11 Affine Weyl groups

We can pass from the Weyl group of a crystallographic root system and form an infinite group that has more information about the root system, and yet still possesses a structure analogous to that of the Weyl group. Notably, it has a Coxeter group structure. This group is called the affine Weyl group. Affine Weyl groups have a number of uses. They will be used in Chapter 12 to analyze subroot systems of crystallographic root systems. They are even useful for understanding ordinary Weyl groups. This will be demonstrated in §11-6.

11-1 The affine Weyl group

Let Δ be a crystallographic root system. For each $\alpha \in \Delta$, we have defined the hyperplane

$$H_\alpha = \{t \in \mathbb{E} \mid (\alpha, t) = 0\}$$

and the associated reflection

$$s_\alpha \cdot x = x - \frac{2(\alpha, x)}{(\alpha, \alpha)}\alpha = x - (\alpha, x)\alpha^\vee,$$

where $\alpha^\vee$ is the coroot

$$\alpha^\vee = \frac{2\alpha}{(\alpha, \alpha)}.$$

We now generalize these concepts. For each $k \in \mathbb{Z}$ and $\alpha \in \Delta$, we can define the hyperplane

$$H_{\alpha,k} = \{t \in \mathbb{E} \mid (\alpha, t) = k\}$$

and the reflection $s_{\alpha,k}$ in the hyperplane $H_{\alpha,k}$

$$s_{\alpha,k} \cdot x = x - (\alpha, x)\alpha^\vee + k\alpha^\vee.$$

Observe that the case $k = 0$ gives the hyperplane H_α and its associated reflection s_α as discussed above. By introducing these extra reflections, the ordinary Weyl group can be extended to the affine Weyl group. We define the *affine Weyl group* by

Definition: $W_{\text{aff}}(\Delta) =$ the group generated by $\{s \mid \alpha \in \Delta, k \in \mathbb{Z}\}$.

As we shall see in §11-3, $W_{\text{aff}} = W_{\text{aff}}(\Delta)$ is a Coxeter group. In the remainder of this section, we establish that W_{aff} has a semidirect product decomposition in terms of the Weyl group W and the coroot lattice $\mathcal{Q}^\vee$. (The coroot lattice was defined and discussed in §9-2. Here we shall be interpreting it as translations on $\mathbb{E}$.)

Proposition $W_{\text{aff}} = \mathcal{Q}^\vee \rtimes W$.

Proof We need to show:

(i) W and $\mathcal{Q}^\vee$ are subgroups of W_{aff};
(ii) $W_{\text{aff}} = \mathcal{Q}^\vee W$;
(iii) $\mathcal{Q}^\vee \cap W = \{0\}$;
(iv) $\mathcal{Q}^\vee$ is normal.

(i): We have already shown that $W \subset W_{\text{aff}}$. Regarding $\mathcal{Q}^\vee$, any $d \in \mathbb{E}$ defines a translation

$$T(d)\colon \mathbb{E} \to \mathbb{E}$$

$$T(d)(x) = x + d.$$

Moreover, $T(d)T(d') = T(d+d')$. Thus $\mathcal{Q}^\vee$ gives a group of translations on $\mathbb{E}$. We have the identity

$$(*) \qquad s_{\alpha,k} = T(k\alpha^\vee)s_\alpha$$

(check the effect of RHS on $H_{\alpha,k}$ and on 0). Thus for all $\alpha \in \Delta$ and $k \in \mathbb{Z}$, $T(k\alpha^\vee) = s_{\alpha,k}s_\alpha^{-1} \in W_{\text{aff}}$. In particular, $\{T(\alpha_1^\vee), \ldots, T(\alpha_\ell^\vee)\} \subset W_{\text{aff}}$. So $\mathcal{Q}^\vee \subset W_{\text{aff}}$.

(ii): This follows from the above identity $(*)$.

(iii): Any nonzero element of $\mathcal{Q}^\vee$ is an element of infinite order while W is a finite group.

(iv): For any $d \in \mathcal{Q}^\vee$ and any $\varphi \in W$, we have the identity

$$\varphi T(d)\varphi^{-1} = T(\varphi \cdot d). \qquad \blacksquare$$

The rest of this chapter will be devoted to demonstrating that W_{aff} satisfies a number of properties analogous to those holding for finite Euclidean reflection groups. We shall show that W_{aff} has a Coxeter group structure that is a natural extension of the Coxeter group structure of W obtained in Chapter 6. In the process, we shall also show that there are precise analogues of previous structure theorems concerning the action of W on root systems and Weyl chambers.

11-2 The highest root

Let $\Delta \subset \mathbb{E}$ be an irreducible crystallographic root system and $\Sigma = \{\alpha_1, \ldots, \alpha_\ell\}$ a fundamental system of Δ. In this section, we shall explain how to choose the *highest root* α_0 of Δ with respect to Σ. *Highest* refers to a partial order we can define on $\mathbb{E}$. Let

$$\mathcal{Q} = \mathbb{Z}\alpha_1 + \cdots + \mathbb{Z}\alpha_\ell$$

be the root lattice as defined in §9-2. Choose $\mathcal{Q}^+ \subset \mathcal{Q}$ by the rule

$$\mathcal{Q}^+ = \{x_1\alpha_1 + \cdots + x_\ell\alpha_\ell \mid x_i \geq 0 \text{ for all } i\}.$$