^1; TW\right) \longrightarrow L^p\left(\mathbb{R}\times S^1;\mathbb{R}^m\right)$$

$$u \xrightarrow{\tilde{\mathcal{F}}} \frac{\partial u}{\partial s} + J(u)\left(\frac{\partial u}{\partial t} - X_t(u)\right)$$

- From now on, let $\mathcal{F}$ denote $\tilde{\mathcal{F}}$.

Add a Tangent

- Take the vector

$$Y(s,t) := (y_1(s,t), \cdots) \in \mathbb{R}^{2n} \subset \mathbb{R}^m$$

  - View Y as a vector in $\mathbb{R}^m$ tangent to W, given by $Y = \sum_{i=1}^{2n} y_i Z_i$.
- Plug $u + Y$ into the equation for $\mathcal{F}$, directly yielding

Add a Tangent

$$\begin{aligned}
\mathcal{F}(u) &= \frac{\partial u}{\partial s} + J(u)\frac{\partial u}{\partial t} - J(u)X_t(u) \\
\mathcal{F}(u+Y) &= \frac{\partial (u+Y)}{\partial s} + J(u+Y)\frac{\partial (u+Y)}{\partial t} - J(u+Y)X_t(u+Y)
\end{aligned}$$

Extract Linear Part

Extract the part that is linear in Y and collect terms:

$$
\begin{aligned}
&(d\mathcal{F})_u(Y) \\
&= \frac{\partial Y}{\partial s} + (dJ)_u(Y)\frac{\partial u}{\partial t} + J(u)\frac{\partial Y}{\partial t} - (dJ)_u(Y)X_t - J(u)\left(dX_t\right)_u(Y) \\
&= \left(\frac{\partial Y}{\partial s} + J(u)\frac{\partial Y}{\partial t}\right) \\
&\quad + \left((dJ)_u(Y)\frac{\partial u}{\partial t} - (dJ)_u(Y)X_t - J(u)\left(dX_t\right)_u(Y)\right)
\end{aligned}
$$

.

- This is a sum of two differential operators:
  - One of order 1, one of order 0 (Perspective 1)
  - The Cauchy-Riemann operator, and one of order zero (Perspective 2, not immediate from this form)

Leibniz Rule

– Now compute in charts. Need a lemma:

Lemma (Leibniz Rule)

For any source space X and any maps

$$J : X \longrightarrow \mathrm{End}(\mathbb{R}^m)$$
$$Y, v : X \longrightarrow \mathbb{R}^m$$

we have

$$(dJ)(Y) \cdot v = d(Jv)(Y) - Jdv(Y).$$

Sketch: Proof of Leibniz Rule

Differentiate the map

$$\begin{aligned} J \cdot v : X &\longrightarrow \mathbb{R}^m \\ x &\mapsto J(x) \cdot v(x) \end{aligned}$$

to obtain

$$\begin{aligned} &J(x+Y)v(x+y) \\ &= (J(x) + (dJ)_x(Y)) \ \cdot \ (v(x) + (dv)_x(Y)) + \cdots \\ &= J(x) \cdot v(x) + J(x) \cdot (dv)_x(Y) + (dJ)_x(Y) \cdot v(x) \\ &\quad + (dJ)_x(Y) \cdot (dv)_x(Y) + \cdots \end{aligned}$$

$$\implies d(J \cdot v)_x(Y) = (dJ)_x(Y) \cdot v(x) + J(x) \cdot (dv)_x(Y).$$

■

Decompose by Order

Using the chart ι defined by $\{Z_i\}$ to write $Y = \sum_{i=1}^{2n} y_i Z_i$ and thus

$$(d\mathcal{F})_u(Y) = O_0 + O_1$$

where O_0 are order 0 terms ("they do not differentiate the y_i") and the O_1 are order 1 terms:

$$O_1 = \sum_{i=1}^{2n} \left(\frac{\partial y_i}{\partial s} Z_i + \frac{\partial y_i}{\partial t} J(u) Z_i \right)$$

$$O_0 = \sum_{i=1}^{2n} y_i \left(\frac{\partial Z_i}{\partial s} + J(u)\frac{\partial Z_i}{\partial t} + (dJ)_u(Z_i)\frac{\partial u}{\partial t} - J(u)(dX_t)_u Z_i - (dJ)_u(Z_i) X_t \right).$$

Order One

- Study O_1 first, which (claim) reduces to

$$O_1 = \sum_{i=1}^{2n} \left(\frac{\partial y_i}{\partial s} + J_0 \frac{\partial y_i}{\partial t} \right) Z_i = \bar{\partial}(y_1, \cdots, y_{2n}).$$

  where J_0 is the standard complex structure on $\mathbb{R}^{2n} = \mathbb{C}^n$
- The second equality follows from the assumption that the Z_i are symplectic and orthonormal.
- Note that this writes $(d\mathcal{F})_u(Y) = O_0 + O_{CR}$, a sum of an order zero and a Cauchy-Riemann operator.

Recap

Note that since we've computed in charts, we have actually computed the differential of $\mathcal{F}_u$ in the following diagram

$$W^{1,p}(\mathbb{R} \times S^1; \mathbb{R}^{2n}) \xrightarrow{\iota} \mathcal{P}^{1,p}(x, y$$

u

Order 0 Term is Linear

$$
\begin{aligned}
(d\mathcal{F})_u &= \left(\frac{\partial Y}{\partial s} + J(u) \frac{\partial Y}{\partial t} \right) \\
&\quad + \left((dJ)_u(Y) \frac{\partial u}{\partial t} - (dJ)_u(Y) X_t - J(u) \left(dX_t\right)_u (Y) \right) \\
&:= \bar{\partial} Y + SY
\end{aligned}
$$

where $S \in C^\infty(\mathbb{R} \times S^1; \mathrm{End}(\mathbb{R}^n))$ is a linear operator of order 0.

Order 0 Symmetry in the Limit

Theorem (8.4.4, CR + Symmetric in the Limit)

If u solves Floer's equation, then

$$(d\mathcal{F})_u = \bar{\partial} + S(s,t)$$

where

- *S is linear*
- *S tends to a symmetric operator as $s \longrightarrow \pm\infty$, and*
- $$\frac{\partial S}{\partial s}(s,t) \overset{s \longrightarrow \pm\infty}{\longrightarrow} 0 \quad \text{uniformly in } t$$

Proof

Omitted – S is exactly O_0 from before:

$$\begin{aligned} O_0 &= \sum_{i=1}^{2n} y_i \left(\frac{\partial Z_i}{\partial s} + J(u)\frac{\partial Z_i}{\partial t} + (dJ)_u(Z_i)\frac{\partial u}{\partial t} \right. \\ &\qquad \left. - J(u)(dX_t)_u Z_i - (dJ)_u(Z_i)X_t \right) \\ &= \sum_{i=1}^{2n} y_i \left( )\frac{\partial Z_i}{\partial s} + (dJ)_u(Z_i)\left(\frac{\partial u}{\partial t} - (Z_i)X_t\right) \right. \\ &\qquad \left. + J(u)\frac{\partial Z_i}{\partial t} - J(u)(dX_t)_u Z_i \right). \end{aligned}$$

- The term in blue vanishes as $s \longrightarrow \pm\infty$
  - Using the fact that u is a solution
  - Uses $\frac{\partial u}{\partial s} \longrightarrow 0$ uniformly (as do its derivatives?)
- Suffices to show the remaining part is symmetric in the limit

Proof

- Write the remaining part as

$$A(y_1, \cdots, y_{2n}) = \cdots \implies A_{ij} = A_{ji}$$

  using inner product calculations
- Uses the fact the Z_i needed to be chosen to be unitary and symplectic.

$\blacksquare$

asdas

Write O_1 as a map $Y \mapsto S \cdot Y$, so $S \in C^\infty(\mathbb{R} \times S^1; \mathrm{End}(\mathbb{R}^{2n}))$ and define the symmetric operators

$$S^{\pm} := \lim_{s \longrightarrow \pm\infty} S(s, \cdot) \quad \text{respectively}$$

Theorem

The equation

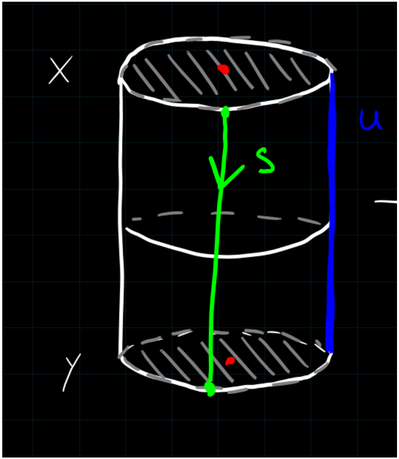
$$\partial_t Y = J_0 S^{\pm} Y$$

linearizes Hamilton's equation

$$\frac{\partial z}{\partial t} = X_t(z) \quad \text{at} \quad \begin{cases} x = \lim_{s \longrightarrow -\infty} u & \text{for } S^- \\ y = \lim_{s \longrightarrow \infty} u & \text{for } S^+ \end{cases} \quad \text{respectively.}$$

Image

Reminder the x, y were the top/bottom pieces of the original cylinder/sphere:



Proof Sketch

- Use the fact that $\frac{\partial Y}{\partial t} = (dX_t)_x Y$
- Expand $\sum \frac{\partial y_i}{\partial t} Z_i$ in the Z_i basis (roughly) to write $\frac{\partial y_i}{\partial t} = \sum b_{ij} y_j$ for some coefficients b_{ij}.
- Collect terms into a matrix/operator $B^{\mp}$ for x, y respectively to write

$$\frac{\partial Y}{\partial t} = B^{-} \cdot Y$$

- Write $(d\mathcal{F})_u = \bar{\partial} + S$ where S is zero order and symmetric in the limit

Proof Sketch

- Get the corresponding operator A in coordinates
- Expand in a basis (roughly) as $A(\sum y_i Z_i) = \sum s_{ij} y_j Z_i$
- Check that $s_{ij} = \pm b_{i \pm n, j}$
- This implies

$$S^{-} = -J_0 B^{-} \quad S^{+} = -J_0 B^{+} \implies \frac{\partial Y}{\partial t} = J_0 S^{\pm} Y$$

Final Remarks

- Given a solution u, we have a right $\mathbb{R}$-action, so for $s \in \mathbb{R}$,

$$\begin{aligned} u \cdot s \in C^\infty(\mathbb{R} \times S^1; W) \\ (\sigma, t) \mapsto u(\sigma + s, t) \end{aligned}$$

  is also a solution, so $\mathcal{F}(u \cdot s) = 0$ for all s.

  In other words: we can flow solutions?

Final Remarks

Punchline: $\frac{\partial u}{\partial s}$ **is a solution of the linearized equation**, since

$$0 = \frac{\partial}{\partial s}\mathcal{F}(u \cdot s) = (d\mathcal{F})_u\left(\frac{\partial u}{\partial s}\right).$$

- Along any nonconstant solution connecting x and y, $\dim\ker(d\mathcal{F})_u \geq 1$.