Complex Analysis Qualifying Exam 2019 Fall—Solutions
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1. Show that $\displaystyle\int_0^\infty \frac{x^{a-1}}{1+x^n}dx = \frac{\pi}{n\sin\frac{a\pi}{n}}$ using complex analysis, $0 < a < n$. Here n is a positive integer.

Solution. Let γ be the perimeter of the wedge-shaped region consisting of the oriented segments:

$$\begin{aligned}\gamma_1 &= \{t \mid \epsilon \le t \le R\}\\ \gamma_2 &= \{Re^{it} \mid 0 \le t \le b\}\\ \gamma_3 &= -\{te^{2\pi i/n} \mid \epsilon \le t \le R\}\\ \gamma_4 &= -\{\epsilon e^{it} \mid 0 \le t \le b\}\end{aligned}$$

Note that the function $f(z) = \frac{z^{a-1}}{1+z^n}$ is meromorphic and single valued in the region $0 < \arg z < \frac{2\pi}{n}$ with a single simple pole at $z = e^{\pi i/n}$ with residue

$$\lim_{z\to e^{\pi i/n}} \frac{z^{a-1}}{1+z^n}\cdot(z - e^{\pi i/n}) = \lim_{z\to e^{\pi i/n}} \frac{(a-1)z^{a-2}(z-e^{\pi i/n}) + z^{a-1}}{nz^{n-1}} = -\frac{e^{a\pi i/n}}{n}.$$

Since $a < n$ we have

$$\frac{z^{a-1}}{1+z^n} = o(R^{-1}) \text{ on } \gamma_2$$

so that $\int_{\gamma_2} f(z)dz \to 0$ as $R \to \infty$. Likewise, since $0 < a$ we have

$$\frac{z^{a-1}}{1+z^n} = o(\epsilon^{-1}) \text{ on } \gamma_4$$

so that $\int_{\gamma_4} f(z)dz \to 0$ as $\epsilon \to 0$. Finally,

$$\int_{\gamma_3} f(z)dz = -e^{2\pi ia/n}\int_{\gamma_1} f(z)dz.$$

Thus, using

$$\int_\gamma f(z)dz = 2\pi i\left(-\frac{e^{a\pi i/n}}{n}\right)$$

and taking the limit, we have

$$\left(1 - e^{2\pi ia/n}\right)\int_0^\infty \frac{x^{a-1}}{1+x^n}dx = -\frac{2\pi i e^{a\pi i/n}}{n}$$

whence

$$\int_0^\infty \frac{x^{a-1}}{1+x^n}dx = \frac{\pi}{n}\cdot\frac{2i}{e^{\pi ia/n} - e^{-\pi ia/n}} = \frac{\pi}{n\sin\frac{a\pi}{n}}.$$

□

2. Prove that the distinct complex numbers z_1, z_2 and z_3 are the vertices of an equilateral triangle if and only if

$$z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_2z_3 + z_3z_1.$$

Solution. Note the above condition can be written

$$(z_3 - z_1)^2 - (z_3 - z_1)(z_2 - z_1) + (z_2 - z_1)^2 = 0$$

which is evidently preserved by scaling and translating. Thus we may assume $z_1 = 0$ and $z_2 = 1$, in which case the condition is $z_3^2 - z_3 + 1 = 0$ and equivalent to $-z_3$ being a third root of unity which is not 1, which is equivalent to the triangle being equilateral. □

3. Let γ be piecewise smooth simple closed curve with interior Ω_1 and exterior Ω_2. Assume $f'(z)$ exists in an open set containing γ and Ω_2 and $\lim_{z\to\infty} f(z) = A$. Show that

$$\frac{1}{2\pi i}\int_\gamma \frac{f(\xi)}{\xi - z}\,d\xi = \begin{cases} A, & \text{if } z \in \Omega_1, \\ -f(z) + A, & \text{if } z \in \Omega_2 \end{cases}$$

Solution. Consider $g(\xi) = f(1/\xi)$, which extends continuously and thus holomorphically over $\xi = 0$. Letting γ' be the pullback of $-\gamma$ under $h : \xi \mapsto \xi^{-1}$, we have

$$\frac{1}{2\pi i}\int_\gamma \frac{f(\xi)}{\xi - z}d\xi = -\frac{1}{2\pi i}\int_{\gamma'} \frac{g(\xi)}{\xi^{-1} - z}\frac{-d\xi}{\xi^2} = \begin{cases} A & \text{if } z \in \Omega_1 \\ -f(z) + A & \text{if } z \in \Omega_2 \end{cases}$$

using the residue formula, as in the first case the integrand's only pole in the interior of γ' is at 0 with residue $-A$ and in the second case at 0 and z^{-1} with residues $-A$ and $f(z)$, respectively. Note that γ' has clockwise orientation. □

4. Let $f : \mathbb{C} \to \mathbb{C}$ be an injective analytic (also called univalent) function. Show that there exist complex numbers $a \neq 0$ and b such that $f(z) = az + b$.

Solution. By the Casorati–Weierstrass theorem, if f has an essential singularity at infinity, the image of any neighborhood of infinity is dense in $\mathbb{C}$. As the image is also open, it follows that if f is injective, it cannot have an essential singularity at infinity, as the intersection of two open dense subsets is nonempty. Thus f and is therefore polynomial, so by the fundamental theorem of algebra, the injectivity of f implies it is degree one. □

5. Find a conformal map from $D = \{z : \ |z| < 1,\ |z - 1/2| > 1/2\}$ to the unit disk $\Delta = \{z : \ |z| < 1\}$.

Solution. $f(z) = \frac{z-i}{z+i}$ maps the upper half-plane conformally to the unit disk Δ, and moreover maps the line Im $z = 1$ to the circle $|z - 1/2| = 1/2$. Thus, the inverse $f^{-1}(z) = i \cdot \frac{1+z}{1-z}$ maps D to the strip $0 < \text{Im } z < 1$. The strip in turn is mapped to the upper half-plane via $g(z) = e^{\pi z}$. Thus, the required map is $f \circ g \circ f^{-1}$.

□

6. A holomorphic mapping $f : U \to V$ is a local bijection on U if for every $z \in U$ there exists an open disc $D \subset U$ centered at z so that $f : D \to f(D)$ is a bijection. Prove that a holomorphic map $f : U \to V$ is a local bijection if and only if $f'(z) \neq 0$ for all $z \in U$.

   Solution. We show for $z \in U$, f is a local bijection in a neighborhood of z if and only if $f'(z) = 0$. We may and do assume $z = 0$ and $f(z) = 0$.

   We may assume $f(z) = z^n + z^{n+1}g(z)$ in a neighborhood of 0 where $n \geq 1$ is the order of vanishing and g is holomorphic. Then $f'(0) = 0$ iff $n > 1$. On a small enough radius r disk Δ_r around 0 we have $|z^{n+1}g(z)| < \frac{1}{2}|z^n|$, so for $|w| < r^n/2$ we have

$$|z^{n+1}g(z) - w| < r^n = |z^n|$$

   on $\partial\Delta_r$. By Rouché's theorem z^n and $f(z) - w$ have the same number of zeroes in Δ_r, that is, n. Thus, if $n = 1$ then f is bijective on any disk contained in $f^{-1}(\Delta_{r/2}) \cap \Delta_r$. Conversely, if $n > 1$, then in any neighborhood of 0 we can find z_0 with $|f(z_0)| < r^n/2$ such that $f'(z_0) \neq 0$, in which case $w = f(z_0)$ must have more than one preimage and f is not locally bijective.

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