Non-archimedean Quantum K-theory and Gromov-Witten Invariants

Tony Yue YU

Caltech

arXiv 2001.05515 + work in progress with M. Porta

Plan:
1. Motivations from mirror symmetry
2. Review of derived non-archimedean geometry
3. Representability theorem
4. Moduli stack of non-archimedean stable maps and Gromov compactness
5. Numerical enumerative invariants
6. Geometric relations between the derived moduli stacks

1. Motivations from mirror symmetry

Def: A smooth projective variety $X/\mathbb{C}$ is called Calabi-Yau if its canonical bundle K_X is trivial, i.e. it has a nowhere vanishing holomorphic volume form.

Examples: Elliptic curve, abelian variety, K3 surface, hypersurface of degree $d+1$ in $\mathbb{CP}^d$.

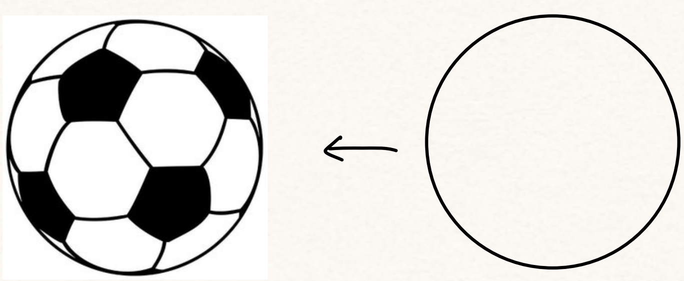
Mirror Symmetry: conjectural duality between Calabi-Yau varieties:

$$\text{Any CY variety } X \longleftrightarrow \exists \text{ mirror variety } \check{X}$$

such that a list of deep geometric relations hold between X and $\check{X}$, involving: Hodge structures, Gromov-Witten invariants, Fukaya categories, derived category of coherent sheaves, SYZ torus fibrations, etc.

A more careful study $\rightsquigarrow$ mirror symmetry is not really a duality between individual CY varieties, but rather a duality between "maximally degenerating families" of CY varieties.

Example: Type III degeneration of K3 surfaces



In general, an algebraic family of varieties over a punctured disk $\circ t$

$\rightsquigarrow$ variety $/ \mathbb{C}((t))$ field of formal Laurent series.

non-archimedean field: norm $|x| = e^{-\mathrm{val}_t}$

$$|x+y| \leq \max\{|x|, |y|\}$$

Non-archimedean geometry: analog of complex geometry over non-archimedean fields.

$\rightsquigarrow$ More general, more symmetric formulation of mirror symmetry as a duality of non-archimedean Calabi-Yau manifolds (with maximal degenerations)

Advantages:

1) Working formally without worrying about complex analytic convergence.
2) Existence of SYZ torus fibration is proved (Nicaise-Xu-Yu 2019)
3) New ways for counting curves with boundaries $\rightsquigarrow$ wall-crossing formulas

These considerations motivate an analog of Gromov-Witten theory in non-archimedean geometry.

Classical approach to Gromov-Witten theory: Perfect obstruction theory by Behrend-Fantechi, Li-Tian

Our approach in the non-archimedean setting: we develop a theory of derived non-archimedean geometry $\rightsquigarrow$ non-archimedean quantum K-invariants

$\rightsquigarrow$ non-archimedean Gromov-Witten invariants

2. Review of derived non-archimedean geometry

Q: What is a derived non-archimedean analytic space?

Recall the definition of a derived scheme:

A derived scheme is a pair $(X, \mathcal{O}_X)$ consisting of a topological space X and a sheaf $\mathcal{O}_X$ of simplicial commutative rings on X, satisfying the following conditions:

(1) The ringed space $(X, \pi_0(\mathcal{O}_X))$ is a scheme.

(2) For each $j \geqslant 0$, the sheaf $\pi_j(\mathcal{O}_X)$ is a quasi-coherent sheaf of $\pi_0(\mathcal{O}_X)$-modules.

In order to adapt the above definition to (non-archimedean) analytic geometry, we need to find a way to impose additional analytic structures on the sheaf $\mathcal{O}_X$, e.g. • a notion of norms on the sections of $\mathcal{O}_X$

• compose the sections of $\mathcal{O}_X$ with convergent power series

Our first attempt: Enhance simplicial commutative rings with non-archimedean analytic structures. "simplicial commutative affinoid / Banach algebras"

Difficult: Banach structure and simplicial structure do not mix well.

(works by Ben-Bassat, Kremnizer, Bambozzi, ⋯)

Our strategy: Use the theory of pregeometry and structured topos of Lurie.

Idea: Use the language of ∞-category / ∞-topos to generate derived sheaves starting from simple classical objects, bypassing any model-dependent constructions (e.g. simplicial algebras, dg-algebras).

Def: A pregeometry is a category $\mathcal{T}$ equipped with a class of admissible morphisms and a Grothendieck topology generated by admissible morphisms, such that

(1) $\mathcal{T}$ admits finite products

(2) The class of admissible morphisms is closed under composition, pullback and retract.

along any morphism always exist

(3)
$$\begin{tikzcd} & Y \arrow[dr, "g"] & \\ X \arrow[ur, "f"] \arrow[rr, "h"'] & & Z \end{tikzcd}$$
g, h admissible $\Rightarrow f$ admissible

Examples: k non-archimedean field

- $\mathcal{T}_{\text{ét}}(k) :=$ category of smooth k-varieties, étale maps, étale topology
- $\mathcal{T}_{\text{an}}(k) :=$ category of smooth k-analytic spaces, étale maps, étale topology

Def: $\mathcal{T}$ pregeometry, $\mathcal{X}$ ∞-topos (e.g. the category of sheaves of spaces on a given topological space)

A $\mathcal{T}$-structure on $\mathcal{X}$ is a functor $\mathcal{O}: \mathcal{T} \to \mathcal{X}$ s.t.

(1) it preserves finite products.

(2) it sends pullbacks of admissible morphisms in $\mathcal{T}$ to pullbacks in $\mathcal{X}$.

(3) it sends coverings in $\mathcal{T}$ to effective epimorphisms in $\mathcal{X}$.

The idea behind this abstract definition:

We can think of a $\mathcal{T}_{\text{an}}(k)$-structure $\mathcal{O}$ as a sheaf of derived rings equipped with an analytic structure:

(1) Let $\mathcal{F} := \mathcal{O}(\mathbb{A}^1) \in \mathcal{X}$

sum $+ : \mathbb{A}^1 \times \mathbb{A}^1 \longrightarrow \mathbb{A}^1 \xrightarrow{\text{product-preserving}} + : \mathcal{F} \times \mathcal{F} \longrightarrow \mathcal{F}$

multiplication $\bullet : \mathbb{A}^1 \times \mathbb{A}^1 \longrightarrow \mathbb{A}^1 \xrightarrow{\text{product-preserving}} \bullet : \mathcal{F} \times \mathcal{F} \longrightarrow \mathcal{F}$

Therefore, we can intuitively think of $\mathcal{F}$ as a sheaf of simplicial commutative rings.

(2) The sheaf $\mathcal{F}$ is also equipped with norms:

Let $\mathbb{D}^1 \subset \mathbb{A}^1$ closed unit disk

Recall: $\mathcal{O}$ sends pullbacks of admissible morphisms in $\mathcal{T}$ to pullbacks in $\mathcal{X}$

$\rightsquigarrow$ $\mathcal{O}(\mathbb{D}^1) \hookrightarrow \mathcal{F}$ is a monomorphism

Therefore, we can think of $\mathcal{O}(\mathbb{D}^1)$ as the subsheaf of $\mathcal{F}$ consisting of functions of norm ≤ 1.

(3) $\forall$ convergent power series f on $\mathbb{D}^1 \xrightarrow{\text{functoriality}}$ morphism $\mathcal{O}(\mathbb{D}^1) \to \mathcal{F}$

We think of it as composition with f

Now we are ready to give the definition of derived non-archimedean analytic space.

Def: A derived k-analytic space X is a pair $(\mathcal{X}, \mathcal{O}_X)$ consisting of a (hypercomplete) ∞-topos $\mathcal{X}$ and a $\mathcal{T}_{an}(k)$-structure $\mathcal{O}_X$ on $\mathcal{X}$ s.t.

(1) $(\mathcal{X}, \pi_0(\mathcal{O}_X^{alg}))$ is equivalent to the ringed ∞-topos associated to the étale site of a k-analytic space.

(2) For every $j \geq 0$, $\pi_j(\mathcal{O}_X^{alg})$ is a coherent sheaf of $\pi_0(\mathcal{O}_X^{alg})$-modules.

2. Representability theorem

Q: How do derived analytic spaces appear in nature?

A: Via the representability theorem

Representability theorem (Porta-Y):
Let F be an analytic moduli functor (i.e. a sheaf over the étale site of derived analytic spaces). The followings are equivalent:

1) F has the structure of a derived analytic space
2) F is compatible with Postnikov towers, has an analytic cotangent complex, and its truncation is an analytic space.

Rem: The representability theorem has two important implications:

1) Philosophical: Our notion of derived analytic space is natural and sufficiently general. I.e. any reasonable analytic moduli functor has the structure of a derived analytic space.
2) Practical: The conditions are easy to verify in practice. So the theorem gives plenty of down-to-earth examples of derived analytic spaces.

Rem: We say that a moduli functor F is compatible with Postnikov towers if it is infinitesimally cohesive and nilcomplete.

- infinitesimally cohesive: F sends squarezero extensions to pullbacks
- nilcomplete: $F(X) \xrightarrow{\sim} F(t_{\leq n} X)$

Rem: We also proved a generalization of the representability theorem for non-archimedean analytic stacks.

Here is an application of the representability theorem:

Theorem (Existence of derived mapping stacks, Porta-Y):

Rigid k-analytic spaces

$$\begin{tikzcd} X \arrow[dr, "\substack{\text{proper}\\ \text{flat}}"'] & & Y \arrow[dl, "\substack{\text{separated}\\ \text{lci}}"] \\ & S & \end{tikzcd}$$

Then the ∞-mapping functor

$$\begin{aligned} \mathrm{Map}_S(X, Y) : \mathrm{dAn}_{/S} &\longrightarrow \mathcal{S} \quad \text{spaces} \\ T &\longmapsto \mathrm{Map}_T(X_T, Y_T) \end{aligned}$$

is representable by a derived k-analytic space separated and locally of finite presentation over S.

3. Moduli stack of non-archimedean stable maps and Gromov compactness

Fix X a smooth rigid k-analytic space.

Def: Let T be a derived k-analytic space. An n-pointed genus g stable map into X over T consists of an n-pointed genus g prestable curve $[C \to T, (s_i)]$ over T and a map $f: C \to X$, s.t. every geometric fiber $[C_t, (s_i(t)), f_t : C_t \to X]$ is a stable map, in the sense that its automorphism group is a finite analytic group.

Representability theorem $\rightsquigarrow$ derived enhancement of the moduli stack of non-archimedean analytic stable maps

Theorem (Porta-Y): The derived moduli stack $\mathbb{R}\overline{M}_{g,n}(X)$ of n-pointed genus g stable maps into X is representable by a derived k-analytic stack locally of finite presentation and $\underbrace{\text{derived lci}}_{\text{cotangent complex is perfect and in tor-amplitude } [1,-\infty)}$.

Theorem (Non-archimedean Gromov compactness, Y): Assume further more that X is proper and equipped with a Kähler structure. Given any curve class β, the substack $\overline{M}_{g,n}(X,\beta) \subset \overline{M}_{g,n}(X)$ is a proper k-analytic stack, hence $\mathbb{R}\overline{M}_{g,n}(X,\beta)$ is a proper derived k-analytic stack.

4. Numerical enumerative invariants

Q: Given the compactness, how do we obtain numerical enumerative invariants from the derived structure?

Two ways classically $\begin{cases} \text{K-theory} \rightsquigarrow \text{quantum K-invariants (Givental-Lee)} \\ \text{intersection theory} \rightsquigarrow \text{Gromov-Witten invariants (Behrend-Fantechi)} \end{cases}$

K-theory works similarly in non-archimedean geometry:

Def: The non-archimedean quantum K-invariants are the maps

$$\begin{aligned} K^X_{g,n,\beta} : K_0(X)^{\otimes n} &\longrightarrow K_0(\overline{M}_{g,n}) \\ a_1 \otimes \cdots \otimes a_n &\longmapsto st_*(ev_1^* a_1 \otimes \cdots \otimes ev_n^* a_n) \end{aligned}$$

where

$$\begin{tikzcd} \mathbb{R}\overline{M}_{g,n}(X,\beta) \arrow[r, "ev_i", "\text{evaluation at } s_i"'] \arrow[d, "st", "\text{stabilization of domain}"'] & X \\ \overline{M}_{g,n} \end{tikzcd}$$

However, intersection theory (in the sense of Fulton's book) does not work in non-archimedean geometry (nor complex analytic geometry).

Reason: there are not enough cycles to have moving lemma, or to have Chern classes from vector bundles.

Solution: work with cohomological theories.

Choices:
1. Étale cohomology $\rightsquigarrow$ invariants in $\mathbb{Q}_\ell$, independence of ℓ
2. de Rham cohomology $\rightsquigarrow$ invariants in k, still not ideal
3. Berkovich integral étale cohomology $\rightsquigarrow$ invariants in $\mathbb{Q}$
   functorial properties not sufficiently developed
   only works over $\mathbb{C}((t))$.
4. Rigid analytic motivic cohomology by Ayoub $\rightsquigarrow$ invariants in $\mathbb{Q}$
   works over general non-archimedean fields
   six functor formalism recently developed by Ayoub-Gallauer-Vezzani

Roughly, any k-analytic space X

$\rightsquigarrow$ $\mathrm{RigSH}_{ét}(S;\mathbb{Q})$ ∞-category of étale k-analytic motives over X with rational coefficients.

Six functors $\otimes$, $\underline{\mathrm{Hom}}$, f^*, f_*, $f_!$, f^*

For any $a: X \to S$ k-analytic space / stack, we have

Motivic cohomology: $H^q(X/S, \mathbb{Q}(r)) = \mathrm{Hom}_{\mathrm{RigSH}_{ét}(S;\mathbb{Q})}(1_S, a_* a^* \mathbb{Q}(r)[q])$

Motivic Borel-Moore homology:

$$H_q^{BM}(X/S, \mathbb{Q}(r)) := \mathrm{Hom}_{\mathrm{RigSH}_{ét}(S;\mathbb{Q})}(1_S(r)[q], a_* a^! \mathbb{Q}), \quad q, r \in \mathbb{Z}.$$

Next we apply a derived analog of deformation to the normal cone following Khan:

Theorem (Khan-Rydh): For any derived lci morphism $f: X \to Y$ of derived k-analytic stacks, there exists a derived lci derived k-analytic stack $D_{X/Y}$ over $Y \times \mathbb{A}^1$, and a derived lci morphism $X \times \mathbb{A}^1 \to D_{X/Y}$ over $Y \times \mathbb{A}^1$, whose fiber over $\mathbb{G}_m = \mathbb{A}^1 \setminus 0$ is $X \times \mathbb{G}_m \to Y \times \mathbb{G}_m$, and the fiber over $0 \in \mathbb{A}^1$ is the 0-section $X \to \mathbb{T}_{X/Y}[1]$ to the shifted tangent bundle.

$d :=$ virtual dim of $X \to Y$

$$\rightsquigarrow H_q^{BM}(Y/S, \mathbb{Q}(r)) \underbrace{\xrightarrow{sp_{X/Y}} H_q^{BM}(\mathbb{T}_{X/Y}[1], \mathbb{Q}(r)) \xrightarrow[\text{homot equiv}]{\sim}}_{f^! :=} H_{q+2d}^{BM}(X/S, \mathbb{Q}(r+d))$$

Def (Khan): The virtual fundamental class of $f: X \to Y$ is the class

$$[X/Y] := f^!(1) \in H_{2d}^{BM}(X/Y, \mathbb{Q}(d)) \text{ where } 1 \in H_0^{BM}(Y/Y, \mathbb{Q}).$$

Def: The non-archimedean Gromov-Witten invariants are the maps

$$I^X_{g,n,\beta} : H^q(X, \mathbb{Q}(r))^{\otimes n} \longrightarrow H^q(\overline{M}_{g,n}, \mathbb{Q}(r))$$
$$a_1 \otimes \cdots \otimes a_n \longmapsto PD^{-1}\, st_*\big((ev_1^* a_1 \cdots \otimes ev_n^* a_n) \cap [\mathbb{R}\overline{M}_{g,n}(X,\beta)]\big)$$

where

$$\begin{tikzcd} \mathbb{R}\overline{M}_{g,n}(X,\beta) \arrow[r, "ev_i", "\text{evaluation at } s_i"'] \arrow[d, "st"', "\text{stabilization of domain}"] & X \\ \overline{M}_{g,n} \end{tikzcd}$$

6. Geometric relations between the derived moduli stacks

Next we need to establish all the expected properties of our non-archimedean invariants. They will follow readily from a list of natural geometric relations between the derived moduli stacks.

In order to state the geometric relations, instead of working with n-pointed genus g stable maps, we use a slight combinatorial refinement called (τ,β)-marked stable maps for any A-graph (τ,β) introduced by Behrend-Manin. (It imposes degeneration types on the domains of stable maps as well as more refined curve classes.)

$\rightsquigarrow$ associated moduli stacks $\overline{M}(X,\tau,\beta)$ of (τ,β)-marked stable maps, and their derived enhancements $\mathbb{R}\overline{M}(X,\tau,\beta)$

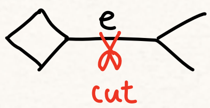
Furthermore, it will be useful to consider the relative situation $\mathbb{R}\overline{M}(X/S,\tau,\beta)$

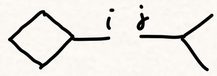
Theorem (Relations of derived moduli stacks, Porta-Y):
Let S be a rigid k-analytic space and X a rigid k-analytic space smooth over S. The derived moduli stack $\mathbb{R}\overline{M}(X/S, \tau, \beta)$ of (τ, β)-marked stable maps into X/S satisfies the following geometric relations with respect to elementary operations on A-graphs:

1) Products: (τ_1, β_1), (τ_2, β_2) A-graphs

$$\mathbb{R}\overline{M}(X/S, \tau_1 \sqcup \tau_2, \beta_1 \sqcup \beta_2) \xrightarrow{\sim} \mathbb{R}\overline{M}(X/S, \tau_1, \beta_1) \underset{S}{\times} \mathbb{R}\overline{M}(X/S, \tau_2, \beta_2)$$

2) Cutting edges: $(\tau, \beta) \rightsquigarrow (\sigma, \beta)$

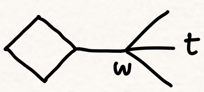


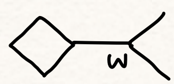


We have a derived pullback diagram

$$\begin{tikzcd}
\mathbb{R}\overline{M}(X/S, \tau, \beta) \arrow[r] \arrow[d, "ev_e"] & \mathbb{R}\overline{M}(X/S, \sigma, \beta) \arrow[d, "ev_i \times ev_j"] \\
X \arrow[r, "\Delta"] & X \underset{S}{\times} X
\end{tikzcd}$$

3) Universal curve: $(\tau, \beta) \xrightarrow[\text{forget the tail } t]{} (\sigma, \beta)$





We have a derived pullback diagram

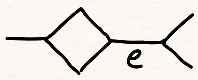
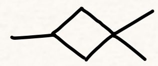
$$\begin{tikzcd}
\mathbb{R}\overline{M}(X/S, \tau, \beta) \arrow[r] \arrow[d] & \mathbb{R}\overline{M}(X/S, \sigma, \beta) \arrow[d] \\
\overline{C}_w^{pre} \arrow[r] & \overline{M}_\sigma^{pre}
\end{tikzcd}$$

universal curve corresponding to w

4) Forgetting tails: Context as above, we have a derived pullback diagram

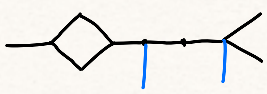
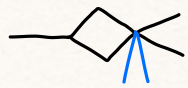
$$\begin{tikzcd}
\mathbb{R}\overline{M}(X/S, \tau, \beta) \arrow[r] \arrow[d] & \overline{M}_\tau \times_{\overline{M}_\sigma} \mathbb{R}\overline{M}(X/S, \sigma, \beta) \arrow[d] \\
\overline{C}_w^{pre} \arrow[r] & \overline{M}_\tau \times_{\overline{M}_\sigma} \overline{M}_\sigma^{pre}
\end{tikzcd}$$

5) Contracting edges: $(\tau, \beta) \rightsquigarrow (\sigma, \beta)$

 contract e 

$\Big\downarrow$ add k tails

$(\tau_\ell^i, \beta_\ell^{ij}) \leftsquigarrow (\tilde{\sigma}, \beta)$

 expand e by a chain of length ℓ 

We have a natural equivalence

$$\operatorname*{colim}_\ell \coprod_{i,j} \mathbb{R}\overline{M}(X/S, \tau_\ell^i, \beta_\ell^{ij}) \xrightarrow{\sim} \overline{M}_\tau \times_{\overline{M}_\sigma} \mathbb{R}\overline{M}(X/S, \tilde{\sigma}, \beta)$$

Rem: The universal curve relation in the particular case where τ is a point:
The forgetful map $\mathbb{R}\overline{M}_{g,n+1}(X/S) \longrightarrow \mathbb{R}\overline{M}_{g,n}(X/S)$ is equivalent to the universal curve $\mathbb{R}\overline{C}_{g,n}(X/S) \longrightarrow \mathbb{R}\overline{M}_{g,n}(X/S)$.
Such an intuitive statement in fact incorporates all the information about virtual counts with respect to forgetting a tail, which is classically expressed and proved in terms of pullback properties of perfect obstruction theories and intrinsic normal cones.

Rem: We take further advantage of the flexibility of our derived approach to introduce a generalized type of Gromov-Witten invariants that allow not only simple incidence conditions for marked points, but also incidence conditions with multiplicities. They satisfy a list of properties parallel to Behrend-Manin axioms. To the best of our knowledge, such invariants are not yet considered in the literature, even in algebraic geometry.