Section 8.6 - 8.8: Setup for Computing the Index

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Outline - Sketch proof of 8.6.3 - Statement of Somewhere Injectivity - Statement of Continuation Principle - Statement of Similarity Principle - 8.7 - 8.8

What we're trying to prove 8.1.5: $(d\mathcal{F})_u$ is a Fredholm operator of index $\mu(x) - \mu(y)$.

1 8.8

- Define

$$L : W^{1,p}\left(\mathbf{R} \times S^1; \mathbf{R}^{2n}\right) \longrightarrow L^p\left(\mathbf{R} \times S^1; \mathbf{R}^{2n}\right) \tag{1}$$

$$Y \longmapsto \frac{\partial Y}{\partial s} + J_0 \frac{\partial Y}{\partial t} + S(s,t)Y \tag{2}$$

- By the end of 8.8: replace the Fredholm operator L by an operator L_1 with the same *index* (not the same kernel/cokernel)
  - Compute the index of ? because we can explicitly describe its kernel and cokernel

- Use the fact $S : \mathbb{R} \times S^1 \longrightarrow \mathrm{Mat}(2n;\mathbb{R})$ and
$$S(s,t) \overset{s\longrightarrow\pm\infty}{\longrightarrow} S^{\pm}(t)$$
which are symmetric.
  - Take corresponding symplectic paths $R^{\pm} : I \longrightarrow \mathrm{Sp}(2n;\mathbb{R})$.
  - If
$$R^{\pm} \in \mathcal{S} := \left\{ R : I \longrightarrow \mathrm{Sp}(2n;\mathbb{R}) \;\middle|\; R(0) = \mathrm{id}, \det(R(1) - \mathrm{id}) \neq 0 \right\},$$
then L is a Fredholm operator
- Theorem 8.8.1: $\mathrm{Ind}(L) = \mu(R^-(t)) - \mu(R^+(t)) = \mu(x) - \mu(y)$.
- Prop 8.8.2: Define an operator
$$\begin{aligned} L_1 : W^{1,p}\left(\mathbf{R} \times S^1; \mathbf{R}^{2n}\right) &\longrightarrow L^p\left(\mathbf{R} \times S^1; \mathbf{R}^{2n}\right) \\ Y &\longmapsto \frac{\partial Y}{\partial s} + J_0 \frac{\partial Y}{\partial t} + S(s)Y \end{aligned}$$
where $S : \mathbb{R} \longrightarrow \mathrm{Mat}(2n;\mathbb{R})$ is a path of diagonal matrices depending on $\mathrm{Ind}(R^{\pm}(t))$; then $\mathrm{Ind}(L_1) = \mathrm{Ind}(L)$.
  - Then $\mathrm{Ind}(L_1) = \mathrm{Ind}(R^-(t)) - \mathrm{Ind}(R^+(t))$.
  - Idea of proof: take a homotopy of operators
$$\begin{aligned} L_\lambda : W^{1,p}\left(\mathbf{R} \times S^1; \mathbf{R}^{2n}\right) &\longrightarrow L^p\left(\mathbf{R} \times S^1; \mathbf{R}^{2n}\right) \\ Y &\longmapsto \frac{\partial Y}{\partial s} + J_0 \frac{\partial Y}{\partial t} + S_\lambda(s,t)Y \end{aligned}$$
which are all Fredholm and all have the same index, then take time 1.
  - Use the fact that $\mathrm{coker}\, L_1 \cong \ker L_1^*$, and we can explicitly write the adjoint of L_1.
  - Get a formula that resembles the Morse case: counting the number of eigenvalues that change sign.
- Summary:
  - Replace L by L_0, which is modified in a neighborhood of zero in the s variable. Use invariance of index under small perturbations.
  - Homotope L_0 to L_1, where S is replaced by a diagonal matrix $S(s)$ that is a constant matrix outside the neighborhood of zero in s. Use invariance of index under homotopy.

2 8.7

- Goal: Toward 8.1.5, show that $L := \bar{\partial} + S(s,t) : W^{1,p} \longrightarrow L^p$ is a Fredholm operator, i.e. the index makes sense (finite-dimensional kernel and cokernel).
- Statement: if $\det(\mathrm{id} - R_1^{\pm}) \neq 0$ then the operator
$$L : W^{1,p}(\mathbb{R} \times S^1; \mathbb{R}^{2n}) \longrightarrow L^p(\mathbb{R} \times S^1; \mathbb{R}^{2n})$$
.

given by $L = \bar{\partial} + S(s,t)$ is Fredholm for every $p > 1$. > Most of the work goes into showing that $\dim(\ker L) < \infty$ and im (L) is closed.

- $\dim \ker L < \infty$:
  - Use elliptic regularity and consequence of Calderón-Zygmund inequality.
    - Elliptic regularity: If $Y \in L^p$ is a weak solution of the linearized Floer equation $LY = 0$, then $Y \in W^{1,p}$ and is smooth.
    - Consequence: If $Y \in W^{1,p}$ and $p > 1$ then $\|Y\|_{W^{1,p}} = O(\|LY\|_{L^p} + \|Y\|_{L^p})$.
  - Case 1: $S(s,t) = S(t)$ doesn't depend on s.
    - Then L is a bijective for every $p > 1$
    - Invertibility allows replacing the weak solution $Y \in L^p(\mathbb{R}\times S^1; \cdot)$ with $Y \in L^p([-M, M]\times S^1, \cdot)$ for $M \gg 0$
    - Restriction $\overline{L} : W^{1,p}(\mathbb{R} \times S^1) \longrightarrow L^p([-M, M] \times S^1)$ is a compact operator, it is "semi-Fredholm"
    - Apply a theorem: if the inequality is satisfies, the kernel is finite-dimensional and the image is closed.

- $\dim \operatorname{coker} L < \infty$:
  - Take

$$L : W^{1,p}(\mathbb{R} \times S^1; \mathbb{R}^{2n}) \longrightarrow L^p(\mathbb{R} \times S^1; \mathbb{R}^{2n})$$

    and consider the adjoint operator

$$L^\star : W^{1,q}\left(\mathbb{R} \times S^1; \mathbb{R}^{2n}\right) \longrightarrow L^q\left(\mathbb{R} \times S^1; \mathbb{R}^{2n}\right)$$

    where $p^{-1} + q^{-1} = 1$, which appeared in 8.5.1.
  - Use the fact that $\operatorname{coker} L = \ker L^\star$
  - Show $\dim \ker L^\star < \infty$ since it satisfies the conditions of 8.7.4.

3 8.6

- Start with $u \in C^\infty(\mathbb{R} \times S^1; W)$ a solution to the equation

$$\frac{\partial u}{\partial s} + J_1(t,u)\left(\frac{\partial u}{\partial t} - X(t,u)\right) = 0$$

- "Unwrap the cylinder" to a map $v \in C^\infty(\mathbb{R}^2; W)$ which is a solution to $\left(\frac{\partial}{\partial s} + J\frac{\partial}{\partial t}\right)u = 0$
  - No longer periodic; instead $v(s, t+1) = \phi(v(s,t))$.
  - Built by precomposing u with the inverse flow ψ_t of X_t
  - Conjugate original J with ψ to get J_1

- Define regular points $R(v)$ as $\frac{\partial}{\partial s}v \neq 0$ with injectivity condition: $s \neq s' \implies v(s,t) \neq v(s',t)$.

- Prove "injectivity result": $R(v) \subseteq \mathbb{R}^2$ is dense and open

3.1 8.6.3, Part 1: R(v) is Open

- Prove $R(v)$ is open: contradict zero derivative
  - Proof uses sequential characterization of being a closed set
  - Construct a sequence in the complement converging to a regular point
  - Since first two conditions of $R(v)$ are open, extract a sequence of points failing injectivity
  - Show it is bounded
  - Apply Bolzano-Weierstrass to extract a convergent subsequence
  - Use quotient definition of derivative, show it is zero, contradiction.

3.2 8.6.3, Part 2: R(v) is Dense in R2 (p.258)

3.3 Step 1: Exclude critical points that are also multiple points

- Definition: *Multiple points* are where injectivity fails in s.
  - Characterize $R(v)$ as those in $C(v)^c$ that are not multiples.
- Suffices to show $R(v)$ is dense in $C(v)^c$ – Every $(s,t) \in C(v)^c \subseteq \mathbb{R}^2$ is the limit of $(s_n, t) \in C(v)^c$ where $v(s_n, t) \neq x^{\pm}(t)$.
  - Why? $v(s + \frac{1}{n}, t) = x^+(t) \implies \frac{\partial v}{\partial s} = 0 \implies (s + \frac{1}{n}, t) \in C(v)$.
  - Then suffices to show every $(s_0, t_0) \in \mathbb{R} \times I$ with (importantly)

$$\frac{\partial v}{\partial s}(s_0, t_0) \neq 0 \quad \text{and} \quad v(s_0, t_0) \neq x^{\pm}(t_0). \tag{3}$$

    {#eq:eq1} is the limit of a sequence of points in $R(v)$.
  - Proceed by assuming this is not the case, toward a contradiction.

3.3.1 A Small Ball Avoids Critical Points in the Image

- Surround every point (s_0, t_0) by a ball $B_\varepsilon(s_0, t_0)$ missing $R(v)$
- We can choose ε small enough and $M \gg 1$ big enough (defining $\mathbf{M} = [-M, M] \subset \mathbb{R}$) such that

  1. Translate to far enough to get a point outside the image of the ball:

$$(s,t) \in \mathbf{M}^c \times [t_0 - \varepsilon, t_0 + \varepsilon] \subset \mathbb{R} \times I \implies$$
$$v(s,t) \bigcap v(B_\varepsilon(s_0, t_0) = \emptyset \quad \text{and} \quad x^{\pm}(t) \notin v(B_\varepsilon(s_0, t_0)).$$

  - Idea: If not, can cook up sequences that force $v(s_0, t_0) = x^{\pm}(t_0)$, a contradiction to @eq:eq1.

  2. For $t \in B_\varepsilon(t_0)$, $B_\varepsilon(s_0) \hookrightarrow W$ is an injective immersion

  - Combine 1 and 2 to show that
    * v is locally constant
    * $(s_0, t_0) \in C(v)$
    * Every point in $B_\varepsilon(s_0, t_0)$ satisfies [@eq:eq1]

  3.

  - Show
$$|\mathbf{M}_C| := \left|(\mathbf{M} \times I) \bigcap C(v)\right| < \infty$$
since it's the intersection of a compact and a discrete set
  - Perturb (s_0, t_0) so that $(s, t) \in \mathbf{M}_C \implies v(s_0, t_0) \neq v(s, t)$.
    * Possible since $(s_0, t_0) \notin C(v) \implies v$ is non-constant in a neighborhood of (s_0, t_0).
  - Decrease ε so that
$$v(B_\varepsilon(s_0, t_0)) \bigcap v(\mathbf{M}_C) = \emptyset.$$
- This means that in the thick strip containing (s_0, t_0), no critical points land in its image.
- Conclude that we only have to consider injectivity, not critical points that are also multiple points

3.4 Step 2: Failure of Injectivity in Small Boxes

- Define
$$\mathbf{S}_M(t_0) = \{s_1, \cdots, s_N\} = \left\{s_i \in [-M, M] \;\middle|\; v(s_i, t_0) = v(s_0, t_0)\right\},$$
the set of multiple points over s_0.
  - This is finite, since infinite $\implies$ has a limit point $\implies \dfrac{\partial v}{\partial s} = 0$, contradiction.
- Lemma: For every $\varepsilon > 0$ there exists a $\delta > 0$ such that defining
$$\Delta_0 = [s_0 - \delta, s_0 + \delta] \times [t_0 - \delta, t_0 + \delta]$$
then
$$(s, t) \in \Delta \implies \exists s' \in B_\varepsilon(s_j) \text{ s.t. } v(s, t) = v(s', t)$$
for some $1 \leq j \leq N$.
  - In English: for every epsilon, we can find a delta box $\Delta_0 \ni (s_0, t_0)$ such that every point in Δ_0 is a multiple point over *some* point in an epsilon ball around *some* point in $\mathbf{S}_M(t_0)$.
  - Proof idea: otherwise, construct a sequence $(\sigma_n, t_n) \longrightarrow (s_0, t_0)$, use properties 1 and 2 from earlier, extract a limit point σ *not* in $\mathbf{S}_M(t_0)$ which satisfies $v(\sigma, t_0) = v(s_0, t_0)$, a contradiction (since that set contained exactly the multiple finitely many points).
- Fix $\varepsilon' < \varepsilon/2$ form Step 1 and apply the lemma to ε' to produce a δ and a box Δ_0.
- Apply the lemma: shrink the delta box Δ_0 to a closed delta ball $\overline{B}_\delta(s_0, t_0)$.
  - Every point in the delta box is a multiple point over some s_j.
- So partition the ball up: define Σ_j to be all of the multiple points over $s_j \in \mathbf{S}_(t_0)$.
- Take a smaller ρ-ball around some point $(s_\star, t_\star) \in \Sigma_1^\circ$, making sure to choose ε' small enough such that
$$B_\rho(s_\star, t_\star) \bigcap \left([s_1 - \varepsilon', s_1 + \varepsilon'] \times [t_0 - \delta, t_0 + \delta]\right) = \emptyset.$$
In other words, the shaded region is disjoint from the ρ-ball.
- Then every point in the ρ-ball is a multiple point over some point in the box around (s_1, t_0). Pick on such point $(s'_\star, t_\star)$ on the $t_\star$ line.

3.5 Step 3: Final Contradiction

- Construct v_1, v_2 which
  - Satisfy the same Cauchy-Riemann equations
  - Are equal at the origin
  - Have nonzero derivative at the origin.
- We want to show they are equal on $\mathbb{R}^2$
- Constructing them: use points from step 2 to translate
  - Obtain $(s_\star, t_\star)$ and $(s'_\star, t_\star)$ from previous step.
  - Define

$$\begin{aligned} v_1(s,t) &= v\left(s+s_\star, t+t_\star\right) &\implies v_1(z) = v(z+w_1) \\ v_2(s,t) &= v\left(s+s'_\star, t+t_\star\right) &\implies v_2(z) = v(z+w_2) \end{aligned}$$

  - Satisfy the same CR equations
  - By construction, they coincide at $(0,0)$ since $v(s_\star, t_\star) = v(s'_\star, t_\star)$.
  - Derivatives at the origin are nonzero, coming from the fact that $\dfrac{\partial v}{\partial s}(s_\star, t_\star) \neq 0$.
- Now work at zero: for every $(s,t) \in B_\rho(0,0)$ there exists a multiple point $s' \in B_{2\varepsilon'}(0)$ over s.
- Use the following extension lemma, consequence of **Continuation Principle**: in this situation, with $X_t \equiv 0$ on $B_\varepsilon(\mathbf{0})$, then

$$z \in B_\varepsilon(\mathbf{0}) \implies v_1(z) = v_2(z).$$

- Define

$$\begin{aligned} \mathcal{F} : \mathrm{C}^\infty\left(\mathbf{R}\times S^1; \mathbf{R}^{2n}\right) &\longrightarrow \mathrm{C}^\infty\left(\mathbf{R}\times S^1; \mathbf{R}^{2n}\right) \\ w &\longmapsto \left(\frac{\partial}{\partial s} + J\cdot\frac{\partial}{\partial t}\right) w \end{aligned}$$

- Since v_1, v_2 satisfy the same CR equation, $\mathcal{F}(v_1) = \mathcal{F}(v_2)$
- Linearize $\mathcal{F}$ as we did for the Floer operator to obtain

$$(d\mathcal{F})_{\dots}(Y) = \left(\frac{\partial}{\partial s} + J_0\cdot\frac{\partial}{\partial t} + \tilde{S}\right) Y.$$

  where $\tilde{S} : I \times \mathbb{R}^2 \longrightarrow \mathrm{End}(\mathbb{R}^{2n})$
- Set $Y = v_1 - v_2$, then

$$S = \int_{[0,1]} \tilde{S} \implies S\left(\frac{\partial}{\partial s} + J_0\cdot\frac{\partial}{\partial t} + S\right) Y = 0$$

- From above, $Y \equiv 0$ in $B_\varepsilon(\mathbf{0})$, apply **Continuation Principle** to obtain $v_1 = v_2$ on $\mathbb{R}^2$
- Inductive argument to show

$$\forall k \in \mathbb{Z}, \quad v(s,t) = v(k(s'_\star - s_\star), t) \overset{k\longrightarrow\infty}{\longrightarrow} x^\pm(t),$$

  which is the desired contradiction. ■

BREAK

3.6 The Continuation Principle

- Take the perturbed CR equation

$$\left(\frac{\partial}{\partial s} + J_0 \cdot \frac{\partial}{\partial t} + S\right) Y = 0 \quad \text{where} \quad S : \mathbb{R}^2 \longrightarrow \mathrm{End}(\mathbb{R}^{2n})$$

  where J_0 is the standard complex structure on $\mathbb{R}^{2n}$.

- Define an *infinite-order zero* z of an arbitrary function f as

$$z_0 \in Z_\infty \iff \sup_{B_r(z_0)} \frac{|f(z)|}{r^k} \overset{r \to 0}{\longrightarrow} 0 \quad \forall k \in \mathbb{Z}^{\geq 0},$$

  or for a smooth function,

$$z_0 \in Z_\infty \iff f^{(k)}(z_0) = 0 \quad \forall k \in \mathbb{Z}^{\geq 0}.$$

- Statement: If Y solves CR on $U \subset \mathbb{R}^2$ then the set

$$C \coloneqq \left\{ (s,t) \in U \;\middle|\; Y \text{ is an infinite-order zero at } (s,t) \right\}.$$

- Explanation: for f smooth, Z_∞ is closed. For f holomorphic, it is clopen.
  - From complex analysis: for a connected domain Ω, the only clopen subsets are $\emptyset, \Omega$, so nonempty and $f = g$ on a connected subset implies $f = g$ on Ω.
  - In particular, $Y = 0$ on $U' \subseteq U$ implies $Y = 0$ on U.

- Prove is a consequence of the **Similarity Principle**

3.7 Similarity Principle

- Statement:
  - Recall definition of perturbed CR equation:

$$\left(\frac{\partial}{\partial s} + J_0 \cdot \frac{\partial}{\partial t} + S\right) Y = 0 \quad \text{where} \quad S : \mathbb{R}^2 \longrightarrow \mathrm{End}(\mathbb{R}^{2n})$$

  - Let
  - $Y \in C^\infty(B_\varepsilon; \mathbb{C}^n)$, or more generally $W^{1,p}(B_\varepsilon; \mathbb{C}^n)$, be a solution
  - $S \in C^\infty(\mathbb{R}^2, \mathrm{End}(\mathbb{R}^2))$ or more generally $L^p(B_\varepsilon; \mathrm{End}_{\mathbb{R}}(\mathbb{R}^{2n}))$
  - $p > 2$

  Then there exist

$$\begin{aligned} &\delta \in (0, \varepsilon), \\ &\sigma \in C^\infty(B_\delta, \mathbb{C}^n) \\ &A \in W^{1,p}(B_\delta, \mathrm{GL}(\mathbb{R}^{2n})). \end{aligned}$$

  such that

$$\forall (s,t) \in B_\delta, \quad Y(s,t) = A(s,t) \cdot \sigma(s + it) \quad \text{and} \quad J_0 A(s,t) = A(s,t) J_0.$$

- Used to prove:
  - $C(v)$ is discrete
  - "Extension" lemma used to prove $R(v)$ is dense
- Some ideas from proof:
  - Matrix A will look like the fundamental matrix of solutions to the equation
  - Compactify to get $B_\delta \subset S^2$, if $Y : S^2 \longrightarrow \mathbb{C}^n$ then we can consider the section

$$\begin{array}{c} \left(A^{0,1}T^\star S^2\right)^n = \Lambda^{0,1}T^\star S^2 \otimes \mathbb{C}^n \\ \big\downarrow \quad \curvearrowleft \bar\partial Y \\ X \end{array}$$

- $\overline{Y} = 0$ makes Y a holomorphic sphere in $\mathbb{C}^n$.

3.8 Odds and Ends

- Theorem: the following is a surjective Fredholm operator for every $p > 1$:

$$\bar\partial : W^{1,p}\left(S^2, \mathbb{C}^n\right) \longrightarrow L^p\left(\Lambda^{0,1}T^\star S^2 \otimes \mathbb{C}^n\right).$$

  - Computation will show that $\dim\ker\bar\partial = \dim\operatorname{coker}\bar\partial = 2n$, so $\operatorname{Ind}\bar\partial = 0$.