Chapter 9

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1 | Background, Notation, Setup

Goals

Theorem 1.1 ***(Arnold Conjecture (Symplectic Morse Inequalities?)).***
Let (W, ω) be a compact symplectic manifold and

$$H : W \to \mathbb{R}$$

a time-dependent Hamiltonian with nondegenerate 1-periodic solutions. Then

$$\# \{\text{1-Periodic trajectories of } X_H\} \geq \sum_{k \in \mathbb{Z}} \dim_? HM_k(W; \mathbb{Z}/2\mathbb{Z}).$$

Here $HM_*(W)$ is the Morse homology, and *nondegenerate* means the differential of the flow at time 1 has no fixed vectors.

Important Ideas for This Chapter:

Theorem 1.2 ***(Use Broken Trajectories to Compactify).***
$\mathcal{L}(x, y)$ is compact, where the compactification is given by adding in

$$\partial \mathcal{L}(x, y) = \{\text{"Broken Trajectories"}\}$$

Theorem 1.3 ***(Gluing Yields a Chain Complex).***

$$\partial^2 = 0$$

Strategy:

In the background, have a Hamiltonian $H : W \to \mathbb{R}$. Basic idea: cook up a gradient flow.

1. Define the action functional $\mathcal{A}_H$

   On an infinite-dimensional space, critical points are periodic solutions of H

2. Construct the chain complex (graded vector space) CF_*.

   Uses analog of the *index* of a critical point.

3. Define the vector field X_H using $-\text{grad } \mathcal{A}_H$.

   This will be used to define ∂ later.

4. Count the trajectories of X_H
5. Show finite-energy trajectories connect critical points of $\mathcal{A}_H$.
6. Show *Gromov Compactness* for space of trajectories of finite energy
7. Define ∂

   Uses another compactness property

8. Show space of trajectories is a manifold, plus analog of "Smale property"
9. **Show that** $\partial^2 = 0$ using a gluing property
10. Show that HF_* doesn't depend on $\mathcal{A}_H$ or X_H
11. Show $HF_* \cong HM_*$, and compare dimensions of the vector spaces CM_* and CF_*.

Ingredients:

- (W, ω, J) with $\omega \in \Omega^2(W)$ is a symplectic manifold
  - With $J : T_pW \to T_pW$ an almost complex structure, so $J^2 = -\mathrm{id}$.
- $H \in C^\infty(W; \mathbb{R})$ a Hamiltonian
  - X_H the corresponding symplectic gradient.
  - Defined by how it acts on tangent vectors in T_xM:

$$\omega_x(\,\cdot\,, X_H(x)) = (dH)_x(\,\cdot\,).$$

  - Zeros of vector field X_H correspond to critical points of H:

$$X_H(x) = 0 \iff (dH)_x = 0.$$

  - Take the associated flow, assumed 1-periodic:

$$\psi^t \in C^\infty(W, W) \qquad \psi^1 = \mathrm{id},$$

  - Critical points of H are periodic trajectories.
- $u \in C^\infty(\mathbb{R} \times S^1; W)$ is a solution to the Floer equation.
- The Floer equation and its linearization:

$$\begin{aligned} \mathcal{F}(u) &= \frac{\partial u}{\partial s} + J\frac{\partial u}{\partial t} + \mathrm{grad}\,_u(H) = 0 \\ (d\mathcal{F})_u(Y) &= \frac{\partial Y}{\partial s} + J_0\frac{\partial Y}{\partial t} + S \cdot Y \end{aligned}$$

$$Y \in u^*TW,\ S \in C^\infty(\mathbb{R} \times S^1; \mathrm{End}(\mathbb{R}^{2n})).$$

- $\mathcal{L}W$ is the *free loop space* on W, i.e. space of contractible loops on W, i.e. $C^\infty(S^1; W)$ with the C^∞ topology
  - Elements $x \in \mathcal{L}W$ can be viewed as maps $S^1 \to W$.
  - Can extend to maps from a closed disc, $u : \overline{\mathbb{D}}^2 \to M$.
  - Loops in $\mathcal{L}W$ can be viewed as maps $S^2 \to W$, since they're maps $I \times S^1 \to W$ with the boundaries pinched:

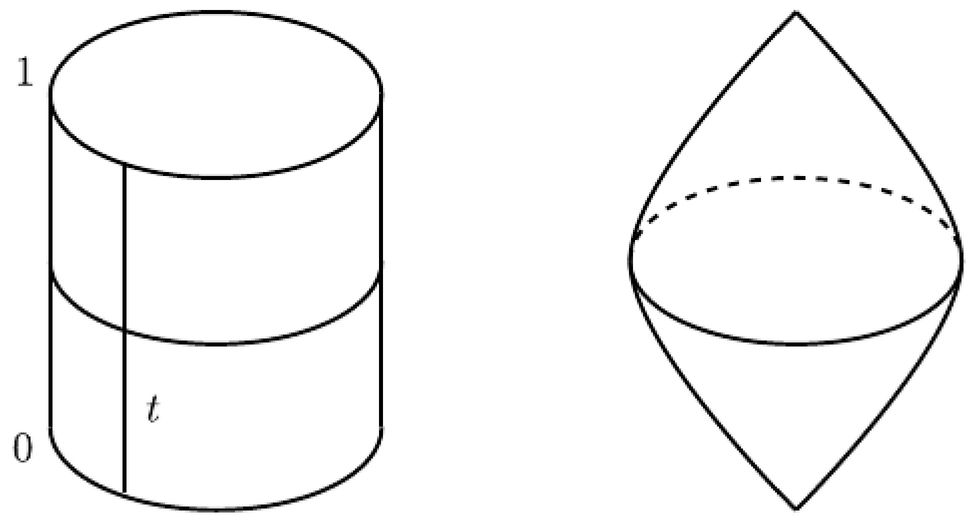

Figure 1: Loops in $\mathcal{L}W$

- The action functional is given by

$$\begin{aligned} \mathcal{A}_H : \mathcal{L}W &\to \mathbb{R} \\ x &\mapsto -\int_{\mathbb{D}} u^*\omega + \int_0^1 H_t(x(t))\ dt \end{aligned}$$

  - Example: $W = \mathbb{R}^{2n} \implies A_H(x) = \int_0^1 (H_t\ dt - p\ dq)$.
  - A correspondence

$$\begin{Bmatrix}\text{Solutions to the} \\ \text{Floer equation}\end{Bmatrix} \iff \begin{Bmatrix}\text{Trajectories} \\ \text{of grad } \mathcal{A}_H\end{Bmatrix}.$$

- x, y periodic orbits of H (nondegenerate, contractible), equivalently critical points of $\mathcal{A}_H$.

- Assumption of *symplectic asphericity*, i.e. the symplectic form is zero on spheres. Statement: for every $u \in C^\infty(S^2, W)$,

$$\int_{S^2} u^*\omega = 0 \quad \text{or equivalently} \quad \langle \omega,\ \pi_2 W\rangle = 0.$$

- Assumption of *symplectic trivialization*: for every $u \in C^\infty(S^2; M)$ there exists a symplectic trivialization of the fiber bundle u^*TM, equivalently

$$\langle c_1 TW,\ \pi_2 W\rangle = 0.$$

Locally a product of base and fiber, transition functions are symplectomorphisms.

- Maslov index: used the fact that
  - Every path in $\gamma : I \to \mathrm{Sp}(2n, \mathbb{R})$ can be assigned an integer coming from a map $\tilde{\gamma} : I \to S^1$ and taking (approximately) its winding number.
- $\mathcal{M}(x, y)$, the moduli space of contractible finite-energy solutions to the Floer equation connecting x, y.
  - After perturbing H to get transversality, get a manifold
    - Dimension:

$$\dim \mathcal{M}(x, y) = \mu(x) - \mu(y).$$

  - How we did it:
    - Describe as zeros of a section of a vector bundle over $\mathcal{P}^{1,p}(x, y)$

      (Banach manifold modeled on the Sobolev spaces $W^{1,p}$),
    - Apply Sard-Smale to show $\mathcal{M}(x, y)$ is the inverse image of a regular value of some map.
  - Needed tangent maps to be Fredholm operators, proved in Ch. 8 and used to show transversality.
    - Showed $(d\mathcal{F})_u$ is a Fredholm operator of index $\mu(x) - \mu(y)$.

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2 | Reminder of Goals

Overall Goal:

Theorem 2.1 ***(Symplectic Morse Inequalities).***

$$\#\left\{\text{1-Periodic trajectories of } X_H\right\} \geq \sum_{k \in \mathbb{Z}} HM_k(W; \mathbb{Z}/2\mathbb{Z}).$$

Important Ideas for This Chapter:

Theorem 2.2 ***(Using Broken Trajectories to Compactify).***
$\mathcal{L}(x, y)$ is compact,

$$\partial \mathcal{L}(x, y) = \{\text{"Broken Trajectories"}\}$$

Theorem 2.3 ***(Using Gluing to Get a Chain Complex).***

$$\partial^2 = 0$$

3 | 9.1 and Review

- Defined moduli space of (parameterized) **solutions**:

  $\mathcal{M}(x, y) = \{\text{Contractible finite-energy solutions connecting } x, y\}$

  $$\begin{aligned} \mathcal{M} &= \{\textbf{All} \text{ contractible finite-energy solutions to the Floer equation}\} \\ &= \bigcup_{x,y} \mathcal{M}(x, y). \end{aligned}$$

- The moduli space of (unparameterized) **trajectories** connecting x, y:

  $$\mathcal{L}(x, y) \coloneqq \mathcal{M}(x, y)/\mathbb{R}.$$

  - Use the quotient topology, define sequentially:

    $$\tilde{u}_n \overset{n\to\infty}{\longrightarrow} \tilde{u} \qquad \iff \qquad \exists \{s_n\} \subset \mathbb{R} \text{ such that } u_n(s_n + s, \cdot\,) \overset{n\to\infty}{\longrightarrow} u(s, \cdot\,).$$

  - When $|\mu(x) - \mu(y)| = 1$, get a compact 0-manifold, so the number of trajectories

    $$n(x, y) \coloneqq \#\mathcal{L}(x, y)$$

    is well-defined.

- $C_k(H) \coloneqq \mathbb{Z}/2\mathbb{Z}[\{\text{Periodic orbits of } X_H \text{ of Maslov index } k\}]$.
  - Finitely many since they are nondegeneracy implies they are isolated.

Remark 1.
Some notation:

$$\begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathcal{M}(x, z) \\ & & \big\downarrow \pi \\ & & \mathcal{L}(x, z) \end{array}$$

Hats will generally denote maps induced on quotient.

- Defined a differential

$$\begin{aligned}\partial : C_k(H) &\to C_{k-1}(H)\\ x &\mapsto \sum_{\mu(y)=k-1} n(x,y)y\end{aligned}$$

$$\begin{aligned}n(x,y) &:= \#\left\{\text{Trajectories of grad } \mathcal{A}_H \text{ connecting } x,y\right\} \mod 2\\ &= \#\mathcal{L}(x,y) \mod 2.\end{aligned}$$

- Examined ∂^2:

$$\begin{aligned}\partial^2 : C_k(H) &\to C_{k-2}(H)\\ x &\mapsto \partial(\partial(x))\\ &= \partial\left(\sum_{\mu(y)=\mu(x)-1} n(x,y)y\right)\\ &= \sum_{\mu(y)=\mu(x)-1} n(x,y)\partial(y)\\ &= \sum_{\mu(y)=\mu(x)-1} n(x,y)\left(\sum_{\mu(z)=\mu(y)-1} n(y,z)z\right)\\ &= \sum_{\mu(y)=\mu(x)-1}\ \sum_{\mu(z)=\mu(y)-1} n(x,y)n(y,z)\, z\\ &= \sum_{\mu(z)=\mu(y)-1}\left(\sum_{\mu(y)=\mu(x)-1} n(x,y)n(y,z)\right) z \qquad \text{(finite sums, swap order)},\end{aligned}$$

so it suffices to show

$$\sum_{\mu(y)=\mu(x)-1} n(x,y)n(y,z) = 0 \quad \text{when} \quad \mu(z) = \mu(x) - 2.$$

Easier to examine parity, so we'll show it's zero mod 2.

- When $\mu(z) = \mu(x) - 2$, $\mathcal{L}(x, z)$ is a non-compact 1-manifold, so we compactify by adding in *broken trajectories* to get $\overline{\mathcal{L}}(x, y)$.
- We'll then have

  $$\overline{\mathcal{L}}(x, z) = \mathcal{L}(x, z) \cup \partial\overline{\mathcal{L}}(x, z), \qquad \partial\overline{L}(x, z) = \bigcup_{\mu(y)=\mu(x)-1} \mathcal{L}(x, y) \times \mathcal{L}(y, z),$$

  which "space-ifies" the equation we want.
- We'll show $\partial\overline{\mathcal{L}}(x, z)$ is a 1-manifold, which must have an even number of points, and thus

  $$\sum_{\mu(y)=\mu(x)-1} n(x, y)n(y, z) = \#\big(\partial\overline{\mathcal{L}}(x, z)\big) \equiv 0 \mod 2.$$

Image here of relations between spaces!

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4 | Three Important Theorems

4.1 First Theorem: Convergence to Broken Trajectories

- Recall: *broken trajectories* are unions of intermediate trajectories connecting intermediate critical points.
- Shown last time: a sequence of trajectories can converge to a broken trajectory, i.e. there are broken trajectories in the closure of $\mathcal{L}(x, z)$.
- This theorem describes their behavior:

Theorem 4.1 ***(9.1.7: Convergence to Broken Trajectories).***
Let $\{u_n\}$ be a sequence in $\mathcal{M}(x, z)$, then there exist

- A subsequence $\left\{u_{n_j}\right\}$
- Critical points $\{x_0, x_1, \cdots, x_{\ell+1}\}$ with $x_0 = x$ and $x_{\ell+1} = z$
- Sequences $\left\{s_n^1\right\}, \left\{s_n^2\right\}, \cdots, \left\{s_n^\ell\right\}$.
- Elements $u^k \in \mathcal{M}(x_k, x_{k+1})$ such that for every $0 \leq k \leq \ell$,

$$u_{n_j} \cdot s_n^k \overset{n\to\infty}{\longrightarrow} u^k.$$

- Upshots:
  - Every sequence upstairs has a subsequence which (after reparameterizing) converges
  - This descends to actual convergence after quotienting by $\mathbb{R}$?
  - Yields uniqueness of limits in $\mathcal{L}(x, z)$, thus a separated topology
  - Sequentially compact $\iff$ compact since $\mathcal{L}(x, z)$ is a metric space?

Corollary 4.2 ***(Compactness).***
$\overline{\mathcal{L}}(x, z)$ is compact.

4.2 Second Theorem: Compactness of $\overline{\mathcal{L}}(x,z)$

Definition 4.2.1 (Regular Pair)**.**
For an almost complex structure J and a Hamiltonian H, the pair (H,J) is **regular** if the Floer map $\mathcal{F}$ is transverse to the zero section in the following vector bundle:

$$E_u := \{\text{Vector fields tangent to } M \text{ along } u\} \longrightarrow C^\infty(\mathbb{R}\times S^1; TM)$$

$$\begin{array}{c} C^\infty(\mathbb{R}\times S^1; TM) \\ \mathcal{F} \uparrow \downarrow \uparrow \mathbf{0} \\ C^\infty(\mathbb{R}\times S^1; M) \end{array}$$

Most of chapter 9 is spent proving this theorem:

Theorem 4.3 ***(9.2.1).***
Let (H,J) be a regular pair with H nondegenerate and x,z be two periodic trajectories of H such that

$$\mu(x) = \mu(z) + 2.$$

Then $\overline{\mathcal{L}}(x,z)$ is a compact 1-manifold with boundary with

$$\partial\overline{\mathcal{L}}(x,z) = \bigcup_{y\in\mathcal{I}(x,z)} \mathcal{L}(x,y)\times\mathcal{L}(y,z)$$

$$\text{where} \qquad \mathcal{I}(x,z) = \left\{ y \,\middle|\, \mu(x) < \mu(y) < \mu(z) \right\}.$$

Note: possibly a typo in the book? Has x,y on the LHS.

Corollary 4.4.

$$\partial^2 = 0.$$

4.3 Third Theorem: Gluing

Theorem 4.5 ***(9.2.3: Gluing).***
Let x, y, z be three critical points of $\mathcal{A}_H$ with three consecutive indices

$$\mu(x) = \mu(y) + 1 = \mu(z) + 2.$$

and let

$$(u, v) \in \mathcal{M}(x, y) \times \mathcal{M}(y, z) \quad \rightsquigarrow \quad (\hat{u}, \hat{v}) \in \mathcal{L}(x, y) \times \mathcal{L}(y, z).$$

Then

1. There exists a $\rho_0 > 0$ and a differentiable map

$$\psi : [\rho_0, \infty) \to \mathcal{M}(x, z)$$

such that $\widehat{\psi}$, the induced map on the quotient

$$\begin{array}{ccc} [\rho_0, \infty) & \xrightarrow{\psi} & \mathcal{M}(x, z) \\ & \searrow^{\widehat{\psi}} & \downarrow \pi \\ & & \mathcal{L}(x, z) \end{array}$$

is an embedding that satisfies

$$\widehat{\psi}(\rho) \overset{\rho \to \infty}{\longrightarrow} (\hat{u}, \hat{v}) \in \overline{\mathcal{L}}(x, z).$$

2. ("Uniqueness") For any sequence $\{\ell_n\} \subseteq \mathcal{L}(x, z)$,

$$\ell_n \overset{n \to \infty}{\longrightarrow} (\hat{u}, \hat{v}) \quad \Longrightarrow \quad \ell_n \in \operatorname{im}(\widehat{\psi}) \text{ for } n \gg 0.$$

- We already know that $\overline{\mathcal{L}}(x, z)$ is compact and $\mathcal{L}(x, z)$ is a 1-manifold, so we look at neighborhoods of boundary points.
- Why unique: will show that the broken trajectory $(\hat{u}, \hat{v})$ is the endpoint of an embedded interval in $\overline{\mathcal{L}}(x, z)$.
  - Then show that any other sequence converging to $(\hat{u}, \hat{v})$ must approach via this interval, otherwise could have cuspidal points:

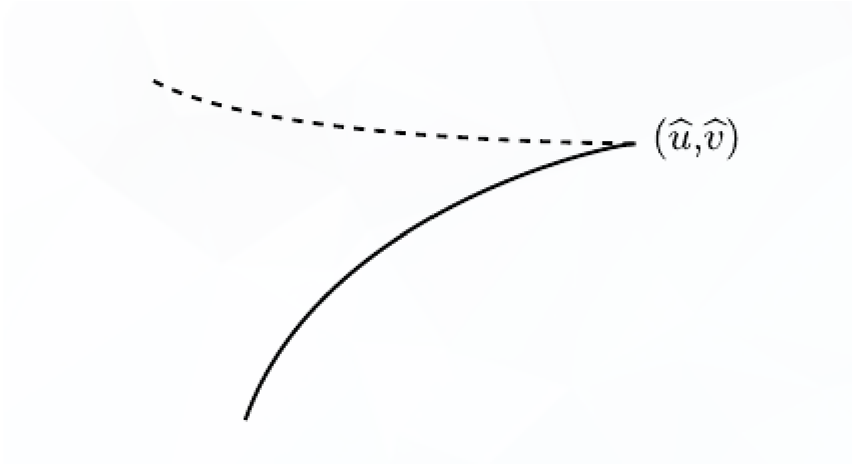

Figure 2: Cuspidal Point on Boundary

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5 | Gluing Theorem

Broken into three steps:

1. **Pre-gluing**:

- Get a function w_ρ which interpolates between u and v in the parameter ρ.
  - Not exactly a solution itself, just an "approximation".

2. **Newton's Method**:

- Apply the Newton-Picard method to w_p to construct a true solution

$$\psi : [-\rho, \infty) \to \mathcal{M}(x, z)$$
$$\rho \mapsto \exp_{w_p}(\gamma(p))$$

$$\text{for some } \gamma(p) \in W^{1,p}(w_p^* TW) = T_{w_p}\mathcal{P}(x, z)$$

- GIF of Newton's Method

3. **Project and Verify Properties**:

- Check that the projection $\widehat{\psi} = \pi \circ \psi$ satisfies the conditions from the theorem.

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6 | 9.3: Pre-gluing, Construction of w_ρ

- Choose (once and for all) a bump function β on $B_\varepsilon(0)^c \subset \mathbb{R} \to [0,1]$ which is 1 on $|x| \geq 1$ and 0 on $|x| < \varepsilon$
- Split into positive and negative parts $\beta^{\pm}(s)$:

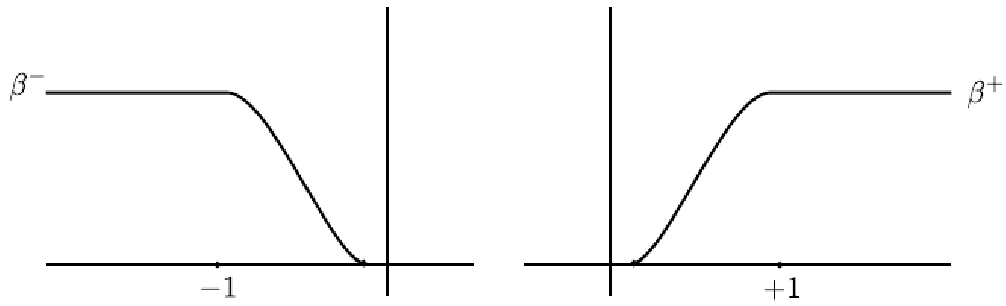

Figure 3: Bump away from zero

- Define an interpolation w_ρ from u to v in the following way: let
  - $\exp[\,\cdot\,] := \exp_{y(t)}(\,\cdot\,)$ and
  - $\ln(\,\cdot\,) := \exp_{y(t)}^{-1}(\,\cdot\,)$,

  then

$$w_\rho : x \to z$$

$$w_\rho(s,t) := \begin{cases} u(s+\rho,t) & s \in (-\infty,-1] \\ \exp\left[\beta^-(s)\ln(u(s+\rho,t)) + \beta^+(s)\ln(u(s-\rho,t))\right] & s \in [-1,1] \\ u(s-\rho,t) & s \in [1,\infty) \end{cases}.$$

- Why does this make sense?

$$|s| \leq 1 \implies u(s \pm \rho, t) \in \left\{ \exp_{y(t)} Y(t) \;\middle|\; \sup_{t \in S^1} \|Y(t)\| \leq r_0 \right\} \subseteq \operatorname{im} \exp_{y(t)}(\,\cdot\,),$$

so we can apply $\exp_{y(t)}^{-1}(\,\cdot\,)$.

- Can make $|s| \leq 1$ for large ρ, since

$$u(s,t) \overset{s\to\infty}{\longrightarrow} y(t)$$
$$v(s,t) \overset{s\to-\infty}{\longrightarrow} y(t).$$

  - So pick a ρ_0 such that this holds for $\rho > \rho_0$.
  - Might have to increase ρ_0 later in the proof, so $\rho > \rho_0$ just means $\rho \gg 0$.

- Some properties:

  - $w_\rho \in C^\infty(x,z)$ and is differentiable in ρ.
  - $s \in [-\varepsilon, \varepsilon] \implies w_\rho(s,t) = y(t)$.

$$w_\rho(s-\rho,t) \overset{\rho\to\infty}{\longrightarrow} u(s,t) \quad \text{in} \quad C^\infty_{\text{loc}}$$

$$w_\rho(s,t) \overset{\rho\to\infty}{\longrightarrow} y(t) \quad \text{in} \quad C^\infty_{\text{loc}}.$$

- Now carry out the linearized version on tangent vectors, to which we will apply Newton-Picard:

  - Let $Y \in T_u\mathcal{P}(x,y)$
  - Let $Z \in T_v\mathcal{P}(x,y)$
  - Replace w_ρ with the interpolation

$$Y\#_\rho Z \in T_{w_\rho}\mathcal{P}(x,y) = W^{1,p}(w_\rho^* TW).$$

  defined by

$$(Y\#_\rho Z)(s,t) = \begin{cases} Y(s+\rho,t) & s \in (-\infty,-1] \\ \exp_T\left[\beta^-(s)\ln_T(Y(s+\rho,t)) + \beta^+(s)\ln_T(Z(s-\rho,t))\right] & s \in [-1,1] \\ Z(s-\rho,t) & s \in [1,\infty) \end{cases},$$

  where the subscript T indicates taking tangents of the exponential maps at appropriate points.

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7 | 9.4: Construction of ψ.

7.1 Summary

- Newton-Picard method, general idea:
  - Allows finding zeros of f given an approximate zero x_0, using the extra information of the 1st derivative f'.
  - Original method and variant: find the limit of a sequence

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \qquad x_{n+1} = x_n - \frac{f(x_n)}{f'(x_0)}.$$

  - Second variant more useful: only need derivative at one point:

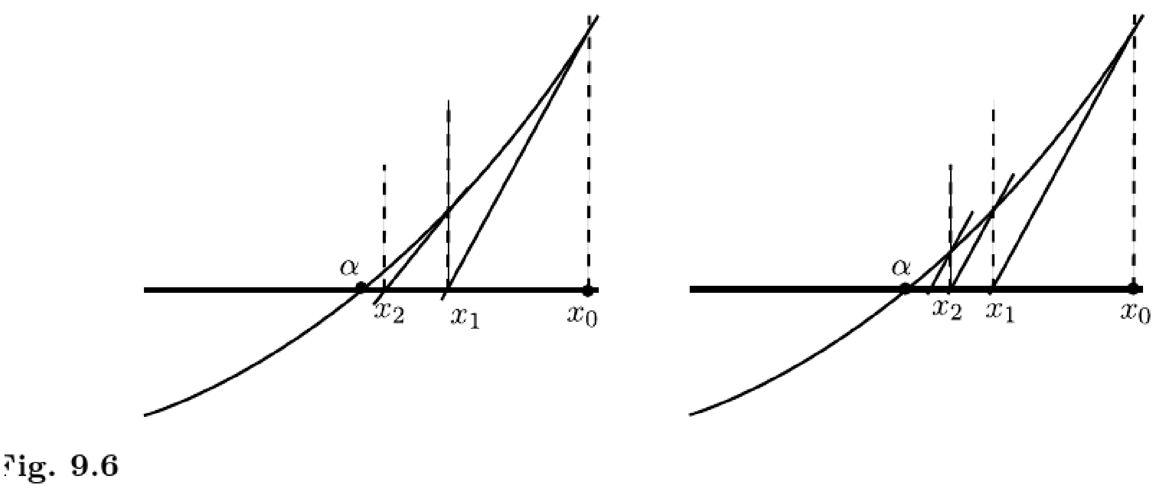

Figure 4: Newton Method Variants

- Pregluing function $w_\rho \in C^\infty_\searrow(x,z)$ from previous section
  - Exponential decay
- Want to construct true solution $\psi_\rho \in \mathcal{M}(x,z)$, so $\mathcal{F}(\psi_p) = 0$.
  - Suffices to get a weak solution
  - Automatic continuity + elliptic regularity $\implies$ strong solution
- Define $\mathcal{F}_\rho$ as $\mathcal{F} \circ \exp_{w_\rho}$ expanded bases Z_i from trivialization of TW.
- $L_\rho = (d\mathcal{F}_\rho)_0$ will be the linearization of the Floer operator at zero.

- Adapting Newton-Picard to operators:
  - L_ρ won't be invertible on entire space, but

$$\frac{1}{f'(x_0)} \iff L_\rho^{-1},$$

  - Decompose

$$T_{w_\rho}\mathcal{P}(x,z) = W^{1,p}(w_\rho^* TW) = W^{1,p}(\mathbb{R} \times S^1; \mathbb{R}^{2n}) = \ker(L_\rho) \oplus W_\rho^\perp,$$

    where L_ρ will have a right inverse on $W_\rho^\perp$.
    * Where does $W_\rho^\perp$ come from? Essentially the kernel of some linear functional given by an integral:

$$W_\rho^\perp := \left\{ Y \in W^{1,p} \;\middle|\; \int_{\mathbb{R}\times S^1} \langle Y, \ \cdots \rangle \, ds\, dt = 0, \ \text{plus conditions} \right\}.$$

  - Run Newton-Picard in $W_\rho^\perp$
- Will obtain for every $\rho \geq \rho_0$ an element $\gamma(\rho) \in W_\rho^\perp$ with

$$\mathcal{F}_\rho(\gamma(\rho)) = 0.$$

- Where does γ come from? Intersection-theoretic interpretation on page 320:

$$\left(\exp_{w_\rho}\right)^{-1}\mathcal{M}(x,z)\cap W_\rho^\perp \subseteq T_{w_\rho}\mathcal{P}(x,z) \qquad \rightsquigarrow \gamma$$
$$\mathcal{M}(x,z)\cap\left\{\exp_{w_\rho} W_\rho^\perp \;\middle|\; \rho\geq\rho_0\right\}\subseteq \mathcal{P}(x,z) \qquad \rightsquigarrow \psi(\rho),$$

which we get by exponentiating.

- This gives a codimension 1 subspace in $\mathcal{M}(x,z)$, which we take to be $\psi(\rho)$:

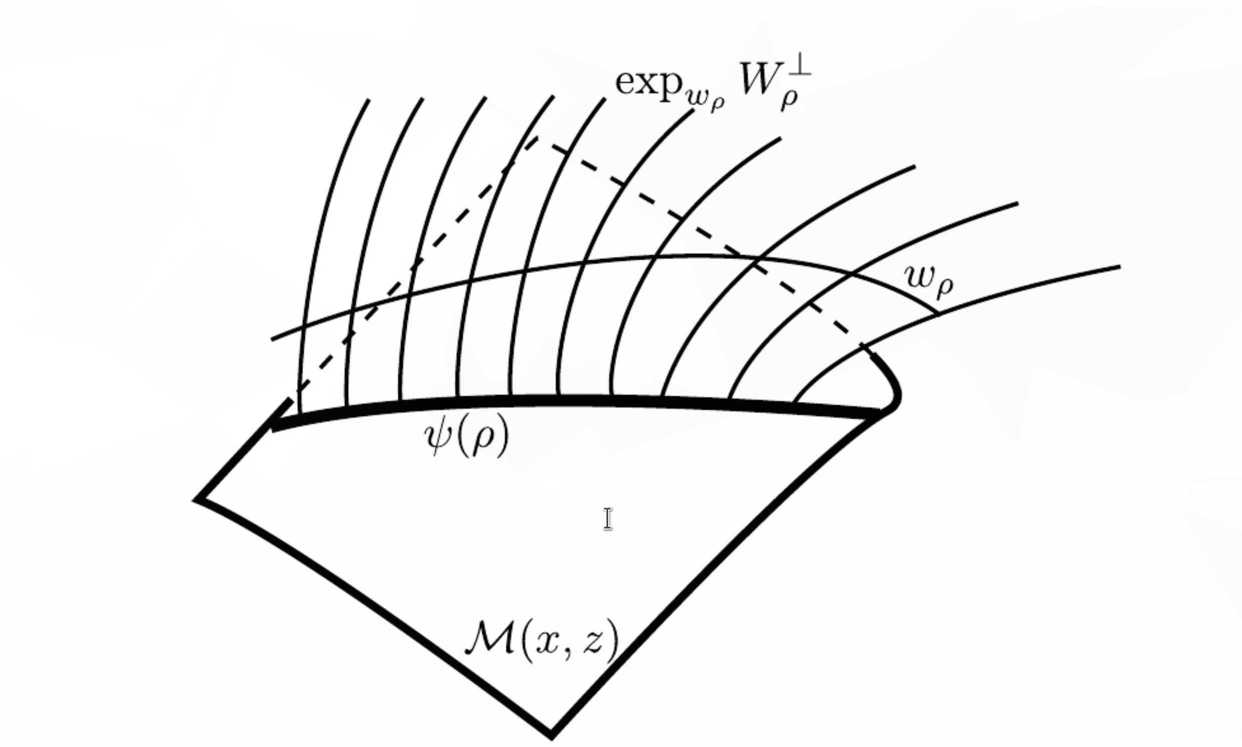

Figure 5: Intersection interpretation

7.1 Summary

Schematic picture here for $\gamma, \psi(\rho)$.

- Apply the implicit function theorem to show differentiability of γ in ρ.
- Use a trivialization Z_i^ρ of TW to get a vector field along w_ρ
  - This is exactly an element of $T_{w_\rho}\mathcal{P}(x,z)$
- Exponentiate to get an element of $\mathcal{M}(x,z)$:

$$\psi(\rho) := \exp_{w_\rho}(\gamma(\rho)).$$

- **Final Result**: project onto $\mathcal{L}(x,z)$ to get $\widehat{\psi}$.

Checking Properties:

- Existence: show $\widehat{\psi}$ is a proper injective immersion $\implies$ embedding.
- Uniqueness: show the broken trajectory $(\hat{u},\hat{v})$ is the endpoint of an embedded interval in $\overline{\mathcal{L}}(x,z)$.
  - Show that any other sequence converging to $(\hat{u},\hat{v})$ must approach via this interval, otherwise could have cuspidal points:

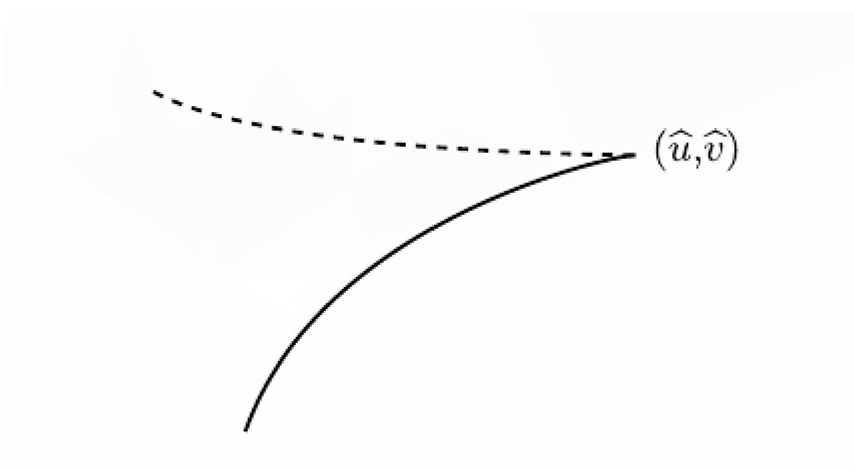

Figure 6: Cuspidal Point on Boundary

Probably not worth going farther than this! Extremely detailed analysis.

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