Mathematics Department
The University of Georgia
Math 8150 Homework Assignment 4

Due during lecture on 4/27/2020. Late homework will not be accepted

Complex Analysis, by Elias M. Stein and Rami Shakarchi,
3.9: 2, 3.
4.4: 1, 2, 3.
8.5: 1, 2, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17,
Additional Problems

1. Assume a real valued continuous function $u(x, y)$ has mean value property (MVP) in a region Ω. That is, for each a in Ω there exists $r_0 > 0$ such that

$$u(a) = \frac{1}{2\pi} \int_0^{2\pi} u(a + re^{it})\, dt \ \text{ for } \ 0 < r \leq r_0\,.$$

This exercise guides to a proof of the fact that MVP implies harmonicity using Poisson formula for the solution of the Dirichlet problem in any disc given in class.

(1) Assume that $\overline{\Omega}_1$ is a compact region in Ω. Prove that u cannot attain maximum (resp. minimum) value in in the interior Ω_1 unless it is a constant on Ω_1. (Avoid circular reasoning: you cannot use the equivalence of harmonicity and MVP and the maximum/minmum modulus principle for harmonic functions.)

(2) Let $\overline{D}_r(a) \in \Omega$ be a disk in Ω. Let $g(z) = u(x, y)$ for $z \in \partial D_r(a)$ and let $U(z)$ be the solution of the Dirichlet problem in $\overline{D}_r(a)$ with boundary values $g(z)$. Show that $u(z) - U(z) = 0$ in $\overline{D}_r(a)$ and conclude that u is harmonic in Ω.

[Note: The same method idea in (1) also yields the uniqueness of the solution of the Dirichlet problem in a disc.]

2. (1) Let $f(z) \in H(\mathbb{D})$, $\mathrm{Re}(f(z)) > 0$, $f(0) = a > 0$. Show that

$$\left|\frac{f(z) - a}{f(z) + a}\right| \leq |z|, \quad |f'(0)| \leq 2a.$$

(2) Show that the above is still true if $\mathrm{Re}(f(z)) > 0$ is replaced with $\mathrm{Re}(f(z)) \geq 0$.

3. Assume $f(z)$ is analytic in $\mathbb{D}$ and $f(0) = 0$ and is not a rotation (i.e. $f(z) \neq e^{i\theta}z$). Show that $\sum_{n=1}^{\infty} f^n(z)$ converges uniformly to an analytic function on compact subsets of $\mathbb{D}$, where $f^{n+1}(z) = f(f^n(z))$.

4. Let $\Omega = \{z : \ |z - 1| < \sqrt{2}, \ |z + 1| < \sqrt{2}\}$. Find a bijective conformal map from Ω to the upper half plane $\mathbb{H}$.

5. Find the fractional linear transformation that maps the circle $|z| = 2$ into $|z+1| = 1$, the point -2 into the origin, and the origin into i.

6. Let $\Omega = \mathbb{D} \backslash (-1, -1/2]$. Find a bijective conformal map from Ω to the unit disk $\mathbb{D}$. How do you find the most general form of all such maps (you don't have to explicitly describe the general form, just explain the strategy for obtaining it)?

7. Let $\Omega = \mathbb{C} \backslash [0, \infty)$. Is there an analytic isomorphism from Ω to $\mathbb{C}$? If yes, exhibit one such isomorphism. If no, explain why.

8. (1) Show that if f is analytic in an open set containing the disc $|z - a| \leq R$, then

$$|f(a)|^2 \leq \frac{1}{\pi R^2} \int_0^{2\pi} \int_0^R |f(a + re^{i\theta})|^2 r dr d\theta$$

(2) Let Ω be a region and $M > 0$ a fixed positive constant. Let $\mathcal{F}$ be the family of all analytic functions f on Ω such that $\iint\limits_{\Omega} |f(z)|^2 dxdy \leq M$. Show that $\mathcal{F}$ is a normal family.