Mathematics Department
The University of Georgia
Math 8150 Homework Assignment 1

Due during lecture on 1/24/2020. Late homework will not be accepted

1. Describe geometrically the sets of points z in the complex plane defined by the following relations:

   (a) $|z-1| = 1$. (b) $|z-1| = 2|z-2|$. (c) $1/z = \bar{z}$.

   (d) $\mathrm{Re}(z) = 3$ (e) $\mathrm{Im}(z) = a$ with $a \in \mathbb{R}$. (f) $\mathrm{Re}(z) > a$ with $a \in \mathbb{R}$.

   (g) $|z-1| < 2|z-2|$.

2. Prove that $|z_1 + z_2| \geq ||z_1| - |z_2||$ and explain when equality holds.

3. Prove that the equation $z^3 + 2z + 4 = 0$ has its roots outside the unit circle. [Hint: what is the maximum value of the modulus of the first two terms if $|z| \leq 1$?]

4. (a) Prove that if $|w_1| = c|w_2|$ where $c > 0$, then $|w_1 - c^2 w_2| = c|w_1 - w_2|$.

   (b) Prove that if $c > 0$, $c \neq 1$ and $z_1 \neq z_2$, then $|\dfrac{z - z_1}{z - z_2}| = c$ represents a circle. Find its center and radius. [Hint: an easy way is to use part (a)]

5. (a) Let z, w be complex numbers, such that $\bar{z}w \neq 1$. Prove that

$$|\frac{w-z}{1-\bar{w}z}| < 1 \quad \text{if } |z| < 1 \text{ and } |w| < 1,$$

   and also that

$$|\frac{w-z}{1-\bar{w}z}| = 1 \quad \text{if } |z| = 1 \text{ or } |w| = 1.$$

   (b) Prove that for fixed w in the unit disk $\mathbb{D}$, the mapping

$$F : z \mapsto \frac{w-z}{1-\bar{w}z}$$

   satisfies the following conditions:

   (i) F maps $\mathbb{D}$ to itself and is holomorphic.

   (ii) F interchanges 0 and w, namely, $F(0) = w$ and $F(w) = 0$.

   (iii) $|F(z)| = 1$ if $|z| = 1$.

   (iv) $F : \mathbb{D} \mapsto \mathbb{D}$ is bijective. [Hint: Calculate $F \circ F$.]

6. Use n-th roots of unity (i.e. solutions of $z^n - 1 = 0$) to show that

$$2^{n-1} \sin\frac{\pi}{n} \sin\frac{2\pi}{n} \cdots \sin\frac{(n-1)\pi}{n} = n\ .$$

   [Hint: $1 - \cos 2\theta = 2\sin^2\theta,\ \sin 2\theta = 2\sin\theta\cos\theta$.]

7. Prove that $f(z) = |z|^2$ has a derivative only at $z = 0$, but nowhere else.

8. Let $f(z)$ be analytic in a domain. Prove that $f(z)$ is a constant if it satisfies any of the following conditions:

   (a) $|f(z)|$ is constant;

   (b) $\mathrm{Re}(f(z))$ is constant;

   (c) $\arg(f(z))$ is constant;

   (d) $\overline{f(z)}$ is analytic;

   How do you generalize (a) and (b)?

9. Let $f(z)$ be analytic. Show that $\overline{f(\bar{z})}$ is also analytic.

10. (a) Show that in polar coordinates, the Cauchy-Riemann equations take the form

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad \text{and} \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

    (b) Use these equations to show that the logarithm function defined by

$$\log z = \log r + i\theta \ \text{ where } z = re^{i\theta} \text{ with } -\pi < \theta < \pi$$

    is a holomorphic function in the region $r > 0$, $-\pi < \theta < \pi$. Also show that $\log z$ defined above is not continuous in $r > 0$.

11. Prove that the distinct complex numbers z_1, z_2 and z_3 are the vertices of an equilateral triangel if and only if

$$z_1^2 + z_2^2 + z_3^2 = z_1 z_2 + z_2 z_3 + z_3 z_1.$$