Mathematics Department
The University of Georgia
Math 8150 Homework Assignment 3

Due during lecture on 3/6/2020. Late homework will not be accepted

Complex Analysis, by Elias M. Stein and Rami Shakarchi,
3.8: 1, 2, 4, 5, 6, 7, 8, 9,10, 14, 15, 17, 19.

1. Prove that if
$$\sum_{n=-\infty}^{\infty} c_n(z-a)^n \quad \text{and} \quad \sum_{n=-\infty}^{\infty} c_n'(z-a)^n$$
are Laurent series expansions of $f(z)$, then $c_n = c_n'$ for all n.

2. Expand $\dfrac{1}{1-z^2} + \dfrac{1}{3-z}$ in a series of the form $\sum_{-\infty}^{\infty} a_n z^n$. How many such expansions are there? In which domain is each of them valid?

3. Let $P(z)$ and $Q(z)$ be polynomials with no common zeros. Assume $Q(a) = 0$. Find the principal part of $P(z)/Q(z)$ at $z = a$ if the zero a is (i) simple; (ii) double. Express your answers explicitly using P and Q.

4. Let $f(z)$ be a non-constant analytic function in $|z| > 0$ such that $f(z_n) = 0$ for infinite many points z_n with $\lim_{n\to\infty} z_n = 0$. Show that $z = 0$ is an essential singularity for $f(z)$. (An example of such a function is $f(z) = \sin(1/z)$.)

5. Let f be entire and suppose that $\lim_{z\to\infty} f(z) = \infty$. Show that f is a polynomial.

6. (1) Show without using 3.8.9 in the textbook by Stein and Shakarchi that
$$\int_0^{2\pi} \log|1 - e^{i\theta}| d\theta = 0.$$
(2) Show the above identity is equivalent to the one in 3.8.9 of the textbook.

7. Evaluate $\displaystyle\int_0^{\infty} \frac{x^{a-1}}{1+x^3} dx$, $0 < a < 4$.

8. (1) Prove the fundamental theorem of algebra using Rouché's theorem.

   (2) Prove the fundamental theorem of algebra using the maximum modulus principle.

9. Assume $f(z)$ is analytic in region D and $gamma$ is a rectifiable curve in D with interior in D. Prove that if $f(z)$ is real for all $z \in \Gamma$, then $f(z)$ is a constant.

10 Evaluate $\displaystyle\int_0^{\frac{\pi}{2}} \frac{d\theta}{a + \sin^2\theta}$, $a > 0$.

11. Find the number of roots of $z^4 - 6z + 3 = 0$ in $|z| < 1$ and $1 < |z| < 2$ respectively.

12. Prove that $z^4 + 2z^3 - 2z + 10 = 0$ has exactly one root in each open quadrant.

13. Prove that the equation $z \tan z = a, \;\; a > 0$, has only real roots in $\mathbb{C}$.

14. Let f be analytic on a bounded region Ω and continuous on the closure $\overline{\Omega}$. Assume $f(z) \neq 0$. Show that $f(z) = e^{i\theta}M$ (where θ is a real constant) if $|f(z)| = M$ (a constant) for $z \in \partial\Omega$.