# Flat families of schemes - Algebraic Geometry exam

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All rings mentioned are unitary and commutative. To indicate that  $I \subseteq R$  is an ideal of the ring R, we will often use the notation  $I \triangleleft R$ . For other conventions used throughout, see [Har77].

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## 1 Flatness

The notion of *flatness* was introduced by J. P. Serre in his famous paper [Ser56].

**Definition 1.1.** Let R be a ring. A R-module M is flat if the functor  $\otimes_R M : \mathfrak{Mod}_R \to \mathfrak{Mod}_R$  is exact.

*Remark.* Recall that  $\otimes_R M$  is always right exact, so the only thing to test to check flatness of a R-module M is that for any exact sequence  $0 \to N_1 \to N_2$  also  $0 \to N_1 \otimes_R M \to N_2 \otimes_R M$  is exact.

A simple example of a flat R-module is R itself, or more in general any free R-module.

**Example 1.2.** Let S be a multiplicatively closed subset of the ring R, and consider the R-module  $S^{-1}R$ , the localization of R aver S. Then  $S^{-1}R$  is a flat R-module. Indeed, consider an exact sequence of R-modules  $0 \to M \to N$ , and assume that an element  $\frac{x}{s}$  of  $S^{-1}M$  becomes 0 in  $S^{-1}N$ . By definition this means that there is some  $s' \in S$  such that s'x = 0 in N, but since M is a R-submodule of N, we also have s'x = 0 in M. But then  $\frac{x}{s}$  was already 0 in  $S^{-1}M$ , thus  $0 \to S^{-1}M \to S^{-1}N$  is exact.

*Remark.* If M is a flat R-module and  $t \in R$  is not a zero divisor, then  $M \xrightarrow{\cdot t} M$  is injective; indeed, since  $0 \to R \xrightarrow{\cdot t} R$  is exact, also  $0 \to R \otimes_R M \xrightarrow{\cdot t} R \otimes_R M$  is.

**Lemma 1.3.** For a *R*-module *M*, the following are equivalent:

- 1. M is a flat R-module;
- 2. for every ideal  $I \triangleleft R$ , the map  $I \otimes_R M \rightarrow M$  defined by  $j \otimes m \mapsto jm$  is injective.

This lemma gives us an equivalent definition of flatness in terms of ideals of R; as we shall see, this will be an important tool in studying properties of flatness.

*Proof.* The implication  $1 \Rightarrow 2$  is immediate, so we just prove that 2 implies 1. Let  $f : N_1 \to N_2$  be an injective morphism of R-modules, and consider  $f \otimes id : N_1 \otimes_R M \to N_2 \otimes_R M$ . To prove that  $f \otimes id$  is injective it is of course sufficient to show that it is injective when restricted to any submodule of the form  $N'_1 \otimes_R M$ , where  $N'_1$  is a finitely generated R-submodule of  $N_1$ . From now on we may assume then that  $N_1$  and  $N_2$  are finitely generated R-modules, and that  $f : N_1 \to N_2$  is the inclusion map.

In this context there are  $n \in \mathbb{Z}_{>0}$  and *R*-submodules  $L_2 \subseteq L_1 \subseteq \mathbb{R}^n$  such that  $N_2 = \mathbb{R}^n/L_2$ ,  $N_1 = L_1/L_2$ . We wish to show that

$$\frac{L_1}{L_2} \otimes_R M \longrightarrow \frac{R^n}{L_2} \otimes_R M$$

is injective. Notice that there are natural maps  $L_2 \otimes_R M \to R^n \otimes_R M \cong M^n$  and  $L_1 \otimes_R M \to M^n$ ; if we manage to show that these are injective, then also  $\frac{L_1}{L_2} \otimes_R M \to \frac{R^n}{L_2} \otimes_R M$  will be injective, since under this hypothesis  $\frac{L_1}{L_2} \otimes_R M \cong \frac{L_1 \otimes_R M}{L_2 \otimes_R M}$ ,  $\frac{R^n}{L_2} \otimes_R M \cong \frac{M^n}{L_2 \otimes_R M}$  and these isomorphisms are such that

$$\begin{array}{cccc} \underbrace{L_1}{L_2} \otimes_R M & \longrightarrow & \underbrace{R^n}{L_2} \otimes_R M \\ & & & \downarrow \wr & & & \downarrow \wr \\ & & & & \downarrow \iota \\ \underbrace{L_1 \otimes_R M}{L_2 \otimes_R M} & \longrightarrow & \underbrace{M^n}{L_2 \otimes_R M} \end{array}$$

is commutative. To conclude then it will be enough to prove that for any *R*-submodule *K* of  $\mathbb{R}^n$ , the natural map  $K \otimes_{\mathbb{R}} M \to M^n$  is injective. We do this by induction on *n*. For n = 1 *K* is an ideal in *R*, so  $K \otimes_{\mathbb{R}} M \to M$  is injective by our hypothesis. Assume now that n > 1 and that the thesis is valid for n - 1. Define

$$K' = K \cap \left(\underbrace{\{0\} \times \dots \times \{0\}}_{n-1 \text{ times}} \times R\right)$$

so that  $K/K' \subseteq \mathbb{R}^{n-1}$ . Then we have the following commutative diagram with exact rows:

tensoring with M we get another diagram with exact rows

where the first and third vertical arrows are injections, by our induction hypothesis and the base case. Then it follows that also the vertical arrow in the middle is an injection, so we have the thesis.  $\Box$ 

*Remark.* Notice that condition 2 in Lemma 1.3 is actually equivalent to

2'. for every finitely generated  $I \triangleleft R$ , the map  $I \otimes_R M \rightarrow M$  is injective.

Indeed, if J is an ideal of R (not necessarily finitely generated) and  $m \in \ker(J \otimes_R M \to M)$ , there is a finitely generated  $I \triangleleft R$  such that  $I \subseteq J$  and  $m \in \ker(I \otimes_R M \to M)$ . Assuming that 2' holds we deduce that m = 0.

**Lemma 1.4.** Let R be a principal ideal domain. Then a R-module M is flat if and only if it is torsion free.

*Proof.* By Lemma 1.3, M is flat if and only if for every  $\mathfrak{a} \triangleleft R$ ,  $\mathfrak{a} \otimes M \to M$  is injective. Since R is a PID, this is equivalent to saying that for all  $t \in R$ , the multiplication  $M \xrightarrow{t} M$  is injective, i.e. M is torsion-free.

**Proposition 1.5.** Let R be a ring, and let  $0 \to M' \to M \to M'' \to 0$  be a short exact sequence of R-modules. If M'' is flat, then for any R-module N the sequence  $0 \to N \otimes_R M' \to N \otimes_R M \to N \otimes_R M'' \to 0$  is exact.

*Proof.* Let N be a R-module, and fix a short exact sequence of R-modules  $0 \to K \to F \to N \to 0$  with F a free R-module; then we can define the following commutative diagram:



and notice that is has exact rows and columns. Indeed, the last column is exact because M'' is flat, and the middle row is exact since F is free (hence flat). From this diagram it can be easily checked that  $N \otimes_R M' \to N \otimes_R M$  is injective by diagram chasing.

A corollary of this is that flatness is stable by extensions. The proof of this is again by diagram chasing.

**Corollary 1.6** (Stability by extensions). Let  $0 \to M' \to M \to M'' \to 0$  be an exact sequence of *R*-modules. If M' and M'' are both flat, then also M is; moreover, if M and M'' are flat, then also M' is.

**Proposition 1.7** (Base change and transitivity). Let M be a flat A-module. Then

- 1. if  $\varphi : A \to B$  is a ring homomorphism, then  $M \otimes_A B$  is a flat B-module;
- 2. if  $\psi: R \to A$  makes A into a flat R-module, then M considered as a R-module is also flat.

*Proof.* For the first claim, let N be any B-module and notice that  $N \otimes_B (M \otimes_A B) \cong N \otimes_A M$ , if we consider N as an A-module via  $\varphi$ . Then, if  $0 \to N_1 \to N_2$  is an exact sequence of B-modules from the flatness of M it follows that also  $0 \to N_1 \otimes_B (M \otimes_A B) \to N_2 \otimes_B (M \otimes_A B)$  is exact.

For the second claim, we can proceed analogously. It is simply enough to notice that if N is a R-module then  $N \otimes_R M \cong N \otimes_R (A \otimes_A M) \cong (N \otimes_R A) \otimes_A M$ .

**Proposition 1.8** (Local criteria). A *R*-module *M* is flat if and only if  $M_{\mathfrak{p}}$  is a flat  $R_{\mathfrak{p}}$ -module for every  $\mathfrak{p} \in \operatorname{Spec}(R)$ . Moreover, a finitely generated module over a local Noetherian ring is flat if and only if it is free.

*Proof.* Proposition 3.G in [Mat87] and Proposition 3.J in [Mat87].

#### **1.1** Flat morphisms of schemes

**Definition 1.9.** Let  $f: X \to Y$  be a morphism of schemes, and let  $\mathscr{F}$  be a sheaf of  $\mathcal{O}_X$ -modules. We say that  $\mathscr{F}$  is f-flat at  $x \in X$  if the stalk  $\mathscr{F}_x$ , seen as a  $\mathcal{O}_{Y,f(x)}$ -module via  $f^{\sharp}: \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ , is flat. If  $\mathscr{F}$  is f-flat at every  $x \in X$ , we say that  $\mathscr{F}$  is f-flat. We will also say that  $\mathscr{F}$  is f-flat at some  $y \in Y$  if it is f-flat at each  $x \in X$  such that f(x) = y.

We are particularly interested in the case  $\mathscr{F} = \mathcal{O}_X$ ; in this case we will simply say that f is flat at  $x \in X$  or at  $y \in Y$ . It's also quite common to use phrases like "let  $X \to Y$  be a flat family of schemes", without naming the morphism.

The idea behind this condition is that a flat morphism of schemes  $X \to Y$  describes a family of schemes parametrized by the base Y which is, in some sense, *continuous*. Why this is the case seems to be a bit of a mystery (these are Hartshorne's words), but nonetheless flat morphisms exhibit a number of nice properties that we would expect from a continuous family of geometric objects.

The previous definition describes actually a *relative* notion of flatness. The corresponding absolute (i.e. that does not depend on a chosen morphism) definition is the following: a sheaf of  $\mathcal{O}_X$ -modules  $\mathscr{F}$  is *flat* if and only if each stalk  $\mathscr{F}_x$  is a flat  $\mathcal{O}_{X,x}$ -module.

**Lemma 1.10.** Let X be a Noetherian scheme, and let  $\mathscr{F}$  be a coherent sheaf of  $\mathcal{O}_X$ -modules. Then  $\mathscr{F}$  is flat if and only if it is locally free.

As usual, the notion of flatness for affine schemes is completely algebraic.

**Lemma 1.11.** Let  $\varphi : A \to B$  be a morphism of rings, and let  $f : \operatorname{Spec}(B) \to \operatorname{Spec}(A)$  be the corresponding map of schemes. Let M be a B-module. Then  $\tilde{M}$  is f-flat if and only if M is a flat A-module.

As a partial justification for the definition of flatness, consider an affine scheme X = Spec(A)of finite type over an algebraically closed field k. Any ideal  $I \triangleleft A$  defines the closed subscheme Z = Spec(A/I). Let  $\tilde{M}$  be a quasicoherent sheaf on X; as usual, we may regard sections of  $\tilde{M}$  as "*M*-valued functions" on X. The restriction of  $\tilde{M}$  to Z is the coherent sheaf on Z defined by the A/I-module  $M \otimes_A A/I$ , and because M is A-flat we know that  $M \otimes_A A/I \cong M/(IM)$ . In other words, a (local) section of M over X restricts to 0 in Z if and only if over Z it took values in IM. Thus we can say that  $\tilde{M}$  is flat when its sections "restrict nicely" to closed subsets of X.

#### 1.1.1 Generic flatness and the open nature of flatness

Now we show that, under some reasonable hypothesis, the failure of flatness can only happen on a "small" subset of the base. This is a consequence of the "Generic freeness Lemma" of Grothendieck [GD65, Lemme 6.9.2].

**Theorem 1.12** (Generic freeness). Let A be an integral Noetherian ring, B a finitely generated Aalgebra and M a finitely generated B-module. Then there is some  $f \in A \setminus \{0\}$  such that  $M_f$  is a free  $A_f$ -module.

As a corollary we easily get that each sufficiently regular morphism of schemes is "generically flat".

**Corollary 1.13** (Generic flatness). Let  $f : X \to Y$  be a morphism of finite type of schemes, with Y locally Noetherian and integral. For any coherent  $\mathcal{O}_X$ -module  $\mathscr{F}$ , there is an open set  $U \subseteq Y$  such that  $\mathscr{F}_{\uparrow f^{-1}(U)}$  is flat over U.

*Proof.* By taking an affine open cover of Y, we see that it will be enough to prove the theorem in the case when Y is affine. Assume then that Y = Spec(A) for a Noetherian integral domain A. Since f is of finite type, we can cover X by a finite number of open affine sets  $\{\text{Spec}(B_i) \mid i = 1, ..., N\}$  such that each  $B_i$  is a finitely generated A-algebra.

If we prove that for each i = 1, ..., N there is an open set  $U_i \subseteq Y$  such that  $\mathscr{F}_{|\operatorname{Spec}(B_i) \cap f^{-1}(U_i)}$  is flat, then it will be enough to take  $U = \bigcap_{i=1}^N U_i$  to get an open set with the desired property. Hence we may also assume that  $X = \operatorname{Spec}(B)$  for some finitely generated A-algebra B. By assumption there is a finitely generated *B*-module *M* such that  $\mathscr{F} = \tilde{M}$ . Then Theorem 1.12 tells us that there is some nonzero  $a \in A$  such that  $M_a$  is a free  $A_a$ -module, and this is precisely what we had to prove.

A result that has a similar flavour is that the set of points at which a coherent sheaf is flat is open, c.f. [GD66, Théorème 11.1.1].

**Proposition 1.14** (Open nature of flatness). Let  $f : X \to Y$  be a morphism locally of finite type, with Y locally Noetherian, and let  $\mathscr{F}$  be a coherent  $\mathcal{O}_X$ -module. Then  $\{x \in X \mid \mathscr{F} \text{ is } f - \text{flat at } x\}$  is an open subset of X.

Since this is again a local question, it is enough to prove it for X = Spec(B), Y = Spec(A),  $\mathscr{F} = \tilde{M}$ for a finitely generated *B*-module *M* and  $f^{\sharp} : A \to B$  making *B* into a finitely generated *A*-algebra. Then the proposition follows again by an algebraic lemma.

**Lemma 1.15.** Let A be a Noetherian ring, B a finitely generated A-algebra, M a finitely generated B-module. choose  $q \in \operatorname{Spec}(B)$ , and let  $\mathfrak{p} \triangleleft A$  be the preimage of  $\mathfrak{q}$  under the natural map  $A \rightarrow B$ . If  $M_{\mathfrak{q}}$  is a flat  $A_{\mathfrak{p}}$ -module, then there is  $g \in B \setminus \mathfrak{q}$  such that for all prime ideals  $\mathfrak{q}' \triangleleft B$  with  $\mathfrak{q} \subseteq \mathfrak{q}'$  and  $g \notin \mathfrak{q}'$ ,  $M_{\mathfrak{q}'}$  is a flat  $A_{\mathfrak{p}'}$ -module, where  $\mathfrak{p}'$  is the preimage of  $\mathfrak{q}'$ .

For a proof, we refer to [GD66, 11.1.1.1].

#### 1.1.2 One-parameter families

Before studying flatness in full generality, we show that in some special (but nonetheless interesting) cases, flatness can be described as an actual continuity property, in the sense that in a flat family each fibre is the "limit" of all the other neighbouring fibres. To do this we will need the concept of *associated points* of a scheme, which is explained in some detail in the Appendix (Section 3).

**Theorem 1.16.** Let  $f : X \to Y$  be a morphism of locally Noetherian schemes, with Y a regular and integral scheme of dimension 1. Let  $y \in Y$  be a closed point. Then f is flat at y if and only if no associated point of X maps to y.

Note that, under these hypothesis, Y has exactly one non-closed point, its generic point.

*Proof.* Since Y is regular of dimension 1 and y is a closed point of Y,  $\dim(\mathcal{O}_{Y,y}) = \dim(Y) = 1$ , so  $\mathcal{O}_{Y,y}$  is a Noetherian local integral domain of dimension 1: it must be a PID<sup>1</sup>.

Assume that  $x \in X$  is such that f(x) = y, and that f is flat at x. Let  $u \in \mathcal{O}_{Y,y}$  be a generator of  $\mathfrak{m}_y$ . By hypothesis u does not divide zero in  $\mathcal{O}_{Y,y}$ , and if we look at the map of local rings  $f_y^{\sharp} : \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ we see that  $f_y^{\sharp}(u) \in \mathfrak{m}_x$  is an element that does not divide 0, by the Remark at page 2. Hence x cannot be an associated point of X, by Lemma 3.16.

For the other direction, assume that  $f^{-1}(y)$  does not contain any associated point of X. Let  $x \in X$  be such that f(x) = y. By Lemma 1.4 we know that f is flat at x if and only if  $\mathcal{O}_{X,x}$  is a torsion-free module, i.e. for all nonzero  $t \in \mathcal{O}_{Y,y}$ ,  $f_y^{\sharp}(t)$  is not a zero divisor in  $\mathcal{O}_{X,x}$ .

Assume, on the contrary, that there is some  $t \in \mathcal{O}_{Y,y}$  such that  $f_y^{\sharp}(t)$  is a zero divisor in  $\mathcal{O}_{X,x}$ . Let u be a generator of  $\mathfrak{m}_y$ , and let  $n \in \mathbb{Z}_{\geq 0}$  be the unique<sup>2</sup> integer such that  $t = au^n$  for some  $a \in \mathcal{O}_{Y,y}$ . Then  $f_y^{\sharp}(au^n)b = 0$  for some nonzero  $b \in \mathcal{O}_{X,x}$ , and  $f_y^{\sharp}(a)b \neq 0$ ; indeed, if  $f^{\sharp}(a)$  were a zero divisor, a would be an element of  $\mathfrak{m}_y$ , implying that  $t = a'u^{n+1}$  for some a'. Then also  $f^{\sharp}(u)$  is a zero divisor in  $\mathcal{O}_{X,x}$ , and so  $f^{\sharp}(u)$  is contained in some associated prime  $\mathfrak{p}$  of  $\mathcal{O}_{X,x}$ , by Lemma 3.4. But this means that there is an associated point  $x' \in X$  such that f(x') = y, against our assumption.

**Corollary 1.17.** Let  $f : X \to Y$  be a morphism of locally Noetherian schemes, with Y a regular and integral scheme of dimension 1. Then f is flat if and only if every associated point of X maps to the generic point of Y.

<sup>&</sup>lt;sup>1</sup>Every regular local domain of dimension 1 is a DVR, and in particular it is a PID[Har77, Theorem 6.2A, chapter 1].

<sup>&</sup>lt;sup>2</sup>The uniqueness follows from the fact that  $\mathcal{O}_{Y,y}$ , being a DVR, is a UFD.

Proof. Let  $\eta$  be the generic point of Y. By the previous Theorem, the implication "f flat  $\Rightarrow$  the associated points of X map to  $\eta$ " is immediate. For the other direction, we already know from Theorem 1.16 that f is flat over each  $x \in X$  that maps to a closed point of Y. So it is enough to show that f is flat at those  $x \in X$  that are in the preimage of  $\eta$ . But the ring  $\mathcal{O}_{Y,\eta}$  is a field, so that for every x such that  $f(x) = \eta$  the ring  $\mathcal{O}_{X,x}$  is a vector space over  $\mathcal{O}_{Y,\eta}$  via  $f_{\eta}^{\sharp}$ . Of course a vector space is always flat over its base field, so we have the thesis.

**Corollary 1.18.** Let  $f: X \to Y$  be a morphism of locally Noetherian schemes, with Y regular and integral of dimension 1. Let  $p \in Y$  be a closed point, and let Z be a closed subscheme of  $X \setminus f^{-1}(p)$ , so that Z is a locally closed subscheme of X. If  $f_{|Z}$  is flat, then there is a unique closed subscheme  $\overline{Z}$ of X such that  $f_{|\overline{Z}}$  is flat and  $\overline{Z}$  coincides with Z on  $X \setminus f^{-1}(p)$ . In other words, there is a unique flat limit of Z in X.

*Proof.* Let Z be the scheme-theoretic closure of Z in X. By Lemma 3.17, the associated points of  $\overline{Z}$  are just the associated points of Z, and all of these are mapped to the generic point of Y by the flatness hypothesis and the previous theorem. This indeed means that  $\overline{Z}$  is flat over Y, again by our previous result. Moreover, if W is any other closed subscheme of X containing Z such that  $Z = W \cap (X \setminus f^{-1}(p))$ , we have that (as sets)  $W = \overline{Z} \cup (W \cap f^{-1}(p))$ . Since these two sets are both closed and nontrivial, we found that W has at least an additional irreducible component that maps to p. So the generic point of this irreducible component is an associate point of W that maps to a closed point of Y; in other words,  $f_{\uparrow W}: W \to Y$  is not flat.

**Corollary 1.19.** Let  $f: X \to Y$  be a morphism of locally Noetherian schemes, with Y a regular and integral scheme of dimension 1. Let  $Z \subseteq X$  a closed subscheme. Then  $f_{\uparrow Z}$  is flat if and only if for each closed point  $y \in Y$ , the fibre of Z over y is the flat limit of  $Z \cap (X \setminus f^{-1}(y))$ .

*Proof.* The implication " $\Rightarrow$ " is immediate from Corollary 1.18. For the other implication we use Theorem 1.16 as follows: fix  $x \in Z$ , and let y = f(x). If y is the generic point of Y then we know that  $f_{\uparrow Z}$  is flat at x, from the proof of Corollary 1.17. If instead y is a closed point of Y, by hypothesis we know that Z is the scheme-theoretic closure in X of  $Z \cap (X \setminus f^{-1}(y))$ , so by Lemma 3.17 the associated points of Z are contained in  $Z \cap (X \setminus f^{-1}(y))$ . In particular, x is not an associated point of Z. But then  $f_{\uparrow Z}$  is flat at x, by Theorem 1.16.

**Example 1.20.** Let  $\Bbbk$  be an algebraically closed field, and consider the closed subscheme  $X \subseteq \mathbb{A}^3_{\Bbbk}$  defined by the ideal (xy - t) in  $\Bbbk[x, y, t]$ . Then we have a map  $X \to T = \text{Spec}(\Bbbk[t])$ , induced by the natural ring map  $\Bbbk[t] \to \Bbbk[x, y, t]$ . This morphism of schemes is flat, by the criterion of Theorem 1.17: our scheme X is integral, so its unique associated point is its generic point, which maps to the generic point of T.

If we identify the closed points of T with elements of  $\Bbbk$ , we notice that the closed fibres of  $X \to T$  are all nonsingular hyperbolas, except for the one over the point 0 which is the union of two lines. Hence the flat limit of the family  $X \setminus X_0 \to T \setminus \{0\}$  is a "singular degeneration" of the general fibre.

**Example 1.21.** Let  $X_1$  be a closed subscheme of  $\mathbb{P}^3_{\Bbbk}$ , and consider for each  $a \in \Bbbk^*$  the automorphism  $\sigma_a$  of  $\mathbb{P}^3_{\Bbbk}$  defined by  $\sigma_a([x_0 : x_1 : x_2 : x_3]) = [x_0 : x_1 : x_2 : ax_3]$ . Then by putting  $X_a := \sigma_a(X_1)$  we obtain a subscheme of  $\mathbb{P}^3_{\Bbbk} \times \mathbb{A}^1_{\Bbbk}$  which is flat over  $\mathbb{A}^1_{\Bbbk} \setminus \{0\}$ , since it is isomorphic to the product  $X_1 \times (\mathbb{A}^1_{\Bbbk} \setminus \{0\})$ . By Corollary 1.18 we can extend this to a flat family over all  $\mathbb{A}^1_{\Bbbk}$  by taking the scheme-theoretic closure.

We do the computation for a twisted cubic curve,  $X_1 = V(x_0x_1 - x_3^2 + x_0^2, x_1x_3 - x_2x_0)$ . A picture of what happens is given in Figure 1; see also Example 1.22 and Example 4.7. We restrict ourselves to the affine open set  $x_0 \neq 0$  of  $\mathbb{P}^3_{\Bbbk}$ , and use coordinates  $x = x_1/x_0$ ,  $y = x_2/x_0$  and  $z = x_3/x_0$  on this affine set. For each  $a \neq 0$ ,  $X_a$  is then given by  $V(x - z^2a^{-2} + 1, xza^{-1} - y)$  and the ideal defining the family in the open set D(t) of  $\operatorname{Spec}(\Bbbk[x, y, z, t])$  is  $I = (x - z^2t^{-2} + 1, xzt^{-1} - y) \triangleleft \Bbbk[x, y, z, t]_t$ . We can find what is  $I \cap \Bbbk[x, y, z, t]$  by a Gröbner basis calculation (performed with Mathematica), since  $I \cap \Bbbk[x, y, z, t]$  is the saturation with respect to t of the ideal  $(t^2x - z^2 + t^2, xz - ty) \triangleleft \Bbbk[x, y, z, t]$ . We discover that

$$I \cap \Bbbk[x, y, z, t] = (t^3y + t^2z - z^3, t^2 + t^2x - z^2, -ty + xz, tx + tx^2 - yz, x^2 + x^3 - y^2)$$



Figure 1: The twisted cubic squashed on the z = 0 plane (Example 1.21).

and in particular the fibre over the closed point 0 is defined in  $\mathbb{A}^3_{\mathbb{k}}$  by the ideal

$$(z^2, xz, yz, y^2 - x^2(x+1))$$

We see that the resulting scheme is contained in the plane z = 0, and has the same support as a nodal cubic. For  $x \neq 0$  and  $y \neq 0$  moreover the local ring of this  $X_0$  has no nilpotents, hence  $X_0$  is reduced away from the node (0, 0, 0). At the node instead we have a nilpotent element in the ring of  $X_0$ , since z is not in the defining ideal of  $X_0$  but  $z^2$  is.

The general member of this family is a smooth cubic, but the flat limit for  $t \to 0$  is singular, with an embedded point.

**Example 1.22.** A very similar example is obtained by using the same automorphisms of  $\mathbb{A}^3_{\mathbb{k}}$ , and choosing as the starting point the scheme defined by the ideal  $(-2x+z^2-4, x^2+y^2-2x)$ . This scheme is just the curve obtained by intersecting a cylinder and a sphere, it's known as "Viviani's curve". The family of morphisms  $\sigma_a$  of the previous example "flattens" this curve onto the plane z = 0, intuitively giving a circle for a going to 0. A picture of the process is given in Figure 2.

Doing the same computation we find that the family of schemes is described by the family of ideals

$$I_a = (-2x + \frac{z^2}{a^2} - 4, x^2 + y^2 - 2x)$$

and describes a subscheme of  $\text{Spec}(\Bbbk[x, y, z, t])$  which is flat over  $\text{Spec}(\Bbbk[t])$ . Indeed the family  $X_a$  for  $a \neq 0$  is given by the fibres of the morphism

$$\operatorname{Spec}\left(\frac{\Bbbk[x, y, z, t]_t}{(-2x + \frac{z^2}{t^2} - 4, x^2 + y^2 - 2x)}\right) \subseteq D(t) \subseteq \operatorname{Spec}(\Bbbk[x, y, z, t]) \longrightarrow \operatorname{Spec}(\Bbbk[t]).$$

According to our theory, this family admits a flat limit; this is computed again via Gröbner basis, and it is given by the spectrum of the ring

$$\frac{\Bbbk[x,y,z,t]}{(5t^4+4t^4y^2-6t^2z^2+z^4,t^2+2t^2x-z^2,5t^2+4t^2y^2-5z^2+2xz^2,-2x+x^2+y^2)}$$

We see that again the fibre over the closed point 0 is described by the circle  $-2x + x^2 + y^2 = 0$  with some nonreduced structure given by the nilpotent element z; this time, however, the nonreducedness is everywhere.

One reason for studying in detail one-parameter families is that they can be used to check flatness of much more general families, c.f. [EH00, Lemma II - 30].

**Theorem 1.23.** Let  $\Bbbk$  be a field, Y a reduced locally Noetherian  $\Bbbk$ -scheme and  $f : X \to Y$  a morphism of finite type. Then for any closed point  $y \in Y$ , f is flat at y if and only if for any regular, integral, locally Noetherian  $\Bbbk$ -scheme Y', any closed point  $y' \in Y$  and any morphism  $\varphi : Y' \to Y$  such that  $\varphi(y') = y$ , the morphism

$$X' = X \times_Y Y' \xrightarrow{f'} Y'$$

is flat at y'.

*Proof.* The implication " $\Rightarrow$ " is immediate from the previous results and the fact that flatness is stable under arbitrary base changes. The other direction, which is much more complicated, in given by [RG71, Corollaire 4.2.10].

#### **1.2** Some properties of flatness

A very nice property of flatness is that it is stable under arbitrary base changes.

**Proposition 1.24.** Let  $f : X \to Y$  be a morphism of schemes, and let  $\mathscr{F}$  be a f-flat  $\mathcal{O}_X$ -module. Then

1. for any base change

$$\begin{array}{ccc} X \times_Y Y' & \stackrel{\varphi}{\longrightarrow} X \\ & \downarrow^{f'} & & \downarrow^f \\ & Y' & \stackrel{g}{\longrightarrow} Y \end{array}$$

also  $\varphi^* \mathscr{F}$  is f'-flat;

2. if  $h: Y \to Z$  is a flat morphism of schemes, then  $\mathscr{F}$  is  $(h \circ f)$ -flat.

Before going on, we define a relative version of the cohomology of a sheaf.

**Definition 1.25.** Let  $f: X \to Y$  be a morphism of schemes. The higher direct image functors  $R^i f_*$  are the right derived functors of  $f_*: \mathfrak{Ab}(x) \to \mathfrak{Ab}(Y)$ .



Figure 2: Viviani's curve squashed on the z = 0 plane (Example 1.22).

This definition is a bit obscure, but it is not too hard to check some properties that help the intuition. For the details we refer to [Har77, §8, chapter III].

**Proposition 1.26.** Let  $f: X \to Y$  be a morphism of schemes, and let  $\mathscr{F}$  be a sheaf of  $\mathcal{O}_X$ -modules. Then for every  $i \ge 0$ ,  $R^i f_* \mathscr{F}$  is the sheaf associated to the presheaf

$$\Big\{V \mapsto H^i(f^{-1}(V), \mathscr{F}_{\restriction f^{-1}(V)})\Big\}.$$

Moreover, if X is Noetherian,  $\mathscr{F}$  is quasi-coherent and Y is affine,  $R^i f_* \mathscr{F} \cong H^{\widetilde{i}}(X, \mathscr{F})$ .

First nice property of flatness: cohomology commutes with flat base changes.

**Proposition 1.27.** Let  $f: X \to Y$  be a separated morphism of finite type of Noetherian schemes, and let  $\mathscr{F}$  be a quasi-coherent sheaf on X. Let  $u: Y' \to Y$  be a flat morphism, with also Y' Noetherian, and consider the fibred product



Then there is a natural isomorphism between  $u^*R^if_*\mathscr{F}$  and  $R^if'_*(v^*\mathscr{F})$ , for all  $i \geq 0$ .

Proof. Since affine open sets form a basis for the topology on Y', we may assume that Y' and Y are affine,  $Y' = \operatorname{Spec}(A')$  and  $Y = \operatorname{Spec}(A)$ . By Proposition 1.26 then we have to exhibit an isomorphism between  $H^i(X', v^*\mathscr{F})$  and  $H^i(X, \mathscr{F}) \otimes_A A'$ , where  $X' := X \otimes_Y Y'$ . Notice that X is separated (since it is separated over an affine scheme), and since  $\mathscr{F}$  is quasi-coherent we can compute  $H^i(X, \mathscr{F})$  by Čech cohomology. For the same reasons we can compute  $H^i(X', v^*\mathscr{F})$  by Čech cohomology, since by base extension also f' is separated and of finite type (hence X' is also separated and Noetherian).

Let  $\mathcal{U}$  be an open cover of X by affine sets; since the fibred product of affine schemes is again affine,  $v^{-1}(U)$  is again affine, and so  $v^{-1}(\mathcal{U})$  is an open cover of X'. We compute Čech cohomologies with respect to these two covers. By definition, the Čech complex  $\mathcal{C}(v^{-1}(\mathcal{U}), v^*\mathscr{F})$  is just  $\mathcal{C}(\mathcal{U}, \mathscr{F}) \otimes_A A'$ ; indeed, for every  $U \in \mathcal{U}$  we have  $v^*\mathscr{F}(v^{-1}(U)) = \mathscr{F}(U) \otimes_A A'$  (to check this, simply notice that U is affine). So we have to compute the cohomology of the complex

$$\mathcal{C}^{0}(\mathcal{U},\mathscr{F})\otimes_{A}A'\longrightarrow \mathcal{C}^{1}(\mathcal{U},\mathscr{F})\otimes_{A}A'\longrightarrow \mathcal{C}^{2}(\mathcal{U},\mathscr{F})\otimes_{A}A'\longrightarrow \ldots$$

but since A' is a flat A-module, tensoring with A' preserves the cohomology of any sequence of A-modules.

#### **1.2.1** Flatness and dimension of fibres

This theorem tells us that, when considering flat families, the dimension of the fibres of the family behaves according to our "geometric intuition". As it is given here, it is a slight generalization of Proposition 9.5 in [Har77, chapter III].

**Theorem 1.28.** Let  $f : X \to Y$  be a flat morphism of locally Noetherian schemes. Choose  $x \in X$ , and let y = f(x). Then

$$\dim(\mathcal{O}_{X,x}) = \dim(\mathcal{O}_{X_y,x}) + \dim(\mathcal{O}_{Y,y})$$

where  $X_y$  is the fibre over y.

*Proof.* We can make a base change  $Y' \to Y$ , with  $Y' = \text{Spec}(\mathcal{O}_{Y,y})$ , and consider the new morphism  $f': X' = X \otimes_Y Y' \to Y'$ . Then f' is still flat, and if  $x' \in X'$  is a preimage of x under the map  $X' \to X$ , we have

$$\dim(\mathcal{O}_{X',x'}) = \dim(\mathcal{O}_{X,x})$$

and also

$$\dim(\mathcal{O}_{X'_{y},x'}) = \dim(\mathcal{O}_{X_{y},x})$$
$$\dim(\mathcal{O}_{Y',y}) = \dim(\mathcal{O}_{Y,y}).$$

Indeed, making the base change  $Y' \to Y$  just means that we are restricting the morphism  $f: X \to Y$  to an arbitrarily small neighbourhood of y in Y, and since all the numbers involved in the theorem are can be computed locally, they are left untouched by this restriction.

So we may assume that Y = Spec(A) for a local Noetherian ring A, with y being the maximal ideal of A. We proceed by induction on  $\dim(Y)$ . If  $\dim(Y) = 0$ , it means that every element of yis nilpotent. Now, choose an affine neighbourhood Spec(B) of x in X, and notice that the fibre  $X_y$ around x is defined by  $\text{Spec}(B \otimes_A \Bbbk(y))$ . The ideal sheaf of  $X_y$  in X is thus defined by the kernel of the map  $B \to B \otimes_A \Bbbk(y)$ , and this can be computed as follows: consider the short exact sequence

$$0 \to y \to A \to \Bbbk(y) \to 0$$

and tensoring with B we get by flatness another short exact sequence

$$0 \to yB \to B \to \Bbbk(y) \otimes_A B \to 0$$

hence the ideal is  $yB \triangleleft B$ . But since each element of y is nilpotent this ideal is contained in the nilradical of B, hence  $\dim(\mathcal{O}_{X_y,x}) = \dim_{X,x}$  and the thesis follows.

Assume now that  $\dim(Y) > 0$ . We can make a base extension to  $Y_{\text{red}}$ , and the numbers in question do not change; hence we may assume that Y is reduced. Since its dimension is greater than 1, there is a prime ideal  $\mathfrak{p} \subseteq A$  such that  $\{0\} \subseteq \mathfrak{p} \subset y$ , so that we can find some  $t \in y$  that is not contained in any minimal prime of A by the Prime Avoidance Lemma 3.6; notice that, since SpecA is Noetherian, it has a finite number of irreducible components (hence A has finitely many minimal primes). In particular t is not a zero divisor, by Lemma 3.2. Let Y' = Spec(A/(t)), and make the base extension  $Y' \to Y$ , obtaining



and notice as usual that f' is also a flat morphism. Moreover, if Spec(B) is, as before, an open affine neighbourhood of x in X, we have that x has a preimage in X' and that this preimage has an open affine neighbourhood isomorphic to  $\text{Spec}(B/(f^{\sharp}(t)))$ . This follows readily by noticing that the following diagram is a push-forward

$$\begin{array}{ccc} A & & \stackrel{f^{\sharp}}{\longrightarrow} & B \\ \downarrow & & \downarrow \\ A/(t) & \longrightarrow & B \otimes_A A/(t) \end{array}$$

and that  $f^{\sharp}(t) \subseteq x$ ,  $B \otimes_A A/(t) \cong B/(f^{\sharp}(t))$ .

Since f is flat, also  $f^{\sharp}(t)$  is not a zero divisor in  $\mathcal{O}_{X,x}$ , and so by Lemma 3.3 and Lemma 3.1 we have

$$\dim(\mathcal{O}_{Y',y}) = \dim(\mathcal{O}_{Y,y}) - 1$$
$$\dim(\mathcal{O}_{X',x}) = \dim(\mathcal{O}_{X,x}) - 1.$$

By the induction hypothesis then we find

$$\dim(\mathcal{O}_{X',x}) = \dim(\mathcal{O}_{X'_{u},x}) + \dim(\mathcal{O}_{Y',y})$$

and to get the thesis it is then enough to notice that fibres of X' and X over y are naturally isomorphic, hence  $\dim(\mathcal{O}_{X'_{y},x}) = \dim(\mathcal{O}_{X_{y},x})$ . **Corollary 1.29.** Let  $f : X \to Y$  be a flat morphism of schemes of finite type over a field  $\Bbbk$ , with Y irreducible. Then the following are equivalent:

- 1. every irreducible component of X has dimension equal to  $\dim(Y) + n$ ;
- 2. for every  $y \in Y$ , every irreducible component of  $X_y$  has dimension n.

*Proof.*  $(1 \Rightarrow 2)$ . Fix  $y \in Y$ , and let Z be an irreducible component of  $X_y$  with the reduced induced subscheme structure. If  $x \in Z$  is a closed point that is not contained in any other irreducible component of Z then  $\dim(Z) = \dim(\mathcal{O}_{Z,x})$ , by Exercise 3.20 in [Har77, chapter II]. Notice also that by that exercise we have<sup>3</sup>

$$\dim(\mathcal{O}_{X,x}) = \dim X - \dim(\overline{\{x\}})$$
$$\dim(\mathcal{O}_{Y,y}) = \dim Y - \dim(\overline{\{y\}}).$$

Since x is a closed point of the fibre  $X_y$ , the residue field  $\Bbbk(x)$  is a finite algebraic extension of  $\Bbbk y$ , and again by the same exercise in Hartshorne this means that  $\dim(\overline{\{x\}}) = \dim(\overline{\{y\}})$ . Since Z has dimension equal to  $\dim(Y) + n$  by hypothesis, we get the thesis from  $\dim(\mathcal{O}_{Z,x}) = \dim(\mathcal{O}_{X,x}) - \dim(\mathcal{O}_{Y,y})$  (this is what Theorem 1.28 tells us).

 $(2 \Rightarrow 1)$ . Let Z be an irreducible component of X, and let  $x \in Z$  be a closed point that is not contained in any other irreducible component of X. By Theorem 1.28 we have

$$\dim(\mathcal{O}_{X,x}) = \dim(\mathcal{O}_{Y,f(y)}) + \dim(\mathcal{O}_{X_{f(y)},x}).$$

Our hypothesis is that  $\dim(\mathcal{O}_{X,x}) = n$ , and we know that  $\dim(Z) = \dim(\mathcal{O}_{X,x})$ . Moreover, f(x) must be a closed point of Y, since x is closed in X. So  $\dim(Y) = \dim(\mathcal{O}_{Y,y})$ , and  $\dim(Z) = \dim(Y) + n$  as we wanted to prove.

## 2 Hilbert polynomials and flatness

**Definition 2.1.** Let X be a projective scheme over the field k, and let  $\mathscr{F}$  be a coherent sheaf on X. Then we define the *Euler characteristic* of  $\mathscr{F}$  as

$$\chi(\mathscr{F}) := \sum_{i=0}^{\infty} (-1)^i \dim_k H^i(X, \mathscr{F}).$$

*Remark.* This is a good definition, since each  $H^i(X, \mathscr{F})$  is a finite-dimensional vector space over k and for  $i > \dim(X)$  we have  $H^i(X, \mathscr{F}) = \{0\}$ .

By using the long exact sequence in cohomology associated to a short exact sequence of sheaves, we easily find that  $\chi$  is "additive": if  $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$  is an exact sequence of coherent sheaves on X, then

$$\chi(\mathscr{F}) = \chi(\mathscr{F}') + \chi(\mathscr{F}'').$$

**Theorem 2.2.** Let X be a projective scheme over the algebraically closed field  $\Bbbk$ , together with a very ample invertible sheaf  $\mathcal{O}_X(1)$ . For any coherent sheaf  $\mathscr{F}$  on X, define  $\mathscr{F}(n) := \mathscr{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(n)$ , where  $\mathcal{O}_X(n) = \mathcal{O}_X(1)^{\otimes n}$ . Then for all coherent sheaves  $\mathscr{F}$  on X there is a polynomial  $P(z) \in \mathbb{Q}[z]$  such that

$$\chi(\mathscr{F}(n)) = P(n), \ \forall n \in \mathbb{Z}_{>0}.$$

*Proof.* We proceed by induction on the dimension d of the support of  $\mathscr{F}$ . If d = 0 then  $\mathscr{F}$  is supported on a finite number of points, and by Grothendieck's vanishing theorem  $\chi(\mathscr{F}(n))$  is constant. Indeed, if p is a point in the support of  $\mathscr{F}$  then  $H^i(\{p\}, \mathscr{F}) = \{0\}$  for i > 0, while  $H^0(\{p\}, \mathscr{F}) = \mathscr{F}_p$ . Assume then that d > 0, and that the thesis holds for all d' < d.

Let  $j: X \to \mathbb{P}^r_{\Bbbk}$  be a closed immersion of  $\Bbbk$ -schemes such that  $\mathcal{O}_X(1) = j^*\mathcal{O}(1)$ . Recall that if  $\mathscr{F}$  is coherent on X then also  $j_*\mathscr{F}$  is coherent on  $\mathbb{P}^r_{\Bbbk}$ , and their cohomologies are the same by [Har77, Lemma 2.10, chapter III]. Then it will be sufficient to prove the theorem for  $X = \mathbb{P}^r_{\Bbbk}$  and  $\mathcal{O}_X(1) = \mathcal{O}(1)$ .

<sup>&</sup>lt;sup>3</sup>Notice that dim( $\mathcal{O}_{X,x}$ ) = codim({x}, X), in the notation of [Har77, Exercise 3.20, chapter II].

Write  $X = \operatorname{Proj}(\mathbb{k}[x_0, \ldots, x_r])$ , and let  $x \in H^0(X, \mathcal{O}(1))$  be such that the hyperplane  $H = \{x = 0\}$ does not contain any irreducible component of  $\operatorname{supp}(\mathscr{F})$ , so that  $H \cap \operatorname{supp}(\mathscr{F})$  has dimension strictly less than d. Let  $\mu : \mathcal{O}_X(-1) \to \mathcal{O}(X)$  be the map given by multiplication by x; this map fits into an exact sequence of sheaves on X

$$0 \longrightarrow \mathcal{O}_X(-1) \xrightarrow{\mu} \mathcal{O}_X \longrightarrow \mathcal{O}_H \longrightarrow 0$$

where  $\mathcal{O}_H$  is the structure sheaf of H considered as a sheaf on X. Tensoring with  $\mathscr{F}$  gives another exact sequence

$$0 \longrightarrow \mathscr{K} \longrightarrow \mathscr{F}(-1) \xrightarrow{\mu \otimes 1} \mathscr{F} \longrightarrow \mathscr{R} \longrightarrow 0$$

and since  $\mathcal{O}_X(1)$  preserves exact sequences (it is locally free) we find for all n an exact sequence

$$0 \longrightarrow \mathscr{K}(n) \longrightarrow \mathscr{F}(n-1) \longrightarrow \mathscr{F}(n) \longrightarrow \mathscr{R}(n) \longrightarrow 0.$$

It is immediate to check that the additivity property of  $\chi$  implies that  $\chi(\mathscr{F}(n)) - \chi(\mathscr{F}(n-1)) = \chi(\mathscr{R}(n)) - \chi(\mathscr{K}(n))$ , so it will be enough to prove that there is a numerical polynomial  $Q \in \mathbb{Q}[z]$  such that  $\chi(\mathscr{R}(n)) - \chi(\mathscr{K}(n)) = Q(n)$  to have the thesis (c.f. [Har77, Proposition 7.3, chapter I]).

This last part follows from the inductive hypothesis, since the supports of  $\mathscr{K}$  and  $\mathscr{R}$  have dimension strictly smaller than d. Indeed,  $\mathscr{Q}_p$  is zero whenever  $p \notin \operatorname{supp}(\mathscr{F})$  or x is a unit in  $\mathcal{O}_{X,p}$ , so  $\operatorname{supp}(\mathscr{Q}_p) \subseteq$  $H \cap \operatorname{supp}(\mathscr{F})$ . the same thing holds for  $\mathscr{K}$ : if x is a unit in  $\mathcal{O}_{X,p}$ , the multiplication by x cannot have a nontrivial kernel in  $\mathscr{F}_p$ .

*Remark.* The requirement of  $\Bbbk$  to be algebraically closed is just so we can guarantee that  $\Bbbk$  is infinite. This is needed to make sure that there is an hyperplane in  $\mathbb{P}^n_{\Bbbk}$  not containing any irreducible component of  $\supp(\mathscr{F})$ . However, we can avoid making this additional hypothesis: indeed, for an arbitrary field  $\Bbbk$ , its algebraic closure  $\bar{\Bbbk}$  is a flat  $\Bbbk$ -module. Then the morphism of schemes  $\operatorname{Spec}(\bar{\Bbbk}) \to \operatorname{Spec}(\Bbbk)$  is flat; let the following diagram be the base change via this morphism



and recall that Proposition 1.27 tells us that for all  $i \ge 0$ 

$$H^i(X,\mathscr{F})\otimes_{\Bbbk}\bar{\Bbbk}\cong H^i(X',\pi^*\mathscr{F}).$$

Hence for the dimensions of the cohomology groups (vector spaces, in this case) we have

$$\dim_{\bar{\Bbbk}} H^{i}(X', \pi^{*}\mathscr{F}) = \dim_{\bar{\Bbbk}} \left( H^{i}(X, \mathscr{F}) \otimes_{\mathbb{k}} \bar{\Bbbk} \right) = \dim_{\mathbb{k}} H^{i}(X, \mathscr{F})$$

so Theorem 2.2 holds also for k not algebraically closed.

Remark. Recall that for a projective scheme X over a Noetherian scheme and a coherent sheaf  $\mathscr{F}$ , there is some  $N \in \mathbb{Z}_{\geq 0}$  such that  $\dim(\mathscr{F}(n)) = 0$  for all  $n \geq N$ . Then the previous theorem tells us that there is a polynomial P(z) with rational coefficients such that  $\dim(\mathscr{F}(n)) = P(n)$  for all sufficiently big n. Traditionally this is known as the *Hilbert polynomial* of  $\mathscr{F}$ . The Hilbert polynomial of X is defined as the Hilbert polynomial of the sheaf  $\mathcal{O}_X(1)$ ; it carries a lot of informations about X, for example the dimension, the degree and the arithmetic genus.

**Lemma 2.3.** Consider an exact sequence of coherent modules on  $X = \mathbb{P}^n_A$ ,

$$\mathscr{F}_1 \to \mathscr{F}_2 \to \cdots \to \mathscr{F}_{n-1} \to \mathscr{F}_n.$$

Then, for all m sufficiently large we have an exact sequence

$$\Gamma(X,\mathscr{F}_1(m)) \to \Gamma(X,\mathscr{F}_2(m)) \to \cdots \to \Gamma(X,\mathscr{F}_{n-1}(m)) \to \Gamma(X,\mathscr{F}_n(m)).$$

*Proof.* By induction on n. For n = 3, expand the sequence as

$$0 \to \ker_1 \to \mathscr{F}_1 \to \operatorname{ran}_1 \to 0 \to \operatorname{ran}_1 \to \mathscr{F}_2 \to \operatorname{ran}_2 \to 0 \to \operatorname{ran}_2 \to \mathscr{F}_3 \to \mathscr{Q} \to 0$$

From each short exact piece we can apply Serre's vanishing Theorem [Har77, Theorem 5.2, chapter III] to find exact sequences in cohomology, like

$$0 \to H^0(X, \ker_1(m)) \to H^0(X, \mathscr{F}_1(m)) \to H^0(X, \operatorname{ran}_1(m)) \to 0.$$

Putting together the various pieces so obtained, we get the thesis. For n > 3, the same reasoning can be applied: just split the sequence at the last element.

**Theorem 2.4.** Let T be an integral Noetherian scheme, and let X be a closed subscheme of  $\mathbb{P}_T^n$ . Then X is flat over T if and only if the Hilbert polynomials  $P_t$  of the fibres  $X_t$  (seen as closed subschemes of  $\mathbb{P}_{\Bbbk(t)}^n$ ) do not depend upon  $t \in T$ .

*Proof.* By considering  $\mathcal{O}_X$  as a coherent sheaf on  $\mathbb{P}_T^n$ , we see that it will be enough to prove that for any coherent sheaf  $\mathscr{F}$  on  $X = \mathbb{P}_T^n$ ,  $\mathscr{F}$  is flat over T if and only if for all  $t \in T$  the Hilbert polynomial of  $\mathscr{F}_t$  equals the Hilbert polynomial of  $\mathscr{F}_0$ , where  $0 \in T$  is the generic point. Up to performing a base change  $\operatorname{Spec}(\mathcal{O}_{T,t}) \to T$  we may also assume that  $T = \operatorname{Spec}(A)$  for a local Noetherian domain A.

We will show that the following are equivalent:

- $\mathscr{F}$  is flat over T;
- $H^0(X, \mathscr{F}(m))$  is a free A-module of finite rank, for all sufficiently large m;
- the Hilbert polynomial  $P_t$  of  $\mathscr{F}_t$  on  $X_t = \mathbb{P}^n_{\Bbbk(t)}$  is independent of t.

 $(1 \Rightarrow 2)$ . Let  $\mathcal{U}$  be the usual open affine cover of projective space. We can compute the cohomology groups  $H^i(X, \mathscr{F}(m))$  by means of the Čech complex  $\mathcal{C}(\mathcal{U}, \mathscr{F}(m))$ , and since  $\mathscr{F}$  is flat each term of the complex is a flat A-module. Moreover, for m >> 0 all the higher cohomology groups  $H^i(X, \mathscr{F}(m))$ vanish, for i > 0. In other words, by taking m large enough we have an exact sequence

$$0 \longrightarrow H^0(X, \mathscr{F}(m)) \longrightarrow \mathcal{C}^0(\mathcal{U}, \mathscr{F}(m)) \longrightarrow \ldots \longrightarrow \mathcal{C}^n(\mathcal{U}, \mathscr{F}(m)) \longrightarrow 0$$

where all the terms except the first are A-flat. But then also  $H^0(X, \mathscr{F}(m))$  must be A-flat, by Proposition 1.6. Since it is also finitely generated and A is a local Noetherian ring, from Proposition 1.8 we find that  $H^0(X, \mathscr{F}(m))$  must be a free A-module.

 $(2 \Rightarrow 1)$ . Let  $S = A[x_0, \dots, x_n]$ , so that  $X = \operatorname{Proj}(S)$ . Since 2 holds, there is some  $m_0$  such that  $H^0(X, \mathscr{F}(m))$  is free for  $m \ge m_0$ . If we define  $M = \bigoplus_{m \ge m_0} H^0(X, \mathscr{F}(m))$ , notice that M equals  $\Gamma_*(\mathscr{F}) = \bigoplus_{m \in \mathbb{Z}} \Gamma(X, \mathscr{F}(m))$  for degrees  $m \ge m_0$ , so  $\tilde{M} = \Gamma_*(\mathscr{F})$ . On the other hand, we know that  $\widetilde{\Gamma_*(\mathscr{F})} = \mathscr{F}$ . Since by hypothesis M is a free A-module, this means that  $\mathscr{F}$  is flat.

 $(2 \Rightarrow 3)$ . The Hilbert polynomial  $P_t$  of the fibre over  $t \in T$  is characterized by the property  $P_t(m) = \dim_{\mathbb{K}(t)} H^0(X_t, \mathscr{F}_t(m))$  for large enough m; then it will be enough to prove for every  $t \in T$  that, for m large enough,  $P_t(m) = \operatorname{rank}_A H^0(X, \mathscr{F}(m))$ . This will follow from equation (1), which we prove without assuming 2.

We propose to show that

$$H^{0}(X_{t},\mathscr{F}_{t}(m)) \cong H^{0}(X,\mathscr{F}) \otimes_{A} \Bbbk(t).$$

$$\tag{1}$$

Choose  $t \in T$ ,  $t = \mathfrak{p}$  for a certain  $\mathfrak{p} \in \operatorname{Spec}(A)$ . Since  $A_{\mathfrak{p}}$  is a flat A-module, by performing the base change  $\operatorname{Spec}(A_{\mathfrak{p}}) \to T$  we reduce to the case in which t is a closed point of T. Indeed, recall from Proposition 1.27 that cohomology commutes with flat base changes.

Since A is Noetherian, the maximal ideal of t is finitely generated, say by r elements. Then we can find an exact sequence

$$A^{\oplus r} \longrightarrow A \longrightarrow \Bbbk(t) \longrightarrow 0 \tag{2}$$

from which we can get an exact sequence

$$\mathscr{F}^{\oplus r} \longrightarrow \mathscr{F} \longrightarrow \mathscr{F}_t \longrightarrow 0.$$

Then, for m sufficiently large we have, by Lemma 2.3 another exact sequence

$$H^0(X,\mathscr{F}(m))^{\oplus r} \longrightarrow H^0(X,\mathscr{F}(m)) \longrightarrow H^0(X_t,\mathscr{F}_t(m)) \longrightarrow 0$$

We can also tensor the sequence in (2) with  $H^0(X, \mathscr{F}(m))$  to obtain

$$H^0(X,\mathscr{F}(m))^{\oplus r} \longrightarrow H^0(X,\mathscr{F}(m)) \longrightarrow H^0(X,\mathscr{F}(m)) \otimes_A \Bbbk(t) \longrightarrow 0$$

so comparing the two sequences we get the desired result.

 $(3 \Rightarrow 2)$ . By [Har77, Lemma 8.9, chapter II], to check that an A-module M is free it is enough to check that  $\dim_{\Bbbk} M \otimes_A \Bbbk = \dim_K M \otimes_A K$ , where  $\Bbbk$  is the residue field of A and K is its quotient field. Our hypothesis is that the Hilbert polynomial of the fibre over the generic point  $\eta$  and the closed point  $\xi$  of A coincide, i.e.

$$\dim_{\mathbb{K}} H^0(X_{\xi}, \mathscr{F}_{\xi}(m)) = \dim_{K} H^0(X_{\eta}, \mathscr{F}_{\eta}(m))$$

for all large enough m. Of course, this equation together with (1) gives the thesis.

*Remark.* By examining the previous proof, it is clear that the implications  $1 \Rightarrow 2$  and  $2 \Rightarrow 3$  hold even if T is not integral. Indeed, the integrality hypothesis is only needed to appeal to [Har77, Lemma 8.9, chapter II]. So we have that for any connected scheme T, if X is a closed subscheme of  $\mathbb{P}_T^n$  flat over T, then the Hilbert polynomials of the fibres of  $X \to T$  are all the same.

Moreover, the hypothesis of Theorem 2.4 can be still a bit relaxed: it is enough that T is connected and reduced to have the thesis.

## 3 Appendix

Here we collect some of the results that are needed to study flatness.

#### 3.1 Algebraic facts

**Lemma 3.1.** Let R be a commutative ring, and let  $\mathfrak{p}$  be a minimal prime of R. Then each element of  $\mathfrak{p}$  is a zero divisor.

*Proof.* Consider the localization  $R_{\mathfrak{p}}$ . Since  $\mathfrak{p}$  is minimal in R,  $\mathfrak{p}R_{\mathfrak{p}}$  is the nilradical of  $R_{\mathfrak{p}}$ . This means that for each  $r \in \mathfrak{p}$  there are  $s \in R \setminus \mathfrak{p}$  and  $n \in \mathbb{Z}_{\geq 0}$  such that  $r^n s = 0$ .

**Lemma 3.2.** Let R be a reduced ring. Then each zero divisor is contained in some minimal prime of R.

*Proof.* Let x be a zero divisor in R, and let  $y \in R \setminus \{0\}$  be such that xy = 0. Since R is reduced, the intersection of all minimal prime ideals of R is 0. Hence there is some  $\mathfrak{p} \triangleleft R$  minimal such that  $y \notin \mathfrak{p}$ , and from this it follows that  $x \in \mathfrak{p}$ .

**Lemma 3.3.** Let R be a local Noetherian ring with maximal ideal  $\mathfrak{m}$ , and let  $f \in \mathfrak{m}$  be an element that is not contained in any minimal prime ideal of A. Then  $\dim(A/fA) = \dim(A) - 1$ .

This is Theorem 5.15 in [Liu02, chapter 2]. C.f. Theorem 1.8A and Theorem 1.11A in [Har77, chapter I].

**Lemma 3.4.** For any ring R, the set of zero divisors of R is the union of all the associated primes of R.

*Proof.* See [Mat87, chapter 2, Corollary 2].

**Lemma 3.5.** If R is a Noetherian ring, the number of associated primes in R is finite.

Proof. See [Mat87, Proposition 7.G].

**Lemma 3.6** (Prime avoidance). Let  $I_1, \ldots, I_n$  be prime ideals of the ring R, and let  $J \triangleleft R$  be an ideal such that  $J \not\subseteq I_i$  for all  $i = 1, \ldots, n$ . Then there exists an element  $x \in J$  such that  $x \notin \bigcup_{i=1}^n I_i$ .

*Remark.* Lemma 3.6 actually holds even if at most two of the ideals  $I_1, \ldots, I_n$  are not prime, but we will not need it in this stronger form.

*Proof.* By induction on n. For n = 1 the claim is obvious. Assume that n > 1 and the thesis holds for n - 1; without loss of generality, we may also assume that there are no inclusions among  $I_1, \ldots, I_n$ .

By the inductive hypothesis we can find  $x \in J \setminus (I_1 \cup \cdots \cup I_{n-1})$ . If  $x \notin I_n$  we are done, so assume that  $x \in I_n$ . Notice that if the product ideal  $JI_1 \ldots I_{n-1}$  were contained in  $I_n$  then we would have  $J \subseteq I_n$ , against our hypothesis; then there is some  $y \in JI_1 \ldots I_{n-1} \setminus I_n$ , and x + y is an element of J not contained in  $I_n$ .

#### 3.2 Scheme-theoretic closure

Let X be a scheme, and let  $V \subseteq X$  be a closed subset. We define the *reduced induced subscheme* structure on V as follows: for X = Spec(A) affine, consider

$$\mathfrak{a} := \bigcap \{ \mathfrak{p} \in \operatorname{Spec}(A) \mid \mathfrak{p} \in V \}$$

which is the largest ideal of A such that  $V = V(\mathfrak{a})$ . Then  $\mathfrak{a}$  defines a scheme structure on V, and with this structure  $V(\mathfrak{a})$  is a reduced scheme: indeed,  $\mathfrak{a} = \bigcap \{\mathfrak{p} \in \operatorname{Spec}(A) \mid \mathfrak{a} \subseteq \mathfrak{p}\}$ . When X is not affine, we take a covering  $\mathcal{U} = \{U_i \mid i \in I\}$  of V consisting of open affines of X.

We should check that the scheme structures on each  $V \cap U_i$  glue together to define a global scheme structure on V. By affine communication, it is enough to check that for any distinguished open set D(f) of  $U_i = \operatorname{Spec}(A)$ , the reduced structure on  $V \cap U_i$  induced by  $\operatorname{Spec}(A)$  gives the reduced structure on  $V \cap D(f)$  induced by  $\operatorname{Spec}(A_f)$ , when restricted to D(f). This is quite easy to check: let  $\mathfrak{a}$  be the ideal defining the reduced structure on  $V \cap U_i$ , and let  $Z_i := \operatorname{Spec}(A/\mathfrak{a})$ . Then  $Z_i \cap D(f) \cong \operatorname{Spec}(A_f/(\mathfrak{a}A_f))$ . Notice also that the reduced structure on  $V \cap D(f)$  is defined by the ideal  $\mathfrak{b} = \bigcap \{\mathfrak{p} \in \operatorname{Spec}(A_f) \mid \mathfrak{p} \in V \cap D(f)\}$ , which is equal to  $\mathfrak{a}A_f$ ; this proves the claim, so the reduced induced structure on V is well-defined.

From the definition of the reduced induced structure, we find immediately that it is the "smallest" closed subscheme structure on the given closed subset.

**Lemma 3.7** (Universal property of the reduced induced structure). Let V be a closed subset of a scheme X, and consider Y as a closed subscheme of X by endowing it with the reduced induced scheme structure. Then for any other closed subscheme Y' of X with the same underlying topological space as Y, the inclusion  $Y \hookrightarrow X$  factors through Y'.

These ideas can be generalized to define a "scheme-theoretic" notion of *image* (sometimes also called *range*) of a morphism of schemes.

**Definition 3.8.** Let  $f: X \to Y$  be a morphism of schemes, and let  $\mathscr{I}$  be the largest quasi-coherent subsheaf of ker $(f^{\sharp}: \mathcal{O}_Y \to f_*\mathcal{O}_X)$ . We define the scheme-theoretic image of f to be the closed subscheme of Y defined by  $\mathscr{I}$ . We denote it by  $\overline{f(X)}$ .

An alternative terminology that can also be used is *schematic image*, but we will stick to the former one.

*Remark.* The scheme-theoretic image is indeed a closed subscheme of Y, see [Har77, chapter II, Proposition 5.9]. However, in general it may fail to have some desirable properties; for example, it may not have as underlying topological space the closure of ran(f) (for an example of this behaviour see [Vak, Example 4, §8.3]). However, it can be easily seen that under some reasonable hypothesis, the scheme-theoretic image of f behaves exactly as we would expect.

**Example 3.9.** Let  $f : X \to Y$  be a morphism of schemes, and assume that Y = Spec(B) is affine. Then  $\overline{f(X)}$  is the closed subscheme of Y defined by the ideal

$$\left\{ b \in B \mid f^{\sharp}(b) = 0 \in \mathcal{O}_X(X) \right\}.$$

To make an explicit computation, consider the map  $\varphi : \Bbbk[x, y] \to \Bbbk[t, t^{-1}]$  defined by  $\varphi(p(x, y)) = p(t, 0)$ . This corresponds to the inclusion of the scheme  $\operatorname{Spec}(\Bbbk[t, t^{-1}])$  (the line without a point) into  $\mathbb{A}^2_{\Bbbk}$ , and the scheme-theoretic image in  $\mathbb{A}^2_{\Bbbk}$  is defined by the ideal  $\{p(x, y) \in \Bbbk[x, y] \mid p(t, 0) = 0\}$ , i.e. the ideal  $\{y\}$ . The scheme-theoretic image is just the whole line.

As a slight variation, we could consider the inclusion of the line without a point into the projective plane  $\mathbb{P}^2_{\mathbb{k}}$ . The target scheme is not affine, but we could easily make the computation affine set by affine set and then glue together the results to obtain a quasi-coherent sheaf of ideals on  $\mathbb{P}^2_{\mathbb{k}}$ .

There are some conditions that allow us to compute the scheme-theoretic image of a morphism affine-locally. In this case, scheme-theoretic images exhibit many pleasant properties. For a proof of the following theorem, see [Vak, Theorem 8.3.4].

**Theorem 3.10.** Let  $f: X \to Y$  be a morphism of schemes. Assume that either X is reduced or f is quasi-compact; then ker $(f^{\sharp}: \mathcal{O}_Y \to f_*\mathcal{O}_X)$  is a quasi-coherent sheaf of ideals on Y, and the underlying set of  $\overline{f(X)}$  is the closure of  $\{f(x) \mid x \in X\}$ .

We are interested in the scheme-theoretic image mostly because it allows us to talk about the scheme-theoretic closure of a subscheme.

**Definition 3.11.** Let X be a scheme. A *locally closed subscheme* of X is a morphism of schemes  $f: Z \to X$  that can be factored as

$$Z \xrightarrow{f_1} U \xrightarrow{f_2} X$$

where  $f_1$  is a closed immersion and  $f_2$  is an open immersion. In other words, a locally closed subscheme is a closed subscheme of an open subscheme. If  $f : Z \to X$  is a locally closed subscheme, we define the *closure* of Z in X to be the scheme-theoretic image of f.

The following lemma tells us that, when considering locally closed subschemes of a locally Noetherian scheme, the hypothesis of Theorem 3.10 are satisfied.

**Lemma 3.12.** Let X be a locally Noetherian scheme, and let  $f : Z \to X$  be a locally closed subscheme of X. Then f is quasi-compact.

*Proof.* Write f as the composition of an open immersion and a closed one,  $Z \hookrightarrow U \hookrightarrow X$ . We know that a closed immersion is quasi-compact and that the composition of quasi-compact morphisms is again quasi-compact. Then to prove the lemma it will be enough to show that  $U \hookrightarrow X$  is quasi-compact.

Let  $V = \operatorname{Spec}(A)$  be an open affine subscheme of X; notice that A is a Noetherian ring, since X is locally Noetherian. We can write  $V \cap U = \bigcup_{i \in I} D(f_i)$  for some collection of  $f_i \in A$ , so that  $V \setminus (V \cap U) = \bigcap_{i \in I} V(f_i)$ . Since A is Noetherian the ideal J generated by  $\{f_i \mid i \in I\}$  is finitely generated, and so there are  $f_1, \ldots, f_n \in J$  such that  $U \cap V = \bigcup_{j=1}^n D(f_j)$ .

Exercise 3.13. It will be useful to see an explicit description of the schematic closure, and by the previous results it is enough to do this on an affine scheme. Consider a locally closed subscheme  $Z \hookrightarrow U \hookrightarrow \operatorname{Spec}(A)$  where A is a Noetherian ring, and let  $f_1, \ldots, f_n$  be such that  $U = \bigcup_{i=1}^n D(f_i)$ . Since Z is a closed subscheme of U, for each  $i = 1, \ldots, n$  there is an ideal  $\mathfrak{a}_i \triangleleft A_{f_i}$  such that  $Z \cap D(f_i) = \operatorname{Spec}(A_{f_i}/\mathfrak{a}_i)$ . Moreover  $\mathfrak{a}_i$  and  $\mathfrak{a}_j$  define the same ideal in  $A_{f_if_j}$ , for all  $i \neq j$ . Let  $\overline{\mathfrak{a}}_i$  be  $\mathfrak{a}_i \cap A$ , and set  $\mathfrak{a} := \bigcap_{i=1}^n \overline{\mathfrak{a}}_i$ . Then the scheme-theoretic closure of Z in  $\operatorname{Spec}(A)$  is  $\overline{Z} := \operatorname{Spec}(A/\mathfrak{a})$ . It is easy to check that for all  $i, \mathfrak{a}_{f_i} = \mathfrak{a}_i$ . Indeed, consider  $\frac{x}{f_i^k} \in \mathfrak{a}_i$ . Since  $\frac{xf_j^k}{(f_if_j)^k} \in \mathfrak{a}_j A_{f_if_j}$  by hypothesis, for all j there is some  $k_j$  such that  $xf_i^{k_j} \in \overline{\mathfrak{a}}_j$ . Then if  $m := \max\{k_1, \ldots, k_n\}$  we have that  $xf_i^m \in \mathfrak{a}$ , and so  $\mathfrak{a}_{f_i} = \mathfrak{a}_i$ .

#### **3.3** Associated points

Next we come to the definition of the "most important points of a scheme" (Vakil's words).

**Definition 3.14.** Let R be a ring, and let M be a R-module. A prime ideal  $\mathfrak{p} \triangleleft R$  is said to be associated to M if there is some  $m \in M$  such that  $\mathfrak{p} = \operatorname{Ann}(m)$ , i.e.  $\mathfrak{p} = \{r \in R \mid rm = 0\}$ . If X is a scheme, any  $x \in X$  is said to be an associated point of X if the maximal ideal  $\mathfrak{m}_x$  in  $\mathcal{O}_{X,x}$  is associated to  $\mathcal{O}_{X,x}$ , considered as a module over itself.

The typical example of associated points of a scheme are the generic points of the irreducible components of a locally Noetherian scheme. Indeed, let  $\eta$  be the generic point of an irreducible component Y of X. If Spec(A) is an open affine neighbourhood of  $\eta$ , then  $Y \cap \text{Spec}(A)$  is defined by a prime  $\mathfrak{p} \triangleleft A$ , minimal with respect to inclusion. Then  $\eta = (0) \in \text{Spec}(A/\mathfrak{p})$ , and  $\mathcal{O}_{X,\eta} = A_\mathfrak{p}$  is a local ring of Krull dimension 0, i.e. the maximal ideal  $\mathfrak{m}_\eta$  is the only prime ideal of  $\mathcal{O}_{X,\eta}$ . But this means that  $\mathfrak{m}_\eta$  is the nilradical of the ring. As we will see later, for a Noetherian ring this is enough to conclude that  $\mathfrak{m}_\eta$  is an associated prime of  $\mathcal{O}_{X,\eta}$ .

**Lemma 3.15.** When X = Spec(A) is a Noetherian affine scheme, a point  $\mathfrak{p} \in X$  is associated precisely when  $\mathfrak{p}$  is an associated prime of A.

*Proof.* It is easy to see that if  $\mathfrak{p} = \operatorname{Ann}(x)$  is prime then  $\mathfrak{p}A_{\mathfrak{p}} = \operatorname{Ann}([x, 1])$ . Conversely, assume that  $\mathfrak{p} \triangleleft A$  is a prime ideal such that  $\mathfrak{p}A_{\mathfrak{p}} = \operatorname{Ann}([x, 1])$  for some  $x \in A$ . Since A is Noetherian, there are  $y_1, \ldots, y_r \in A$  such that  $\mathfrak{p} = (y_1, \ldots, y_r)$ . Moreover, for every  $i = 1, \ldots, r$  there is some  $g_i \in A \setminus \mathfrak{p}$  such that  $y_i g_i x = 0$ . Consider then  $\operatorname{Ann}(xg_1 \ldots g_r)$ ; of course  $\mathfrak{p} \subseteq \operatorname{Ann}(xg_1 \ldots g_r)$ , and if  $y \in \operatorname{Ann}(xg_1 \ldots g_r)$  then  $[y, 1] \in \operatorname{Ann}([x, 1])$  since each of the  $g_i$  is invertible in  $A_{\mathfrak{p}}$ . But then  $[y, 1] \in \mathfrak{p}A_{\mathfrak{p}}$ , so  $y \in \mathfrak{p}$ .  $\Box$ 

*Remark.* Actually the implication

 $\mathfrak{p} \triangleleft A$  is an associated prime  $\Rightarrow \mathfrak{p} \in \operatorname{Spec}(A)$  is an associated point

holds for any ring; the converse, however, holds just for Noetherian rings; see Example 4.2 below. This is one of the reasons why we will talk about associated points just for locally Noetherian schemes. Another reason for considering associated points just for Noetherian schemes is given by the following lemma.

**Lemma 3.16.** Let R be a Noetherian local ring with maximal ideal  $\mathfrak{m}$ . Then  $\mathfrak{m}$  is an associated prime of R if and only if each element of  $\mathfrak{m}$  is a zero divisor.

*Proof.* This is an immediate consequence of Lemma 3.4, Lemma 3.5 and Lemma 3.6.

From now on we will actually just be interested in prime ideals  $\mathfrak{p} \triangleleft R$  associated to R, so we will just say "let  $\mathfrak{p} \triangleleft R$  be an associated prime".

*Remark.* Lemma 3.16 does not hold for local rings that are not Noetherian (c.f. [Har77, Definition at page 257]), see Examples 4.3 and 4.4 below.

The next lemma tells us that the scheme-theoretic closure of a locally closed subscheme does not add any extra nonreduced structure to the original scheme.

**Lemma 3.17.** Let X be a locally Noetherian scheme, and consider a locally closed subscheme  $Z \to X$ . Then the associated points of Z are precisely the associated points of the scheme-theoretic closure of Z in X.

*Proof.* Let  $U \subseteq X$  be an open subscheme such that  $Z \subset U$  is a closed subscheme, and let Z be the scheme-theoretic closure of Z in X. Of course any associated point of Z is also an associated point of  $\overline{Z}$ , so we just have to show that the associated points of  $\overline{Z}$  come from associated points of Z.

The question we have to study is local, so we may assume that  $\overline{Z} = \text{Spec}(A)$  is affine, and that  $Z = \bigcup_{i=1}^{n} D(f_i)$  is an open subscheme of  $\overline{Z}$  that is scheme-theoretic dense in  $\overline{Z}$ . This happens precisely when the  $f_i$  are such that for any  $a \in A$ , if for all i = 1, ..., n there is  $m_i \in \mathbb{Z}_{\geq 0}$  such that  $f_i^{m_i} a = 0$ , then a = 0.

Let  $\mathfrak{q} \triangleleft A$  be an associated point of  $\overline{Z}$ ; then there is some  $a \in A$  such that  $\mathfrak{q} = \operatorname{Ann}(a)$ . Since the closure of Z is  $\overline{Z}$ , at least one of the  $f_i$  cannot be in  $\mathfrak{q}$ ; indeed, if  $f_1, \ldots, f_n \in \mathfrak{q}$  then  $f_i a = 0$  for all i, implying that a = 0. But  $\operatorname{Ann}(0) = A$ . This means that  $\mathfrak{q} \in D(f_i)$  for some i, i.e.  $\mathfrak{q} \in Z$ .

## 4 More examples

**Example 4.1** (Computation of an affine closure). Let X be the crossing of two affine lines,  $X = \operatorname{Spec}(A)$  with  $A = \Bbbk[x, y]/(xy)$ , and consider the scheme  $Z = \operatorname{Spec}(\Bbbk[x, x^{-1}])$ . The map  $\Bbbk[x, y]/(xy) \to \Bbbk[x, x^{-1}]$  defined as  $f(x, y) + (xy) \mapsto f(x, 0)$  induces on Z a structure of open subscheme of X, since it is an isomorphism between Z and  $D(x + (xy)) \subset X$ . What is the closure of Z in X?

We just have to compute the ideal  $\mathfrak{a} \triangleleft A$  of all the elements  $\alpha \in A$  that are sent to zero by the map  $A \rightarrow A_{x+(xy)}$ . For all  $\alpha \in A$ ,  $[\alpha, 1] = [0, 1]$  in  $A_{x+(xy)}$  if and only if there is some n such that  $(x + (xy))^n \alpha = 0$ . This happens precisely when  $\alpha \in (y + (xy)) \triangleleft A$ , so  $\mathfrak{a} = (y + (xy))$ . Then the closure of Z in X is  $\operatorname{Spec}(A/(y + (xy))) = \operatorname{Spec}(\Bbbk[x])$ ; this is just the affine line.

**Example 4.2** (Failure of Lemma 3.15 for non-Noetherian rings). Consider the ideal

$$I = (x_i y_i \mid i \in \mathbb{Z}_{\geq 0}) \lhd \Bbbk[x_1, y_1, x_2, y_2, x_3, \dots].$$

Then  $A := \Bbbk[x_1, y_1, x_2, y_2, \dots]/I$  is not a Noetherian ring, and in this ring we can see that Lemma 3.15 fails. Indeed, consider the prime ideal  $\mathfrak{p} = (\bar{x}_1, \bar{x}_2, \bar{x}_3, \dots)$ , where for any  $a \in \Bbbk[x_1, y_1, \dots]$  we denote by  $\bar{a}$  its image in A under the canonical projection. Then  $A_{\mathfrak{p}}$  is a field, and  $\mathfrak{p}A_{\mathfrak{p}} = 0$ ; in particular,  $\mathfrak{p}A_{\mathfrak{p}} = \operatorname{Ann}(\alpha)$  for any  $\alpha \in A_{\mathfrak{p}} \setminus \{0\}$ , while  $\mathfrak{p}$  is not an associated prime of A. Indeed, assume that  $m \in A$  is such that  $\mathfrak{p} \subseteq \operatorname{Ann}(m)$ , and let r be such that  $m \in (\bar{x}_1, \bar{y}_1, \dots, \bar{x}_r, \bar{y}_r)$ . Then it is easy to check that  $\bar{x}_{r+1}m = 0$  if and only if m = 0.

**Example 4.3** (Failure of Lemma 3.16 for non-Noetherian rings). Let  $\Bbbk$  be an algebraically closed field, let I be the ideal  $(x_0^2, x_0 - x_1^2, x_1 - x_2^2, x_2 - x_3^2, ...) \triangleleft \Bbbk[x_0, x_1, x_2, x_3, ...]$  and consider the ring

$$R = \frac{\mathbb{k}[x_0, x_1, x_2, x_3, \dots]}{I}.$$

If we let  $y_0 := x_i + I$  for  $i \ge 0$  then we have that  $y_0^2 = 0$ ,  $y_{k+1}^2 = y_k$  and  $y_k^{2^k} = y_0$  for all  $k \in \mathbb{Z}_{\ge 0}$ . The ring R satisfies the following properties:

- 1. R is not Noetherian;
- 2. R is a local ring of finite (Krull) dimension;
- 3. if  $\mathfrak{m}$  is the maximal ideal of R, then every element of  $\mathfrak{m}$  is a zero divisor, but  $\mathfrak{m}$  is not an associated prime of R.

1. The increasing sequence of ideals  $(y_0) \subset (y_1) \subset (y_2) \subset \ldots$  does not stabilize. Indeed, if this were not the case there would be some  $f \in R$  and some  $k \in \mathbb{Z}_{>0}$  such that  $fy_k = y_{k+1}$ . But  $(fy_k)^{(2^{k+1})} = 0$ , while  $y_{k+1}^{(2^{k+1})} = y_0 \neq 0$ .

2. Let  $\mathfrak{m}$  be the ideal of R generated by  $y_0, y_1, y_2, \ldots$ . Each element of  $\mathfrak{m}$  is nilpotent, and all the elements in  $R \setminus \mathfrak{m}$  are invertible. Then R is indeed a local ring, with maximal ideal  $\mathfrak{m}$ . From the fact that each element of  $\mathfrak{m}$  is nilpotent it follows readily that if  $\mathfrak{p} \subseteq \mathfrak{m}$  is a prime ideal of  $R, \mathfrak{p}$  must equal  $\mathfrak{m}$ .

3. Suppose that  $x \in R$  is such that mx = 0 for all  $m \in \mathfrak{m}$ . Since  $x \in \mathfrak{m}$ , we can write it as

$$x = x_0 y_k^{a_0} + \dots + x_n y_k^{a_n}$$

for some k, with  $1 \le a_0 < \cdots < a_n \le 2^k$  and  $x_0, \ldots, x_n \in \mathbb{k}$ . Thus multiplying x by  $y_k^{(2^k - a_0)}$  we obtain

$$0 = x y_k^{(2^k - a_0)} = x_0 \varepsilon$$

so  $x_0 = 0$ . Then we may multiply x by  $y_k^{(2^k - a_1)}$  to find  $x_1 = 0$ , and recursively we get  $x_j = 0$  for all j, i.e. x = 0.



Figure 3: Two different schemes obtained from Example 4.5.

**Example 4.4** (Failure of Lemma 3.16 for non-Noetherian rings). A very similar reasoning can be carried out with the ring

$$\frac{\Bbbk[x_1, x_2, x_3, \dots]}{(x_1^2, x_2^2, x_3^2, \dots)}.$$

**Example 4.5** (A flat morphism). Let k be an algebraically closed field, and let  $f(x, y) \in k[x, y]$ . Consider the ideal  $I = (z^2 - f(x, y)) \triangleleft k[x, y, z]$ , the scheme X = Spec(k[x, y, z]/I) and the map  $X \rightarrow \mathbb{A}^2_{\mathbb{k}}$  induced by the inclusion  $k[x, y] \rightarrow k[x, y, z]/I$ . Then X is flat over  $\mathbb{A}^2_{\mathbb{k}}$ ; indeed, if we write R := k[x, y], the *R*-module k[x, y, z]/I is freely generated by 1 and z, so that  $k[x, y, z]/I \cong R \oplus R$  as a *R*-module. For a couple of particular cases, see Figure 3.

**Example 4.6** (A morphism that is not flat). Let  $\Bbbk$  be an algebraically closed field, and consider the affine scheme  $X = \operatorname{Spec}(\Bbbk[x, y, z]/(xz - y))$  together with the map  $f : X \to \mathbb{A}^2_{\Bbbk}$  defined by the inclusion  $f^{\sharp} : \Bbbk[z, y] \to \Bbbk[x, y, z]/(xz - y)$ . See also Figure 4.

Then X is irreducible and of dimension 2, so by Corollary 1.29 the fibre over each  $p \in \mathbb{A}^2_{\mathbb{k}}$  should be 0-dimensional. But the fibre over  $p = (x, y) \in \mathbb{A}^2_{\mathbb{k}}$  is a whole copy of  $\mathbb{k}[z]$ , hence f cannot be flat.

**Example 4.7** (Another flat family of cubics). Consider the twisted cubic C given by the 3-uple immersion of  $\mathbb{P}^1_{\Bbbk}$  in  $\mathbb{P}^3_{\Bbbk}$ ,

$$\mathbb{P}^1_{\Bbbk} \to \mathbb{P}^3_{\Bbbk}$$
$$[s:u] \mapsto [s^3: s^2u: su^2: u^3].$$

Then C is defined in  $\mathbb{P}^3_\Bbbk$  by the homogeneous ideal

$$I_h = (x_0 x_3 - x_1 x_2, x_1 x_3 - x_2^2, x_0 x_2 - x_1^2).$$

Consider the family of curves  $C = \{C_a \mid a \neq 0\}$  described by the family of homogeneous ideals

$$I_h(a) = (x_0 x_3 - \frac{1}{a} x_1 x_2, x_1 x_3 - \frac{1}{a^2} x_2^2, \frac{1}{a} x_0 x_2 - x_1^2)$$

Each member of the family is a closed subscheme of the fibre, over the closed point  $(t-a) \in \text{Spec}(\Bbbk[t])$ , of the natural morphism of schemes

$$\mathbb{P}^{3}_{\Bbbk} \times (\mathbb{A}^{1}_{\Bbbk} \setminus \{(t)\}) = \mathbb{P}^{3}_{\Bbbk} \times \operatorname{Spec}(\Bbbk[t, t^{-1}]) \longrightarrow \mathbb{A}^{1}_{\Bbbk} = \operatorname{Spec}(\Bbbk[t]).$$



Figure 4: The scheme of Example 4.6.

According to the theory developed for Corollary 1.18, this family admits a unique flat limit in  $\mathbb{P}^3_{\mathbb{k}} \times \mathbb{A}^1_{\mathbb{k}}$ ; to compute it, we consider the restriction of the family to the open affine subset  $U_3 = \{x_3 \neq 0\}$  of  $\mathbb{P}^3_{\mathbb{k}}$ . Setting  $x = \frac{x_0}{x_3}$ ,  $y = \frac{x_1}{x_3}$  and  $z = \frac{x_2}{x_3}$ , the curve  $C_a$  is described in  $U_3$  by the ideal

$$I(a) = \left(x - \frac{yz}{a}, y - \frac{z^2}{a^2}, \frac{xz}{a} - y^2\right)$$

i.e. the family of curves is described by (see also Figure 5):

$$\mathcal{C} \cap U_3 = \operatorname{Spec}\left(\frac{\mathbb{K}[x, y, z, t]_t}{\left(x - \frac{yz}{t}, y - \frac{z^2}{t^2}, \frac{xz}{t} - y^2\right)}\right)$$

We want to find the ideal  $J \lhd \Bbbk[x, y, z, t]$  such that

$$J = \left(x - \frac{yz}{t}, y - \frac{z^2}{a^2}, \frac{xz}{t} - y^2\right) \cap \mathbb{k}[x, y, z, t]$$

in other words, J is the saturation with respect to t of the ideal

$$J' = (tx - yz, t^2y - z^2, xz - ty^2) \lhd \mathbb{k}[x, y, z, t].$$

To compute this saturation we use the method of Gröbner bases: using the program Mathematica we find a Gröbner bases for the saturation of J':

$$J=(t^2y-z^2,tx-yz,xz-ty^2,x^2-y^3)\lhd \Bbbk[x,y,z,t].$$

Hence

$$\overline{\mathcal{C}} \cap U_3 = \operatorname{Spec}\left(\frac{\Bbbk[x, y, z, t]}{(t^2y - z^2, tx - yz, xz - ty^2, x^2 - y^3)}\right)$$

and the fibre  $C_0$  over the closed point  $0 = (t) \in \text{Spec}(\Bbbk[a])$  is

Spec 
$$\left(\frac{\mathbb{k}[x,y,z]}{(z^2,yz,xz,x^2-y^3)}\right)$$
.

We see that this fibre is the cuspidal cubic in the plane z = 0, with some additional structure: indeed, the point x = y = 0 has some extra nonreduced structure, coming from the presence of  $z^2$  in the defining ideal of  $C_0$ .



Figure 5: The family of cubics of Example 4.7.

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