Section 8.6 - 8.8: Setup for Computing the Index

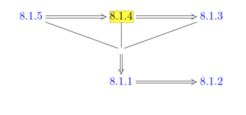
May 21, 2020

Intro

Outline

What we're trying to prove:

- 8.1.5: $(d\mathcal{F})_u$ is a Fredholm operator of index $\mu(x) - \mu(y)$.



Used to show:

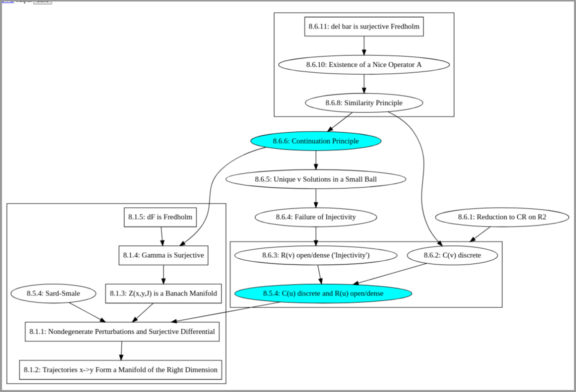
- 8.1.4: $\Gamma : W^{1,p} \times C^\infty_\varepsilon \longrightarrow L^p$ has a continuous right-inverse and is surjective
- 8.1.3: $\mathcal{Z}(x, y, J)$ is a Banach manifold
- 8.1.1: $h \in \mathcal{H}_{\text{reg}}$, $H_0 + h$ nondegenerate and

$$(d\mathcal{F})_u \text{ surjective } \forall\ u \in \mathcal{M}(H_0 + h, J).$$

- 8.1.2: For $h \in \mathcal{H}_{\text{reg}}$ and $x, y \sim \{\text{pt}\}$ of H_0,

$$\dim_{\text{mfd}} \mathcal{M}(x, y, H_0 + h) = \mu(x) - \mu(y).$$

Map



8.1.4 :$\Gamma : W^{1,p} \times C^{\infty}_{\varepsilon} \longrightarrow L^p$ cts right-inverse 8.1.3 : $\mathcal{Z}(x, y, J)$a Banach manifold

8.1.1 :$h \in \mathcal{H}_{\text{reg}}, H_0 + h$ nondegenerate and $(d\mathcal{F})_u$ surjective $\forall\ u \in \mathcal{M}(H_0 + h, J)$

8.1.2 :$h \in \mathcal{H}_{\text{reg}}$ and $x, y \sim \{\text{pt}\}$ of $H_0 \implies \dim_{\text{mfd}} \mathcal{M}(x, y, H_0 + h) = \mu(x) - \mu(y)$.

Destination

What we're working toward now:

- 8.1.5: $(d\mathcal{F})_u$ is a Fredholm operator of index $\mu(x) - \mu(y)$.

Outline for today:

- 8.6: Filling in lemmas used in previous sections
  - Sketch proof of 8.6.3, statement of **Somewhere Injectivity**
  - Statement of **Continuation Principle**
  - Statement of **Similarity Principle**
- High-level outlines
  - 8.7: Proving the operator is Fredholm
  - 8.8: Computing its index

Review From Last Time

- $u \in C^\infty(\mathbb{R} \times S^1; W)$ is a solution to the Floer equation
- $C(u)$ the critical points and $R(u)$ the regular points of u:

$$C(u) := \left\{ (s_0, t_0) \in \mathbb{R} \times S^1 \;\middle|\; \frac{\partial u}{\partial s}(s_0, t_0) = 0 \right\}$$

$$R(u) := \left\{ (s_0, t_0) \in C(u)^c \;\middle|\; v(s,t) \neq x^{\pm}(t), \quad s \neq s_0 \implies u(s, t_0) \neq u(s_0, t_0) \right\}.$$

WTS: $C(u)$ is discrete and $R(u)$ is open/dense in $\mathbb{R} \times S^1$

- Strategy: “unwrap” u to an easier solution v on $\mathbb{R}^2$.

Strategy

Strategy:

1. "Unwrapping" u. Replace

$$\frac{\partial u}{\partial s} + J(t,u)\left(\frac{\partial u}{\partial t} - X(t,u)\right) = 0 \quad \text{where} \quad u \in C^\infty(\mathbb{R} \times S^1; W)$$

with a Cauchy-Riemann equation on $\mathbb{R}^2$:

$$\frac{\partial v}{\partial s} + J\frac{\partial v}{\partial t} = 0 \quad \text{where} \quad v \in C^\infty(\mathbb{R}^2; W)$$

(Reduces to showing statements for v instead of u)

2. Show $R(v) \subset \mathbb{R}^2$ is open (short)
3. Show $R(v) \subset \mathbb{R}^2$ is dense (main obstacle)

Main Ingredients:

- Continuation Principle
- Similarity Principle

8.6 Review

Review: 8.6.1, Unwrapping/Reduction

If u is a solution to the following equation:

$$\frac{\partial u}{\partial s} + J(t,u)\left(\frac{\partial u}{\partial t} - X(t,u)\right) = 0$$

Then there exist

- An almost complex structure J_1
- A diffeomorphism φ on W ?
- A map $v \in C^\infty(\mathbb{R}^2; W)$

where

$$\frac{\partial v}{\partial s} + J_1(v)\frac{\partial v}{\partial t} = 0$$
$$v(s,t+1) = \varphi(v(s,t))$$
$$C(u) = C(v) \quad \text{and} \quad R(u) = R(v).$$

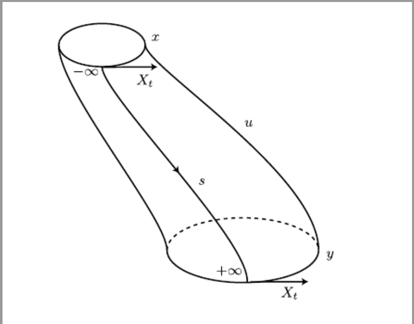
- Recall definition of v:

$$v(s,t) := \psi_t^{-1}(u(s,t))$$

Toward 8.6.3, Injectivity: $R(v)$ is Open

$$C(u) \coloneqq \left\{ (s_0, t_0) \in \mathbb{R} \times S^1 \;\middle|\; \frac{\partial u}{\partial s}(s_0, t_0) = 0 \right\}$$

$$R(u) \coloneqq \left\{ (s_0, t_0) \in C(u)^c \;\middle|\; v(s,t) \neq x^{\pm}(t), \quad s \neq s_0 \implies u(s, t_0)$$



Sketch of Proof: $R(v)$ is Open

$$C(u) := \left\{ (s_0, t_0) \in \mathbb{R} \times S^1 \;\middle|\; \frac{\partial u}{\partial s}(s_0, t_0) = 0 \right\}$$

$$R(u) := \left\{ (s_0, t_0) \in C(u)^c \;\middle|\; v(s,t) \neq x^{\pm}(t), \quad s \neq s_0 \implies u(s, t_0) \neq u(s_0, t_0) \right\}.$$

Proving $R(v)$ is open: contradict zero derivative.

- Use sequential characterization of being a closed set
- Construct a sequence in $R(v)^c$ converging to a point in $R(v)$.
- First two conditions of $R(v)$ are open, so extract a sequence failing injectivity
- Show it is bounded
- Apply Bolzano-Weierstrass to extract a convergent subsequence
- Use quotient definition of $\frac{\partial v}{\partial s}$, show it is zero, contradiction.

■

Toward 8.6.3, Injectivity: $R(v)$ is Dense

Define **multiple points**:

- Set $\overline{\mathbb{R}} = \mathbb{R} \coprod \{\pm\infty\}$, extend $v : \mathbb{R}^2 \longrightarrow W$ to

$$v : \overline{\mathbb{R}} \times \mathbb{R} \longrightarrow W$$
$$v(\pm\infty, t) = x^{\pm}(t).$$

- Define the set of *multiple points over* (s_0, t_0):

$$M(s_0, t_0) := \left\{ (s', t_0) \in \mathbb{R}^2 \;\middle|\; s \neq s' \in \overline{\mathbb{R}}, \quad v(s', t_0) = v(s_0, t_0) \right\}$$

- *Multiple points* are where injectivity fails in s.
- Characterizes $R(v) \subset C(v)^c$ as points which don't admit multiples.

8.6.3: $R(v)$ is dense in $\mathbb{R}^2$

Step 1: Exclude critical points $\bigcap$ multiple points

- Suffices to show $R(v)$ is dense in $C(v)^c$

$$(s,t) \in C(v)^c \implies \exists (s_n, t) \subset C(v)^c \overset{n\longrightarrow\infty}{\Longrightarrow} (s,t) \ \text{ with } v(s_n,t) \neq x^{\pm}(t) \ \forall n.$$

- Why? E.g.

$$v(s+\frac{1}{n},t) = x^+(t) \implies \frac{\partial v}{\partial s} = 0 \implies (s+\frac{1}{n},t) \in C(v).$$

- Then suffices to show every $(s_0,t_0) \in \mathbb{R} \times [0,1]$ with (importantly)

$$\frac{\partial v}{\partial s}(s_0,t_0) \neq 0 \quad \text{and} \quad v(s_0,t_0) \neq x^{\pm}(t_0). \tag{1}$$

  is the limit of a sequence of points in $R(v)$.
- Proceed by assuming this is not the case, toward a contradiction.

Step 1: Exclude critical points $\bigcap$ multiple points

$$\frac{\partial v}{\partial s}(s_0, t_0) \neq 0 \quad \text{and} \quad v(s_0, t_0) \neq x^{\pm}(t_0). \tag{2}$$

A Small Ball Avoids Critical Points in the Image

- Surround (s_0, t_0) by a ball $B_\varepsilon(s_0, t_0) \subset R(v)^c$
- We can choose ε small enough and $M \gg 1$ big enough, defining

$$\mathbf{M} = [-M, M] \subset \mathbb{R},$$

such that several properties hold:

Step 1: Exclude critical points $\bigcap$ multiple points

$$\frac{\partial v}{\partial s}(s_0, t_0) \neq 0 \quad \text{and} \quad v(s_0, t_0) \neq x^{\pm}(t_0).$$

1. Translate to far enough to get a point outside the image of the ball:

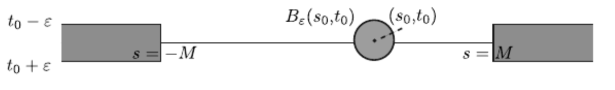
$$(s, t) \in \mathbf{M}^c \times [t_0 - \varepsilon, t_0 + \varepsilon] \subset \mathbb{R} \times I \implies$$
$$v(s, t) \bigcap v(B_\varepsilon(s_0, t_0) = \emptyset \quad \text{and} \quad x^{\pm}(t) \notin v(B_\varepsilon(s_0, t_0)).$$

   - Idea: else, cook up sequences forcing $v(s_0, t_0) = x^{\pm}(t_0)$, contradicting open conditions

2. For $t \in B_\varepsilon(t_0)$, $B_\varepsilon(s_0) \hookrightarrow W$ is an injective immersion

Step 1: Exclude critical points $\bigcap$ multiple points

1. Translated boxes that miss the image of $B_\varepsilon(s_0, t_0)$ *and* contain no multiple points over (s_0, t_0)
2. $B_\varepsilon(s_0) \hookrightarrow W$ immersively



. 8.3

Combine 1 and 2 to show that

- v is locally constant
- $(s_0, t_0) \in C(v)$
- Every point in $B_\varepsilon(s_0, t_0)$ satisfies open conditions

$$\frac{\partial v}{\partial s}(s_0, t_0) \neq 0 \quad \text{and} \quad v(s_0, t_0) \neq x^{\pm}(t_0).$$

Step 1: Exclude critical points $\bigcap$ multiple points

3

$$|\mathbf{M}_C| := \left|(\mathbf{M} \times I) \bigcap C(v)\right| < \infty$$

since it's the intersection of a compact and a discrete set

- Perturb (s_0, t_0) so that

$$(s, t) \in \mathbf{M}_C \implies v(s_0, t_0) \neq v(s, t).$$

  - Possible since $(s_0, t_0) \notin C(v) \implies v$ is non-constant in a neighborhood of (s_0, t_0).

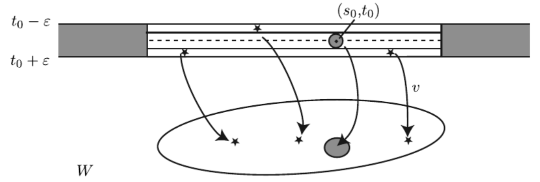
- Decrease ε so that

$$v(B_\varepsilon(s_0, t_0)) \bigcap v(\mathbf{M}_C) = \emptyset.$$

Step 1: Exclude critical points $\bigcap$ multiple points

$$v(B_\varepsilon(s_0, t_0)) \bigcap v(\mathbf{M}_C) = \emptyset.$$

This means that in the thick strip containing (s_0, t_0), no critical points land in its image.



8.4

Conclude that we *only* have to consider injectivity, not critical points that are also multiple points.

Step 2: Failure of Injectivity in Small Boxes

- Define the set of multiple points over s_0:

$$\mathbf{S}_M(t_0) = \{s_1, \cdots, s_N\} = \left\{ s_i \in [-M, M] \;\middle|\; v(s_i, t_0) = v(s_0, t_0) \right\},$$

  - Finite, since infinite $\implies$ limit point $\implies \frac{\partial v}{\partial s} = 0.$ $\Rightarrow\!\Leftarrow$
- Lemma: For every $\varepsilon > 0$ there exists a $\delta > 0$ such that defining

$$\Delta_0 = [s_0 - \delta, s_0 + \delta] \times [t_0 - \delta, t_0 + \delta]$$

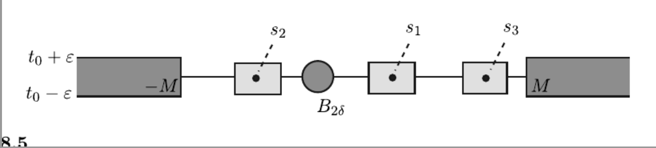
then

$$(s, t) \in \Delta_0 \implies \exists s' \in B_\varepsilon(s_j) \text{ s.t. } v(s, t) = v(s', t)$$

for some $1 \leq j \leq N$.

Step 2: Failure of Injectivity in Small Boxes

- In words: for every ε, we can find a δ-box $\Delta_0 \ni (s_0, t_0)$ such that every point in Δ_0 is a multiple point over *some* point in an epsilon ball around *some* point in $\mathbf{S}_M(t_0)$.
- Fix $\varepsilon' < \varepsilon/2$ form Step 1 and apply the lemma to ε' to produce δ and Δ_0.
- Apply the lemma: shrink Δ_0 to a closed delta ball $\overline{B}_\delta(s_0, t_0)$.
  - **Upshot**: Every point in Δ_0 is a multiple point over some s_j.



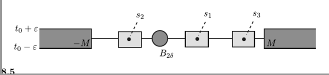
Step 2: Failure of Injectivity in Small Boxes

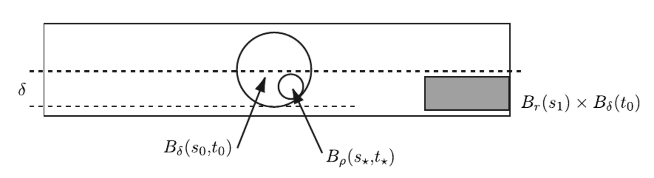
$$\mathbf{S}_M(t_0) = \{s_1, \cdots, s_N\} = \Big\{ s_i \in [-M, M] \;\Big|\; v(s_i, t_0) = v(s_0, t_0) \Big\}$$

- So partition the ball up: define Σ_j: all multiple points over $s_j \in \mathbf{S}_M(t_0)$.
- Take smaller ρ-ball some $(s_\star, t_\star) \in \Sigma_1^\circ$, choose ε' small enough such that

$$B_\rho(s_\star, t_\star) \bigcap \left([s_1 - \varepsilon', s_1 + \varepsilon'] \times [t_0 - \delta, t_0 + \delta] \right) = \emptyset.$$

Upshot: the shaded region is disjoint from the ρ-ball.





Step 3: Final Contradiction

- Construct v_1, v_2 which
  - Satisfy the same Cauchy-Riemann equations
  - Are equal at the origin
  - Have nonzero derivative at the origin.
- We want to show they are equal on $\mathbb{R}^2$
- Constructing them: use points from step 2 to translate
  - Obtain $(s_\star, t_\star)$ and $(s'_\star, t_\star)$ from previous step.
  - Define

$$\begin{aligned} v_1(s,t) &= v\left(s + s_\star, t + t_\star\right) \quad \Longrightarrow \quad v_1(z) = v(z + w_1) \\ v_2(s,t) &= v\left(s + s'_\star, t + t_\star\right) \quad \Longrightarrow \quad v_2(z) = v(z + w_2) \end{aligned}$$

  - Satisfy the same CR equations
  - By construction, they coincide at $(0,0)$ since $v(s_\star, t_\star) = v(s'_\star, t_\star)$.
  - Derivatives at the origin are nonzero, coming from the fact that $\frac{\partial v}{\partial s}(s_\star, t_\star) \neq 0$.

Step 3: Final Contradiction

- Now work at zero: for every $(s, t) \in B_\rho(0, 0)$ there exists a multiple point $s' \in B_{2\varepsilon'}(0)$ over s.
- Use the following extension lemma, consequence of **Continuation Principle**: in this situation, with $X_t \equiv 0$ on $B_\varepsilon(\mathbf{0})$, then

$$z \in B_\varepsilon(\mathbf{0}) \implies v_1(z) = v_2(z).$$

- Define

$$\begin{aligned} \mathcal{F} : \mathrm{C}^\infty\left(\mathbf{R} \times S^1; \mathbf{R}^{2n}\right) &\longrightarrow \mathrm{C}^\infty\left(\mathbf{R} \times S^1; \mathbf{R}^{2n}\right) \\ w &\longmapsto \left(\frac{\partial}{\partial s} + J \cdot \frac{\partial}{\partial t}\right) w \end{aligned}$$

- Since v_1, v_2 satisfy the same CR equation, $\mathcal{F}(v_1) = \mathcal{F}(v_2)$

Step 3: Final Contradiction

$$\begin{aligned} \mathcal{F} : \mathrm{C}^\infty\left(\mathbf{R}\times S^1;\mathbf{R}^{2n}\right) &\longrightarrow \mathrm{C}^\infty\left(\mathbf{R}\times S^1;\mathbf{R}^{2n}\right) \\ w &\longmapsto \left(\frac{\partial}{\partial s} + J\cdot\frac{\partial}{\partial t}\right) w \end{aligned}$$

Linearize $\mathcal{F}$ as we did for the Floer operator to obtain

$$(d\mathcal{F})_{\dots}(Y) = \left(\frac{\partial}{\partial s} + J_0\cdot\frac{\partial}{\partial t} + \tilde{S}\right) Y.$$

where $\tilde{S} : I \times \mathbb{R}^2 \longrightarrow \mathrm{End}(\mathbb{R}^{2n})$

Step 3: Final Contradiction

- Set $Y = v_1 - v_2$, then

$$S = \int_{[0,1]} \tilde{S} \implies S\left(\frac{\partial}{\partial s} + J_0 \cdot \frac{\partial}{\partial t} + S\right) Y = 0$$

- From above, $Y \equiv 0$ in $B_\varepsilon(\mathbf{0})$, apply **Continuation Principle** to obtain $v_1 = v_2$ on $\mathbb{R}^2$
- Inductive argument to show

$$\forall k \in \mathbb{Z}, \quad v(s,t) = v(k(s'_\star - s_\star), t) \overset{k \longrightarrow \infty}{\longrightarrow} x^{\pm}(t),$$

which contradicts an open condition. ■

The Continuation Principle

- Take the perturbed CR equation

$$\left(\frac{\partial}{\partial s} + J_0 \cdot \frac{\partial}{\partial t} + S\right) Y = 0 \quad \text{where} \quad S : \mathbb{R}^2 \longrightarrow \text{End}(\mathbb{R}^{2n})$$

  where J_0 is the standard complex structure on $\mathbb{R}^{2n}$.
- Define an *infinite-order zero* z of an arbitrary function f as

$$z_0 \in Z_\infty \iff \sup_{B_r(z_0)} \frac{|f(z)|}{r^k} \overset{r \longrightarrow 0}{\longrightarrow} 0 \quad \forall k \in \mathbb{Z}^{\geq 0},$$

  or for a smooth function,

$$z_0 \in Z_\infty \iff f^{(k)}(z_0) = 0 \quad \forall k \in \mathbb{Z}^{\geq 0}.$$

The Continuation Principle

- Statement: If Y solves CR on $U \subset \mathbb{R}^2$ then the set

$$C := \left\{ (s,t) \in U \;\middle|\; Y \text{ is an infinite-order zero at } (s,t) \right\}.$$

- Explanation: for f smooth, Z_∞ is closed. For f holomorphic, it is clopen.
  - From complex analysis: for a connected domain Ω, the only clopen subsets are $\emptyset, \Omega$, so nonempty and $f = g$ on a connected subset implies $f = g$ on Ω.
  - In particular, $Y = 0$ on $U' \subseteq U$ implies $Y = 0$ on U.
- Proof is a consequence of the **Similarity Principle**

Similarity Principle

- Recall definition of perturbed CR equation:

$$\left(\frac{\partial}{\partial s} + J_0 \cdot \frac{\partial}{\partial t} + S\right)Y = 0 \quad \text{where} \quad S : \mathbb{R}^2 \longrightarrow \text{End}(\mathbb{R}^{2n})$$

Let

- $Y \in C^\infty(B_\varepsilon; \mathbb{C}^n)$, or more generally $W^{1,p}(B_\varepsilon; \mathbb{C}^n)$, be a solution
- $S \in C^\infty(\mathbb{R}^2, \text{End}(\mathbb{R}^2))$ or more generally $L^p(B_\varepsilon; \text{End}_\mathbb{R}(\mathbb{R}^{2n}))$
- $p > 2$

Then there exist

$$\delta \in (0, \varepsilon),$$
$$\sigma \in C^\infty(B_\delta, \mathbb{C}^n)$$
$$A \in W^{1,p}(B_\delta, \mathrm{GL}(\mathbb{R}^{2n})).$$

such that

$$\forall (s,t) \in B_\delta, \qquad Y(s,t) = A(s,t) \cdot \sigma(s + it) \quad \text{and} \quad J_0 A(s,t) = A(s,t) J_0.$$

Similarity Principle

Used to prove:

- $C(v)$ is discrete
- "Extension" lemma used to prove $R(v)$ is dense

Some ideas from proof:

- Matrix A will look like the fundamental matrix of solutions to an equation
- Compactify to get $B_\delta \subset S^2$, if $Y : S^2 \longrightarrow \mathbb{C}^n$ then we can consider $\overline{Y}$ as a section of the bundle

$$\left(A^{0,1} T^\star S^2\right)^n = \Lambda^{0,1} T^\star S^2 \otimes \mathbb{C}^n.$$

- $\overline{Y} = 0$ makes Y a holomorphic sphere in $\mathbb{C}^n$.

8.7

8.7 Outline

What we're trying to prove

- 8.1.5: $(d\mathcal{F})_u$ is a Fredholm operator of index $\mu(x) - \mu(y)$.
- Setup

$$S^{\pm}(t) = \lim_{s \longrightarrow \pm\infty} S(s,t),$$

$$R_t^{\pm} \text{a solution to the IVP}$$

$$\frac{\partial}{\partial t} R = J_0 S^{\pm} R, \quad R_0^{\pm} = \text{id}.$$

- Statement: if $\det(\text{id} - R_1^{\pm}) \neq 0$ then the operator

$$L : W^{1,p}(\mathbb{R} \times S^1; \mathbb{R}^{2n}) \longrightarrow L^p(\mathbb{R} \times S^1; \mathbb{R}^{2n})$$

$$L = \bar{\partial} + S(s,t) \qquad .$$

  is Fredholm for every $p > 1$.
  (i.e. index makes sense, $\dim\ker L, \dim\operatorname{coker} L < \infty$*)*
- Most of the work: $\dim(\ker L) < \infty$ and im (L) closed.

8.7 Outline: Step 1, $\dim \ker L < \infty$

- Main ingredients:
  - **Elliptic regularity**: For $Y \in L^p(\mathbb{R} \times S^1)$ a weak solution of the linearized Floer equation

$$LY = 0 \implies Y \in W^{1,p}(\mathbb{R} \times S^1) \bigcap C^\infty.$$

  - A consequence of the **Calderón-Zygmund (CZ) Inequality**: For $Y \in W^{1,p}(\mathbb{R} \times S^1; \mathbb{R}^{2n})$ and $p > 1$,

$$\|Y\|_{W^{1,p}(\mathbb{R}\times S^1;\mathbb{R}^{2n})} \leq C\left(\|LY\|_{L^p(\mathbb{R}\times S^1)} + \|Y\|_{L^p(\mathbb{R}\times S^1)}\right) \quad (3)$$

    for some constant C.
- Strategy: split into cases
  - Case 1: $S(s,t) = S(t)$ doesn't depend on s.
  - Case 2: $S(s,t)$ *does* depend on s

8.7 Outline: $\dim \ker L < \infty$

CZ inequality:

$$\|Y\|_{W^{1,p}(\mathbb{R}\times S^1;\mathbb{R}^{2n})} \leq C\Big(\|LY\|_{L^p(\mathbb{R}\times S^1)} + \|Y\|_{L^p(\mathbb{R}\times S^1)}\Big) \qquad (4)$$

Step 1: $S(s,t) = S(t)$ doesn't depend on s, prove improved estimate.

- Consider the "asymptotic operator"

$$D : W^{1,p}(\mathbb{R}\times S^1;\mathbb{R}^{2n}) \longrightarrow L^p(\mathbb{R}\times S^1)$$

$$D = \bar{\partial} + S(t) = \lim_{s\longrightarrow\pm\infty} L := \lim_{s\longrightarrow\pm\infty} \big(\bar{\partial} + S(s,t)\big).$$

- Show for $p > 1$, D is invertible.
- Invertibility improves estimate: replace $\mathbb{R}$ with $[-M, M]$.

8.7 Outline: $\dim \ker L < \infty$

- Step 2: $S(s,t)$ does depend on s
- Improved estimate in Step 1 allows replacing weak soln:

$$Y \in L^p(\mathbb{R} \times S^1; \cdot) \rightsquigarrow Y \in L^p([-M, M] \times S^1, \cdot).$$

- Then restrict

$$L : W^{1,p}(\mathbb{R} \times S^1; \mathbb{R}^{2n}) \longrightarrow L^p(\mathbb{R} \times S^1; \mathbb{R}^{2n})$$
$$L := \bar{\partial} + S(s,t)$$

$$\rightsquigarrow L_M : W^{1,p}(\mathbb{R} \times S^1) \longrightarrow L^p([-M, M] \times S^1).$$

- Since the restriction is a *compact* operator, it is "semi-Fredholm", apply a theorem:

CZ inequality satisfied $\implies$ $\dim \ker L_M < \infty$, im L_M closed.

8.7 Outline: $\dim \ker L < \infty$

- Will need some real/functional analysis to invert operators:
  - "Variation of constants"
  - Hilbert spaces and Spectrum of an operator
  - *Hille-Yosida* theory: existence and uniqueness for operator IVPs, e.g. $\frac{\partial Y}{\partial s} = -AY$
  - Young's Inequality (some convolution integrals)
  - Holder's Inequality
  - Distribution theory
  - Rellich's Theorem (Multiple uses)
  - Hahn-Banach Theorem
  - Riesz Representation Theorem
- Conclude 8.7 by showing L is Fredholm:
  - $\dim \ker L < \infty$ (long)
  - $\dim \operatorname{coker} L < \infty$ (very short)

8.8

8.8: Outline

What we're trying to prove

- 8.1.5: $(d\mathcal{F})_u$ is a Fredholm operator of index $\mu(x) - \mu(y)$.
- Define

$$L : W^{1,p}\left(\mathbf{R} \times S^1; \mathbf{R}^{2n}\right) \longrightarrow L^p\left(\mathbf{R} \times S^1; \mathbf{R}^{2n}\right)$$
$$Y \longmapsto \frac{\partial Y}{\partial s} + J_0 \frac{\partial Y}{\partial t} + S(s,t)Y$$

- 8.7: Shows L is Fredholm
- By the end of 8.8: replace L by L_1 with the same *index*
  - *(not the same kernel/cokernel)*
- Compute Ind L_1: explicitly describe $\ker L_1$, coker L_1.

8.8: Replacing L

$$\begin{aligned} L : W^{1,p}\left(\mathbf{R} \times S^1; \mathbf{R}^{2n}\right) &\longrightarrow L^p\left(\mathbf{R} \times S^1; \mathbf{R}^{2n}\right) \\ Y &\longmapsto \frac{\partial Y}{\partial s} + J_0 \frac{\partial Y}{\partial t} + S(s,t)Y \end{aligned}$$

- Replace in two steps:
  - $L \rightsquigarrow L_0$, modified in a $B_\varepsilon(0)$ in s.
    - Use invariance of index under small perturbations.
  - $L_0 \rightsquigarrow L_1$ by a homotopy, where $S_\lambda : S \rightsquigarrow S(s)$ a diagonal matrix that is a constant matrix *outside* $B_\varepsilon(0)$.
    - Use invariance of index under homotopy.

8.8: Replacing L

$$S(s,t) \overset{s\longrightarrow\pm\infty}{\longrightarrow} S^{\pm}(t)$$

$$R_t^{\pm} \text{a solution to the IVP} \qquad \frac{\partial}{\partial t} R = J_0 S^{\pm} R, \quad R_0^{\pm} = \mathrm{id}.$$

- Use the fact $S : \mathbb{R} \times S^1 \longrightarrow \mathrm{Mat}(2n;\mathbb{R})$ and $S^{\pm}(t)$ are symmetric.
- Take corresponding symplectic paths $R^{\pm} : I \longrightarrow \mathrm{Sp}(2n;\mathbb{R})$.
- L will be a Fredholm operator if

$$R^{\pm} \in \mathcal{S} := \left\{ R : I \longrightarrow \mathrm{Sp}(2n;\mathbb{R}) \;\middle|\; R(0) = \mathrm{id}, \quad \det(R(1) - \mathrm{id}) \neq 0 \right\}.$$

- Theorem 8.8.1:

$$\mathrm{Ind}(L) = \mu(R^{-}(t)) - \mu(R^{+}(t)) = \mu(x) - \mu(y).$$

8.8: $L_0 \rightsquigarrow L_1$

- Prop 8.8.2: Construct an operator

$$L_1 : W^{1,p}\left(\mathbf{R} \times S^1; \mathbf{R}^{2n}\right) \longrightarrow L^p\left(\mathbf{R} \times S^1; \mathbf{R}^{2n}\right)$$
$$Y \longmapsto \frac{\partial Y}{\partial s} + J_0 \frac{\partial Y}{\partial t} + S(s)Y$$

where $S : \mathbb{R} \longrightarrow \mathrm{Mat}(2n; \mathbb{R})$ is a path of diagonal matrices depending on $\mathrm{Ind}(R^{\pm}(t))$; then

$$\mathrm{Ind}(L) = \mathrm{Ind}(L_1) = \mathrm{Ind}(R^{-}(t)) - \mathrm{Ind}(R^{+}(t)).$$

- Idea of proof: take a homotopy of operators

$$L_\lambda : W^{1,p}\left(\mathbf{R} \times S^1; \mathbf{R}^{2n}\right) \longrightarrow L^p\left(\mathbf{R} \times S^1; \mathbf{R}^{2n}\right)$$
$$Y \longmapsto \frac{\partial Y}{\partial s} + J_0 \frac{\partial Y}{\partial t} + S_\lambda(s,t)Y$$

which are all Fredholm and all have the same index, then take time 1.

8.8: Index of L_1

$$
\begin{aligned}
L_1 : W^{1,p}\left(\mathbf{R} \times S^1; \mathbf{R}^{2n}\right) &\longrightarrow L^p\left(\mathbf{R} \times S^1; \mathbf{R}^{2n}\right) \\
Y &\longmapsto \frac{\partial Y}{\partial s} + J_0 \frac{\partial Y}{\partial t} + S(s)Y
\end{aligned}
$$

- Use the fact that

$$\operatorname{coker} L_1 \cong \ker L_1^*,$$

  and we can explicitly write the adjoint of L_1.
- Get a formula that resembles the Morse case
  - *(Counting number of eigenvalues that change sign).*

■