Exercise X.8.3. Evaluate the integral

$$\int_C (z-2)^{-1}(2z+1)^{-2}(3z-1)^{-3}dz,$$

where C is the unit circle with the counterclockwise orientation. (Suggestion: Try to use the preceding exercise.)

Exercise X.8.4. Evaluate the integrals

$$\int_C \frac{\sin z}{2z^2-5z+2}dz, \qquad \int_C \frac{2z^2-5z+2}{\sin z}dz$$

where C is the unit circle with the counterclockwise orientation.

Exercise X.8.5. Extend the residue theorem to the case where the function f has countably many isolated singularities in the domain G.

X.9. Cauchy's Formula

Let G be a domain in $\mathbf{C}$, f a holomorphic function in G, and Γ a simple contour contained with its interior in G. Then

$$f(z_0) = \frac{1}{2\pi i}\int_\Gamma \frac{f(z)}{z-z_0}dz, \qquad z_0 \in \operatorname{int}\Gamma.$$

This is an immediate consequence of the residue theorem: under the given hypotheses, the integrand in the preceding integral is holomorphic in $G\backslash\{z_0\}$, and its residue at z_0 is $f(z_0)$.

Cauchy's formula for the n-th derivative ($n = 1, 2, 3, \ldots$) is a consequence: under the given hypothesis:

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_\Gamma \frac{f(z)}{(z-z_0)^{n+1}}dz, \qquad z_0 \in \operatorname{int}\Gamma.$$

The deduction is contained in the discussion of Cauchy integrals in Section VII.7.

X.10. More Definite Integrals

The exploitation of complex integration to evaluate improper Riemann integrals has been illustrated in Sections VI.12 and VII.4. Now that we have further developed the theory, we can handle many additional integrals. The residue theorem is an especially powerful tool for this purpose.

Example 1. $\displaystyle\int_0^\infty \frac{\sin x}{x}dx.$

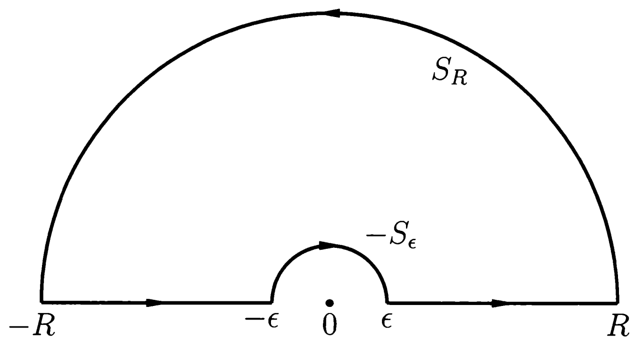

Figure 8. The contour $\Gamma_{\epsilon,R}$ for Example 1.

This is an important integral in Fourier analysis. It is not obvious that the integral converges. Although the integrand is well behaved at the left endpoint of the interval of integration, there is potential trouble at the far end, because $\int_0^\infty \left|\frac{\sin x}{x}\right| dx$ diverges (Exercise X.10.1 below). Nevertheless, as we shall prove on the basis of Cauchy's theorem, the given integral does converge.

For $r > 0$, let S_r denote the semicircle in the upper half-plane with center 0 and radius r, oriented counterclockwise. For $0 < \epsilon < R$, let $\Gamma_{\epsilon,R}$ denote the closed curve consisting of the interval $[\epsilon, R]$ followed by the semicircle S_R followed by the interval $[-R, -\epsilon]$ followed by the semicircle $-S_\epsilon$ (see Figure 8). The function $\frac{e^{iz}}{z}$, whose imaginary part on the real axis equals $\frac{\sin x}{x}$, is holomorphic in $\mathbf{C}\backslash\{0\}$, a domain containing $\Gamma_{\epsilon,R}$ and its interior. By Cauchy's theorem,

$$\int_{\Gamma_{\epsilon,R}} \frac{e^{iz}}{z} dz = 0.$$

(We are using here the general Cauchy theorem. By an argument like that in Section VII.5, the perverse reader could draw the same conclusion using only Cauchy's theorem for convex domains. Now that we have the general Cauchy theorem, such contortions are unnecessary.)

On the real axis, the real part of $\frac{e^{ix}}{x}$ is $\frac{\cos x}{x}$, an odd function (undefined at 0), and the imaginary part, as already noted, is $\frac{\sin x}{x}$, an even function. The preceding equality therefore implies that

$$(*) \qquad 2i \int_\epsilon^R \frac{\sin x}{x} dx = -\int_{S_R} \frac{e^{iz}}{z} dz + \int_{S_\epsilon} \frac{e^{iz}}{z} dz.$$

We shall now take the limit as $R \to \infty$ and $\epsilon \to 0$.

Consider first the integral over S_R. We shall prove that it tends to 0 as $R \to \infty$. This is intuitively plausible, because, in the upper half-plane, $|e^{iz}|$ is very small except in the vicinity of the real axis. Introducing the parametrization $t \mapsto Re^{it}$ $(0 \le t \le \pi)$ of S_R, we can write

$$\int_{S_R} \frac{e^{iz}}{z} dz = i \int_0^\pi e^{iRe^{it}} dt,$$

and so

$$\begin{aligned}\left| \int_{S_R} \frac{e^{iz}}{z} dz \right| &\le \int_0^\pi \left| e^{iRe^{it}} \right| dt \\ &= \int_0^\pi e^{-R\sin t} dt \\ &= 2 \int_0^{\pi/2} e^{-R\sin t} dt.\end{aligned}$$

Using the inequality $\sin t \ge 2t/\pi$ $(0 \le t \le \pi/2)$, we obtain

$$\begin{aligned}\left| \int_{S_R} \frac{e^{iz}}{z} dz \right| &\le 2 \int_0^{\pi/2} e^{-2Rt/\pi} dt \\ &= \frac{\pi}{R}\left(1 - e^{-R}\right).\end{aligned}$$

Because the right side tends to 0 as $R \to \infty$, we can conclude that

$$\lim_{R\to\infty} \int_{S_R} \frac{e^{iz}}{z} dz = 0,$$

as desired.

Consider now the integral over S_ϵ. The function $f(z) = \dfrac{e^{iz} - 1}{z}$ has a removable singularity at the origin; in particular, it stays bounded near the origin. From this, one easily concludes that

$$\lim_{\epsilon\to 0} \int_{S_\epsilon} f(z) dz = 0.$$

Since $\dfrac{e^{iz}}{z} = \dfrac{1}{z} + f(z)$ and

$$\int_{S_\epsilon} \frac{1}{z} dz = \pi i$$

for all ϵ (by a straightforward calculation), we obtain

$$\lim_{\epsilon\to 0} \int_{S_\epsilon} \frac{e^{iz}}{z} dz = \pi i.$$

Combining the two limits just found with the equality $(*)$, we obtain

$$\lim_{\substack{\epsilon\to 0\\ R\to\infty}} \int_{\epsilon}^{R} \frac{\sin x}{x}\,dx = \frac{\pi}{2}.$$

In other words,

$$\int_0^{\infty} \frac{\sin x}{x}\,dx = \frac{\pi}{2}.$$

Example 2. $\displaystyle\int_{-\infty}^{\infty} \frac{\cos x}{1+x^4}\,dx.$

In this example we can see at the outset, using a comparison test, that the integral converges. (Details are left to the reader; prior knowledge of convergence will not be used below.) As in the preceding example, we make use of the function e^{iz}, whose absolute value in the upper half-plane is bounded by 1. The integrand in our integral coincides on the real axis with the real part of the function $\dfrac{e^{iz}}{1+z^4}$. That function is holomorphic in $\mathbf{C}$ except for simple poles at the points $e^{\pi i n/4}$, $n = 1, 3, 5, 7$ (the fourth roots of -1).

For $R > 1$, let Γ_R denote the closed curve consisting of the interval $[-R, R]$ followed by the semicircle S_R (defined as in the preceding example). The singularities of the function $\dfrac{e^{iz}}{1+z^4}$ in the interior of Γ_R are the points $z_1 = e^{\pi i/4}$ and $z_2 = e^{3\pi i/4}$. By the residue theorem,

$$\int_{\Gamma_R} \frac{e^{iz}}{1+z^4}\,dz = 2\pi i\left[\operatorname{res}_{z=z_1}\left(\frac{e^{iz}}{1+z^4}\right) + \operatorname{res}_{z=z_2}\left(\frac{e^{iz}}{1+z^4}\right)\right].$$

To evaluate the residues we use the result of Exercise VIII.12.1: if the functions g and h are holomorphic in an open set containing the point z_0 and h has a simple zero at z_0, then $\operatorname{res}_{z_0} \dfrac{g}{h} = \dfrac{g(z_0)}{h'(z_0)}$. We thus have

$$\begin{aligned}
\operatorname{res}_{z=z_1}\left(\frac{e^{iz}}{1+z^4}\right) &= \frac{\exp(iz_1)}{4z_1^3} \\
&= \frac{\exp\left(-\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)}{4\left(-\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)} \\
&= \frac{1}{4}\left(-\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right)e^{-1/\sqrt{2}}\left(\cos\frac{1}{\sqrt{2}} + i\sin\frac{1}{\sqrt{2}}\right),
\end{aligned}$$

$$\begin{aligned}
\operatorname{res}_{z=z_2}\left(\frac{e^{iz}}{1+z^4}\right) &= \frac{\exp(iz_2)}{4z_2^3} \\
&= \frac{\exp\left(-\frac{1}{\sqrt{2}}-\frac{i}{\sqrt{2}}\right)}{4\left(\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}\right)} \\
&= \frac{1}{4}\left(\frac{1}{\sqrt{2}}-\frac{i}{\sqrt{2}}\right)e^{-1/\sqrt{2}}\left(\cos\frac{1}{\sqrt{2}}-i\sin\frac{1}{\sqrt{2}}\right).
\end{aligned}$$

The residue at z_2 is the negative of the complex conjugate of the residue at z_1, so that the sum of the two residues equals $2i$ times the imaginary part of the residue at z_1. The sum of the two residues therefore equals

$$\frac{i\sqrt{2}}{4}e^{-1/\sqrt{2}}\left(-\cos\frac{1}{\sqrt{2}}-\sin\frac{1}{\sqrt{2}}\right).$$

It follows that

$$\int_{\Gamma_R}\frac{e^{iz}}{1+z^4}dz = \frac{\pi\sqrt{2}}{2}e^{-1/\sqrt{2}}\left(\cos\frac{1}{\sqrt{2}}+\sin\frac{1}{\sqrt{2}}\right).$$

On the real axis, the imaginary part of $\dfrac{e^{ix}}{1+x^4}$ is an odd function. Thus,

$$\int_{\Gamma_R}\frac{e^{iz}}{1+z^4}dz = \int_{-R}^{R}\frac{\cos x}{1+x^4}dx + \int_{S_R}\frac{e^{iz}}{1+z^4}dz.$$

This in combination with the preceding equality gives

$$\int_{-R}^{R}\frac{\cos x}{1+x^4}dx = \frac{\pi\sqrt{2}}{2}e^{-1/\sqrt{2}}\left(\cos\frac{1}{\sqrt{2}}+\sin\frac{1}{\sqrt{2}}\right) - \int_{S_R}\frac{e^{iz}}{1+z^4}dz.$$

We now take the limit as $R\to\infty$. The absolute value of the integrand in the integral over S_R is bounded by $\dfrac{1}{R^4-1}$, so the integral itself is bounded in absolute value by $\dfrac{\pi R}{R^4-1}$, which tends to 0 as $R\to\infty$. We can conclude that

$$\int_{-\infty}^{\infty}\frac{\cos x}{1+x^4}dx = \frac{\pi\sqrt{2}}{2}e^{-1/\sqrt{2}}\left(\cos\frac{1}{\sqrt{2}}+\sin\frac{1}{\sqrt{2}}\right).$$

Example 3. $\displaystyle\int_0^\infty \frac{1}{1+x^a}dx, \quad a>1.$

As in the last example, one can see at the outset, by means of a comparison test, that the integral converges, although that knowledge will not be required below. In the domain $G=\{re^{i\theta}: r>0, \left|\theta-\frac{\pi}{a}\right|<\pi\}$ we consider the branch of the function z^a that takes the value 1 at the point $z=1$. We

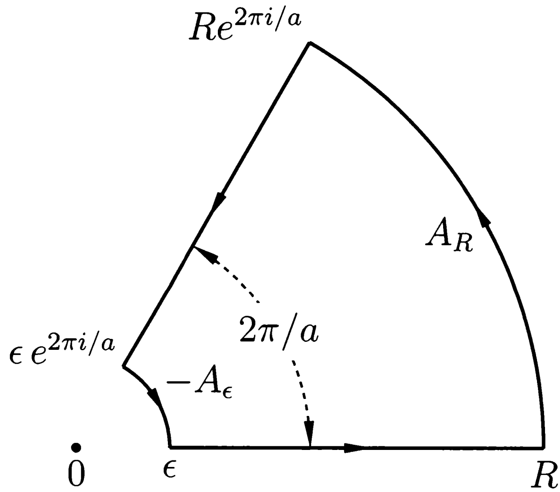

Figure 9. The contour $\Gamma_{\epsilon,R}$ for Example 3.

denote this function simply by z^a. The function $\dfrac{1}{1+z^a}$ is holomorphic in G except for a simple pole at the point $z_0 = e^{\pi i/a}$.

For $r > 0$, let A_r denote the circular arc $\left\{re^{i\theta} : 0 < \theta < \frac{2\pi}{a}\right\}$, oriented counterclockwise. For $0 < \epsilon < 1 < R$, let $\Gamma_{\epsilon,R}$ denote the closed curve consisting of the interval $[\epsilon, R]$ followed by the arc A_R followed by the segment $\left[Re^{2\pi i/a}, \epsilon e^{2\pi i/a}\right]$ followed by the arc $-A_\epsilon$ (see Figure 9). By the residue theorem,

$$\int_{\Gamma_{\epsilon,R}} \frac{1}{1+z^a}dz \;=\; 2\pi i \operatorname{res}_{z=z_0}\left(\frac{1}{1+z^a}\right).$$

We can compute the residue by the same method we used in the preceding example. We obtain

$$\operatorname{res}_{z=z_0}\left(\frac{1}{1+z^a}\right) \;=\; \frac{1}{az_0^{a-1}} \;=\; \frac{-e^{\pi i/a}}{a}.$$

Thus

$$\int_{\Gamma_{\epsilon,R}} \frac{1}{1+z^a}dz \;=\; \frac{-2\pi i e^{\pi i/a}}{a}.$$

Now, the integral of $\dfrac{1}{1+z^a}$ over the interval $[\epsilon, R]$ equals $\displaystyle\int_\epsilon^R \frac{1}{1+x^a}dx$, and the integral of the same function along the segment $\left[\epsilon e^{2\pi i/a}, Re^{2\pi i/a}\right]$ equals $e^{2\pi i/a}\displaystyle\int_\epsilon^R \frac{1}{1+x^a}dx$ (as one sees by using the parametrization $t \mapsto$

$te^{2\pi i/a}$ ($\epsilon \le t \le R$)). We can therefore rewrite the preceding equality as

$$\left(1 - e^{2\pi i/a}\right)\int_\epsilon^R \frac{1}{1+x^a}dx = \frac{-2\pi i e^{\pi i/a}}{a} + \int_{A_\epsilon}\frac{1}{1+z^a}dz - \int_{A_R}\frac{1}{1+z^a}dz.$$

We now take the limit as $\epsilon \to 0$ and $R \to \infty$. The integral over A_ϵ clearly tends to 0 (since the integrand stays bounded). In the integral over A_R, the integrand is bounded in absolute value by $\frac{1}{R^a - 1}$, so the integral itself is bounded in absolute value by $\frac{2\pi R}{a(R^a-1)}$, which tends to 0 as $R \to \infty$. In the limit we thus obtain

$$\left(1 - e^{2\pi i/a}\right)\int_0^\infty \frac{1}{1+x^a}dx = \frac{-2\pi i e^{\pi i/a}}{a},$$

giving (after a short calculation)

$$\int_0^\infty \frac{1}{1+x^a}dx = \frac{\pi}{a\sin\frac{\pi}{a}}.$$

Exercise X.10.1. Prove that the integral $\int_0^\infty \left|\frac{\sin x}{x}\right| dx$ diverges.

Exercise X.10.2. Derive the formula

$$\int_{-\infty}^\infty \frac{\cos x}{\cosh x}dx = \frac{\pi}{\cosh\frac{\pi}{2}}$$

by integrating the function $\frac{e^{iz}}{\cosh z}$ around the rectangle with vertices $-R$, R, $R + \pi i$, $-R + \pi i$, and letting $R \to \infty$.

Exercise X.10.3. Evaluate $\int_0^\infty \frac{\sin^2 x}{x^2}dx$ by integrating the function $\frac{1-e^{2iz}}{z^2}$ around the curves $\Gamma_{\epsilon,R}$ used in Example 1, and taking the limit as $\epsilon \to 0$ and $R \to \infty$.

Exercise X.10.4. Derive the formula

$$\int_0^\infty \frac{\cos ax}{(x^2+b^2)^2}dx = \frac{\pi}{4b^3}(1+ab)e^{-ab}, \qquad a \ge 0,\ b > 0.$$

Exercise X.10.5. Derive the formula

$$\int_0^\infty \frac{x^{a-1}}{1+x^2}dx = \frac{\pi}{2\sin\left(\frac{\pi a}{2}\right)}, \qquad 0 < a < 2.$$

Exercise X.10.6. Evaluate

$$\int_0^{2\pi} \frac{\cos\theta}{a - \cos\theta}d\theta, \qquad a > 1.$$

Exercise X.10.7. Let the function $f = u + iv$ be holomorphic in a domain containing the closed unit disk. Derive the relations

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z} f(e^{i\theta})d\theta \;=\; 2f(z) - f(0), \qquad |z| < 1,$$

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z} \overline{f(e^{i\theta})}d\theta \;=\; \overline{f(0)}, \qquad |z| < 1.$$

From these deduce Herglotz's formula,

$$f(z) \;=\; \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z} u(e^{i\theta})d\theta + iv(0), \qquad |z| < 1,$$

and Poisson's formula,

$$u(z) \;=\; \frac{1}{2\pi}\int_0^{2\pi} \frac{1-|z|^2}{|e^{i\theta}-z|^2} u(e^{i\theta})d\theta, \qquad |z| < 1.$$

(These formulas are of central importance in more advanced function theory.)

X.11. The Argument Principle

Let G be a domain in $\mathbf{C}$ and Γ a simple contour contained with its interior in G. Let f be a holomorphic function in G without zeros on Γ. Then the number of zeros of f in int Γ, taking into account multiplicities, equals

$$\frac{1}{2\pi i}\int_\Gamma \frac{f'(z)}{f(z)}dz.$$

From the discussion in Section IX.3 we know that

$$\frac{1}{i}\int_\Gamma \frac{f'(z)}{f(z)}dz$$

equals the increment experienced by arg $f(z)$ as z makes one circuit of Γ. The argument principle thus states that each zero of f in int Γ accounts for 2π of this increment. Multiple zeros, as indicated in the statement of the principle, are counted as many times as required by their multiplicities.

To establish the argument principle we first note that f can have only finitely many zeros in int Γ. In fact, the set ext Γ is open and contains both $\mathbf{C}\backslash G$ and the unbounded connected component of $\mathbf{C}\backslash\Gamma$. The set $\mathbf{C}\backslash$ ext Γ is thus a closed and bounded (i.e., compact) subset of G. Since the zero set of f has no limit points in G (see Section VII.13), there can be only finitely many zeros of f in $\mathbf{C}\backslash$ ext Γ, and thus there are only finitely many zeros of f in int Γ, as asserted.

Suppose $z_1, \ldots, z_p$ are the zeros of f in int Γ, and let $m_1, \ldots, m_p$ be their respective orders. Let G_1 be the domain obtained by removing from G the zeros of f in ext Γ. Then Γ and its interior are contained in G_1, where the function f'/f is holomorphic except for simple poles at the points $z_1, \dots, z_p$. By the residue theorem,

$$\int_\Gamma \frac{f'(z)}{f(z)} dz = 2\pi i \sum_{k=1}^{p} res_{z_k}\left(\frac{f'}{f}\right).$$

A simple calculation (requested as Exercise VIII.12.3) shows that

$$res_{z_k}\left(\frac{f'}{f}\right) = m_k,$$

and the desired equality follows.

Exercise X.11.1. Evaluate

$$\frac{1}{2\pi i}\int_C \frac{z^{n-1}}{3z^n - 1} dz,$$

where n is a positive integer, and C is the unit circle with the counterclockwise orientation.

Exercise X.11.2. (Argument principle for meromorphic functions.) Let G be a domain in $\mathbf{C}$ and Γ a simple contour contained with its interior in G. Let the function f be holomorphic in G except for isolated poles. (Such an f is said to be meromorphic in G.) Prove that if f has neither zeros nor poles on Γ, then

$$\frac{1}{2\pi i}\int_\Gamma \frac{f'(z)}{f(z)} dz$$

equals the number of zeros that f has in int Γ minus the number of poles that f has in int Γ, taking account of multiplicities.

Exercise X.11.3. Let the function f be holomorphic in a domain containing the closed unit disk. Prove that the increment in arg f around the unit circle equals 2π times the total number of zeros of f in the closed unit disk, provided zeros on the unit circle are counted with one-half their multiplicities. (Part of the problem is to make a reasonable definition of the increment in arg f.)

X.12. Rouché's Theorem

Let G be a domain in $\mathbf{C}$, K a compact subset of G, and f and g holomorphic functions in G such that $|f(z) - g(z)| < |f(z)|$ for every point z in the boundary of K. Then f and g have the same number of zeros in the interior of K, taking into account multiplicities.

Notice that the hypotheses imply that neither f nor g vanishes on the boundary of K. Roughly, the theorem states that the number of zeros a