The above analysis shows that if Z is a 2-dimensional topological field theory, then the vector space $A = Z(S^1)$ is naturally endowed with the structure of a commutative Frobenius algebra over k. In fact, the converse is true as well: given a commutative Frobenius algebra A, one can construct a 2-dimensional topological field theory Z such that $A = Z(S^1)$ and the multiplication and trace on A are given by evaluating Z on a pair of pants and a disk, respectively. Moreover, Z is determined up to unique isomorphism: in other words, the category of 2-dimensional topological field theories is *equivalent* to the category of commutative Frobenius algebras.

Theorem 1.2.16 (Baez-Dolan Cobordism Hypothesis). *Let $\mathcal{C}$ be a symmetric monoidal n-category. Then the evaluation functor*

$$Z \mapsto Z(*)$$

determines a bijective correspondence between (isomorphism classes of) framed extended $\mathcal{C}$-valued topological field theories and (isomorphism classes of) fully dualizable objects of $\mathcal{C}$.

Theorem 1.2.16 asserts that for every fully dualizable object C of a symmetric monoidal n-category $\mathcal{C}$, there is an essentially unique symmetric monoidal functor $Z_C : \mathbf{Cob}_n^{\mathrm{fr}}(n) \to \mathcal{C}$ such that $Z_C(*) \simeq C$. In other words, the symmetric monoidal $\mathbf{Cob}_n^{\mathrm{fr}}(n)$ is *freely generated* by a single fully dualizable object: namely, the object consisting of a single point.

Definition 1.1.5 (Atiyah). Let k be a field. A *topological field theory* of dimension n is a symmetric monoidal functor $Z : \mathbf{Cob}(n) \to \mathbf{Vect}(k)$.

$(\infty, 0)$-cats := Spaces

Definition 1.1.12. Let k be a field. A *commutative Frobenius algebra* over k is a finite dimensional commutative k-algebra A, together with a linear map $\mathrm{tr} : A \to k$ such that the bilinear form $(a, b) \mapsto \mathrm{tr}(ab)$ is nondegenerate.

Definition 1.2.3. A *strict 2-category* is a category enriched over categories. In other words, a strict 2-category $\mathcal{C}$ consists of the following data:

- The objects of $\mathbf{Vect}_2(k)$ are *cocomplete* k-linear categories: that is, k-linear categories $\mathcal{C}$ which are closed under the formation of direct sums and cokernels.

category theory. For each $n \geq 0$, one can define an n-category $\pi_{\leq n}X$, called the *fundamental n-groupoid of* X. Informally, this n-category can be described as follows:

- The objects of $\pi_{\leq n}X$ are the points of X.
- Given a pair of objects $x, y \in X$, a 1-morphism in $\pi_{\leq n}X$ from x to y is a path in X from x to y.
- Given a pair of objects $x, y \in X$ and a pair of 1-morphism $f, g : x \to y$, a 2-morphism from f to g in $\pi_{\leq n}X$ is a homotopy of paths in X (which is required to be fixed at the common endpoints x and y).
- An n-morphism in $\pi_{\leq n}X$ is given by a homotopy between homotopies between ...between paths between points of X. Two such homotopies determined the same n-morphism in $\pi_{\leq n}X$ if they are homotopic to one another (via a homotopy which is fixed on the common boundaries).

Definition 1.3.3. A topological space X is called an *n-type* if the homotopy groups $\pi_k(X, x)$ vanish for all $x \in X$ and all $k > n$.

Between the theory of n-categories in general (which are difficult to describe) and the theory of n-groupoids (which are easy to describe) there are various intermediate levels of complexity.

Definition 1.3.5. Suppose we are given a pair of nonnegative integers $m \leq n$. An *(n, m)-category* is an n-category in which all k-morphisms are assumed to be invertible, for $m < k \leq n$.

Example 1.3.6. An $(n, 0)$-category is an n-groupoid; an (n, n)-category is an n-category.

Variant 1.3.7. In Definition 1.3.5, it is convenient to allow the case $n = \infty$: in this case, an (n, m)-category has morphisms of all orders, but all k-morphisms are assumed to be invertible for $k > m$. It is possible to allow $m = \infty$ as well, but this case will play no role in this paper.

Taking n to ∞ in the formulation of Thesis 1.3.4, we obtain the following:

Thesis 1.3.8. *There is a construction $X \mapsto \pi_{\leq \infty}X$ which establishes a bijection between topological spaces (up to weak homotopy equivalence) and $(\infty, 0)$-categories (up to equivalence).*

Definition Sketch 1.3.11. For $n > 0$, an (∞, n)-category $\mathcal{C}$ consists of the following data:

(1) A collection of objects $X, Y, Z, \ldots$

(2) For every pair of objects $X, Y \in \mathcal{C}$, an $(\infty, n-1)$-category $\mathrm{Map}_{\mathcal{C}}(X, Y)$ of 1-morphisms.

(3) An composition law for 1-morphisms which is associative (and unital) up to coherent isomorphism.

Alternatively, we can characterize the space $\mathcal{B}(M, N)$ up to homotopy equivalence by the following property: there exists a fiber bundle $p : E \to \mathcal{B}(M, N)$ whose fibers are (smooth) bordisms from M to N. This fiber bundle is universal in the following sense: for any reasonable space S, pullback of E determines a bijective correspondence between homotopy classes of maps from S into $\mathcal{B}(M, N)$ and fiber bundles $E' \to S$ whose fibers are (smooth) bordisms from M to N. In particular (taking S to consist of a single point), we deduce that the set of path components $\pi_0 \mathcal{B}(M, N)$ can be identified with the collection of *diffeomorphism classes* of bordisms from M to N. In other words, we have a bijection $\pi_0 \mathcal{B}(M, N) \simeq \mathrm{Hom}_{\mathbf{Cob}(n)}(M, N)$.

Definition Sketch 1.4.6. Let n be a nonnegative integer. The (∞, n)-category $\mathbf{Bord}_n$ is described informally as follows:

- The objects of $\mathbf{Bord}_n$ are 0-manifolds.
- The 1-morphisms of $\mathbf{Bord}_n$ are bordisms between 0-manifolds.
- The 2-morphisms of $\mathbf{Bord}_n$ are bordisms between bordisms between 0-manifolds.
- ...
- The n-morphisms of $\mathbf{Bord}_n$ are bordisms between bordisms between ... between bordisms between 0-manifolds (in other words, n-manifolds with corners).
- The $(n+1)$-morphisms of $\mathbf{Bord}_n$ are diffeomorphisms (which reduce to the identity on the boundaries of the relevant manifolds).
- The $(n+2)$-morphisms of $\mathbf{Bord}_n$ are isotopies of diffeomorphisms.
- ...

relate to those described in §1.2. The topological category $\mathbf{Cob}_t(n)$ of Definition 1.4.5 should really be regarded as an $(\infty, 1)$-category, which may be described more informally as follows:

- The objects of $\mathbf{Cob}_t(n)$ are closed, oriented $(n-1)$-manifolds.
- The 1-morphisms of $\mathbf{Cob}_t(n)$ are oriented bordisms.
- The 2-morphisms of $\mathbf{Cob}_t(n)$ are orientation-preserving diffeomorphisms.
- The 3-morphisms of $\mathbf{Cob}_t(n)$ are isotopies between diffeomorphisms.

Definition 2.1.3. The category $\mathbf{\Delta}$ of *combinatorial simplices* is defined as follows:

- The objects of $\mathbf{\Delta}$ are the nonnegative integers. For each $n \geq 0$, we let $[n]$ denote the corresponding object of $\mathbf{\Delta}$.
- Given a pair of integers $m, n \geq 0$, we define $\mathrm{Hom}_{\mathbf{\Delta}}([m], [n])$ to be the set of nonstrictly increasing maps $f : \{0 < 1 < \ldots < m\} \to \{0 < 1 < \ldots < n\}$.

Let $\mathbf{A}$ be an arbitrary category. A *simplicial object* of $\mathbf{A}$ is a functor from $\mathbf{\Delta}^{op}$ into $\mathbf{A}$.

Remark 2.1.4. We will typically let $A_\bullet$ denote a simplicial object of a category $\mathbf{A}$, and A_n the value of the functor $A_\bullet$ when evaluated at the object $[n] \in \mathbf{\Delta}$.

The most important special case of Definition 2.1.3 is the following:

Definition 2.1.5. A *simplicial set* is a simplicial object in the category of sets.

theory.

Definition 2.1.10. Let $f : X \to Z$ and $g : Y \to Z$ be continuous maps of topological spaces. The *homotopy fiber product* of $X \times_Z^R Y$ is the topological space

$$X \times_Z Z^{[0,1]} \times_Z Y$$

whose points consist of triples (x, y, p), where $x \in X$, $y \in Y$, and $p : [0,1] \to Z$ is a continuous path from $p(0) = f(x)$ to $p(1) = g(y)$.

Definition 2.1.13. Suppose given a commutative diagram of topological spaces

$$\begin{tikzcd} W \arrow[r] \arrow[d] & X \arrow[d] \\ Y \arrow[r] & Z. \end{tikzcd}$$

We say that this diagram is a *homotopy pullback square* (or a *homotopy Cartesian diagram*) if the composite map

$$W \to X \times_Z Y \to X \times_Z^R Y$$

is a weak homotopy equivalence.

Definition 2.1.15. Let $X_\bullet$ be a simplicial space. We say that $X_\bullet$ is a *Segal space* if the following condition is satisfied:

$(*)$ For every pair of integers $m, n \geq 0$, the diagram

$$\begin{tikzcd} X_{m+n} \arrow[r] \arrow[d] & X_m \arrow[d] \\ X_n \arrow[r] & X_0 \end{tikzcd}$$

is a homotopy pullback square.

Warning 2.1.16. Definition 2.1.15 is not completely standard. Some authors impose the additional requirement that the simplicial space $X_\bullet$ be *Reedy fibrant*: this is a harmless technical condition which guarantees, among other things, that each of the maps in the diagram

$$\begin{tikzcd} X_{n+m} \arrow[r] \arrow[d] & X_m \arrow[d] \\ X_n \arrow[r] & X_0 \end{tikzcd}$$

is a Serre fibration of topological spaces. If we assume this condition, then $X_\bullet$ is a Segal space if and only if each of the maps $X_{n+m} \to X_n \times_{X_0} X_m$ is a weak homotopy equivalence.

Example 2.1.21. Let $X_\bullet$ be a Segal space, and let $\delta : X_0 \to X_1$ be the "degeneracy map" induced by the unique nondecreasing function $\{0, 1\} \to \{0\}$. For every point x in X_0, the morphism $[\delta(x)]$ in the homotopy category $\mathrm{h}X_\bullet$ coincides with the identity map $\mathrm{id}_x : x \to x$. In particular, $\delta(x)$ is invertible for each $x \in X_0$.

Definition 2.1.22. Let $X_\bullet$ be a Segal space, and let $Z \subseteq X_1$ denote the subset consisting of the invertible elements (this is a union of path components in X_1; we will consider Z as endowed with the subspace topology). We will say that $X_\bullet$ is *complete* if the map $\delta : X_0 \to Z$ of Example 2.1.21 is a weak homotopy equivalence.

Roughly speaking, a Segal space $X_\bullet$ is complete if every isomorphism in the associated $(\infty, 1)$-category $\mathcal{C}$

$(\infty, 1)$-cat := Complete Segal Space

(∞, n)-cat := n-fold " " "

Let n be a positive integer, which we regard as fixed throughout this section. In §1.4, we argued that it is natural to replace the ordinary bordism category $\mathbf{Cob}(n)$ with an $(\infty, 1)$-category $\mathbf{Cob}_t(n)$, which encodes information about the homotopy types of diffeomorphism groups of n-manifolds. In §2.1, we introduced the notion of a *Segal space*, and argued that complete Segal spaces can be regarded as representatives for $(\infty, 1)$-categories. Our goal in this section is to unite these two lines of thought, giving an explicit construction of

Theorem 2.4.6 (Cobordism Hypothesis: Framed Version). *Let $\mathcal{C}$ be a symmetric monoidal (∞, n)-category with duals. Then the evaluation functor $Z \mapsto Z(*)$ induces an equivalence*

$$\mathrm{Fun}^{\otimes}(\mathbf{Bord}_n^{\mathrm{fr}}, \mathcal{C}) \to \mathcal{C}^{\sim}.$$

In particular, $\mathrm{Fun}^{\otimes}(\mathbf{Bord}_n^{\mathrm{fr}}, \mathcal{C})$ *is an $(\infty, 0)$-category.*

Remark 2.4.7. Theorem 2.4.6 is best regarded as comprised of two separate assertions:

$\mathrm{Fun}^{\otimes}$:= sym. mon. functors.

$\mathcal{C}^{\sim}$:= Discard noninvertible morphism

Example 2.4.15. Let $\mathcal{C}$ be a Picard ∞-groupoid (see Example 2.3.18). Using Thesis 1.3.8, we can identify $\mathcal{C}$ with a topological space X. The symmetric monoidal structure on $\mathcal{C}$ endows X with the structure of an E_∞*-space*: that is, it is equipped with a multiplication operation which is commutative, associative, and unital up to coherent homotopy. The assumption that every object of $\mathcal{C}$ be invertible translates into the requirement that X be *grouplike*: that is, the commutative monoid $\pi_0 X$ is actually an abelian group. It

$(d+1)$ whose boundary is diffeomorphic with $\overline{M} \coprod N$. Cobordism is an equivalence relation on manifolds, and the set of equivalence classes Ω_d has the structure of an abelian group (given by disjoint unions of manifolds). In [22], Thom showed that the calculation of the groups $\{\Omega_d\}_{d \geq 0}$ could be reduced to a problem of homotopy theory. More precisely, there exists a pointed topological space X and a sequence of isomorphisms $\Omega_d \simeq \pi_d X$. Moreover, the space X admits a direct construction in the language of algebraic topology: it is the 0th space of what is now callled the *Thom spectrum* MSO.

Definition 2.4.23. Let G be a topological group acting continuously on a topological space X. The *homotopy fixed set* X^{hG} is defined to be the space of G-equivariant maps $\mathrm{Hom}_G(EG, X)$, where EG is as in Notation 2.4.21.

main ideas; a detailed account will appear elsewhere.

For the reader's convenience, we begin by giving a basic summary of our strategy:

(1) To prove the cobordism hypothesis, we need to show that the (∞, n)-category $\mathbf{Bord}_n$ and its variants can be characterized by universal properties. The first idea is to try to establish these universal properties using induction on n. Roughly speaking, instead of trying to describe $\mathbf{Bord}_n$ by generators and relations, we begin by assuming that we have a similar presentation for $\mathbf{Bord}_{n-1}$; we are then reduced to describing only the generators and relations which need to be adjoined to pass from $\mathbf{Bord}_{n-1}$ to $\mathbf{Bord}_n$. We will carry out this reduction in §3.1.

(2) Theorem 2.4.18 gives us a description of $\mathbf{Bord}_n^{(X,\zeta)}$ for any topological space X and any rank n vector bundle ζ (with inner product) on X. In §3.2, we will see that it suffices to treat only the universal case where X is a classifying space $\mathrm{BO}(n)$ (and ζ is the tautological bundle on X). Roughly speaking, the idea is to consider a topological field theory $Z : \mathbf{Bord}_n^{(X,\zeta)} \to \mathcal{C}$ as an unoriented topological field theory having a different target category, whose value on a manifold M is a collection of $\mathcal{C}$-valued invariants parametrized by the space of (X, ζ)-structures on M. This reduction to the unoriented case is not logically necessary for the rest of the argument, but does result in some simplifications.

(3) In §3.3, we will explain how the cobordism hypothesis (and many other assertions regarding symmetric monoidal (∞, n)-categories with duals) can be reformulated entirely within the setting of $(\infty, 1)$-categories. Again, this formulation is probably not logically necessary, but it does make the constructions of §3.4 considerably more transparent.

(4) The bulk of the argument will be carried out in §3.4. Roughly speaking, we can view $\mathbf{Bord}_n$ as obtained from $\mathbf{Bord}_{n-1}$ by adjoining new n-morphisms corresponding to bordisms between $(n-1)$-manifolds. Using Morse theory, we can break any bordism up into a sequence of handle attachments, which give us "generators" for $\mathbf{Bord}_n$ relative to $\mathbf{Bord}_{n-1}$. The "relations" are given by handle cancellations. The key geometric input for our argument is a theorem of Igusa, which asserts that the space of "framed generalized Morse functions" on a manifold M is highly connected. This will allow us to prove the cobordism hypothesis for a modified version of the (∞, n)-category $\mathbf{Bord}_n$, which we will denote by $\mathbf{Bord}_n^{\mathrm{ff}}$.

(5) In §3.5, we will complete the proof of the cobordism hypothesis by showing that $\mathbf{Bord}_n^{\mathrm{ff}}$ is equivalent to $\mathbf{Bord}_n$. The key ingredients are a connectivity estimate of Igusa (Theorem 3.5.21) and an obstruction theoretic argument which relies on a cohomological calculation (Theorem 3.5.23) generalizing the work of Galatius, Madsen, Tillmann, and Weiss.

Step (i) involves delicate geometric arguments which are very specific to manifolds of dimension 2 (such as the Harer stability theorem). However, step (ii) has an analogue which is true in any dimension. Moreover, it is possible to be much more precise: we can describe not just the rational cohomology of the space $\Omega^2 |\mathbf{Bord}_2^{\mathrm{or}}|$, but the entire homotopy type of the classifying space $|\mathbf{Bord}_2^{\mathrm{or}}|$ itself:

Theorem 2.5.7 (Galatius-Madsen-Tillmann-Weiss, [11]). *Let $n \geq 0$ be an integer. Then the geometric realization $|\mathbf{Bord}_n^{\mathrm{or}}|$ is homotopy equivalent to the 0th space of the spectrum $\Sigma^n \mathrm{MTSO}(n)$. Here $\mathrm{MTSO}(n)$ denotes the Thom spectrum of the virtual bundle $-\zeta$, where ζ is the universal rank n-vector bundle over the classifying space $\mathrm{BSO}(n)$.*

calculates the derived endomorphisms of A as an A-bimodule. Another basic operation is the calculation of the universal trace (i.e., the universal target for a map out of A coequalizing left and right multiplication). The derived version of the universal trace is the Hochschild chain complex (or the Hochschild homology), which

The free loop space of a derived stack X is the internal hom $\mathcal{L}X = X^{S^1} = \mathrm{Map}(S^1, X)$ of maps from the constant stack given by the circle S^1. As a derived stack, the loop space may be described explicitly as the collection of pairs of points in X with two paths between them, or in other words, as the derived self-intersection of the diagonal

$$\mathcal{L}X \simeq X \times_{X \times X} X.$$

problem is that passing to homotopy categories discards essential information (in particular, homotopy coherent structures, homotopy limits and homotopy colimits).

This intermediate regime between model categories and homotopy categories is encoded by the theory of $(\infty,1)$-*categories*, or simply ∞-categories. The notion of ∞-category captures (roughly speaking) the notion of a category whose morphisms form topological spaces and whose compositions and associativity properties are defined up to coherent homotopies. Thus an important distinction between ∞-categories and model categories or homotopy categories is that coherent homotopies are naturally built in to all the definitions. Thus for example all functors are naturally derived and the natural notions of limits and colimits in the ∞-categorical context correspond to *homotopy* limits and colimits in more traditional formulations.

on Joyal's quasi-categories [Jo]. Namely, an ∞-category is a simplicial set, satisfying a weak version of the Kan condition guaranteeing the fillability of certain horns. The underlying simplicial set plays the role of the set of objects while the fillable horns correspond to sequences of composable morphisms. The book [L2] presents

2.1.1. *Enhancing triangulated categories.* The ∞-categorical analogue of the additive setting of homological algebra is the setting of stable ∞-categories [L3]. A stable ∞-category can be defined as an ∞-category with a zero-object, closed under finite limits and colimits, and in which pushouts and pullbacks coincide [L3, 2,4].

Definition 3.7. A stable category $\mathcal{C}$ is said to be *compactly generated* if there is a small ∞-category $\mathcal{C}^\circ$ of compact objects $C_i \in \mathcal{C}$ whose right orthogonal vanishes: if $M \in \mathcal{C}$ satisfies $\mathrm{Hom}_{\mathcal{C}}(C_i, M) \simeq 0$, for all i, then $M \simeq 0$.

chains, or its spectral analogue, topological Hochschild cohomology). In the case of algebras, Deligne's conjecture states that the Hochschild cochain complex has the structure of an $\mathcal{E}_2$-algebra (lifting the Gerstenhaber algebra, or $H_*(\mathcal{E}_2)$-algebra, structure on Hochschild cohomology). There is also a cyclic version of the conjecture which states that the Hochschild cochains for a *Frobenius* algebra possesses the further structure of a framed $\mathcal{E}_2$, or ribbon, algebra. See [T1, K1, KS1, MS, C, KS2]

for various proofs of the Deligne conjecture and [Kau, TZ] for its cyclic version. The Kontsevich conjecture (see [T2, HKV]) generalizes this picture to higher algebras, asserting that the Hochschild cochains on an $\mathcal{E}_n$-algebra have a natural $\mathcal{E}_{n+1}$-structure.

brief primer). Recall that stacks and higher stacks arise naturally from performing quotients (and more complicated colimits) on schemes. Thus we correct the notion of forming quotients by passing to stacks. Likewise, derived stacks arise naturally from taking fiber products (and more complicated limits) on schemes and stacks. Thus we correct the notion of imposing an equation by passing to derived stacks.

Cts functor:
Preserves colim

Definition 3.3.

(1) An object M of a stable ∞-category $\mathcal{C}$ is said to be *compact* if $\mathrm{Hom}_{\mathcal{C}}(M, -)$ commutes with all coproducts (equivalently, with all colimits).
(2) An object M of a stable symmetric monoidal ∞-category $\mathcal{C}$ is said to be (strongly) *dualizable* if there is an object $M^\vee$ and unit and trace maps

$$1 \xrightarrow{u} M \otimes M^\vee \xrightarrow{\tau} 1$$

such that the composite map

$$M \xrightarrow{u \otimes \mathrm{id}} M \otimes M^\vee \otimes M \xrightarrow{\mathrm{id} \otimes \tau} M$$

is the identity.

satisfying the appropriate conditions (since one already has an evaluation map). If an object $M \in \mathcal{C}$ is dualizable, then we can turn internal Hom from M into the tensor product with $M^\vee$ in the sense that there is a canonical equivalence

$$\mathcal{H}om(M, -) \simeq M^\vee \otimes (-).$$

In particular, this implies that $\mathcal{H}om(M, -)$ preserves colimits and $M \otimes -$ preserves limits: