# 8.8 Part 2, Computing the Index of $L$
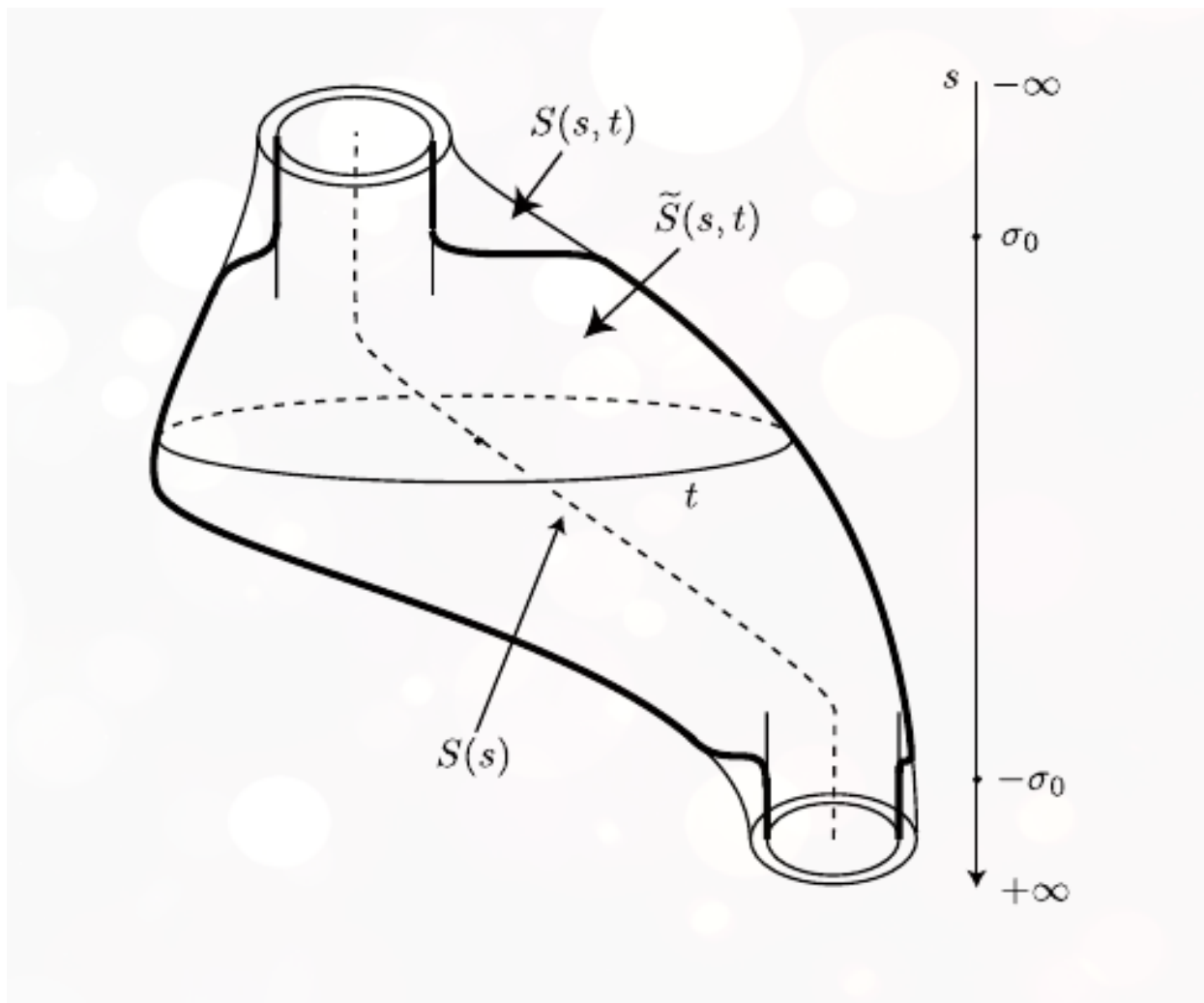
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What we're trying to prove:

- 8.1.5: $(d\mathcal{F})_u$ is a Fredholm operator of index $\mu(x) - \mu(y)$.

- Define

$$L : W^{1,p}\left(\mathbb{R} \times S^1; \mathbb{R}^{2n}\right) \longrightarrow L^p\left(\mathbb{R} \times S^1; \mathbb{R}^{2n}\right)$$

$$Y \longmapsto \frac{\partial Y}{\partial s} + J_0 \frac{\partial Y}{\partial t} + S(s,t)Y$$

where

$$S : \mathbb{R} \times S^1 \longrightarrow \mathrm{Mat}(2n; \mathbb{R})$$

$$S(s,t) \overset{s \to \pm\infty}{\Longrightarrow} S^{\pm}(t).$$

- 8.7: Shows $L$ is Fredholm

- By the end of 8.8: replace $L$ by $L_1$ with the same *index*

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- – *(not the same kernel/cokernel)*

- Compute Ind $L_1$: explicitly describe $\ker L_1, \operatorname{coker} L_1$.

- Replace in two steps:

  - $L \rightsquigarrow L_0$, modified outside $B_{\sigma_0}(0)$ in $s$.
    * Replace $S(s,t)$ by a matrix

    $$\tilde{S}(s,t) = \begin{cases} S^-(t) & s \leq -\sigma_0 \\ S^+(t) & s \geq \sigma_0 \end{cases}.$$

    * Idea: approximate by cylinders at infinity.
    * Use invariance of index under small perturbations.
  - $L_0 \rightsquigarrow L_1$ by a homotopy, where $S_\lambda : S \rightsquigarrow S(s)$ a diagonal matrix that is a constant matrix *outside $B_\varepsilon(0)$*.
    * Use invariance of index under homotopy.

## 0.1  Main Results

- Theorem 8.8.1:

$$\operatorname{Ind}(L) = \mu\left(R^-(t)\right) - \mu\left(R^+(t)\right) = \mu(x) - \mu(y).$$

- Prop 8.8.2: Reducing $L$ to $L_1$ Construct an operator

$$L_1 : W^{1,p}\left(\mathbb{R} \times S^1; \mathbb{R}^{2n}\right) \longrightarrow L^p\left(\mathbb{R} \times S^1; \mathbb{R}^{2n}\right)$$
$$Y \longmapsto \frac{\partial Y}{\partial s} + J_0 \frac{\partial Y}{\partial t} + S(s)Y$$

where $S : \mathbb{R} \longrightarrow \operatorname{Mat}(2n; \mathbb{R})$ is a path of diagonal matrices depending on $\operatorname{Ind}(R^\pm(t))$; then

$$\operatorname{Ind}(L) = \operatorname{Ind}(L_1) = \operatorname{Ind}(R^-(t)) - \operatorname{Ind}(R^+(t)).$$

- Prop 8.8.3: Reducing $L_1$ to $R^\pm$. Let $k^\pm := \operatorname{Ind}(R^\pm)$; then $\operatorname{Ind}(L_1) = k^- - k^+$.

- Lemma 8.8.4: $\operatorname{Ind}(L_0) = \operatorname{Ind}(L)$.

- Han's Talk:

  - Prop 8.8.3, using Lemma 8.8.5

- Me

  - Proof of 8.8.5

## 0.2  8.8.5:

Used in the proof of 8.8.3, $\operatorname{Ind}(L_1) = k^- - k^+$.

Setup:

We have

$$J_0^1 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \implies J_0 = \begin{bmatrix} J_0^1 & & & \\ & J_0^1 & & \\ & & \ddots & \\ & & & J_0^1 \end{bmatrix} \in \bigoplus_{i=1}^n \mathrm{Mat}(2;\mathbb{R}).$$

We had a path of diagonal matrices:

$$S(s) := \begin{pmatrix} a_1(s) & 0 \\ 0 & a_2(s) \end{pmatrix}, \quad \text{with } a_i(s) :=_? \begin{cases} a_i^- & \text{if } s \leq -s_0 \\ a_i^+ & \text{if } s \geq s_0 \end{cases}.$$

Statement: let $p > 2$ and define

$$F : W^{1,p}\left(\mathbb{R} \times S^1; \mathbb{R}^2\right) \longrightarrow L^p\left(\mathbb{R} \times S^1; \mathbb{R}^2\right)$$
$$Y \mapsto \frac{\partial Y}{\partial s} + J_0 \frac{\partial Y}{\partial t} + S(s)Y.$$

This looks like $L_1$ for $n = 1$?

1. Suppose $a_1(s) = a_2(s)$ and define $a^\pm := a_1^\pm = a_2^\pm$. Then

$$\dim \ker F = 2 \cdot \#\left\{\ell \in \mathbf{Z} | a^- < 2\pi\ell < a^+\right\}$$
$$\dim \ker F^\star = 2 \cdot \#\left\{\ell \in \mathbf{Z} | a^+ < 2\pi\ell < a^-\right\}.$$

2. Suppose $\sup_{s \in \mathbb{R}} \|S(s)\| < 1$, then

$$\dim \ker F = \#\left\{i \in \{1,2\} \;\middle|\; a_i^- < 0 \text{ and } a_i^+ > 0\right\}$$
$$\dim \ker F^\star = \#\left\{i \in \{1,2\} \;\middle|\; a_i^+ < 0 \text{ and } a_i^- > 0\right\}.$$

Remark: Resembles formula for computing index in Morse case, number of eigenvalues that change sign.

Remark: Proof will proceed by explicitly computing kernel.

## 0.3 Proof

### 0.3.1 Assertion 1

**Assertion 1**: Suppose $a_1(s) = a_2(s)$ and define $a^\pm := a_1^\pm = a_2^\pm$. Then

$$\dim \ker F = 2 \cdot \#\left\{\ell \in \mathbf{Z} | a^- < 2\pi\ell < a^+\right\}$$
$$\dim \ker F^\star = 2 \cdot \#\left\{\ell \in \mathbf{Z} | a^+ < 2\pi\ell < a^-\right\}.$$

Step 1: Transform to Cauchy-Riemann Equations

- Write $a(s) = a_1(s) = a_2(s)$.

- Start with equation on $\mathbb{R}^2$,

$$Y(s,t) = (Y_1(s,t), Y_2(s,t))$$

- Replace with equation on $\mathbb{C}$:

$$Y(s,y) = Y_1(s,t) + iY_2(s,t)$$

.

- Rewrite the PDE $F(Y) = 0$ as $\bar{\partial}Y + S(s)Y = 0$, i.e.

$$\frac{\partial}{\partial s}\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} + \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\frac{\partial}{\partial t}\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} + \begin{pmatrix} a(s) & 0 \\ 0 & a(s) \end{pmatrix}\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = 0.$$

- Change of variables: let $Y = B\tilde{Y}$ where $B \in \mathrm{GL}(1, \mathbb{C})$ satisfies $\left(\bar{\partial} + S\right)B = 0$ to obtain $\bar{\partial}\tilde{Y} = 0$.

  - Can choose $B = \begin{bmatrix} b(s))0 \\ 0 \quad b(s) \end{bmatrix}$ where $\frac{\partial b}{\partial s} = -a(s)b(s)$.

  - Explicitly, we can take the integral $b(s) = e^{\int_0^s -a(t)\ dt} = e^{-A(s)}$

- Remark: for some constants $C_i$, we have

$$A(s) = \begin{cases} C_1 + a^- s & s \leq -\sigma_0 \\ C_2 + a^+ s & s \geq \sigma_0 \end{cases}.$$

- Remark: the new $\tilde{Y}$ satisfies CR. It is continuous and $L^1_{\mathrm{loc}}$ and thus by elliptic regularity $C^\infty$. Its real/imaginary parts are $C^\infty$ and harmonic.

Step 2: ?

- Identify $s + it \in \mathbb{R} \times S^1$ with $u = e^{2\pi z}$

- Apply Laurent's theorem to $\tilde{Y}(u)$ on $\mathbb{C} \setminus \{0\}$ to obtain an expansion of $\tilde{Y}$ in $z$.

- Deduce that the solutions of the system are given by

$$\tilde{Y}(s + it) = \sum_{\ell \in \mathbf{Z}} c_\ell e^{(s+it)2\pi\ell}.$$

  where $c_\ell \in \mathbb{C}$ and this sequence converges for all $s, t$.

- Write in real coordinates as

$$\tilde{Y}(s,t) = \sum_{\ell \in \mathbf{Z}} e^{2\pi s\ell}\left(\alpha_\ell\begin{pmatrix} \cos 2\pi\ell t \\ \sin 2\pi\ell t \end{pmatrix} + \beta_\ell\begin{pmatrix} -\sin 2\pi\ell t \\ \cos 2\pi\ell t \end{pmatrix}\right).$$

- Return to $Y = B\tilde{Y}$:

$$Y(s,t) = \sum_{\ell \in \mathbf{Z}} e^{2\pi s\ell}\left(\alpha_\ell\begin{pmatrix} e^{-A(s)}\cos 2\pi\ell t \\ e^{-A(s)}\sin 2\pi\ell t \end{pmatrix} + \beta_\ell\begin{pmatrix} -e^{-A(s)}\sin 2\pi\ell t \\ e^{-A(s)}\cos 2\pi\ell t \end{pmatrix}\right).$$

- For $s \geq s_0$, for some constants $K_i$ we can write

$$Y(s,t) = \sum_{\ell \in \mathbf{Z}} \begin{pmatrix} e^{(2\pi\ell - a^-)s + K} \left(\alpha_\ell \cos 2\pi\ell t - \beta_\ell \sin 2\pi\ell t\right) \\ e^{(2\pi\ell - a^-)s + K'} \left(\alpha_\ell \sin 2\pi\ell t + \beta_\ell \cos 2\pi\ell t\right) \end{pmatrix}.$$

  and for $s \geq s_0$

$$Y(s,t) = \sum_{\ell \in \mathbf{Z}} \begin{pmatrix} e^{(2\pi\ell - a^+)s + C} \left(\alpha_\ell \cos 2\pi\ell t - \beta_\ell \sin 2\pi\ell t\right) \\ e^{(2\pi\ell - a^+)s + C'} \left(\alpha_\ell \sin 2\pi\ell t + \beta_\ell \cos 2\pi\ell t\right) \end{pmatrix}.$$

- Then $Y \in L^p \iff$ the exponential terms die at infinity. Forces the conditions:

  - $\ell \neq 0 \implies \alpha_\ell = \beta_\ell = 0$ or $2\pi\ell < a^+$.
  - $\ell = 0 \implies \left(a_0 = 0 \text{ or } a^+ > 0\right)$ and $\left(\beta_0 = 0 \text{ or } a^+ > .0\right)$.

This further forces

$$\begin{cases} \alpha_\ell = \beta_\ell = 0 \text{ or } a^- < 2\pi\ell < a^+ & \ell \neq 0 \\ \left(\alpha_0 = 0 \text{ or } a^- < 0 < a^+\right) \text{ and } \left(\beta_0 = 0 \text{ or } a^- < 0 < a^+\right) & \ell = 0 \end{cases}.$$

- Finitely many such $\ell$ that satisfy these conditions
- Sufficient conditions for $Y(s,t) \in W^{1,p}$.

$$F : W^{1,p}\left(\mathbb{R} \times S^1; \mathbb{R}^2\right) \longrightarrow L^p\left(\mathbb{R} \times S^1; \mathbb{R}^2\right)$$
$$Y \mapsto \frac{\partial Y}{\partial s} + J_0 \frac{\partial Y}{\partial t} + S(s)Y.$$

I.e. $F = \dfrac{\partial}{\partial s} + J_0 \dfrac{\partial}{\partial t} + S(s)$.

- Compute dimension of space of solutions:

$$\dim \operatorname{Ker} F = 2\#\left\{\ell \in \mathbf{Z}^\star | a^- < 2\pi\ell < a^+\right\} \qquad = 2\#\left\{\ell \in \mathbf{Z} | a^- < 2\pi\ell < a^+\right\}.$$

Test:

$$\mathbb{1}\left[\{x\}\right].$$

Use this to deduce $\dim \ker F^*$:

- $Y \in \ker F^* \iff Z(s,t) := Y(-s,t)$ is in the kernel of the operator

$$\tilde{F} : W^{1,q}\left(\mathbb{R} \times S^1; \mathbb{R}^2\right) \longrightarrow L^p\left(\mathbb{R} \times S^1; \mathbb{R}^2\right)$$
$$Z \mapsto \frac{\partial Z}{\partial s} + J_0 \frac{\partial Z}{\partial t} + S(-s)Y.$$

- Obtain $\ker F^* \cong \ker \tilde{F}$.

$\blacksquare_1$

### 0.3.2 Assertion 2

**Assertion 2**: Suppose $\sup\limits_{s\in\mathbb{R}}\|S(s)\| < 1$, then

$$\dim\operatorname{Ker} F = \#\left\{ i \in \{1,2\} \ \middle| \ a_i^- < 0 \text{ and } a_i^+ > 0 \right\}$$

$$\dim\operatorname{Ker} F^\star = \#\left\{ i \in \{1,2\} \ \middle| \ a_i^+ < 0 \text{ and } a_i^- > 0 \right\}.$$

We use the following lemma

- Lemma 8.8.7:

$$\sup_{s\in\mathbb{R}}\|S(s)\| < 1 \implies \text{ the elements in } \ker F, \ \ker F^* \text{ are independent of } t.$$

- Proof: see Proposition 10.1.7, in subsection 10.4.a.

- We know (?)

$$\mathbf{a}(s) := \begin{bmatrix} a_1(s) \\ a_2(s) \end{bmatrix}, \quad \mathbf{Y} := \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} \in \ker F \implies \frac{\partial \mathbf{Y}}{\partial s} = -\mathbf{a}(s)\mathbf{Y}.$$

- Therefore we can solve to obtain

$$\mathbf{Y}(s) = \mathbf{c}\exp\left(-\mathbf{A}(s)\right), \quad \mathbf{A}(s) = \int_0^s -\mathbf{a}(\sigma) \ d\sigma.$$

- Explicitly,

$$\begin{cases} \dfrac{\partial Y_1}{\partial s} = -a_1(s)Y_1 \\ \dfrac{\partial Y_s}{\partial s} = -a_s(s)Y_2 \end{cases} \implies Y_i(s) = c_i e^{-A_i(s)}, \quad A_i(s) = \int_0^s -a_i(\sigma) \ d\sigma.$$

- As before,

$$A_i(s) = \begin{cases} C_{1,i} + a_i^- \cdot s & s \leq -\sigma_0 \\ C_{2,i} + a_i^+ \cdot s & s \geq \sigma_0 \end{cases}.$$

- Thus $Y_i \in W^{1,p} \iff 0 \in (a_i^-, a_i^+)$, establishing the formula for $\dim\ker F$.

- Get formula for $\dim\ker F^*$ using $\tilde{F}$ as before.

$$\blacksquare_2$$

(End of Chapter)

# 1  8.8.3:

- Previously: we replaced $S$ with a path of diagonal matrices only depending on $s$.

- Obtained an operator $L_1$.

- Statement: Prop 8.8.3: Reducing $L_1$ to $R^\pm$. Let $k^\pm := \operatorname{Ind}(R^\pm)$; then $\operatorname{Ind}(L_1) = k^- - k^+$.

## 1.1  Proof

- From section 8.7: coker $L_1 \cong \ker L_1^*$:

- Recall definition of $L_1$:

$$L_1 : W^{1,p}\left(\mathbb{R} \times S^1; \mathbb{R}^{2n}\right) \longrightarrow L^p\left(\mathbb{R} \times S^1; \mathbb{R}^{2n}\right)$$
$$Y \longmapsto \frac{\partial Y}{\partial s} + J_0 \frac{\partial Y}{\partial t} + S(s)Y$$

- We can write the adjoint:

$$L_1^\star : W^{1,q}\left(\mathbb{R} \times S^1; \mathbb{R}^{2n}\right) \longrightarrow L^q\left(\mathbb{R} \times S^1; \mathbb{R}^{2n}\right)$$
$$Z \longmapsto -\frac{\partial Z}{\partial s} + J_0 \frac{\partial Z}{\partial t} + S^T(s)Z$$

  where $\dfrac{1}{p} + \dfrac{1}{q} = 1$ are conjugate exponents.

- Now apply Lemma 8.8.3, which computes the kernels explicity

- Consider four cases, depending on parity of $k^\pm - n$, show all 4 lead to $\mathrm{Ind}(L_1) = k^- - k^+$. $S_{k^\pm}$ are certain diagonal matrices, and

1. $k^- \equiv k^+ \equiv n \mod 2$.
2. $k^- \equiv n, k^+ \equiv n-1 \mod 2$
3. $k^- \equiv n-1, k^+ \equiv n \mod 2$.
4. $k^- \equiv k^+ \equiv n-1 \mod 2$

| $k^-$ | $k^+$ | $n$ |
|-------|-------|-----|
| ✓ | ✓ | ✓ |
| ✓ |   | ✓ |
|   | ✓ | ✓ |
| ✓ | ✓ |   |

Case 1

$$S_{k^-} = \begin{bmatrix} -\pi & & & & & & \\ & -\pi & & & & & \\ & & \ddots & & & & \\ & & & -\pi & & & \\ & & & & -\pi & & \\ & & & & & (n-1-k^-)\pi & \\ & & & & & & (n-1-k^-)\pi \end{bmatrix}$$

$$S_{k^+} = \begin{bmatrix} -\pi & & & & & & \\ & -\pi & & & & & \\ & & \ddots & & & & \\ & & & -\pi & & & \\ & & & & -\pi & & \\ & & & & & (n-1-k^+)\pi & \\ & & & & & & (n-1-k^+)\pi \end{bmatrix}.$$

## 1.1  Proof

Case 2

$$S_{k-} = \begin{bmatrix} -\pi & & & & & & \\ & -\pi & & & & & \\ & & \ddots & & & & \\ & & & -\varepsilon\pi & & & \\ & & & & -\varepsilon\pi & & \\ & & & & & (n-1-k^-)\pi & \\ & & & & & & (n-1-k^-)\pi \end{bmatrix}$$

$$S_{k+} = \begin{bmatrix} -\pi & & & & & & \\ & -\pi & & & & & \\ & & \ddots & & & & \\ & & & \varepsilon & & & \\ & & & & -\varepsilon & & \\ & & & & & (n-2-k^+)\pi & \\ & & & & & & (n-2-k^+)\pi \end{bmatrix}.$$

Case 3: switches order of case 2.

$$S_{k-} = \begin{bmatrix} -\pi & & & & & & \\ & -\pi & & & & & \\ & & \ddots & & & & \\ & & & \varepsilon & & & \\ & & & & -\varepsilon & & \\ & & & & & (n-2-k^-)\pi & \\ & & & & & & (n-2-k^-)\pi \end{bmatrix}$$

$$S_{k+} = \begin{bmatrix} -\pi & & & & & & \\ & -\pi & & & & & \\ & & \ddots & & & & \\ & & & -\varepsilon\pi & & & \\ & & & & -\varepsilon\pi & & \\ & & & & & (n-1-k^+)\pi & \\ & & & & & & (n-1-k^+)\pi \end{bmatrix}.$$

Case 4

$$S_{k-} = \begin{bmatrix} -\pi & & & & & & \\ & -\pi & & & & & \\ & & \ddots & & & & \\ & & & 1 & & & \\ & & & & -1 & & \\ & & & & & (n-2-k^-)\pi & \\ & & & & & & (n-2-k^-)\pi \end{bmatrix}$$

$$S_{k+} = \begin{bmatrix} -\pi & & & & & & \\ & -\pi & & & & & \\ & & \ddots & & & & \\ & & & 1 & & & \\ & & & & -1 & & \\ & & & & & (n-s-k^+)\pi & \\ & & & & & & (n-2-k^+)\pi \end{bmatrix}.$$