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A Complete moduli space for $K3$ surfaces of degree 2

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1. Introduction

A nonsingular, projective surface, V, over $\mathbf{C}$ is called a $K3$ surface if $H^1(V, \underline{o}_V) = 0$ and the canonical divisor class of V is trivial. It is called a $K3$ surface of degree 2 if V carries a line bundle L with $L \cdot L = 2$. V is said to be generic if its Picard number $\rho(V)$ $(=$ the rank of $\mathrm{Pic}(V))$ is equal to one.

If L is a line bundle on a generic $K3$ surface V such that $L \cdot L = 2$, the linear system $|L|$ has no fixed components and maps V onto $\mathbf{P}_2$ as a double cover of $\mathbf{P}_2$, ramified over a nonsingular sextic curve [7]. Conversely, there exists a unique sextic double plane V corresponding to a given sextic curve; the minimal desingularization of V is a $K3$ surface if and only if the singularities of V consist of isolated, rational double points. The points corresponding to nonsingular sextic double planes form an open, dense subset U_0 in the moduli space of $K3$ surfaces of degree 2. The object of this paper is to describe a completion $\mathfrak{M}$ of U_0 such that $\mathfrak{M}$ contains the moduli space of $K3$ surfaces of degree 2 as an open subset U. Let us call a two-dimensional, projective scheme V a (singular) $K3$ surface if it can be deformed into a nonsingular $K3$ surface. We will associate with each point on the boundary of U in $\mathfrak{M}$, a unique, singular, $K3$ surface.

We construct $\widetilde{\mathfrak{M}}$ via the geometric invariant theory [8]. Let $\widetilde{\mathfrak{M}}$ be the quotient of the space H_6^{ss} of semistable sextic curves by PGL_3. Let $\mathfrak{f}\colon H_6^{ss} \to \widetilde{\mathfrak{M}}$ be the quotient morphism. We recall (Proposition 2.1, § 2) that if $f\colon X \to \mathrm{Spec}\,\mathbf{C}[[t]]$ is a flat, projective morphism such that the generic geometric fiber of f is isomorphic to a nonsingular sextic double plane, then there exists a flat, projective morphism $f'\colon X' \to \mathrm{Spec}\,\mathbf{C}[[t]]$ and a map $\rho\colon \mathrm{Spec}\,\mathbf{C}[[t]] \to \mathrm{Spec}\,\mathbf{C}[[t]]$ such that (i) the generic fiber of f' and the generic fiber of the pull-back of f via ρ are isomorphic and (ii) the geometric fibers of f' are

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double planes ramified over semistable sextic curves belonging to minimal orbits. Let $\widetilde{\mathfrak{M}}_0$ be the open subset of $\widetilde{\mathfrak{M}}$ representing the sextic curves which do not have either a multiple component or a point of multiplicity $\geqq 4$ or consecutive triple points. $\widetilde{\mathfrak{M}}_0$ represents sextic double planes whose singularities consist of isolated rational double points and hence $\widetilde{\mathfrak{M}}_0$ represents nonsingular $K3$ surfaces which carry a line bundle L of degree 2 such that the linear system $|L|$ has no fixed components. There is a unique point δ in $\widetilde{\mathfrak{M}} - \widetilde{\mathfrak{M}}_0$ such that the unique closed orbit, O_δ, in $\mathfrak{f}^{-1}(\delta)$ is the orbit of a sextic curve consisting of a nonsingular quadric with multiplicity three. The sextic double planes corresponding to the points in $\widetilde{\mathfrak{M}} - \delta$ have only insignificant limit singularities [12]. If a one-parameter family of nonsingular projective surfaces specializes to a sextic double plane with only insignificant limit singularities, the mixed Hodge structure of the sextic double plane determines the Hodge components of the limit mixed Hodge structure of the family. We classify such sextic double planes according to their mixed Hodge structure (§ 3). Moreover, we show that a point x in $\widetilde{\mathfrak{M}} - \widetilde{\mathfrak{M}}_0 - \delta$ cannot correspond to a nonsingular $K3$ surface (Proposition 3.1). We associate with x, the sextic double plane which is ramified over a sextic curve belonging to the unique closed orbit in $\mathfrak{f}^{-1}(x)$.

The point δ represents all the semistable sextic double planes which have significant limit singularities. The occurrence of these surfaces reflects the fact that $\widetilde{\mathfrak{M}}$ cannot represent all $K3$ surfaces of degree 2 because some $K3$ surfaces carry a line bundle L of degree 2 which is ample, but not very ample. If V is such a $K3$ surface, then $|L|$ has a fixed component, D, which is a nonsingular rational curve. $|L| - D$ is linearly equivalent to $2C$ where C is a nonsingular, elliptic curve [7]. $|L|$ maps V onto $\mathbf{P}_1$. Consider a family of $K3$ surfaces, $f\colon X \to \operatorname{Spec} \mathbf{C}[[t]]$ such that f is *smooth* and its generic geometric fiber is a generic $K3$ surface of degree 2. Let $\mathfrak{L}$ be a line bundle on X such that $\mathfrak{L}$ induces an ample line bundle of degree 2 on the geometric fibers of f. Let X_0 be the fiber of f over the closed point of $\operatorname{Spec} \mathbf{C}[[t]]$. Let L_0 be the restriction of $\mathfrak{L}$ to X_0. Suppose that L_0 is not very ample and let D be the fixed component of $|L_0|$. Let $\varphi_{\mathfrak{L}}\colon X \dashrightarrow \mathbf{P}_2 \times \operatorname{Spec} \mathbf{C}[[t]]$ be the rational map defined by $\mathfrak{L}$. X must be blown up along D in order to extend $\varphi_{\mathfrak{L}}$ to a morphism $\varphi'_{\mathfrak{L}}\colon X' \to \mathbf{P}_2 \times \operatorname{Spec} \mathbf{C}[[t]]$. Let

$$\begin{tikzcd}
X' \arrow[rr, "\pi"] \arrow[dr, "\varphi'_{\mathfrak{L}}"'] & & Y \arrow[dl] \\
& \mathbf{P}_2 \times \operatorname{Spec} \mathbf{C}[[t]] &
\end{tikzcd}$$

be the Stein factorization. Y is a double cover of $\mathbf{P}_2 \times \operatorname{Spec} \mathbf{C}[[t]]$ whose branch locus is a family of sextic curves specializing to a nonsingular conic with multiplicity three. Thus, all $K3$ surfaces carrying a line bundle which is ample, but not very ample, are represented by a single point δ in $\widetilde{\mathfrak{M}}$. Therefore, the space $\widetilde{\mathfrak{M}}$ must be blown up at δ. This is done, in effect, by blowing up H_6^{ss} along O_δ. Instead of actually working with O_δ, we proceed as follows (§ 5). Fix a point Q on O_δ. Let G_Q denote the stabilizer group of Q in PGL_3. We define a G_Q-invariant subspace, M, of H_6, which is 'normal' to O_δ at Q. Let $\mathfrak{A}'$ be the categorical quotient of M^{ss} by G_Q. Let ∂ be the image of Q in $\mathfrak{A}'$. We show that there is a canonical map $\mathfrak{A}' \to \widetilde{\mathfrak{M}}$ which is étale at ∂. We now blow up M^{ss} at Q. (The blow-up is defined by a weight filtration.) The action of G_Q extends to the blown-up M^{ss} and thus induces a blow-up of $\mathfrak{A}'$ at ∂ and, hence, of $\widetilde{\mathfrak{M}}$ at δ. In $\mathbf{P}_5$, let Σ_4^0 denote a cone over a rational, normal, quartic curve contained in a hyperplane. The points on the exceptional divisor in the blown-up $\widetilde{\mathfrak{M}}$ correspond to double covers of Σ_4^0, ramified over the vertex and a section of Σ_4^0 by a cubic hypersurface in $\mathbf{P}_5$ which does not pass through the vertex. The moduli space $\mathfrak{N}$ of such surfaces is constructed in Section 4. The semistable surfaces have, in this case, only insignificant limit singularities. We prove (Theorem 6.1) that if $f\colon X \to \operatorname{Spec} \mathbf{C}[[t]]$ is a family of sextic double planes such that the generic geometric fiber of f is nonsingular and the singular fiber of f is triply ramified over a nonsingular conic, then there exists a flat, *projective* morphism, $f'\colon X' \to \operatorname{Spec} \mathbf{C}[[t]]$ and a map $\rho\colon \operatorname{Spec} \mathbf{C}[[t]] \to \operatorname{Spec} \mathbf{C}[[t]]$ such that (i) the generic fiber of f' and the generic fiber of the pull-back of f via ρ are isomorphic and (ii) the fiber of f' over the closed point of $\operatorname{Spec} \mathbf{C}[[t]]$ is a double cover of Σ_4^0, has insignificant limit singularities and belongs to a minimal orbit. Among the double covers of Σ_4^0, only the surfaces whose singularities consist of rational double points correspond to elliptic $K3$ surfaces carrying an ample line bundle of degree 2 which is not very ample. The remaining surfaces cannot correspond to nonsingular $K3$ surfaces.

The surjectivity of the period map for $K3$ surfaces of degree 2 follows as a corollary to our construction.

A preliminary version of these results was announced in [13]. An analysis of $K3$ surfaces of degree 4 by the technique described in this paper will appear elsewhere [14].

E. Horikawa has analyzed sextic double planes in order to prove the surjectivity of the period map in this case [6]. My work differs from Horikawa's in two respects. To show that the points in $\widetilde{\mathfrak{M}} - \widetilde{\mathfrak{M}}_0 - \delta$ cannot correspond to nonsingular $K3$ surfaces, Horikawa relies on Borel's extension theorem for the period map. Secondly, Horikawa does not construct the actual blow-up $\mathfrak{M}$ of $\widetilde{\mathfrak{M}}$, but proves a slightly weaker version of our Theorem 6.1 by direct computations. We prove the theorem by applying the geometric invariant theory.

I want to take this opportunity to express my thanks to my thesis adviser, Professor Michael Artin.

2. Stability of sextic curves

We recall some facts from the geometric invariant theory [8]. Let X be a projective scheme over an algebraically closed field k of characteristic zero. Let G be a reductive algebraic group acting on X. Fix a G-linearized, ample invertible sheaf on X. Let X^{ss} (respectively, X^{s}) be the set of semistable (respectively, stable) points of X. Then, a universal categorical quotient $\mathfrak{h}: X^{ss} \to Y$ of X^{ss} by G exists. Y is a projective scheme and $\mathfrak{h}$ is an affine, universally submersive (consequently, surjective) morphism. Moreover, there is an open set $Y^0 \subset Y$ such that $X^s = \mathfrak{h}^{-1}(Y^0)$ and such that the restriction $\mathfrak{h}^0: X^s \to Y^0$ is a universal geometric quotient of X^s by G.

Let R be a discrete valuation ring over k with an algebraically closed residue field and let t be a local parameter in R. Let $S = \operatorname{Spec} R$, $o =$ the closed point of S and $\zeta =$ the generic point of S. For $n \geqq 1$, let $\rho_n: S \to S$ be the map which takes t to t^n. Let f be a ζ-valued point of Y^0. Then, there exists an integer $n > 0$ and a section $g: \zeta \dashrightarrow X^s$ such that the following diagram

$$\begin{tikzcd} & & X^s \arrow[d, "\mathfrak{h}^0"] \\ \zeta \arrow[r, "\rho_{n,\zeta}"'] \arrow[urr, "g"] & \zeta \arrow[r, "f"'] & Y^0 \end{tikzcd}$$

where $\rho_{n,\zeta}$ is the restriction of ρ_n to ζ, commutes. The section g is unique in the following sense. Given another positive integer n' and another section g' such that the analogous diagram commutes, there exists an integer m such that n and n' divide m and such that the pull-back of g via $\rho_{m/n}$ and the pull-back of g' via $\rho_{m/n'}$ belong to the same orbit. This follows from the fact that the geometric fibers of $\mathfrak{h}^0$ contain exactly one orbit.

We get only fiberwise uniqueness when we consider sections of $\mathfrak{h}$. Recall that every geometric fiber of $\mathfrak{h}$ contains a unique closed orbit, called the minimal orbit, and it lies in the closure of every orbit in the fiber.

PROPOSITION 2.1. *Let $f: S \to Y$ be a map such that $f(\zeta) \in Y^0$. There exists a positive integer n and a map $g: S \to X^{ss}$ such that the following* *diagram*

$$\begin{tikzcd} & & X^{ss} \arrow[d, "\mathfrak{h}"] \\ S \arrow[r, "\rho_n"'] \arrow[urr, "g"] & S \arrow[r, "f"'] & Y \end{tikzcd}$$

commutes. Moreover, we may assume that $g(o)$ is a point in a minimal orbit.

Proof. Let $X_S = X^{ss} \times_Y S$ and let $\mathfrak{h}_S\colon X_S \to S$ be the canonical morphism. $\mathfrak{h}_S$ is submersive; that is, a subset U in S is open if and only if $\mathfrak{h}_S^{-1}(U)$ is open in X_S. In particular, $\mathfrak{h}_S^{-1}(o)$ is not open and $\mathfrak{h}_S^{-1}(\zeta)$ is not closed. Therefore, there exists an irreducible component W in the closure of $\mathfrak{h}_S^{-1}(\zeta)$ such that the restriction map $\psi\colon W \to S$ is surjective and hence equidimensional. It follows from the proof of the lemma on page 14 in [8] that there exists an integer n and a map $g\colon S \to W \subset X^{ss}$ such that the above diagram commutes. Moreover, if x is a closed point in $\psi^{-1}(o)$, we may assume that $g(o) = x$. Since $\psi^{-1}(o)$ must contain the minimal orbit in $\mathfrak{h}_S^{-1}(f(o))$, the proposition follows.

Q.E.D.

Remark 2.2. Let $R = k[[t]]$. If two sections, g and g', satisfy the conclusion of Proposition 2.1, they do not necessarily belong to a common orbit, even after they are pulled back over some extension ρ_m. The trouble lies with positive-dimensional stabilizer groups of semistable points. Let G_0 denote the stabilizer of $g(o)$ which is a point in a minimal orbit. Assume that G_0 is of positive dimension. Let λ: Spec $k[t, t^{-1}] \to G_0$ be a nontrivial one-parameter subgroup of G_0. For a suitable integer m, λ transforms $g \circ \rho_m$ into a map $g'\colon S \to X^{ss}$ such that $g'(o) = g(o)$. However, λ cannot extend to an S-valued point of G_0.

Let $H_6 = |H^0(\mathbf{P}_2, \underline{o}_{\mathbf{P}_2}(6))|$ = the space of plane sextic curves. Consider the canonical action of PGL_3 on H_6. In his book [8], Mumford has given explicit stability computations for plane quartic curves. We use his method and notation for determining semistable sextic curves. We assume that by a suitable choice of coordinates, a given one-parameter subgroup λ of PGL_3 is diagonalized and has the form

$$\begin{bmatrix} t^{r_0} & 0 & 0 \\ 0 & t^{r_1} & 0 \\ 0 & 0 & t^{r_2} \end{bmatrix}$$

such that $\Sigma r_i = 0$ and $r_0 \geqq r_1 \geqq r_2$. We then denote λ by the triple (r_0, r_1, r_2). We use also the following notation:

i, j: positive integers,
a, a_1, a_2, a_3: complex numbers,
$f_i(\cdots)$: a form of degree i in variables indicated in parentheses,
x_0, x_1, x_2: homogeneous coordinates in $\mathbf{P}_2$,
$x = x_1/x_0$: $y = x_2/x_0$.

We define a weight filtration on $\mathbf{C}[x, y]$ by assigning to x weight 1 and to y weight 2. We describe the sextics by means of their homogeneous equation $F(x_0, x_1, x_2) = 0$ or the inhomogeneous equation $f(x, y) = 0$.

We first describe the unstable sextics. These are determined by considering all one-parameter subgroups λ and the corresponding sets $H_\lambda^- = \{p \in H_6 : \mu(p, \lambda) < 0\}$. The two maximal sets are as follows:

(a) a_{03} = the coefficient of y^3 in $f(x, y) \neq 0$. Let $(r_0, r_1, r_2) = (4i + j, i, -5i - j)$. $H_\lambda^- : f(x, y) = y^3 +$ terms of weight >6, that is, the sextics which have a line as a component and have a triple point on that line such that the triple point remains a triple point with a threefold tangent under a quadratic transformation.

(b) $a_{03} = 0$. Let $(r_0, r_1, r_2) = (5i - j, -i, j - 4i)$, $i > j/3 > 0$.

$$H_\lambda^- : f(x, y) = y^3 f_1(x, y) + f_5(x, y) + f_6(x, y) ,$$

that is, the sextics with either a quadruple point which has a threefold or a fourfold tangent or a singular point of multiplicity $\geqq 5$.

Next, we determine the semistable sextics and their minimal orbits. A closed point p in H_6^{ss} is stable if and only if $\mathfrak{f}^{-1}(\mathfrak{f}(p))$ consists of the minimal orbit. If O and O' are two orbits in H_6^{ss} such that $O' \subset$ the closure $\bar{O}$ of O in H_6^{ss}, then, there exist a one-parameter subgroup, λ: Spec $\mathbf{C}[t, t^{-1}] \rightarrow$ PGL_3, a point p in O and a point p' in O' such that $\lim_{t\to 0} p^{\lambda(t)} = p'$ and $\mu_\lambda(p) = \mu_\lambda(p') = 0$ where μ_λ is the numerical function defined on H_6 by λ [8]. For each one-parameter subgroup λ, let $H_\lambda = \{p \in H_6 : \mu_\lambda(p) = 0\}$ and $\bar{H}_\lambda$ = the points in H_λ which are fixed under the action of λ. Let $H_\lambda^{ss} = H_\lambda \cap H_\lambda^{ss}$ and $\bar{H}_\lambda^{ss} = \bar{H}_\lambda \cap H_\lambda^{ss}$. If $p \in H_\lambda$, then $\lim_{t\to 0} p^{\lambda(t)} \in \bar{H}_\lambda$. A semistable point p does not belong to a minimal orbit in H_6^{ss} if and only if there exists a one-parameter subgroup λ such that $p \in H_\lambda^{ss} - \bar{H}_\lambda^{ss}$ and such that p and $\lim_{t\to 0} p^{\lambda(t)}$ do not belong to the same orbit. If we partially order the sets H_λ^{ss} by the relation:

$H_{\lambda_1}^{ss} > H_{\lambda_2}^{ss}$ if and only if $H_{\lambda_1}^{ss} \supset H_{\lambda_2}^{ss}$ and for every point $p \in \bar{H}_{\lambda_2}^{ss}$, the closure of the orbit of p in H_6^{ss} contains a point of $\bar{H}_{\lambda_1}^{ss}$;

then, in order to determine the minimal orbits in H_6^{ss}, it is enough to determine all the maximal sets H_λ^{ss}. These are as follows where we have parametrized H_λ^{ss} and $\bar{H}_\lambda^{ss}$ by polynomials f and $\bar{f}$ respectively.

1. $r_0 = r_1$.

$f = y^2 f'$ where f' is a polynomial in x, y of degree $\leqq 4$. H_λ^{ss} consists of sextics which have a line as a component with multiplicity 2.

$\bar{f} = y^2 \bar{f}'$ where $\bar{f}'$ is a polynomial in x of degree $\leqq 4$.

2. $r_1 = 0$.

$f = a_1 y_2 + a_2 y^2 x^2 + a_3 y x^4 + a_4 x^6 +$ terms of weight >6. If $a_1 = 0$, the sextic belongs to Case 5 below. If $a_1 \neq 0$, the sextic has consecutive triple points at the origin, $x = y = 0$.

$\bar{f} = a_1 y^3 + a_2 y^2 x^2 + a_3 y x^4 + a_4 x^6$. If $a_1 = 0$, the sextic has a quadruple point at the origin and belongs to Case 5 below. If $a_3 = a_4 = 0$, the sextic has a double line and belongs to Case 4 below.

3. $r_1 = r_2$.

$f = f_4(x, y) + f_5(x, y) + f_6(x, y)$. The sextics have a quadruple point at the origin.

$\bar{f} = f_4(x, y)$.

4. $r_0 \neq r_1 \neq r_2$, $r_1 > 0$; let $(r_0, r_1, r_2) = (4i + j, i, -5i - j)$.

$f = ay^3 + x^2 y^2 +$ terms of weight >6.

$\bar{f} = x^2 y^2$. The sextic consists of three distinct lines, each with multiplicity 2.

5. $r_0 \neq r_1 \neq r_2$, $r_1 < 0$; let $(r_0, r_1, r_2) = (5i - j, -i, j - 4i)$, $i > j/3 > 0$.

$f = y^2 f_2(x, y) + f_5(x, y) + f_6(x, y)$ such that $f_2(x, 0) \neq 0$. The sextics have a quadruple point which has a tangent of multiplicity equal to 2.

$\bar{f} = x^2 y^2$.

From the above lists, we conclude the following theorems.

THEOREM 2.3. *A sextic curve C is properly stable if and only if the following conditions are satisfied*:

(i) *C does not have a multiple line as a component.*

(ii) *C does not have consecutive triple points.*

(iii) *C does not have a point of multiplicity $\geqq 4$.*

THEOREM 2.4. *The semistable sextic curves belonging to minimal orbits are as follows.*

Group I. *Reduced sextics which have neither consecutive triple points nor a point of multiplicity $\geqq 4$.*

Group II. *A sextic curve defined by one of the following equations*:

(1) $(x_0 x_2 + a_1 x_1^2)(x_0 x_2 + a_2 x_1^2)(x_0 x_2 + a_3 x_1^2) = 0$ *where a_1, a_2, a_3 are distinct complex numbers.*

(2) $x_2^2 f_4(x_0, x_1) = 0$ *where* $f_4(x_0, x_1)$ *has no multiple factors.*

(3) $(x_0x_2 + x_1^2)^2 f_2(x_0, x_1, x_2) = 0$ *such that the quadric curves defined by the equations* $x_0x_2 + x_1^2 = 0$ *and* $f_2(x_0, x_1, x_2) = 0$ *intersect in four distinct points.*

(4) $f_3^2(x_0, x_1, x_2) = 0$ *where* $f_3(x_0, x_1, x_2) = 0$ *defines a nonsingular curve.*

Group III. *A sextic curve defined by one of the following equations*:

(1) $(x_0x_2 + x_1^2)^2(x_0x_2 + ax_1^2) = 0$ *where* $a \neq 1$.

(2) $x_0^2x_1^2x_2^2 = 0$.

Group IV. *The sextic curve defined by the equation* $(x_0x_2 + x_1^2)^3 = 0$.

Remark 2.5. Let $R = \mathbf{C}[[t]]$, $S = \operatorname{Spec} R$, $o =$ the closed point of S and $\zeta =$ the generic point of S. Let $g: X \to S$ be a family of sextic curves such that the generic geometric fiber is smooth. Let x_0 be a coordinate in $\mathbf{P}_2$ and let $A =$ the complement of the line $(x_0 = 0)$. X is a divisor on $\mathbf{P}_2 \times S$. Let $f = 0$ be a local equation of X in $A \times S$. Let K be the function field of $\mathbf{P}_2 \times S$. Let Y be the normalization of $\mathbf{P}_2 \times S$ in $K(\sqrt{f})$. Y is a double cover of $\mathbf{P}_2 \times S$ ramified over X. It is unique since $\mathbf{P}_2 \times S$ is simply connected. Let Y' be the normalization of $\mathbf{P}_2 \times S$ in $K(\sqrt{tf})$. Then, the restrictions of Y and Y' to $\mathbf{P}_2 \times \zeta$ are the two, nonisomorphic, double covers of $\mathbf{P}_2 \times \zeta$ ramified over $X \times_S \zeta$. Y' is ramified over $X \cup (\mathbf{P}_2 \times \{0\})$. However, the normalizations of the pull-backs of Y and Y' to $\operatorname{Spec} R[\sqrt{t}\,]$ are isomorphic.

3. Monodromy of families of sextic double planes

In this section, we explain the classification of sextic double planes according to the four groups of semistable sextic curves. By *a family of surfaces over a connected scheme W*, we mean a flat, projective morphism $g: X \to W$ such that the geometric generic fibers of g are smooth, connected and two-dimensional. Let $S = \operatorname{Spec} \mathbf{C}[[t]]$ and let o denote its closed point. If $h: Y \to S$ is a family of surfaces over S, we let Y_0 denote $h^{-1}(o)$. Let A be a connected, nonsingular curve over $\mathbf{C}$. Let s_o be a closed point of A. If $g: X \to A$ is a family of surfaces over A, we let X_o denote $g^{-1}(s_o)$. A family of surfaces $h: Y \to S$ is called *a local modification of g at s_0* if there exists a map $\rho: S \to A$ such that $\rho(o) = s_o$ and such that the generic fiber of h and the generic fiber of the pull-back of g via ρ are isomorphic.

PROPOSITION 3.1. *Let $g: X \to A$ be a family of sextic double planes such that X_o is ramified over a semistable curve C belonging to a minimal orbit. If C belongs to Group* I *of sextic curves, then X_0 has only rational double* *points as singularities and it is birationally a $K3$ surface. If C belongs to Group* II *or* III *of sextic curves and if $h\colon Y \to S$ is a local modification of g at s_o, then every component of Y_o is birationally ruled.*

Proof. The first assertion is clear. Suppose that C belongs to Group II or III. Let $\rho\colon S \to A$ be the map as described above. Let $g'\colon X' \to S$ be the pull-back of g via ρ. There exists a blow-up $Y' \to Y$ such that the rational map $Y \to X'$ extends to a morphism $Y' \to X'$ and such that Y' is nonsingular. Then Y' is a resolution of singularities of X'. Every component of Y'_o is birationally ruled since every component of X_o is birationally ruled and since X_o has only insignificant limit singularities. Q.E.D.

For a further classification of the singular fibers, we look at the monodromy of a family of surfaces, $g\colon X \to A$. Let t be a local parameter at s_o. For some $\varepsilon > 0$, let B be the disc $\{t\colon |t| < \varepsilon\}$ in A. Choose ε sufficiently small so that g is smooth over $B - s_o$. Let s be a point in B other than s_0. Let $X_s = g^{-1}(s)$. The fundamental group $\Pi = \pi_1\ (B - s_o, s)$ acts on the cohomology group $H^2(X_s, \mathbf{Q})$. Let γ denote the image of Π in $\mathrm{Aut}(H^2(X_s, \mathbf{Q}))$; it is called the local monodromy group of the family at s_o. Let T be a generator of γ; T is the local Picard-Lefschetz transformation at s_o.

By passing to a ramified covering of A, we may assume that (i) the fiber X_o over s_o is a double plane ramified over a sextic curve C which belongs to a minimal orbit and (ii) T is unipotent. We prove the following theorem.

THEOREM 3.2. *Let $g\colon X \to A$ be a family of surfaces (not necessarily sextic double planes) such that the special fiber X_o over s_o is isomorphic to a double plane, ramified over a sextic curve C which belongs to a minimal orbit and such that the local Picard-Lefschetz transformation T at s_o is unipotent. Let $N = \ln T$. Let $\nu =$ the exponent of nilpotence of N; $\nu = \min\{i\colon N^i = 0\}$. Then,*

$\nu = 1$ *if C belongs to Group* I *of sextic curves (see Theorem* 2.4*),*

$\nu = 2$ *if C belongs to Group* II *of sextic curves,*

$\nu = 3$ *if C belongs to Group* III *of sextic curves.*

Proof. Recall that the weight filtration of the limit mixed Hodge structure on $H^2(X_s, \mathbf{C})$ is induced by N [11]. Let h_m denote the dimension of $W_m(H^2(X_s, \mathbf{C}))/W_{m-1}(H^2(X_s, \mathbf{C}))$. Then, $\nu = \max\{m\colon h_m \neq 0\} - 1$. We compute the numbers h_m via the dual complex of X_o which we now define. We give a simpler version of the construction given in [12].

Let Z be a two-dimensional, reduced, projective scheme over $\mathbf{C}$ such that if P is a singular point of Z, then $\hat{\underline{o}}_{Z,P} \approx \mathbf{C}[[x, y, z]]/(f)$ where f is one of the following elements in $\mathbf{C}[[x, y, z]]$:

f_1: $z^2 + \alpha$ where $\alpha \in \mathbf{C}[x, y]$ is such that the singularity at P is an isolated rational double point;

f_2: yz;

f_3: $z^2 + y^2x$;

f_4: $z^2 + (y + a_1x^2)(y + a_2x^2)(y + a_3x^2)$ + terms in x, y of weight >6 where $a_1, a_2, a_3 \in \mathbf{C}$, at least two a_i's are distinct and where we define the weight of a nonzero monomial in x, y by assigning to x weight 1 and to y weight 2;

f_5: $z^2 + (y + a_1x)(y + a_2x)(y + a_3x)(y + a_4x)$ + terms in x, y of degree >4 where $a_1, a_2, a_3, a_4 \in \mathbf{C}$ and no three a_i's are the same;

f_6: an element such that the projective tangent cone of $\underline{o}_{Z,P}$ is a plane cubic curve whose singularities consist of only ordinary double points.

Let Z' = the disjoint union of the irreducible components of Z.

Z^* = a resolution of singularities of Z'.

Let π: $Z^* \to Z$ be the canonical projection.

Let Δ = the singular locus of Z,

Δ_0 = the subset of Δ consisting of points of types f_4, f_5 and f_6,

Δ_1 = the subset of Δ_0 consisting of simple elliptic singularities (A singular point is a simple elliptic singularity if the exceptional divisor in its minimal resolution consists of a nonsingular elliptic curve.),

$\Delta_2 = \Delta_0 - \Delta_1$,

$\{C_i\}$ = the set of one-dimensional, irreducible components of Δ,

C_i^* = the normalization of C_i,

$C = \bigcup C_i$,

C^* = the disjoint union $\coprod C_i^*$,

$D = \bigcup\{D_i: D_i$ an irreducible curve contained in $\pi^{-1}(C)$ such that the restriction of π to D_i is a finite map$\}$,

D^* = the disjoint union of the normalized irreducible components of D,

E^* = the set of nonsingular elliptic curves in $\pi^{-1}(\Delta_1)$.

Consider diagrams of the form $Z_1 \overset{d_0}{\underset{d_1}{\rightrightarrows}} Z_0$ where Z_0 and Z_1 are projective schemes over $\mathbf{C}$. We denote the diagram by $Z_.$ and call it a simplicial scheme. We say that the simplicial scheme $Z_.$ is defined over Z if Z_1 and Z_0 are Z-schemes and d_0 and d_1 are Z-morphisms. For a simplicial scheme $Z_.$, we define its dual complex as follows. Let $\wedge_.$ denote the object obtained from $Z_.$ by replacing each connected component of Z_1 and Z_0 by a point.

$$\wedge_\cdot = \wedge_1 \underset{d_1}{\overset{d_0}{\rightrightarrows}} \wedge_0$$

The dual complex $\wedge$ of $Z_\cdot$ is the geometric realization of $\wedge_\cdot$, obtained from the topological sum $(\wedge_1 \times I) \amalg \wedge_0$ (where I denotes the unit interval) by identifying $(e, 0)$ with $d_0(e)$ and $(e, 1)$ with $d_1(e)$ for all $e \in \wedge_1$. We may also define a dual complex of $Z_\cdot$ with coefficients by fixing an integer m and assigning to each vertex $v \in \wedge_0$, the group $H^m(Z_{0,v}, \mathrm{C})$ where $Z_{0,v}$ is the connected component of Z_0 corresponding to v, and to each edge $e \times I$, $e \in \wedge_1$, the group $H^m(Z_{1,e}, \mathrm{C})$ where $Z_{1,e}$ is the connected component of Z_1 corresponding to e. We thus obtain a canonical cochain complex $\cdots 0 \to H^m(Z_0, \mathrm{C}) \xrightarrow{d} H^m(Z_1, \mathrm{C}) \to 0 \cdots$ where $d = d_0^* - d_1^*$.

Now, let $Z_0 = Z^* \amalg \Delta_1 \amalg C^*$ and $Z_1 = E^* \amalg D^*$. Let $Z_\cdot$ be the simplicial scheme

$$\begin{tikzcd} & \Delta_1 \\ E^* \arrow[ur, "d_1"] \arrow[dr, "d_0"] & \\ & Z^* \\ D^* \arrow[ur, "d_0"] \arrow[dr, "d_1"] & \\ & C^* \end{tikzcd}$$

It is defined over Z via canonical maps $\varepsilon_i\colon Z_i \to Z$, $i = 0, 1$. The dual complex Γ of Z is obtained from the dual complex $\wedge$ by attaching a two-cell for each point in Δ_2 as follows. Let $P \in \Delta_2$. Let $Z_{\cdot P}$ be the simplicial scheme $Z_{1,P} \underset{d_1}{\overset{d_0}{\rightrightarrows}} Z_{2,P}$ where $Z_{i,P} = \varepsilon_i^{-1}(P)$. Let $\wedge_P$ be the dual complex of $Z_{\cdot P}$. The canonical morphism $Z_{\cdot P} \to Z_\cdot$ induces a map of dual complexes, $\lambda_P\colon \wedge_P \to \wedge$. If P is an isolated singularity, $\wedge_P$ is just a point. Attach a two-cell by collapsing its boundary onto $\lambda_P(\wedge_P)$. If P is not an isolated singularity, $\wedge_P$ is homeomorphic to a circle. Attach a two-cell via λ_P by identifying its boundary with $\wedge_P$. Note that $\wedge = \mathrm{Sk}_1\Gamma =$ 1-skeleton of Γ. Let $\mathcal{C}^*(\mathrm{Sk}_1\Gamma, \mathcal{H}^1)$ denote the cochain complex

$$\cdots 0 \longrightarrow \underset{\dim 0}{H^1(Z_0, \mathrm{C})} \xrightarrow{\ d\ } \underset{\dim 1}{H^1(Z_1, \mathrm{C})} \longrightarrow 0 \cdots .$$

Let $h^i(\mathrm{Sk}_1\Gamma, \mathcal{H}^1)$ denote the dimension of the i^{th} cohomology group of the cochain complex. Let $h^i(\Gamma, \mathrm{C}) =$ the dimension of $H^i(\Gamma, \mathrm{C})$.

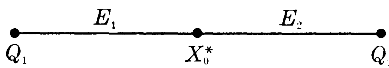
The cohomology of the scheme Z carries mixed Hodge structure [4]. Let $h_m(Z)$ denote the dimension of $W_m(H^2(Z, \mathbf{C}))/W_{m-1}(H^2(Z, \mathbf{C}))$. It follows from Section 1 in [12] that $h_0(Z) = h^2(\Gamma, \mathbf{C})$ and $h_1(Z) = h^1(\mathrm{Sk}_1\Gamma, \mathcal{H}^1)$.

Now let Z be the special fiber X_0. By Theorem 2 in [12], $h_4 = h_0 = h_0(X_0) = h^2(\Gamma, \mathbf{C})$ and $h_3 = h_1 = h_1(X_0) = h^1(\mathrm{Sk}_1\Gamma, \mathcal{H}^1)$. Let $X_0'' =$ the disjoint union of the normalized irreducible components of X_0 and $X_0^* =$ the minimal resolution of singularities of X_0''. Let $\mathbf{C}^m$ denote the direct sum of m copies of $\mathbf{C}$. We retain the rest of the notation from above.

Group I: Γ is just a point.

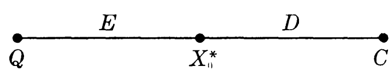
Group II:

(1) $\Delta = \Delta_1 =$ two simple elliptic double points, Q_1 and Q_2. Let $E_i =$ the exceptional divisor in X_0^* over Q_i. X_0^* is birationally a ruled surface with the base curve isomorphic to E_1 and E_2.

Γ:


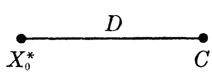
$$\mathcal{C}^*(\mathrm{Sk}_1\Gamma, \mathcal{H}^1): \quad \cdots 0 \longrightarrow \mathbf{C}^2 \xrightarrow{d} \mathbf{C}^4 \longrightarrow 0 \cdots,$$
$$d = \text{diagonal map} \quad h^1(\mathrm{Sk}^1\Gamma, \mathcal{H}^1) = 2 .$$

(2) $\Delta_0 = \Delta_1$. Δ_1 consists of a single point, Q. C consists of a nonsingular rational curve and D is a nonsingular elliptic curve. Let E denote the exceptional divisor above Q. X_0^* is birationally a ruled surface with the base curve isomorphic to D and E.

Γ:


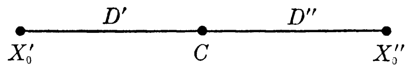
$$\mathcal{C}^*(\mathrm{Sk}_1\Gamma, \mathcal{H}^1): \quad \cdots 0 \longrightarrow \mathbf{C}^2 \xrightarrow{d} \mathbf{C}^4 \longrightarrow 0 \cdots,$$
$$d = \text{diagonal map.} \ h^1(\mathrm{Sk}_1\Gamma, \mathcal{H}^1) = 2 .$$

(3) Δ_0 is empty. Δ consists of a nonsingular, rational curve C. D is a nonsingular, elliptic curve. X_0^* is a rational surface.

Γ:


$$\mathcal{C}^*(\mathrm{Sk}_1\Gamma, \mathcal{H}^1): \quad \cdots 0 \longrightarrow \underset{\text{dim } 1}{\mathbf{C}^2} \longrightarrow 0 \cdots, \qquad h^1(\mathrm{Sk}_1\Gamma, \mathcal{H}^1) = 2 .$$

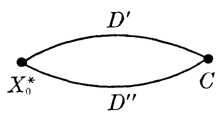
(4) X_0 consists of two irreducible components, X_0' and X_0'', each isomorphic to $\mathbf{P}_2$. X_0' and X_0'' intersect in a nonsingular elliptic curve C. D consists of two connected components, D' and D'', each isomorphic to C.

Γ: 

$$\mathcal{C}^*(\mathrm{Sk}_1\Gamma, \mathcal{H}^1): \quad \cdots 0 \longrightarrow \mathbf{C}^2 \xrightarrow{d} \mathbf{C}^4 \longrightarrow 0 \cdots ,$$
$$d = \text{diagonal map. } h^1(\mathrm{Sk}_1\Gamma, \mathcal{H}^1) = 2 .$$

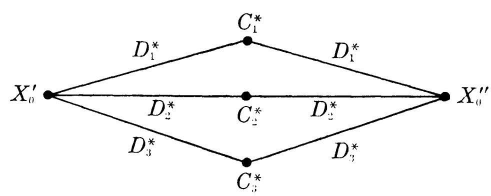
Group III:

(1) $\Delta_0 = \Delta_2 =$ two points, Q_1 and Q_2, each of type f_4. C consists of a nonsingular, rational curve while D^* consists of two curves, D' and D'', each isomorphic to C. X_0^* is a rational surface.

$\mathrm{Sk}_1\Gamma$: 

Γ is obtained from $\mathrm{Sk}_1\Gamma$ by attaching two 2-cells so that Γ is homeomorphic to a sphere. $\mathcal{C}^*(\mathrm{Sk}_1\Gamma, \mathcal{H}^1)$ is trivial.

(2) $\Delta_0 = \Delta_2 =$ three points, Q_1, Q_2, and Q_3, each of type f_5. C^* consists of three nonsingular rational curves and D^* is an unramified double cover of C^*. X_0 consists of two irreducible components, X_0' and X_0'', each isomorphic to $\mathbf{P}_2$.

$\mathrm{Sk}_1\Gamma$: 

Γ is obtained from $\mathrm{Sk}_1\Gamma$ by attaching three 2-cells so that Γ is homeomorphic to a sphere. $\mathcal{C}^*(\mathrm{Sk}_1\Gamma, \mathcal{H}^1)$ is trivial. Theorem 3.2 is now proved.

Remark 3.3. If X_0 is triply ramified over a nonsingular conic, then X_0 does not determine the exponent of nilpotence ν. In fact, ν may be 1, 2 or 3 depending on the choice of the family specializing to X_0 (see Theorem 6.1).

4. $K3$ surfaces which are double covers of Σ_4^0

Let X be a nonsingular $K3$ surface carrying a nonsingular curve E of genus 1 and a nonsingular rational curve F such that $E \cdot F = 1$. Let $L = \mathcal{O}_X(2E + F)$. We have the following description [7, 10]. L is a line bundle of degree 2 such that F is the fixed component of the linear system $|L|$. $|L|$ maps X onto $\mathbf{P}_1$, giving X the structure of an elliptic surface with a section. The linear system $|2L|$ does not have a fixed component and maps X onto Σ_4^0, a cone in $\mathbf{P}_5$ over a rational normal quartic curve contained in a hyperplane.

The morphism $X \to \Sigma_4^0$ defined by $|L|$ factors as follows:

$$X \xrightarrow{\pi} X^* \longrightarrow \Sigma_4^0$$

where π is the contraction of all the nonsingular rational curves, D, such that $D \cdot L = 0$. The singularities of X^* consist of isolated, rational double points and π is the minimal resolution of these singularities. X^* is also the normalization of Σ_4^0 in the function field of X. X^* is a double cover of Σ_4^0, ramified over the vertex of Σ_4^0 and a section of Σ_4^0 by a cubic hypersurface in $\mathbf{P}_5$, not passing through the vertex. Conversely, if B is such a section of Σ_4^0 by a cubic hypersurface, there exists a unique double cover of Σ_4^0, ramified over B and the vertex such that the double cover is normal over the vertex (see Appendix).

We construct a moduli space $\mathfrak{M}$ of such $K3$ surfaces via the geometric invariant theory. Since the group of automorphisms of Σ_4^0 is notreductive, we adopt a somewhat round about construction.

Let $\mathcal{G}$ denote the group of automorphisms of Σ_4^0. $\mathcal{G} \approx G_u \cdot \mathrm{GL}_2/\mu_4$ where $G_u \approx H^0(\mathbf{P}_1, \underline{o}_{\mathbf{P}_1}(4))$, where the action of GL_2 on G_u is induced by the canonical action of GL_2 on $H^0(\mathbf{P}_1, \underline{o}_{\mathbf{P}_1}(1))$ and $\mu_4 =$ the matrices in GL_2 of the form αI, $\alpha^4 = 1$, $I =$ the identity matrix [5]. The action of $\mathcal{G}$ on Σ_4^0 extends to an action of $\mathbf{P}_5$ and $\underline{o}_{\mathbf{P}_5}(1)$ admits a $\mathcal{G}$-linearization as follows. Let $\mathbf{A} = H^0(\mathbf{P}_5, \underline{o}_{\mathbf{P}_5}(1)) \approx H^0(\Sigma_4^0, \underline{o}_{\Sigma_4^0}(1))$. Let $\mathbf{\Theta}$ denote the subspace of $\mathbf{A}$ consisting of sections which vanish at the vertex of Σ_4^0. Let s_∞ denote the exceptional divisor in the minimal resolution of singularity $\Sigma_4 \to \Sigma_4^0$. The action of $\mathcal{G}$ extends canonically to an action on Σ_4 which preserves s_∞. Pick a basis $\{q_0, q_1, \cdots, q_5\}$ of $\mathbf{A}$ such that $\{q_1, \cdots, q_5\}$ is a basis of $\mathbf{\Theta}$. G_u acts on $\mathbf{A}$ via matrices of the form

$$\begin{bmatrix} 1 & \underline{c} \\ 0 & I_5 \end{bmatrix}$$

where $\underline{c}$ is a 5-dimensional row-vector and I_5 is the 5×5 identity matrix. GL_2 acts on $\mathbf{\Theta}$ via the canonical isomorphism $\mathbf{\Theta} \approx H^0(\mathbf{P}_1, \underline{o}_{\mathbf{P}_1}(4))$ induced by restriction of $\mathbf{\Theta}$ to s_∞. By letting GL_2 act trivially on $\mathbf{C} \cdot q_0$, we obtain an action of GL_2 on $\mathbf{A}$. Since μ_4 acts trivially on $H^0(\mathbf{P}_1, \underline{o}_{\mathbf{P}_1}(4))$, we get an action of $\mathcal{G}$ on $\mathbf{A}$. Note that G_u identifies with the unipotent subgroup of $\mathcal{G}$ and thus does not depend on the choice of q_0.

Let G_r denote GL_2/μ_4 and let G_m denote its center. G_m equals T_2/μ_4 where T_2 is the center of GL_2 and hence G_m is isomorphic with the one-dimensional multiplicative group. Let G_s denote the image of SL_2 in G_r. $G_s \approx \mathrm{SL}_2/\mu_2$ where $\mu_2 = \pm I$. Consider the isogeny $T_2 \times \mathrm{SL}_2 \xrightarrow{j} \mathrm{GL}_2$. $j^{-1}(\mu_4) = \mu_4 \times \mu_2$. Hence,

$$G_m \times \mathrm{PGL}_2 \approx (T_2 \times \mathrm{SL}_2)/(\mu_4 \times \mu_2) \approx \mathrm{GL}_2/\mu_4$$

and G_r is isomorphic to the direct product $G_m \times G_s$.

Let $\mathbf{B} = H^0(\Sigma_4^0, \underline{o}_{\Sigma_4^0}(2))$ and $\mathbf{D} = H^0(\Sigma_4^0, \underline{o}_{\Sigma_4^0}(3))$. Both $\mathbf{B}$ and $\mathbf{D}$ have $\mathcal{G}$-invariant filtration which is induced by the order of vanishing of their elements on s_∞. Since G_r is reductive, we actually get a G_r-invariant decomposition of $\mathbf{B}$ and $\mathbf{D}$ as follows. Let $\mathbf{\Phi}$ = the elements of $\mathbf{B}$ whose order of vanishing on s_∞ is 2 and Ξ = the elements of $\mathbf{D}$ whose order of vanishing on s_∞ is 3. Then,

$$\mathbf{B} \approx \mathbf{C} \cdot q_0^2 \oplus q_0 \cdot \mathbf{\Theta} \oplus \mathbf{\Phi} \text{ and } \mathbf{D} \approx \mathbf{C} \cdot q_0^3 \oplus q_0^2 \cdot \mathbf{\Theta} \oplus q_0 \cdot \mathbf{\Phi} \oplus \Xi .$$

Moreover, we have a canonical, G_r-linear identification of $\mathbf{\Theta}$, $\mathbf{\Phi}$ and Ξ with $H^0(\mathbf{P}_1, \underline{o}_{\mathbf{P}_1}(4))$, $H^0(\mathbf{P}_1, \underline{o}_{\mathbf{P}_1}(8))$ and $H^0(\mathbf{P}_1, \underline{o}_{\mathbf{P}_1}(12))$ respectively, by restriction to s_∞. We also have $\mathcal{G}$-linear surjections $\mathrm{Symm}^2(\mathbf{\Theta}) \to \mathbf{\Phi}$ and $\mathrm{Symm}^3(\mathbf{\Theta}) \to \Xi$, where $\mathrm{Symm}^k(\mathbf{V})$ denotes k^{th} symmetric tensor product of a vector space $\mathbf{V}$.

Given a vector space $\mathbf{V}$, we let $\mathbf{V}^*$ denote its dual and let A_V denote the affine space corresponding to $\mathbf{V}$. $A_V = \operatorname{Spec} \mathrm{Symm}(\mathbf{V}^*)$ where $\mathrm{Symm}(\mathbf{V}^*) = \bigoplus_{k \geqq 0} \mathrm{Symm}^k(\mathbf{V}^*)$.

Let $\mathbf{D}_0$ = the elements of $\mathbf{D}$ which vanish at the vertex of Σ_4^0. Let $|\mathbf{D}|^\times = |\mathbf{D}| - |\mathbf{D}_0|$. A closed point of $|\mathbf{D}|^\times$ is uniquely represented by an element f of $\mathbf{D}$ which is of the form $q_0^3 + q_0^2\theta + q_0\phi + \xi$ where $\theta \in \mathbf{\Theta}$, $\phi \in \mathbf{\Phi}$ and $\xi \in \Xi$. $|\mathbf{D}|^\times$ is isomorphic with the affine $A_\Theta \times A_\Phi \times A_\Xi$. Let g be a closed point of G_u which acts trivially on $\mathbf{\Theta}$ and takes q_0 to $q_0 + \theta_0$. g transforms f into

$$q_0^3 + (\theta + 3\theta_0)q_0^2 + (\phi + 2\theta_0\theta + 3\theta_0^2)q_0 + (\xi + \theta_0\phi + \theta_0^2\theta + \theta_0^3) .$$

Therefore, we have a G_u-invariant morphism $\beta\colon A_\Theta \times A_\Phi \times A_\Xi \to A_\Phi \times A_\Xi$ which takes the closed point corresponding to a vector (θ, ϕ, ξ) to the closed point corresponding to the vector $(\phi - \theta^2/3, \xi - \theta\phi/3 + 2\theta^3/27)$. We also have a G_u-invariant section of β defined by the canonical inclusion $\mathbf{\Phi} \oplus \Xi \subset \mathbf{\Theta} \oplus \mathbf{\Phi} \oplus \Xi$.

LEMMA 4.1. *The morphism β defines a geometric quotient of $A_\Theta \times A_\Phi \times A_\Xi$ by G_u.*

Proof. We apply Proposition 0.2 of [8]. Let k be an algebraically closed field over $\mathbf{C}$. Let $f = q_0^3 + q_0^2\theta + q_0\phi + \xi$ represent a k-valued point of $A_\Theta \times A_\Phi \times A_\Xi$ where $\theta \in \mathbf{\Theta} \otimes_{\mathbf{C}} k$, $\phi \in \mathbf{\Phi} \otimes_{\mathbf{C}} k$ and $\xi \in \Xi \otimes_{\mathbf{C}} k$. But, under the action of a k-valued point of G_u which takes q_0 to $q_0 - \theta/3$, f is transformed into an element of $\mathbf{\Phi} \oplus \Xi$. Q.E.D.

Next, we consider the action of G_r on $A_\Phi \times A_\Xi$. Let g be a closed point of G_m. Now g transforms an element (ϕ, ξ) of $\mathbf{\Phi} \oplus \Xi$ into $(a^2\phi, a^3\xi)$. Let

$\Omega = \text{Symm}(\mathbf{\Phi}^* \bigoplus \Xi^*)$ so that $A_\Phi \times A_\Xi = \text{Spec}\,\Omega$. Grade Ω by assigning weight 2 to $\mathbf{\Phi}^*$ and weight 3 to Ξ^*. Proj Ω is a geometric quotient of $A_\Phi \times A_\Xi - (0, 0)$ by G_m. The action of G_s descends to an action on Proj Ω such that the projection $A_\Phi \times A_\Xi - (0, 0) \to \text{Proj}\,\Omega$ commutes with the action of G_s. It remains to determine the set $(\text{Proj}\,\Omega)^{ss}$ of the semistable points of Proj Ω in order to form a compact quotient.

Proj Ω contains G_s-invariant projective spaces $|\mathbf{\Phi}|$ and $|\Xi|$. Let $p_{r_1}: \text{Proj}\,\Omega \dashrightarrow |\mathbf{\Phi}|$ and $p_{r_2}: \text{Proj}\,\Omega \dashrightarrow |\Xi|$ be the rational maps defined by the canonical projections. If $w \in \text{Proj}\,\Omega$, for $i = 1, 2$, let $p_{r_i}(w)$ denote the empty set if p_{r_i} is not defined at w. If $w \in |\mathbf{\Phi}|$ (respectively, $|\Xi|$), let $\bar{w}$ denote its restriction to s_∞.

PROPOSITION 4.2. *Let $w \in \text{Proj}\,\Omega$. Then w is stable if and only if s_∞ does not have a point p such that for $i = 1, 2$, p has multiplicity $\geqq 2(i + 1)$ in $\overline{p_{r_i}(w)}$. Also w is strictly semistable (that is, semistable, but not stable) if and only if there exists a point p in s_∞ such that for $i = 1, 2$, p has multiplicity $= 2(i + 1)$ in $\overline{p_{r_i}(w)}$ if $p_{r_i}(w)$ is not empty; w is strictly semistable and belongs to a minimal orbit if and only if s_∞ has two distinct points such that for $i = 1, 2$, each has multiplicity $= 2(i + 1)$ in $\overline{p_{r_i}(w)}$ if $p_{r_i}(w)$ is not empty.*

Proof. Let Ω_i denote the graded piece of weight i in Ω, and let $\Omega^0 = \bigoplus_{i=0}^\infty \Omega_{6i}$, $\Omega_\Phi^0 = \bigoplus_{i=0}^\infty \text{Symm}^{3i}(\mathbf{\Phi}^*)$ and $\Omega_\Xi^0 = \bigoplus_{i=0}^\infty \text{Symm}^{2i}(\Xi^*)$. Let $V = \text{Spec}\,\Omega^0$, $V_\Phi = \text{Spec}\,\Omega_\Phi^0$ and $V_\Xi = \text{Spec}\,\Omega_\Xi^0$ be the respective affine cones.

$$\text{Proj}\,\Omega^0 \approx \text{Proj}\,\Omega, \quad \text{Proj}\,\Omega_\Phi^0 \approx |\mathbf{\Phi}| \text{ and } \text{Proj}\,\Omega_\Xi^0 \approx |\Xi| .$$

Let λ be a one-parameter subgroup of G_s. Choose coordinates so that the action of λ on V_Φ and V_Ξ is diagonalized. We now apply Proposition 2.3 of [8] to determine the numerical function $\mu(w, \lambda)$. If w is either in $|\mathbf{\Phi}|$ or in $|\Xi|$, then clearly, its stability may be determined by considering the action of G_s on these subspaces. If w is neither in $|\mathbf{\Phi}|$ nor in $|\Xi|$, then, since $\Omega_6 \approx \text{Symm}^3(\mathbf{\Phi}^*) \bigoplus \text{Symm}^2(\Xi^*)$,

$$\mu(w, \lambda) = \max\{\mu(p_{r_1}(w), \lambda), \mu(p_{r_2}(w), \lambda)\} .$$

Now, $|\mathbf{\Phi}|$ is G_s-isomorphic to $|H^0(\mathbf{P}_1, \underline{o}_{\mathbf{P}_1}(8))|$ such that the G_s-linearization of V_Φ is induced by that of $o_{\mathbf{P}_1}(8)$. If w' is a closed point in $|\mathbf{\Phi}|$ and p_0, p_∞ are respectively the attractive and the repulsive fixed points of the action of λ on $\mathbf{P}_1$, then $\mu(w', \lambda) > 0$ (respectively, $=0$, respectively, <0) if the multiplicity of p_∞ in $\bar{w}'$ is less than (respectively, equal to, respectively, greater than) 4. Let λ be parametrized by t and let $w_0' = \lim_{t\to 0} w'^{\lambda(t)}$. If $\mu(w', \lambda) = 0$, then both p_0 and p_∞ have multiplicity equal to 4 in $\bar{w}_0'$. A similar argument applies to a point in $|\Xi|$. The proposition follows.

Q.E.D.

Let $(\mathfrak{M}, \mathfrak{g})$ denote a universal categorical quotient of $(\operatorname{Proj}\Omega)^{ss}$ by G_s.

THEOREM 4.3. *Let ω be a closed point of $\mathfrak{M}$. Let w be a closed point in the minimal orbit above ω in $(\operatorname{Proj}\Omega)^{ss}$. Let B be the corresponding divisor on Σ_4^0. Let X be a double cover of Σ_4^0, ramified over the vertex and B such that X is normal over the vertex. Then, X has only insignificant limit singularities and it is one of the surfaces listed below*:

Case 1: *B is reduced.*

(i) *If w is stable, X has at most rational double points as singularities. If ν is the exponent of nilpotence of a one-parameter family of surfaces specializing to X, then $\nu = 1$.*

(ii) *If w is strictly semistable, the singularities of X consist of two simple elliptic double points*; $\nu = 2$.

Case 2: *B has a component of multiplicity* 2. *B has an equation of the form $(q_0 + \theta)^2(q_0 - 2\theta) = 0$ such that the restriction $\bar{\theta}$ of θ to s_∞ determines a semistable divisor of degree 4 on s_∞. Let s' and s'' be the divisors on Σ_4^0 defined by the equations $q_0 + \theta = 0$ and $q_0 - 2\theta = 0$ respectively. X has a nodal curve over s'.*

(i) *If w is stable, then $\bar{\theta}$ is stable so that s' and s'' intersect transversely in four distinct points*; $\nu = 2$.

(ii) *If w is strictly semistable, then $\bar{\theta}$ is strictly semistable. Hence s' and s'' are tangent to each other at two distinct points*; $\nu = 3$.

Proof. Let $\mathfrak{p}\colon \Sigma_4 \to \mathbf{P}_1$ be the canonical projection, obtained by projecting Σ_4^0 from its vertex. Let l be a fiber of Σ_4. We have isomorphisms $\mathbf{\Theta} \approx H^0(\Sigma_4, \underline{o}_{\Sigma_4}(4l))$, $\mathbf{\Phi} \approx H^0(\Sigma_4, \underline{o}_{\Sigma_4}(8l))$ and $\Xi \approx H^0(\Sigma_4, \underline{o}_{\Sigma_4}(12l))$. Let b be a closed point of B. Choose a basis $\{u, v\}$ of $H^0(\Sigma_4, \underline{o}_{\Sigma_4}(l))$ such that u vanishes at b. Let l_0 and l_∞ denote the divisors defined by the equations $u = 0$ and $v = 0$ respectively. Let q_∞ be a nonzero element in $H^0(\Sigma_4, \underline{o}_{\Sigma_4}(s_\infty))$. Let $x = u/v$ and $y = q_0/q_\infty$. Let $P_i(u, v)$ denote a homogeneous polynomial of degree i in variable u, v. Let $p_i(x) = P_i(u/v, 1)$.

B is defined by an equation of the form $f = q_0^3 + q_0\phi + \xi = 0$ such that either ϕ has multiplicity $\leqq 4$ at b or ξ has multiplicity $\leqq 6$ at b. In the affine $\Sigma_4 - s_\infty - l_\infty$, B is defined by an equation of the form $y^3 + yp_8(x) + p_{12}(x)$ such that either $x^5 \nmid p_8(x)$ or $x^7 \nmid p_{12}(x)$. Over $\Sigma_4 - s_\infty - l_\infty$, X is defined by an equation of the form $z^2 = y^3 + yp_8(x) + p_{12}(x)$.

Suppose that B is reduced. Then, the singularity of X above b is a non-rational double point if and only if B has consecutive triple points at b. Since the y^2-term is missing in the equation of B, if b is a triple point, it must have coordinates $x = y = 0$. Therefore, B has consecutive triple points at b if and only if $x^4 \mid p_8(x)$ and $x^6 \mid p_{12}(x)$. But if $x^4 \mid p_8(x)$ and $x^6 \mid p_{12}(x)$, then w is not stable. Since w belongs to a minimal orbit, we may choose the basis $\{u, v\}$ such that $p_8(x) = a'x^4$ and $p_{12}(x) = a''x^6$. This proves Case 1.

Suppose that B has a multiple component. Since the y^2-term is missing, f cannot be of the form $(q_0 + \theta)^3$. Hence, $f = (q_0 + \theta)^2(q_0 - 2\theta) = q_0^3 - 3q_0\theta^2 - 2\theta^3$ such that if w is stable, then the restriction $\bar{\theta}$ does not have multiplicity $\geqq 2$ at any point of s_∞ and if w is strictly semistable, the equation $\bar{\theta} = 0$ defines a divisor consisting of two distinct points, each with multiplicity equal to 2. It is now easy to check Case 2. Q.E.D.

5. Moduli of $K3$ surfaces of degree 2

In this section, we construct a blow-up $\mathfrak{M} \to \widetilde{\mathfrak{M}}$ with center δ such that the exceptional divisor is isomorphic with $\mathfrak{N}$. Let H_2 denote the space $|H^0(\mathbf{P}_2, \underline{o}_{\mathbf{P}_2}(2))|$. Let G denote the group PGL_3. Let H_2^{ss} = the space of nonsingular (therefore, semistable) conics. Fix $Q \in H_2^{ss}$. Let C be the corresponding conic in $\mathbf{P}_2$. Let $\mathbf{P}_1 \hookrightarrow \mathbf{P}_2$ be an embedding, mapping $\mathbf{P}_1$ onto C. The action of PGL_2 extends via this embedding to an action on $\mathbf{P}_2$. PGL_2, in fact, may be identified with G_Q, the isotropy group of Q in G. Since $\underline{o}_{\mathbf{P}_1}(2m)$ is PGL_2-linear for $m \geqq 1$, $\underline{o}_{\mathbf{P}_2}(m)$ is G_Q-linear. Let q be a nonzero element of $H^0(\mathbf{P}_2, \underline{o}_{\mathbf{P}_2}(2))$ which vanishes on C. Since G_Q is a semisimple group, there exists a unique, G_Q-invariant decomposition $H^0(\mathbf{P}_2, \underline{o}_{\mathbf{P}_2}(2)) \approx \mathbf{C} \cdot q \bigoplus \Theta$ such that $\Theta \approx H^0(\mathbf{P}_1, \underline{o}_{\mathbf{P}_1}(4))$. In turn, we get unique, G_Q-invariant decompositions

$$H^0(\mathbf{P}_2, \underline{o}_{\mathbf{P}_2}(4)) \approx \mathbf{C} \cdot q^2 \bigoplus q \cdot \Theta \bigoplus \Phi \qquad \text{and}$$
$$H^0(\mathbf{P}_2, \underline{o}_{\mathbf{P}_2}(6)) \approx \mathbf{C} \cdot q^3 \bigoplus q^2 \cdot \Theta \bigoplus q \cdot \Phi \bigoplus \Xi$$

such that

$$\Phi \approx H^0(\mathbf{P}_1, \underline{o}_{\mathbf{P}_1}(8)) \quad \text{and} \quad \Xi \approx H^0(\mathbf{P}_1, \underline{o}_{\mathbf{P}_1}(12)) .$$

Let Q also denote the closed point of H_6^{ss} corresponding to the sextic which consists of C with multiplicity three. Let M denote the projective subspace of H_6 corresponding to the vector space $\mathbf{C} \cdot q^3 \bigoplus q \cdot \Phi \bigoplus \Xi$. M is invariant under G_Q and has a G_Q-invariant decomposition into subspaces, Q, $|q\Phi|$ and $|\Xi|$ which span M.

Let $\tilde{A}$ = the affine in M, consisting of points with nonzero q^3-coordinate. A closed point of $\tilde{A}$ corresponds to an element of $H^0(\mathbf{P}_2, \underline{o}_{\mathbf{P}_2}(6))$ which can be written uniquely as $q^3 + q\phi + \xi$ where $\phi \in \Phi$ and $\xi \in \Xi$. $\tilde{A}$ is G_Q-invariant, contained in H_6^{ss} and isomorphic with $A_\Phi \times A_\Xi$. Let $\sigma_Q: G_Q \times \tilde{A} \to \tilde{A}$ denote the action of G_Q on $\tilde{A}$. Let $\tilde{\mathfrak{A}}$ denote the universal categorical quotient of $\tilde{A}$ by G_Q. Let $\tau: \tilde{A} \to \tilde{\mathfrak{A}}$ denote the quotient morphism. Let $\partial = \tau(Q)$. Consider the canonical diagram

$$\begin{tikzcd} \tilde{A} \arrow[r, hook] \arrow[d, "\tau"'] & H_6^{ss} \arrow[d] \\ \tilde{\mathfrak{A}} \arrow[r, "\chi"] & \tilde{\mathfrak{M}} . \end{tikzcd}$$

PROPOSITION 5.1. *The morphism χ is étale at ∂.*

Proof. Via the action of G on H_2, H_6 and $H_2 \times H_6$, we obtain a commutative diagram

$$\begin{tikzcd} G \times \tilde{A} \arrow[r, "\beta_1"] & H_6^{ss} \\ G \times Q \times \tilde{A} \arrow[u, "\wr"] \arrow[r, "\beta_2"] \arrow[d] & H_2^{ss} \times H_6^{ss} \arrow[u] \arrow[d] \\ G \times Q \arrow[r, "\beta_3"] & H_2^{ss} . \end{tikzcd}$$

If we let G act on itself from the left via multiplication, the above diagram is also equivariant under the action of G. Let Z denote the image of β_2. The proposition will follow from Proposition 5.2 below if we show that

(i) There is a unique map $\mathfrak{z}: Z \to \tilde{\mathfrak{A}}$ such that $(Z, \mathfrak{z})$ is the universal categorical quotient of Z by G, and

(ii) The canonical projection $j: Z \to H_6^{ss}$ is étale in a neighborhood of $\beta_2(G \times Q \times Q)$.

To prove the first statement, consider the G_Q-actions

(a) $G \times G_Q \xrightarrow{(1_G, r)} G \times G_Q \xrightarrow{\text{mult.}} G$ where r is the morphism defining the inverse,

(b) $\sigma_Q: G_Q \times \tilde{A} \to \tilde{A}$ and

(c) $G_Q \times Q \to Q$.

These actions define actions of G_Q on $G \times \tilde{A}, G \times Q \times \tilde{A}$ and $G \times Q$ such that the morphisms β_1, β_2, β_3 factor through these actions. (H_2^{ss}, β_3) is the geometric quotient of $G \times Q$ by G_Q. Apply Proposition 0.2 in [8] to β_2. Let k be an algebraically closed field and let x and y be k-valued points of H_2^{ss} and H_6^{ss} respectively. Suppose that the fiber of β_2 over (x, y) is not empty. Let (g_1, Q, w_1) and (g_2, Q, w_2) be two points lying above the point (x, y). There exists a k-valued point g_0 of G_Q such that $g_1 g_0^{-1} = g_2$. Then, $y = g_2 w_2 = g_1 w_1 = g_2 g_0 w_1$ and hence $g_0 w_1 = w_2$. It follows that (Z, β_2) is the geometric quotient of $G \times Q \times \widetilde{A}$ by G_Q. Since the projection $G \times Q \times \widetilde{A} \to \widetilde{\mathfrak{A}}$ is constant on the G_Q-orbits, there is a unique map $\mathfrak{z}: Z \to \widetilde{\mathfrak{A}}$ so that we have a commutative diagram

$$\begin{array}{ccc} G \times \widetilde{A} & \xrightarrow{\beta_2} & Z \\ \downarrow{\scriptstyle p_{r_2}} & & \downarrow{\scriptstyle \mathfrak{z}} \\ \widetilde{A} & \longrightarrow & \widetilde{\mathfrak{A}} \end{array}$$

where β_2, p_{r_2} and τ are quotient morphisms. (p_{r_2} is the projection on the second factor and $(\widetilde{A}, p_{r_2})$ is the geometric quotient of $G \times \widetilde{A}$ by G.) Therefore, a morphism $f: Z \to Y$ which is equivariant under the action of G on Z is equivalent to a morphism $g: G \times \widetilde{A} \to Y$ which is equivariant under the action of G and G_Q. Such a g is equivalent to a G_Q-invariant morphism $h: \widetilde{A} \to Y$. Finally, h is equivalent to a morphism $\widetilde{\mathfrak{A}} \to Y$. This proves Statement (i).

The projection $Z \to H_2^{ss}$ is equivariant under the action of G. By homogeneity under G, it follows that Z is a vector bundle over H_2^{ss} with fibers isomorphic to $\widetilde{A}$. Moreover, by homogeneity, it is enough to show that the projection j is étale at $Q \times Q$. But, j maps $Q \times \widetilde{A}$ and $\beta_2(G \times Q \times Q)$ isomorphically into H_6^{ss}. Therefore, it is enough to show that $\widetilde{A}$ and $\beta_1(G \times Q)$ intersect transversely at Q. Let $O_\delta =$ the orbit of Q in H_6^{ss} and let $\beta: G \to H_6^{ss}$ be the map which takes g to $g \cdot Q$. $d\beta$ maps the tangent vector space T_G of G at the identity surjectively onto the tangent vector space of O_δ at Q. The kernel of $d\beta$ equals the tangent vector space of G_Q at the identity. Let $v \in T_G$. v induces a derivation of the homogeneous coordinate ring of $\mathbf{P}_2$ under which the image of q^3 is $3q^2dq$. Therefore $d\beta(v)$ is a vector along $\widetilde{A}$ if and only if $dq = 0$. But, if $dq = 0$, then, the infinitesimal automorphism of H_6^{ss} induced by v fixes Q and therefore, $d\beta(v) = 0$.

Q.E.D.

PROPOSITION 5.2. *Let X be an algebraic scheme over an algebraically closed field k of characteristic zero. Let G^* be a reductive algebraic group acting on X. Suppose that $\pi: X \to Y$ is the universal categorical quotient of X by G^* and that π is affine. Let V be an invariant closed subset of X and let $W = \pi(V)$. Then*

(i) *W is closed and it is the universal categorical quotient of V by G^*.*

(ii) *Suppose that X is integral and that $W =$ a minimal orbit in X. Let $\hat{V}$ be the completion of X along V and let $\hat{W}$ be the completion of Y along W. Then, the canonical map $\hat{\pi}: \hat{V} \to \hat{W}$ is the categorical quotient of*

$\hat{V}$ *by* G^*.

Proof. Since π is affine, we are immediately reduced to the case where $X = \operatorname{Spec} R$. Let $R_0 =$ the subring of invariants so that $Y = \operatorname{Spec} R_0$. Let J be an invariant ideal in R. Then, $R/(J \cap R_0)$ is the ring of invariants in R/J (Statement 3 on page 29 in [8]). This proves (i). If W is a minimal orbit, then V is a closed point of Y and $J \cap R_0$ is a maximal ideal in R_0. Let $J_0 = J \cap R_0$. Let $\bar{J} = J_0 R$. Note that $\bar{J} \subset J$. Recall that for any ideal $I \subset R_0$, $(IR) \cap R_0 = I$. Therefore, $\bar{J}^n \cap R_0 = (J_0^n R) \cap R_0 = J_0^n$. Let $(J^n)_0 = J^n \cap R_0$. We have a projective system of commutative diagrams:

$$\begin{array}{ccc} R/\bar{J}^n & \twoheadrightarrow & R/J^n \\ \uparrow\cup & & \uparrow\cup \\ R_0/J_0^n & \twoheadrightarrow & R_0/(J^n)_0 \end{array}.$$

Therefore, we have a commutative diagram of complete rings:

$$\begin{array}{ccc} (R, \{\bar{J}^n\})^\wedge & \longrightarrow & (R, \{J^n\})^\wedge \\ \uparrow\cup & & \uparrow\cup \\ \hat{R}_0 = (R_0, \{J_0^n\})^\wedge & \longrightarrow & (R_0, \{(J^n)_0\})^\wedge = \mathfrak{R} \end{array}$$

where the superscript $\wedge$ denotes the completion with respect to the indicated filtration and the rings of invariants of the rings in the top row are the ones directly below. R_0 is integral since R is. Therefore, $\bigcap_{n=1}^{\infty}(J^n)_0 = 0$. A theorem of Chevalley asserts that the topology induced by the maximal ideal in a local ring is the weakest Hausdorff topology induced by a filtration (Ch. VIII, § 5, Theorem 13 in [16]). Therefore, for any integer $n \geqq 1$, there exists an integer m such that $(J^m)_0 \subset J_0^n$ and hence, the completions $\hat{R}_0$ and $\mathfrak{R}$ are isomorphic. Q.E.D.

We now define a blow-up of $\widetilde{\mathfrak{M}}$ by blowing up $\widetilde{\mathfrak{A}}$ at ∂. The coordinate ring of $\tilde{A}$ is isomorphic with $\Omega = \operatorname{Symm}(\Phi^* \oplus \Xi^*)$. Grade Ω by assigning weight 2 to Φ^* and weight 3 to Ξ^*. Then $\Omega = \bigoplus \Omega_i$ where Ω_i is the graded piece of weight i. Let $\Omega^\#$ be the graded ring $\bigoplus_{k \geqq 0} \Omega_k^\#$ where $\Omega_k^\# = \bigoplus_{i \geqq k} \Omega_i$. If we regard Ω as an ungraded ring, $\Omega^\#$ is a graded algebra over Ω. Let $A = \operatorname{Proj} \Omega^\#$ and let $\pi \colon A \to \tilde{A}$ be the canonical projection. π is an isomorphism everywhere except over the point Q in $\tilde{A}$. The exceptional divisor E is isomorphic to $\operatorname{Proj} \Omega$ where we regard Ω as a graded ring. Since the blow-up is equivariant with respect to the action of G_Q, G_Q acts on A. We consider the stability of the points of A via the action of G_Q on $\operatorname{Spec} \Omega^\#$.

Let $\Delta = (\tau \cdot \pi)^{-1}(\partial)$. If y is a closed point of $A - \Delta$, the closure of its orbit lies in $A - \Delta$ since $A - \Delta \approx \pi(A - \Delta) \approx \tilde{A} - \tau^{-1}(\partial)$ and the closure of the orbit of $\pi(y)$ is contained in $\widetilde{A} - \tau^{-1}(\partial)$. y is semistable since $\pi(y)$ is. Suppose that $y \in \Delta - E$. Then $\pi(y)$ lies in $\tau^{-1}(\partial) - Q$ and Q lies in the closure of the orbit of $\pi(y)$. That is, there exists a one-parameter subgroup $\lambda(t)$ of G_Q such that $\lim_{t\to 0}(\pi(y))^{\lambda(t)} = Q$. Therefore, $\pi(y)$ is represented by a sextic form $q^3 + q\phi + \xi$ such that $\lim_{t\to 0}(\phi, \xi)^{\lambda(t)} = (0, 0)$. In other words, (ϕ, ξ) represents an unstable point of $\operatorname{Proj}\Omega$. There exists a positive integer m such that $(\phi, \xi)^{\lambda(t)} = (t^{2m}\phi_t, t^{3m}\xi_t)$ where $\lim_{t\to 0}(\phi_t, \xi_t) = (\phi_0, \xi_0) \neq (0, 0)$ and $\mu((\phi_0, \xi_0), \lambda) > 0$. Therefore, y is unstable. Hence, $\Delta^{ss} = E^{ss}$. It remains to consider the points in E. But the ring Ω is isomorphic with the ring Ω of the previous section with isomorphic group actions. Therefore, the stability of the points of E is described by Proposition 4.2 via the identifications $\Phi \approx H^0(\mathbf{P}_1, \underline{o}_{\mathbf{P}_1}(8))$ and $\Xi \approx H^0(\mathbf{P}_1, \underline{o}_{\mathbf{P}_1}(12))$. Let $\mathfrak{A}$ denote the categorical quotient of A^{ss} by G_Q. $\mathfrak{A}$ is a blow-up of $\widetilde{\mathfrak{A}}$ and induces a blow-up $\mathfrak{M}$ of $\widetilde{\mathfrak{M}}$. By Propositions 5.2, the exceptional divisor in $\mathfrak{A}$ above ∂ is the categorical quotient of E^{ss} and hence, it is isomorphic with $\mathfrak{N}$.

6. Degenerations of $K3$ surfaces of degree 2

In view of Proposition 2.1, Theorem 2.4 and Theorem 3.2, we only need to prove.

THEOREM 6.1. *Let $S = \operatorname{Spec} \mathbf{C}[[t]]$. Let $f: X \to S$ be a family of sextic double planes such that the generic geometric fiber of f is nonsingular and the special fiber X_0 over the closed point, o, of S is a sextic double plane, triply ramified over a nonsingular conic. Then, there exists a flat, projective morphism, $f': X' \to S$ and a map $\rho: S \to S$ such that*

(i) *the generic fiber of f' and the generic fiber of the pull-back of f via ρ are isomorphic;*

(ii) *The special fiber X'_o of f' over o is a double cover of Σ_4^0 and*

(iii) *X'_o is one of the surfaces listed in Theorem* 4.3.

Proof. The theorem is proved as follows. We embed $\mathbf{P}_2 \times S$ in $\mathbf{P}_5 \times S$ via the linear system $H^0(\mathbf{P}_2, \underline{o}_{\mathbf{P}_2}(2)) \otimes \mathbf{C}[[t]]$. Let $\mathcal{B}$ be the ramification divisor of the family in $\mathbf{P}_2 \times S$. We deform $\mathbf{P}_2 \times S$ in $\mathbf{P}_5 \times S$ under the action of a one-parameter subgroup, λ, of the stabilizer of Σ_4^0 in PGL_6 such that we obtain a deformation of the Veronese surface into Σ_4^0. Here, Σ_4^0 is a cone over the embedding of C in a hyperplane in $\mathbf{P}_5$ as a rational normal quartic curve. If λ is chosen appropriately, $\mathcal{B}$ is modified under the action of λ into a family of curves, $\mathcal{B}'$, which specializes to a curve B'_0 on Σ_4^0. B'_0 does not pass the through the vertex of Σ_4^0 and corresponds to a point in $(\operatorname{Proj}\Omega)^{ss}$. Finally, X' is obtained as a double cover ramified over $\mathcal{B}'$ and the vertex of Σ_4^0.

Step I: Deformation of $\mathbf{P}_2$ in $\mathbf{P}_5$. Consider the embedding $\iota: \mathbf{P}_2 \to \mathbf{P}_5$ via the linear system H_2. Let $W = \iota(\mathbf{P}_2)$. W is well-known to be projectively Cohen-Macaulay.[1] Let $D = \iota(C)$. D is a rational normal quartic curve contained in the hyperplane defined by the equation $q = 0$. The action of G_Q on $\mathbf{P}_2$ extends to an action on $\mathbf{P}_5$ under which D is invariant. Choose a basis $\{q_1, \cdots, q_5\}$ of Θ. Let R denote the graded ring $\mathbf{C}[q, q_1, \cdots, q_5]$. For a given positive integer n, let λ_n be the one-parameter subgroup of PGL_6 which acts on R via the transformations: $q \mapsto t^n q$ and for $1 \leqq i \leqq 5$, $q_i \mapsto q_i$. Note that D is invariant under λ_n and that λ_n commutes with G_Q. Let I be the ideal of W in R. Let I_t be the ideal in $R \otimes \mathbf{C}[[t]]$ obtained by replacing q by $t^n q$ in the elements of I. Let $\mathfrak{W}$ be the scheme over S defined by the ideal I_t. Let I_0 be the ideal of the fiber of $\mathfrak{W}$ over o. The generators for I_o are obtained by putting $q = 0$ in the generators for I. Since the hyperplane $q = 0$ cuts out the rational quartic curve D on W, I_o defines the $\lambda_n \times G_Q$-invariant cone, Σ_4^0, over D with its vertex at the point with coordinates $q_1 = q_2 = \cdots = q_5 = 0$. Thus $\lambda_n \times G_Q \subset$ the stabilizer Σ_4^0. Σ_4^0 is projectively normal [2]. Therefore, $\mathfrak{W}$ is flat over S by Proposition III-4.3 and Proposition V-3.5 in [1].

Step II: Lifting of $H^0(\mathbf{P}_2, \underline{o}_{\mathbf{P}_2}(6))$ to $H^0(\mathbf{P}_5, \underline{o}_{\mathbf{P}_5}(3))$. Let J' be the kernel of the $\lambda_n \times G_Q$-invariant restriction $r'\colon \mathrm{Symm}^2\Theta \to H^0(\Sigma_4^0, \underline{o}_{\Sigma_4^0}(2))$. Let Φ' be the $\lambda_n \times G_Q$-invariant complement of J' in $\mathrm{Symm}^2\Theta$. Similarly, let J'' be the kernel of the restriction $r''\colon \mathrm{Symm}^3\Theta \to H^0(\Sigma_4^0, \underline{o}_{\Sigma_4^0}(3))$ and let Ξ' be the $\lambda_n \times G_Q$-invariant complement of J''. We have a $\lambda_n \times G_Q$-invariant isomorphism

$$\mathbf{C} \cdot q^3 \oplus q^2 \cdot \Theta \oplus q \cdot \Phi' \oplus \Xi' \longrightarrow H^0(\Sigma_4^0, \underline{o}_{\Sigma_4^0}(3))$$

which induces the G_r-invariant decomposition of $H^0(\Sigma_4^0, \underline{o}_{\Sigma_4^0}(3))$ described in Section 4. Moreover, the G_Q-linear restrictions $\Phi' \to \Phi$ and $\Xi' \to \Xi$ are isomorphisms since they are isomorphisms when both sides are restricted to the rational quartic curve D. The G_Q-linear restriction

$$\mathbf{C} \cdot q^3 \oplus q^2 \cdot \Theta \oplus q \cdot \Phi' \oplus \Xi' \longrightarrow H^0(W, o_W(3)) \longrightarrow H^0(\mathbf{P}_2, \underline{o}_{\mathbf{P}_2}(6))$$

is an isomorphism which induces the G_Q-invariant decomposition of $H^0(\mathbf{P}_2,$

[1] This follows from a general result of M. Hochster and J. L. Roberts. (See "Rings of Invariants of reductive groups acting on regular rings are Cohen-Macaulay", Adv. in Math. **13** (1974), 115–175.) One may prove the assertion directly as follows. The homogeneous coordinate ring of W is the ring of invariants in $\mathbf{C}[x_0, x_1, x_2]$ under the involution which sends each variable to its negative. It is enough to show that the ring of invariants, $\mathcal{R}_0$, in the ring $\mathcal{R} = \mathbf{C}[[x_0, x_1, x_2]]$ under the involution is Cohen-Macaulay. But this follows from (i) the invariant elements x_0^2, x_1^2, x_2^2, form a regular sequence in $\mathcal{R}$ and (ii) for any ideal $J \subset \mathcal{R}_0$, $(J \cdot \mathcal{R}) \cap \mathcal{R}_0 = J$.

$\mathcal{O}_{\mathbf{P}_2}(6))$ described at the beginning of this section.

Step III: Modification of $\mathfrak{B}$. The family $f\colon X \to S$ induces a map $\tilde{u}\colon S \to \tilde{\mathfrak{A}}$ such that $\tilde{u}(o) = \partial$. Thus map $\tilde{u}$ lifts uniquely to a map $u\colon S \to \mathfrak{A}$. There exists a map $\rho\colon S \to S$ such that the composition $u \cdot \rho$ lifts to a map $v\colon S \to A^{ss}$ and such that $v(o)$ belongs to a minimal orbit. Let $\tilde{v}\colon S \to \tilde{A}$ be the projection of v. Then $\tilde{v}$ corresponds to a family of sextic forms $\psi(t) = q^3 + t^{2n}q\phi_t + t^{3n}\xi_t$ where $n > 0$, $\phi_t \in \Phi \otimes \mathbf{C}[[t]]$, $\xi_t \in \Xi \otimes \mathbf{C}[[t]]$ such that $\lim_{t\to 0}(\phi_t, \xi_t) = (\phi_0, \xi_0) \neq 0$ and defines a semistable point of $\operatorname{Proj}\Omega$, belonging to a minimal orbit. Moreover, by choosing ρ appropriately, we may assume that n is even. Now lift ϕ_t to an element ϕ'_t of $\Phi' \otimes \mathbf{C}[[t]]$ and ξ_t to an element ξ'_t of $\Xi' \otimes \mathbf{C}[[t]]$. The equation $\psi'(t) = q^3 + t^{2n}q\phi'_t + t^{3n}\xi'_t = 0$ defines a family of cubic hypersurfaces in $\mathbf{P}_5$. Under the action of λ_n, the family transforms into a new family which is defined by the equation $\psi''(t) = q^3 + q\phi'_t + \xi'_t = 0$. Let $\mathfrak{B}'$ be the divisor on $\mathfrak{W}$ defined by the equation $\psi''(t) = 0$. Let B'_o be the fiber of $\mathfrak{B}'$ over the closed point of S. B'_o is a section of Σ_4^0 by the cubic hypersurface defined by the equation $q^3 + q\phi'_0 + \xi'_0 = 0$ where $(\phi'_0, \xi'_0) = \lim_{t\to 0}(\phi'_t, \xi'_t) \neq 0$. It follows that B'_o does not pass through the vertex of Σ_4^0 and is a semistable curve on Σ_4^0 belonging to a minimal orbit in the sense of Section 4.

Step IV. Double cover of $\mathfrak{W}$. Let $\eta \in H^0(\mathbf{P}_2, \mathcal{O}_{\mathbf{P}_2}(6))$ such that $\eta = \alpha^6$ for some nonzero $\alpha \in H^0(\mathbf{P}_2, \mathcal{O}_{\mathbf{P}_2}(1))$. Let K denote the function field of $\mathfrak{W}$. X is the normalization of $\mathbf{P}_2 \times S$ in the field $K(\sqrt{\psi(t)/\eta})$. Let $aq^3 + q^2\theta + q\phi + \xi$ be the decomposition of η. Let ϕ', ξ' be the liftings of ϕ, ξ to Φ' and Ξ' respectively. Let $\eta' = aq^3 + q^2\theta + q\phi' + \xi'$. Let $\eta'(t) = at^{3n}q^3 + t^{2n}q^2\theta + t^n\phi' + \xi'$. Note that $\eta'(0) = \xi' \neq 0$. Moreover, $\psi'(t)^{\lambda_n(t)} = t^{3n}\psi''(t)$ and $\eta'^{\lambda_n(t)} = \eta'(t)$. Let $\bar{\psi}(t)$ and $\bar{\eta}(t)$ be the restrictions of $\psi''(t)$ and $\eta'(t)$ to $\mathfrak{W}$. λ_n induces an automorphism of K which transforms $\psi(t)/\eta$ into $t^{3n}\bar{\psi}(t)/\bar{\eta}(t)$. Since n is even, the fields $K(\sqrt{\psi(t)/\eta})$ and $K(\sqrt{\bar{\psi}(t)/\bar{\eta}(t)})$ are isomorphic. Let X' be the normalization of $\mathfrak{W}$ in $K(\sqrt{\bar{\psi}(t)/\bar{\eta}(t)})$. I claim that X' is flat over $\mathfrak{W}$. Certainly, X' is flat over $\mathfrak{W}$ except at a finite number of closed points by Proposition V-3.5 in [1]. Let $\mathfrak{U}$ be the maximal open set in $\mathfrak{W}$ which does not contain the vertex of Σ_4^0 and over which X' is flat. The equation $\bar{\eta}(t) = 0$ defines a divisor on $\mathfrak{U}$ of even multiplicity and hence the branch locus of X' in $\mathfrak{U}$ equals $\mathfrak{U} \cap \mathfrak{B}'$. In particular, the special fiber X'_o is generically reduced and hence reduced since it satisfies the Serre property S_1 (V-2.2 and V-2.3 in [1]). Let K^0 denote the function field of Σ_4^0. $K^0(\sqrt{\bar{\psi}(0)/\bar{\eta}(0)})$ is the function field of X'_o. Let X''_o be the normalization of X'_o. As in Section 4, X''_o is a double cover of Σ_4^0 ramified over the vertex and $\bar{B}'_o$ where $\bar{B}'_o$ is the reduced curve consisting of those components of B_o' which have odd multiplicity in B_o'. It follows that X_o' itself is a double cover. Therefore, X' is a double cover and hence it is Cohen-Macaulay (Theorem VII-4.8 in [1]). Therefore, by Serre's criterion, X_o' is normal over the vertex of Σ_0^0. Q.E.D.

COROLLARY 6.2. *Let $\mathfrak{D}$ be the period space of $K3$ surfaces of degree* 2. *Then, to every point of $\mathfrak{D}$, there corresponds a unique $K3$ surface.*

Proof. The uniqueness has been proved by Piatetskiĭ-Šapiro and Šafarevic [9]. The existence may be proved as follows. The period map embeds the open set in $\mathfrak{M}$ corresponding to the nonsingular sextic double planes into $\mathfrak{D}$ as an open set, U_o. Let x be a point of $\mathfrak{D}$. Let $S = \operatorname{Spec}[[t]]$. Let o be the closed point of S and let ζ be its generic point. Let $f\colon S \to \mathfrak{D}$ be a map such that $f(o) = x$ and $f(\zeta) \subset U_o$. The restriction $f_\zeta\colon \zeta \to \mathfrak{D}$ lifts to a map $g_\zeta\colon \zeta \to \mathfrak{M}$ which then extends to a map $g\colon S \to \mathfrak{M}$. After replacing t by a suitable root of t, we may assume that g determines a family of $K3$ surfaces such that the generic geometric fiber is nonsingular and such that the special fiber is a semistable double cover of $\mathbf{P}_2$ or Σ_4^0. Moreover, we may assume that the special fiber has insignificant limit singularities and belongs to a minimal orbit. Since the limit mixed Hodge structure of the family must be a pure Hodge structure, the monodromy group of the family must be finite. It follows from Theorems 3.2 and 4.3 that the special fiber must be a $K3$ surface with at most rational double points as singularities. Q.E.D.

Appendix

Existence and uniqueness of the double covers of Σ_4^0

Let B be a section of Σ_4^0 by a cubic hypersurface in $\mathbf{P}_5$ which does not pass through the vertex of Σ_4^0. Let $\Sigma_4 \to \Sigma_4^0$ be the minimal resolution of the singularity of Σ_4^0. Let s_∞ be the exceptional curve in Σ_4. Let l be a fiber of the ruled surface Σ_4. There exists an element f in the function field K^0 of Σ_4^0 such that the divisor of f equals $B + s_\infty - 2Z$ where $Z = 2s_\infty + 6l$. Suppose first that B is reduced. Then the normalization X^* of Σ_4 in $K^0(\sqrt{f})$ is flat over Σ_4 by Proposition V-3.5 in [1]. Hence, X^* is a double cover of Σ_4, ramified over B and s_∞. Let $X^* \to X$ be the contraction of the nonsingular rational curve in X^* over s_∞. Then X is the normalization of Σ_4^0 in $K^0(\sqrt{f})$. It is a double cover of Σ_4^0, ramified over B and the vertex. Suppose that there exists f' such that the normalization of Σ_4^0 in $K^0(\sqrt{f'})$ is ramified over B and the vertex. Then the normalization of Σ_4^0 in $K^0(\sqrt{f/f'})$ is an unramified double cover of Σ_4^0 which must be trivial since Σ_4^0 is simply connected.

Therefore, $f = u^2 f'$ for some $u \in K^0$. Hence, $K^0(\sqrt{f'}) \approx K^0(\sqrt{f})$ and X is unique.

Next, suppose that B is not reduced. Let $\bar{B}$ be the reduced curve consisting of those components of B which have odd multiplicity in B. Let X' be the normal double cover of Σ_4^0, ramified over $\bar{B}$ and the vertex. Define a subalgebra $\underline{o}_X$ of $\underline{o}_{X'}$ as follows. Let U be an affine in Σ_4^0. If U does not intersect B, let $\underline{o}_{X|_U} \approx \underline{o}_{X'|_U}$. Suppose that U intersects B. We may assume that U does not contain the vertex. $\underline{o}_{X'|_U} \approx R[z]/(z^2 + h)$ where $R = \Gamma(U, \underline{o}_{\Sigma_4^0})$ and $h \in R$. Therefore, in U, B is defined by an equation of the form $g^{2m}h = 0$. Let $\underline{o}_{X|_U}$ be the subalgebra generated by $g^m z$.

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