Chapter 10

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1 Monday, October 12: Audin Chapter 10 (From Floer to Morse)

1.1 Notation and Setup

- (W, ω, J) a symplectic manifold with an almost complex structure
- $H \in C^\infty(W, \mathbb{R})$ will be either a Morse function or a Hamiltonian
- X will be a vector field, potentially X_H, the symplectic gradient of H:

$$\omega_x(\,\cdot\,, X_H(x)) = (dH)_x(\,\cdot\,).$$

- DH will denote differentials, D^2H will denote Hessians (where they're defined)
- $CM_*(H, J)$ will be the Morse complex associated with a Morse function H, its vector field $\operatorname{grad} H$ the gradient for the metric defined by J, ω.
- $CF_*(H, J)$ will be the Floer complex

Theorem 1.1.1 ***(Main Goal).***
There exists a nondegenerate Hamiltonian that is sufficiently small in the C^2 topology for which both the Floer and Morse complexes are well-defined, and

$$CF_*(H, J) \cong CM_{*+n}(H, J) = CM_*(H, J)[n].$$

Orbits = critical pts!

1.2 Strategy

Need to show two things:

1. $CF_* = CM_*[n]$, and
2. $\partial_F = \partial_M$.

$\mathrm{Ind}(x) = \mu(x) + n$

$\dim(W) = 2n$?

Thm section 7.2.2 p.199

1.2.1 Equality of Complexes

Definition 1.2.1 (Nondegenerate 1: Critical Points of a Function)**.**
For a function $f \in C^\infty(W, \mathbb{R})$, define a bilinear form

$$\begin{aligned}(D^2 f)_p : T_pW \otimes T_pW &\to \mathbb{R} \\ (\mathbf{v}, \mathbf{w}) &\mapsto \langle \mathbf{x},\ (Y \cdot f)(p) \rangle\end{aligned}$$

for some vector field Y extending $\mathbf{y}$.
A critical point p is **nondegenerate** iff $(D^2 f)_p$ is a nondegenerate quadratic form.

Definition 1.2.2 (Nondegenerate 2: Critical Points of Periodic Trajectories)**.**
For a Hamiltonian system H, a periodic solution x is **nondegenerate** iff $1 \notin \operatorname{Spec}(d\psi_1)$, i.e. 1 is not an eigenvalue of the differential, i.e.

$$\det\left(\operatorname{id} - \left(D\psi^1\right)_{x(0)}\right) \neq 0.$$

Example 1.2.1.
Motivation:

$$H = \frac{1}{2}\sum a_{ij}p_ip_j + \sum b_{ij}p_iq_j + \frac{1}{2}\sum c_{ij}q_iq_j \implies X_H\mathbf{p} = A\mathbf{p}, A \approx D^2H(0).$$

Yields a flow $\psi_t = e^{tA}$, then if $\psi_1 = e^A$ doesn't have eigenvalue 1, A doesn't have eigenvalue zero, and the quadratic form H is nondegenerate, so the critical point of H at zero is nondegenerate.

Proposition 1.2.1 ***(5.4.5)*.**
Definition 2 implies definition 1: if x is a critical point of H which is nondegenerate as a periodic solution of the Hamiltonian system, then $a \in \operatorname{crit}(H)$ is nondegenerate as a critical point of the *function* H.

- Can start with an H_0 and rescale to define $H := H_0/k$
- When H sufficiently small in the C^2 sense (close sups of 1st and 2nd derivatives), the only periodic trajectories are constant
  - Use prop: def 2 implies def 1, conclude that H is Morse.
  - As a result,

$$\operatorname{crit}(\mathcal{A}_H) \iff \operatorname{crit}(H) \iff \{\text{Constant trajectories}\}.$$

  - Use remark 5.4.6: for the Hessian of H, $\operatorname{Spec}(D^2H) \cap 2\pi\mathbb{Z} = \emptyset$
  - Yields *Index Comparison Formula*:

$$\operatorname{Ind}_H(x) = \mu(x) + n.$$

Goal by end of Ch. 10:

- Show that all Floer solutions connecting two consecutive critical points are *also* Morse trajectories,

- Regularity: $d\mathcal{F}_u$ is surjective along these trajectories
  - Implies $\mathcal{M}^{(H,J)}(x,y)$ is a manifold, allows defining Floer complex
- Index Comparison Formula **yields equality of vector spaces**, up to a dimension shift.

1.2.2 Equality of Differentials

- Next need to show both differentials ∂_M, ∂_F can be defined, and they coincide
- Defining ∂_M:
  - Need a vector field X adapted to H
  - X needs to satisfy Smale condition (genericness)
- Recall the **Smale Condition**: all stable and unstable manifolds of critical points meet transversely,

$$W^{\mathrm{u}}(a) \pitchfork W^{\mathrm{v}}(b) \qquad \forall a, b \in \mathrm{crit}(H).$$

The Smale condition forbids the existence of flow lines such as those shown in Figure 2.15. Figure 2.16 shows the trajectories of a neighboring field satisfying the Smale condition (obtained by the general method explained in the following subsection).

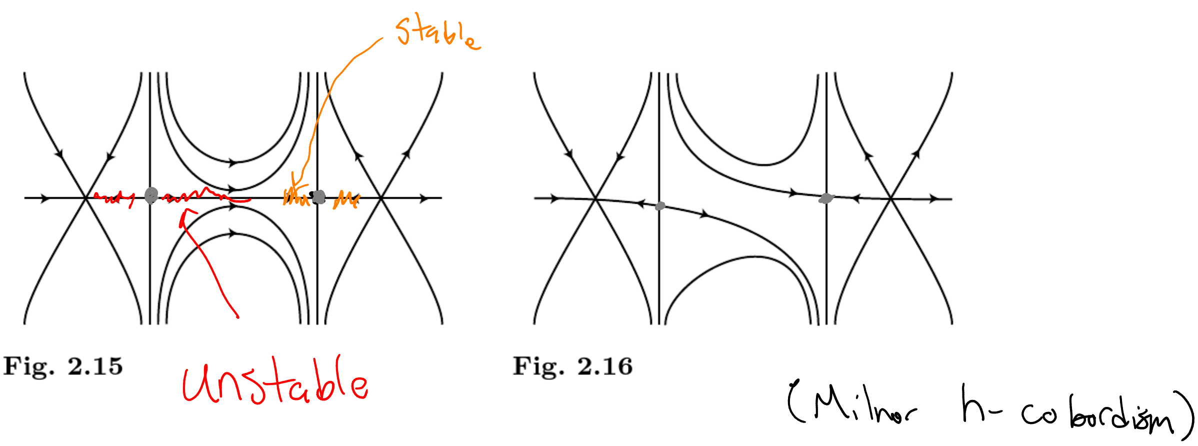

Fig. 2.15 **Fig. 2.16**

Figure 1: LHS: violates Smale condition. RHS: okay!

- Goal: given fixed data for the Floer theory, relate it to Morse data (define the Morse complex).
- Strategy: running ideas backwards, getting theorems for Morse functions similar to what we did when linearizing the Floer operator

1.2.3 Define Morse Differentials

- To define ∂_M: need to relate trajectories of X to solutions of Floer equation:

$$\left\{\begin{array}{c}\text{Solutions to}\\ \frac{\partial u}{\partial s} + X(u) = 0\end{array}\right\} \iff \left\{\begin{array}{c}\text{Solutions to}\\ \frac{\partial u}{\partial s} + J(u)\frac{\partial u}{\partial t} + \mathrm{grad} H(u) = 0\end{array}\right\}.$$

To do this: need $X = \mathrm{grad} H$ for the metric induced by J, ω.

Definition 1.2.3 (Pseudo-Gradient)**.**
For $f : W \to \mathbb{R}$ a Morse function, a vector field X is a **pseudo-gradient** for f iff

1. $(Df)_p(X_p) \leq 0$ with equality iff $p \in \text{crit}(f)$
2. In a Morse chart about $p \in \text{crit}(f)$, we have $X = -\text{grad}_g f$ for the canonical metric g on $\mathbb{R}^n$.

Definition 1.2.4 (Morse-Smale Pair)**.**
A pair (f, X) of a function and a vector field is a **Morse-Smale pair** iff f is Morse and X is a pseudo-gradient for f satisfying the Smale condition.

Theorem 1.2.1 ***(Theorem to Prove).***
Let H be Morse on (W, ω). Then there exists a dense subset $\mathcal{J}_{\text{reg}}(H)$ of almost complex structures J calibrated by ω such that $(H, -JX_H)$ is Morse-Smale.

Note: transversality result analogous to ones in 8.5

Proof .

$L_u \overset{\text{linearized?}}{\rightsquigarrow} \left(\frac{\partial}{\partial s} + X\right) u$

- Proof in two steps:
  - Step 1: Morse Side, arbitrary morse functions
    - ♢ Linearize the Morse equation $\frac{\partial u}{\partial s} + X(u) = 0$ of the flow of $-X$ along one of its solutions $L_u Y = 0$.
    - ♢ Show that whenever H is Morse and u is a trajectory connecting critical points, L_u is Fredholm and

$$\text{Ind}(L_u) = \text{Ind}_H(y) - \text{Ind}_H(x).$$

    - ♢ Show that for H a nondegenerate Hamiltonian and u a trajectory of JX_H, the operators $(d\mathcal{F})_u$ and L_u are Fredholm of equal index.
    - ♢ Show that X is Smale $\iff$ L_u is surjective.
  - Step 2: Floer Side, specific case of Hamiltonian
    - ♢ Prove the actual result.

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- Now fix an almost complex structure to obtain a Smale vector field X

1.2.4 Compare solutions to Floer equation and trajectories of X

- Goal: for $\mathrm{Ind}(x) - \mathrm{Ind}(y) \leq 2$, get an equality

$$\left\{\begin{matrix}\text{Trajectories of Floer equation}\\ \text{associated to } (H,J) \text{ connecting } x,y\end{matrix}\right\} \iff \left\{\begin{matrix}\text{Trajectories of the Smale}\\ \text{vector field } -JX_H\end{matrix}\right\}.$$

- Solutions to Floer equation that *do not* depend on t are precisely trajectories of $X = -\mathrm{grad}H$.
- Next show that elements in $\ker(d\mathcal{F}_u)$ do not depend on t.
- Corollary: $d\mathcal{F}_u$ is surjective along every trajectory of $\mathrm{grad}H$.
- Then show that replacing $H_k := H/k$ for $k \gg 0$ preserves all critical points and all indices
- Punch line: all the solutions of the Floer equation that we need are time-independent.
  - Statement: For $k \gg 0$, solutions to the Floer equation for H_k connecting $x \to y$ with $\mathrm{Ind}(x) - \mathrm{Ind}(y) \leq 2$ are independent of t.

1.3 Summary

- Take H_k for $k \gg 0$ and $J \in \mathcal{J}_{\mathrm{reg}}$ (dense)
- Then when $\mathrm{Ind}(x) - \mathrm{Ind}(y) \leq 2$, trajectories of Floer equation for (H, J) connecting critical points x, y are trajectories of the Smale vector field $X = -JX_H$.
  - x, y will be critical points for both H and $\mathcal{A}_H$
- Regularity? The linearized Floer operator is surjective along these trajectories
- Implies that $\mathcal{M}^{(H,J)}(x, y)$ is a manifold, so CF_* can be defined.
- Claim: this shows the differentials coincide, and we're done.

1.4 Linearizing the Morse Equation

- Let f be morse on $V \hookrightarrow \mathbb{R}^m$ ($m \gg 0$) with adapted pseudo-gradient field X, then

$$\left\{\begin{matrix}\text{Trajectories}\\ \text{of } X\end{matrix}\right\} \iff \left\{\begin{matrix}\text{Solutions of}\\ \frac{\partial u}{\partial s} + X(u(s)) = 0\end{matrix}\right\}.$$

- Fix a metric g on V such that $X = \operatorname{grad}_g f$.
- Define the space of solutions of finite energy:

$$E(u) := \int_{\mathbb{R}} \left\| \frac{\partial u}{\partial s} \right\|^2 ds$$
$$\mathcal{M} := \left\{ u \in C^\infty(\mathbb{R}, V) \;\middle|\; \frac{\partial u}{\partial s} + \operatorname{grad} f = 0, \quad E(u) < \infty \right\}.$$

- Then $\mathcal{M}$ is compact and equal to $\cup_{x,y} \mathcal{M}(x,y)$, using the fact that if V is compact, *all* trajectories are of finite energy
- Now go to coordinates and linearize the equation of the flow along the solution u to get a linear differential equation
- Yields an equation

$$L_u : W^{1,2}(\mathbb{R}, \mathbb{R}^n) \to L^2(\mathbb{R}, \mathbb{R}^n)$$
$$Y \mapsto \frac{\partial Y}{\partial s} + A(s)Y := L_u Y,$$

  where A is a matrix limiting to $\operatorname{grad}_y^2 f$ and $\operatorname{grad}_x^2 f$ at $s = \pm\infty$
  - Limiting to Hessians of nondegenerate critical points will yield symmetric invertible matrices
  - We then consider $\ker L_u \subseteq \ker(d\mathcal{F}_u)$. Note: we have exponential decay.
- Note: the space of solutions to equation linearized at u is $T_u\mathcal{M}(x,y)$.

1.4.1 Showing L_u is Fredholm

- Bootstrapping: $Y \in \ker(L_u)$ in $W^{1,2}$ is continuous, thus C^1, this C^∞ and form a finite-dimensional vector space.
- Behavior at infinity: reduces to

$$L_u Y = 0 \iff \frac{\partial Y}{\partial s} = -AY$$

  where A is a constant diagonal matrix
  - This is a linear system, so solutions are

$$Y(s) = e^{-As} Y(0) \quad \text{i.e.} \quad y_i(s) = y_i e^{-\lambda_i s}.$$

- Will prove that if u is a trajectory of grad$f$ connecting $x \to y$ then L_u is Fredholm
  - Proof: involves bounding $W^{1,2}$ norm of Y by L^2 norms of $Y, L_u Y$.
  - Lots of integral estimates: Fourier transform, Plancherel, Cauchy-Schwarz
- Integral bound yields: $\dim \ker L_u < \infty$ and $\mathrm{im}(L)_u$ is closed.
- Lemma: $\dim \mathrm{coker} < \infty$.
  - Proof: computer kernel of adjoint

$$L_u^* = -\frac{\partial}{\partial s} + A^*$$

    where the matrix is transposed.
  - Use the fact that

$$Z \in \mathrm{coker}(L_u) \iff Z \in \ker(L_u^*),$$

    i.e. $L_u^* Z = 0$ in the sense of distributions

1.4.2 Computing $\mathrm{Ind}\, L_u$

- Unsurprisingly, will show $\mathrm{Ind}(L_u) = \mathrm{Ind}_f(x) - \mathrm{Ind}_f(y)$.
- Ideas in proof:
  - Will choose two real numbers σ, s to plug into u, and consider *resolvent*: map between tangent spaces to V at $u(\sigma), u(s)$.
  - Look at the tangent spaces at $u(\sigma)$ of the stable and unstable manifolds will be the Floer complex

$$E^{\mathrm{u}}(\sigma) := T_{u(\sigma)} W^{\mathrm{u}}(x)$$
$$E^{\mathrm{s}}(\sigma) := T_{u(\sigma)} W^{\mathrm{s}}(x)$$

.

  - Then $\ker L_u$ is isomorphic to the intersection for all σ.

1.4.3 Smale Condition

- Recall $X = \mathrm{grad}_g f$ for g a metric.
- Statement: the vector field X satisfies the Smale condition $\iff$ all L_u are surjective.

Proof .

- L_u is surjective $\iff$ $\mathrm{coker}(L_u) = 0 \iff \ker(L_u^*)$ is injective
- This is equivalent to

$$T_{u(\sigma)} W^{\mathrm{u}}(x) + T_{u(\sigma)} W^{\mathrm{s}}(x) = T_{u(\sigma)} V.$$

- This is exactly the transversality condition for the stable and unstable manifolds

- We want this for all critical points

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1.5 10.4: Morse and Floer Trajectories Coincide

1.5.1 Comparing Kernels

- Note $\ker(L_u) \subset \ker(d\mathcal{F}_u)$ since

$$\left(\frac{\partial}{\partial s} + S(s)\right)Y = 0 \implies \left(\frac{\partial}{\partial s} + J\frac{\partial}{\partial t} + S(s)\right)Y = 0,$$

  so just need to show reverse inclusion.

- Use a lemma: for $f : [0,1] \to \mathbb{R}$,

$$\|f\|_{L^p([0,1])} \leq \left\|\frac{\partial f}{\partial t}\right\|_{L^p([0,1])},$$

  then apply this to $f(t) := Y(s,t)$ and $p = 2$.

- Yields an equation

$$\left\|\frac{\partial Y}{\partial s}\right\|_{L^2}^2 + \left\|\frac{\partial Y}{\partial t}\right\|_{L^2}^2 \leq \sup_s \|S(s)\|_{\text{op}}^2 \|Y\|_{L^2}^2 \implies \|Y\|_{L^2}^2 \leq \sup_s \|S(s)\|_{\text{op}} \left\|Y_{L^2}^2\right\|$$

  where the sup term being small forces $Y = 0$.

1.5.2 Trajectories are Independent of t

WTS: Trajectories of H_k appearing in the Floer complex are exactly those appearing in the Morse complex. I.e. proving 10.1.9

Idea of proof:

- Contradiction: suppose there exists a sequence $n_k \to \infty$ with time-dependent solutions u_{n_k} connecting $x \to y$ which solve the Floer equation
- Consider case where indices differ by 1: using broken trajectories theorem, extract a subsequence converging to some $v \in \mathcal{M}(x, y, H)$.
  - Show v doesn't depend on t
  - Since $d\mathcal{F}_v$ is surjective, v is in a 1-dim component, and thus an isolated point of $\mathcal{L}(x, y)$
  - Get a contradiction from taking $k \gg 0$ and using

$$v_{n_k}(s,t) = v(s + \sigma_k, t) = v(s + \sigma_k),$$

    which does *not* depend on time
- Consider case where indices differ by 2
  - Use Smale property of the gradient $-JX_H$ of H: trajectories $x \to y$ form a 2-manifold

  - Since trajectories are also in $\mathcal{M}(x, y, H)$, parameterizes a submanifold in a neighborhood of v.

- Show that convergence toward broken orbits in Morse setting corresponds to converges toward broken trajectories in Floer setting
- Use gluing from last chapter: $\widehat{v}_{n_k} \in \mathrm{im}(\widehat{\varphi})$ for $k \gg 0$, contradicting the fact that v_{n_k} doesn't depend on t